Profiles of random trees: plane-oriented recursive trees (Extended Abstract)†

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We summarize several limit results for the profile of random plane-oriented recursive trees. These include the limit distribution of the normalized profile, asymptotic bimodality of the variance, asymptotic approximations of the expected width and the correlation coefficients of two level sizes. We also unveil an unexpected connection between the profile of plane-oriented recursive trees (with logarithmic height) and that of random binary trees (with height proportional to the square root of tree size).

Keywords: Plane-oriented recursive trees, profile of trees, limit distribution, convergence of all moments, total path length, random binary trees

1 Introduction

Plane-oriented recursive trees, abbreviated as PORTs throughout this extended abstract, were introduced in the literature under a few different names such as heap-ordered trees ([4, 17]), nonuniform recursive trees ([20]), scale-free trees ([3, 19]), and have been widely addressed recently due most notably to the stimulating paper [1] by Barabási and Albert on network models. We give without proof in this extended abstract the major phenomena exhibited by the profile of random PORTs, following our recent papers [8, 9, 12]. While bearing many similarities to the profiles of random recursive trees and random binary search trees, the profile of random PORTs gives rise to several different behaviors, as highlighted by the lack of a fixed-point equation for the limit distribution of the normalized profile and its special connection to profile of random binary trees.

PORTs. PORTs are labelled ordered (or plane) trees with the property that labels along any path down from the root are increasing. Such a characterization first appeared in [18] by Prodinger and Urbanek. The total number T_n of such trees of n nodes is given by the $(2n-2)$-nd moment of the standard normal distribution

$$T_n = (2n-3)!! = 1 \cdot 3 \cdots (2n-3) = n!2^{1-n}C_n,$$

where $C_n = \binom{2n-2}{n-1}/n$ denotes the Catalan numbers. By random PORTs, we assume that all PORTs of n nodes are equally likely.

An alternative construction of random PORTs, first given in [20] by Szymański in a more general setting, is as follows. We begin by a tree with one root node labelled 1 and then insert the labels $\{2, \dots, n\}$ successively such that the $(i+1)$-st node (with label $i+1$) is attached to an existing node with d children with probability $(d+1)/(2i-1)$. Note that random recursive trees are constructed similarly but each existing node is chosen with equal probability.

Profile of PORTs. Let $X_{n,k}$ denote the number of nodes at level k (the root being at level 0) in a random PORT of n nodes. By definition and by conditioning on the size of the first subtree, we have the recurrence for $X_{n,k}$

$$X_{n,k} \stackrel{d}{=} X_{J_n,k-1} + X^*_{n-J_n,k},$$

† All proofs of the results in this extended abstract and more detailed discussions are given in the full version of this paper.
‡ Partially supported by National Science Council under the Grant NSC-93-2118-M-001-007.

with $X_{n,0} = 1$ for $n \geq 1$, where the $X^*_{n,k}$ are independent copies of $X_{n,k}$ and

$$\mathbb{P}(J_n = j) = \pi_{n,j} := \frac{2(n-j)C_jC_{n-j}}{nC_n} \qquad (1 \leq j < n).$$

Note that, by the estimate

$$C_n \sim \pi^{-1/2} n^{-3/2} 4^{n-1}, \tag{1}$$

we see that $J_n \xrightarrow{d} J$, where $\mathbb{P}(J = j) = 2C_j/4^j$. It also follows from (1) that J_n *converges in distribution to* J *but without convergence of any integral moment.*

Expected profile. Let $\mu_{n,k} := \mathbb{E}(X_{n,k})$. It is known that (see [2, 17])

$$\sum_{n,k} 4^{-n} C_n \mu_{n,k} u^k z^n = \frac{1}{4}(1-z)^{-u/2} \int_0^z (1-t)^{(u-1)/2}\, \mathrm{d}t,$$

from which we deduce, by singularity analysis (see [11]) and the saddle-point method used in [13], that

$$\mu_{n,k} = \frac{\sqrt{\pi n}}{(1+2\alpha_{n,k})\Gamma(1+\alpha_{n,k})} \cdot \frac{(\frac{1}{2}\log n)^{k-1}}{(k-1)!} \left(1 + O((\log n)^{-1})\right), \tag{2}$$

uniformly for $1 \leq k = O(\log n)$, where, *here and throughout this paper*, Γ is the Gamma function and $\alpha_{n,k} := k/\log n$. See [3, 19] for crude estimates for $\mu_{n,k}$.

