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Convex hull for intersections of random lines

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The problem of finding the convex hull of the intersection points of random lines was studied in [4] and [8], and algorithms with expected linear time were found. We improve the previous results of the model in [4] by giving a universal algorithm for a wider range of distributions.

Keywords: convex hull, random lines

1 Introduction

Numerous problems can be reduced to finding the convex hull of a set of points – halfspace intersection, Delaunay triangulation, etc. An algorithm for finding the convex hull in the plane, known as Graham scan [5], achieves an $O(n \log n)$ running time. This algorithm is optimal in the worst case. Another algorithm [6] for the same problem runs in $O(nh)$ time, where h is the number of hull points, and outperforms the preceding algorithm if h happens to be very small. Kirkpatrick and Seidel [7] designed an algorithm with $O(n \log h)$ runtime, which is always at least as good as the better of the above two algorithms. A simplification of their algorithm has been recently reported by Chan [2].

For some special sets of points, it is possible to improve the results above. We concentrate on the case when the set consists of all intersection points of n lines. A straightforward application of the algorithms above leads to a runtime $O(n^2 \log n)$. Attalah [1] and Ching and Lee [3] independently presented $O(n \log n)$ runtime worst-case algorithms with $O(n)$ space. Ching and Lee [3] also showed that this result is best possible. Devroye and Toussaint [4] and Golin, Langerman and Steiger [8] studied the case when the lines are random with certain distributions. (Note that our model is different from the model of [8].) In both models, they presented algorithms with linear expected time.

Let us concentrate on the model of Devroye and Toussaint. It is convenient to use the representation of lines by points. A line not passing through the origin is uniquely determined by its intersection point with the line perpendicular to it from the origin. It is often useful to define a mechanism for selecting random lines via a mechanism for a random selection of the corresponding intersection points.

In the model of Devroye and Toussaint [4] all lines are independent identically distributed. The polar coordinates of the corresponding points are selected as follows. The distance from the origin is distributed according to some distribution law $\mathcal{R}$ (required to have a finite mean) and the angle is distributed uniformly in $[0, 2\pi)$; the distance and the angle are independent.

As mentioned earlier, their algorithm works in linear expected time. The linearity follows from a result they claim for the set of outer layer points. Here, given a set S, the *outer layer* of S consists of those points $P \in S$ such that at least one quadrant around P does not contain any other point of S. Clearly, any point in the convex hull belongs also to the outer layer. Their theorem asserts that the expected number of outer layer points is bounded above by some constant.

In fact, Devroye and Toussaint proved that, given a distribution $\mathcal{R}$, there exists a constant C such that, denoting by O_n the number of outer layer points arising from n lines, we have $E(O_n) \le C$ for sufficiently large n. If one could find a constant C and an N_0 such that $E(O_n) \le C$ for every $n \ge N_0$, independently of $\mathcal{R}$, the problem would be completely solved. However, we show by means of counter-examples that no such C and N_0 exist.

As indicated above, the result of Devroye and Toussaint [4] regarding the expected number of layer points was proved under the assumption that the distribution $\mathcal{R}$ has a finite mean. Here we construct a

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distribution with an infinite mean such that the expected number of points in the outer layer is $\Omega(n)$. Thus, their approach cannot possibly be extended to such distributions.

In this work we present another algorithm for this model. The algorithm has expected linear time, where the implied constant is global. Moreover, it works for any distribution law $\mathcal{R}$, be it of finite or of infinite mean.

In Section 2 we present the main results, and in the Sections $3, 4$ and 5 – the proofs.

2 The Main Results

Given a point in the plane with polar coordinates (r, θ), denote by $L(r, \theta)$ the line passing through it and perpendicular to the line segment linking the origin with this point. If $r = 0$, then the line is given by the equation $