Rough descriptions of the shapes of random PORTs based on (2). From (2), we see first that $\mu_{n,k} \to \infty$ when

$$k \leq \alpha_+ \log n - \frac{\alpha_+}{2\alpha_+ + 1} \log\log n - \omega_n,$$

where $\alpha_+ \approx 1.79556$ solves the equation $\frac{1}{2} + z - z\log(2z) = 0$ and ω_n is any sequence tending to infinity. Note that α_+ is the leading constant for the expected height derived in [16].

Secondly, the root has about $\sqrt{\pi n}$ subtrees, which is to be compared with $\log n$ for random recursive trees and 2 for random binary search trees; see [2, 12]. The result (2) also says that except for the root each node roughly attracts about $\frac{1}{2}\log n$ new nodes (up to order of subtrees).

Finally, most nodes in a random PORT lie at the levels $k = \frac{1}{2}\log n + O(\sqrt{\log n})$, each of these levels having roughly $n/\sqrt{\log n}$ nodes.

For other results for random PORTs, see the full version of the paper and the references therein.

Limit distribution. Let $\alpha := \lim_{n\to\infty} k/\log n$ if the limit exists. Our first result states that $X_{n,k}/\mu_{n,k}$ converges in distribution to some law when $\alpha \in [0, \frac{1}{2}]$.

Theorem 1. *If* $\alpha \in [0, \frac{1}{2}]$, *then*

$$\frac{X_{n,k}}{\mu_{n,k}} \xrightarrow{d} X(\alpha), \tag{3}$$

with convergence of all moments, where $X(\alpha)$ *is uniquely characterized by its moment sequence* $\xi_m(\alpha) := \mathbb{E}(X(\alpha)^m)$, *which satisfies the recurrence* ($\bar{\alpha} := \alpha + \frac{1}{2}$)

$$\xi_m(\alpha) = \frac{1}{\sqrt{\pi}(2m\bar{\alpha} - (2\alpha)^m - 1)} \sum_{1\leq \ell < m} \binom{m}{\ell} \xi_\ell(\alpha)\xi_{m-\ell}(\alpha)(2\alpha)^\ell \frac{\Gamma(\ell\bar{\alpha} - \frac{1}{2})\Gamma((m-\ell)\bar{\alpha} + \frac{1}{2})}{\Gamma(m\bar{\alpha} - \frac{1}{2})}, \tag{4}$$

for $m \geq 2$ *with* $\xi_1(\alpha) = 1$.

The range $[0, \frac{1}{2}]$ is the best possible for convergence of all moments because for $\alpha > \frac{1}{2}$ only convergence of finite moments (depending on α) holds. Indeed, let ζ_m denote the positive real zero of the polynomial $2m(z + \frac{1}{2}) - (2z)^m - 1$ for $m \geq 2$. Then $\mathbb{E}(X_{n,k}/\mu_{n,k})^j$ converges to $\xi_j(\alpha)$ for $j = 0, \ldots, m$ but not for $j \geq m+1$ when $\alpha \in [0, \zeta_m)$. Note that $X(\alpha)$ is well-defined when $\alpha \in [0, \frac{1}{2}]$ since the (infinite) moment sequence uniquely characterize its distribution. However, when $\alpha \in (\frac{1}{2}, \alpha_+)$, it is unclear how to properly define $X(\alpha)$ since only finite moments are available.

Unlike random recursive trees and binary search trees, no fixed-point equation is known for $X(\alpha)$. Thus the contraction method, as that used in [12], does not directly apply. This leaves open the (anticipated) convergence in distribution of $X_{n,k}/\mu_{n,k}$ for $\alpha \in (\frac{1}{2}, \alpha_+)$.

m	2	3	4	5	6
ζ_m	1.20711	1	0.89217	0.82531	0.77946
m	7	8	9	10	11
ζ_m	0.74589	0.72016	0.69975	0.68312	0.66929

Tab. 1: *Approximate numeric values of* ζ_m*, the positive zeros of the equation* $2m(z+\frac{1}{2})-(2z)^m=1$*, for* $m=2,\dots,11$.

Corollary 1. *If* $\alpha\in[0,\zeta_2)$*, then*

$$\mathbb{V}(X_{n,k})\sim\big(\xi_2(\alpha)-\xi_1(\alpha)^2\big)\left(\frac{\sqrt{\pi n}(\frac{1}{2}\log n)^{k-1}}{(1+2\alpha)\Gamma(1+\alpha)(k-1)!}\right)^2.$$

Note that

$$\begin{aligned}\xi_2(\alpha)-\xi_1(\alpha)^2&=\frac{4\Gamma(1+\alpha)^2}{\sqrt{\pi}(1+4\alpha-4\alpha^2)\Gamma(2\alpha+\frac{1}{2})}-1\\&\sim\left(6-\frac{\pi^2}{2}\right)\left(\alpha-\frac{1}{2}\right)^2\qquad\left(\alpha\sim\frac{1}{2}\right),\end{aligned}$$

so that $\mathbb{V}(X_{n,k})=o(\mu_{n,k}^2)$ when $\alpha=\frac{1}{2}$. See (6) for a more precise asymptotic approximation. On the other hand, $\xi_2(\alpha)-\xi_1(\alpha)^2\to4/\pi-1$ when $\alpha\to0^+$.

The case $\alpha=0$**.** When $\alpha=0$, the right-hand side of recurrence (4) is to be interpreted as the limit when $\alpha\to0^+$, so that it becomes $\xi_1(0)=1$ and for $m\ge2$

$$\xi_m(0)=\frac{m\Gamma(m/2)}{\sqrt{\pi}\Gamma((m+1)/2)}\xi_{m-1}(0),$$

which is solved to be

$$\xi_m(0)=\frac{m!}{\Gamma((m+1)/2)}\pi^{-(m-1)/2}=2^m\pi^{-m/2}\Gamma((m+2)/2)\qquad(m\ge1).$$

It follows that

$$\mathbb{E}(e^{\sqrt{\pi}X(0)y})=\sum_{m\ge0}\frac{\Gamma((m+2)/2)}{m!}(2y)^m=\frac{1}{2}\int_0^\infty te^{ty-t^2/4}\,\mathrm{d}t;$$

thus when $1\le k=o(\log n)$

$$\frac{X_{n,k}}{\sqrt{n}(\frac{1}{2}\log n)^{k-1}/(k-1)!}\xrightarrow{d}\sqrt{\pi}X(0),$$

which is a Rayleigh distribution with density function $te^{-t^2/4}/2$.

The middle range $\alpha=\frac{1}{2}$**.** When $\alpha=\frac{1}{2}$, all $\xi_m(\frac{1}{2})$'s are identically 1, so that $X(\frac{1}{2})=1$. We can refine the convergence in distribution (3) as follows.

Theorem 2. *If* $k=\frac{1}{2}\log n+s_{n,k}$*, where* $|s_{n,k}|\to\infty$ *and* $s_{n,k}=o(\log n)$*, then*

$$\frac{X_{n,k}-\mu_{n,k}}{s_{n,k}\sqrt{\pi n}(\frac{1}{2}\log n)^{k-1}/k!}\xrightarrow{d}Y,$$

where Y *is completely characterized by its moment sequence* $\eta_m:=\mathbb{E}(Y^m)$ *satisfying the recurrence*

$$\eta_m=\frac{\Gamma(m-1)}{2\sqrt{\pi}\Gamma(m-\frac{1}{2})}\sum_{\substack{a+b+c=m\\0\le a,b<m\\0\le c\le m}}\binom{m}{a,b,c}\eta_a\eta_b\int_0^1x^{a-3/2}(1-x)^{b-1/2}\varphi_1(x)^c\,\mathrm{d}x,$$

for $m\ge2$ *with* $\eta_0=1$ *and* $\eta_1=0$*. Here* $\varphi_1(x):=(x\log x+(1-x)\log(1-x)+2x)/(2\sqrt{\pi})$.

If $s_{n,k}=O(1)$*, then the sequence of random variables* $(X_{n,k}-\mu_{n,k})/\sqrt{\mathbb{V}(X_{n,k})}$ *does not converge to a fixed limit law.*

Let $Y_n := \sum_k kX_{n,k}$ denote the total path length of random PORTs.

Theorem 3. *The total path length Y_n satisfies*

$$\frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\pi}\,n} \xrightarrow{d} Y,$$

with convergence of all moments.

Thus the total path length has the same limit law as the profile $X_{n,k}$ when $k \sim \frac{1}{2}\log n$ and $|k - \frac{1}{2}\log n| \to \infty$. Convergence in distribution was given in [15] by a martingale approach but without characterization of Y; see also [17] for the first two moments.

To prove the second part of Theorem 2, the crucial step is to show that

$$\mathbb{E}\left(\frac{X_{n,k} - \mu_{n,k}}{s_{n,k}\sqrt{\pi n}(\frac{1}{2}\log n)^{k-1}/k!}\right)^m \sim p_m(s_{n,k}), \tag{5}$$

for $m \geq 2$, where $p_m(s)$ is a polynomial of degree m. Then the non-convergence follows from the arguments used in [5].

Corollary 2. *If $k = \frac{1}{2}\log n + s_{n,k}$, where $s_{n,k} = o(\log n)$, then*

$$\mathbb{V}(X_{n,k}) \sim \frac{p_2(s_{n,k})}{(\log n)^2}\left(\sqrt{n}\frac{(\frac{1}{2}\log n)^{k-1}}{(k-1)!}\right)^2, \tag{6}$$

where $p_2(s) = c_2 s^2 + 2c_1 s + c_0$, with

$$\begin{cases} c_2 = 6 - \dfrac{\pi^2}{2}, \\ c_1 = -c_2(2\log 2 - 1 + \gamma) + 8 - 7\zeta(3), \\ c_0 = -c_2\left((2\log 2 - 1 + \gamma)^2 - 6\right) - 2c_1(2\log 2 - 1 + \gamma) + 8 - \dfrac{\pi^4}{8}, \end{cases} \tag{7}$$

γ being the Euler constant and $\zeta(3) := \sum_{j\geq 1} j^{-3}$.

Since $p_2(s)$ is a quadratic polynomial with positive leading coefficient, *the variance exhibits asymptotically a bimodal behavior for large n and varying k with a valley at $k = \frac{1}{2}\log n + o(\log n)$*; see Figure 1 and cf. [8].

Covariance of two levels. Define

$$f(u,v) := \frac{16\sqrt{\pi}\,uv}{(1+2u)(1+2v)(1+2u+2v-4uv)\Gamma(u+v+1/2)} - \frac{4\pi}{(1+2u)(1+2v)\Gamma(u)\Gamma(v)},$$

and $p(s,t) := c_2 st + c_1(s+t) + c_0$, with the coefficients given by (7). Note that

$$c_2 = f''_{uv}\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad c_1 = -\tfrac{1}{2}f'''_{uv^2}\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad c_0 = \tfrac{1}{4}f^{(4)}_{u^2v^2}\left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

Also define

$$\begin{cases} c_3 := f'_v(\alpha, \frac{1}{2}) = -\frac{2\sqrt{\pi}(\psi(1+\alpha)-2\alpha+2\log 2-1+\gamma)}{(1+2\alpha)\Gamma(\alpha)}, \\ c_4 := -\frac{1}{2}f''_{v^2}(\alpha, \frac{1}{2}) = -\frac{\sqrt{\pi}((\psi(1+\alpha)-2\alpha)^2+(2\alpha-1)^2-(1-\gamma)^2+4\log(2)(1-\gamma-\log 2)-\psi'(1+\alpha)+\pi^2/2)}{(1+2\alpha)\Gamma(\alpha)}. \end{cases}$$

Let $k, h \geq 1$, $\beta_{n,h} := h/\log n$ and $\beta := \lim_n \beta_{n,h}$ if the limit exists.

Theorem 4. *If $\alpha, \beta \in [0, \zeta_2)$, then the correlation coefficient of $X_{n,k}$ and $X_{n,h}$ satisfies*

$$\rho(X_{n,k}, X_{n,h}) \sim \begin{cases} \dfrac{f(\alpha,\beta)}{\sqrt{f(\alpha,\alpha)f(\beta,\beta)}}, & \text{if } \alpha,\beta \neq \frac{1}{2}; \\ \dfrac{c_3 t_{n,h} + c_4}{\sqrt{f(\alpha,\alpha)p(t_{n,h},t_{n,h})}}, & \text{if } \alpha \neq \frac{1}{2}, \beta = \frac{1}{2}; \\ \dfrac{p(s_{n,k},t_{n,h})}{\sqrt{p(s_{n,k},s_{n,k})p(t_{n,h},t_{n,h})}}, & \text{if } \alpha = \beta = \frac{1}{2}, \end{cases} \tag{8}$$

where $s_{n,k} := k - \frac{1}{2}\log n$ and $t_{n,h} := h - \frac{1}{2}\log n$.

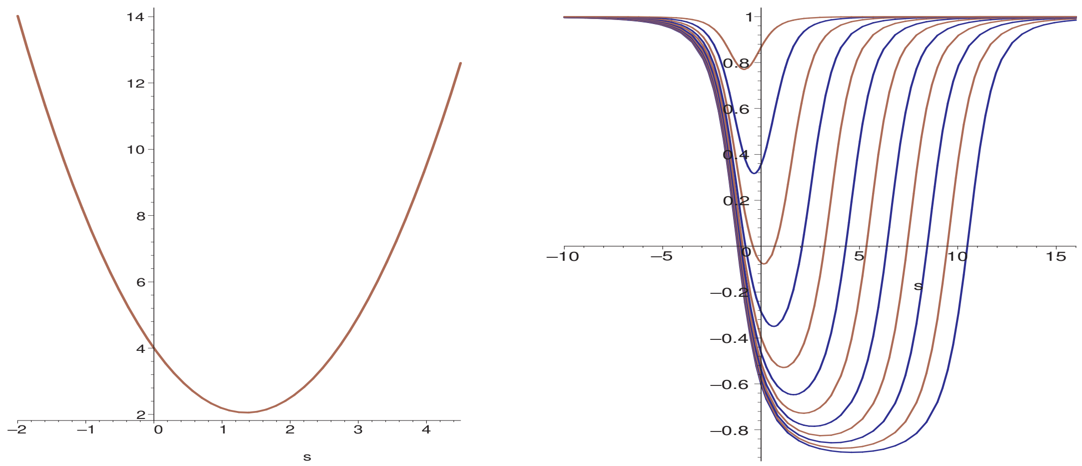


Fig. 1: *The polynomial $p_2(s)$ (left) and the asymptotic correlation coefficient (right) of levels in the range $\frac{1}{2}\log n + O(1)$: $p(s, s+\ell)/\sqrt{p(s,s)p(s+\ell, s+\ell)}$ for $\ell = 1, \dots, 12$ (in increasing order from top to bottom).*

Corollary 3. *The correlation coefficient of $Y_{n,k}$ and $Y_{n,h}$ is asymptotic to 1 (i) if $\alpha = \beta \neq \frac{1}{2}$ $(0 \le \alpha, \beta < \zeta_2)$; or (ii) if both $|s_{n,k}|, |t_{n,h}| \to \infty$ (not necessarily at the same rate) when $\alpha = \beta = \frac{1}{2}$.*

A salient feature of the profile is that the correlation coefficient of neighboring levels is asymptotic to 1 except when $k, h = \frac{1}{2}\log n + O(1)$ (we leave aside the correlation of levels whose distances to the root are $\ge (\zeta_2 - \varepsilon)\log n$ since there are relatively less nodes there). In particular,

$$\min_{s\in\mathbb{R}} \frac{p(s,s+1)}{\sqrt{p(s,s)p(s+1,s+1)}} = 1 - \frac{2(\pi^2-12)}{\pi^6+13\pi^4-664\pi^2+3344+1792\zeta(3)-784\zeta(3)^2} \approx 0.770444\dots;$$

see Figure 1 for a plot of $p(s,s+\ell)/\sqrt{p(s,s)p(s+\ell,s+\ell)}$. This feature is closely connected with the bimodality of the variance of $Y_{n,k}$ and the concentration of the width; see [6].

Corollary 4. *The correlation coefficient $\rho(X_{n,k}, X_{n,h})$ exhibits asymptotically a sharp sign-change at $\beta = \frac{1}{2}$ when $\alpha \in (0, \zeta_2)$ is fixed and β is varying from 0 to ζ_2.*

Two plots of the asymptotic correlation coefficient are given in Figures 2, highlighting in particular the discontinuous sign-change at $\frac{1}{2}$.

Our method of proof relies on the relation

$$\sum_{n,k,h} 4^{-n} C_n \mathbb{E}(X_{n,k}X_{n,h}) u^k v^h z^n = \frac{uv(1-z)^{-(u+v+1)/2}}{(1+u)(1+v)(1+u+v-uv)} + \frac{(1-z)^{-u/2}+(1-z)^{-v/2}-(1-z)^{1/2}}{2(1+u)(1+v)} - \frac{(1-z)^{-uv/2}}{2(1+u+v-uv)}.$$

Then (8) is derived, similarly as in [9], by a uniform estimate for the function on the right-hand side in the u, v plane (by applying the singularity analysis of Flajolet and Odlyzko [11]) and then by extending the saddle point method used in [13].

Width. Define $W_n := \max_k X_{n,k}$. By (2), we easily have the lower bound

$$\mathbb{E}(W_n) \ge \max_k \mu_{n,k} = \frac{n}{\sqrt{\pi \log n}}\left(1 + \Theta\left((\log n)^{-1}\right)\right).$$

This lower bound is indeed tight; a very general approach is recently proposed in [6] to showing that

$$\mathbb{E}(W_n) = \frac{n}{\sqrt{\pi \log n}}\left(1 + \Theta\left((\log n)^{-1}\right)\right).$$

Estimates for higher central moments and concentration of the distribution of the width are also given there. The method proof is direct, correlation-free and relies on the estimates for higher central moments of the profile in the middle range.

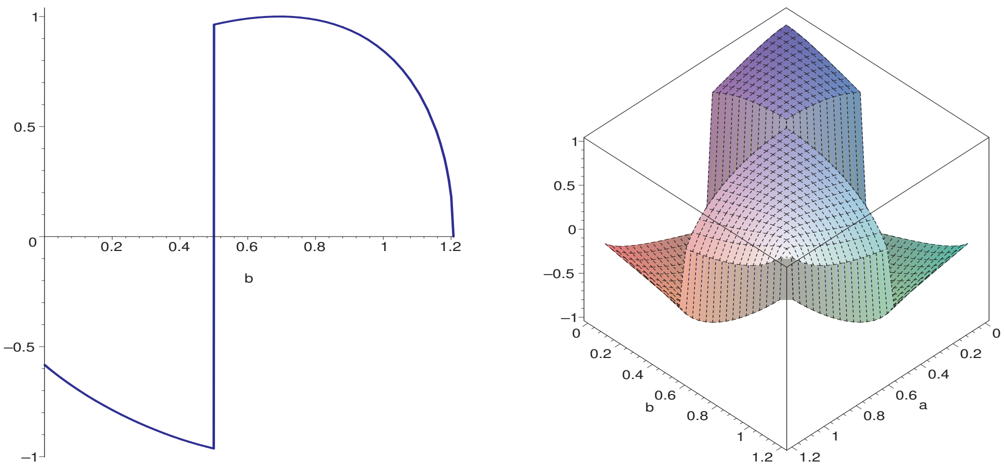
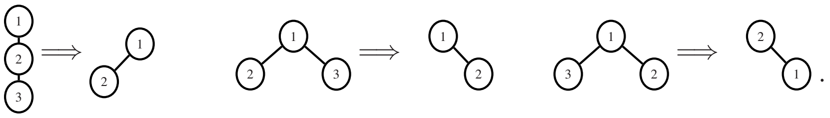


Fig. 2: *Asymptotic correlation coefficient of the number of nodes at two levels. The discontinuity of the sign at* $\frac{1}{2}$ *is visible from both figures. Here* $\alpha = \log 2 \approx 0.69$ *(left) and a 3-dimensional rendering (right) of* $f(\alpha,\beta)/\sqrt{f(\alpha,\alpha)f(\beta,\beta)}$.

An unexpected connection. Profile of recursive trees can essentially be regarded as counting only left-branches in random binary search trees. This is seen by the standard transformation of a multiway tree to a binary tree, called the *natural correspondence* between forests and binary trees in [14, Sec. 2.3.3]. For details, see [12]. Both profiles (of recursive trees and of binary search trees) turn out to behave very similarly. Note that since the order of the subtrees of any node in recursive trees are not distinguished, we can always arrange the subtrees in increasing order of their root labels when we read them off from left to right; then applying the binary-tree transformation on recursive trees results in a binary increasing tree (with labels on any path down from the root still forming an increasing sequence).

We can apply the same transformation to convert a random PORT into a binary tree; see Figure 3 for a plot. While the resulting tree is combinatorially less interesting because the monotonicity property of the labels along paths is destroyed, the profile in such binary trees is identically distributed as the profile of random binary trees although there is no bijection between their shapes; for example, when $n = 3$,



We see that the profiles of the resulting transformed binary trees have the same distribution as those of binary trees of two nodes

but without bijection between shapes.

Intuitively, since the root of random PORTs has already about $\sqrt{\pi n}$ nodes, nodes in the corresponding transformed binary tree is expected to be more dispersed. Thus *the "log-profile phenomena" exhibited by the profile of random PORTs becomes the "square-root profile phenomena" after the transformation*. Such a change of order for the transformation was first observed by Chen and Ni [4] for the expected total path length, which can be proved to have the same Airy limit distribution as that of random binary trees; see [10] for many objects leading to that law. Random trees with square-root height have been extensively studied in the literature; see [7] and the references therein.

A comparison of some shape parameters. We list in Table 2 the asymptotics of some properties related to the profiles of random binary search trees, random recursive trees and random PORTs.

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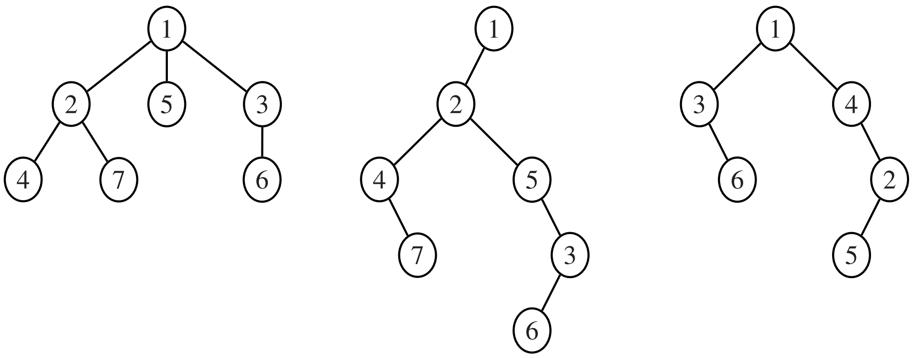

Fig. 3: *A PORT of* 7 *nodes (left), its corresponding transformed binary tree (middle), and the binary tree obtained by removing the root and decreasing all label values by* 1 *(right).*

Property	Binary search trees	Recursive trees	PORTs
root degree $\sim$	$1 +$ Bernoulli$(1 - \frac{2}{n})$	$N(\log n, \log n)$	$\sqrt{\pi n}$ Rayleigh
$\frac{X_{n,k}}{\mu_{n,k}} \xrightarrow{d} X(\alpha)$	$\alpha \in (0.37\ldots, 4.31\ldots)$	$\alpha \in [0, e)$	$\alpha \in [0, \alpha_+)$?
$\frac{X_{n,k}}{\mu_{n,k}} \xrightarrow{m} X(\alpha)$	$\alpha \in [1, 2]$	$\alpha \in [0, 1]$	$\alpha \in [0, 1/2]$
fixed-point eq. for $X(\alpha)$?	yes	yes	no
$\mathbb{E}$(width) $\sim$	$\frac{n}{\sqrt{4\pi \log n}}$	$\frac{n}{\sqrt{2\pi \log n}}$	$\frac{n}{\sqrt{\pi \log n}}$

Tab. 2: *A comparison of some properties of random binary search trees, recursive trees and PORTs. Here Bernoulli*(p) *denotes a Bernoulli random variable with mean* p*,* $N(\mu, \sigma^2)$ *normal with mean* μ *and variance* σ^2*,* $0.37\ldots$ *and* $4.31\ldots$ *are the two positive zeros of the equation* $z = 1 + z\log(z/2)$ *and the symbol* $\xrightarrow{m}$ *stands for convergence of all moments.*

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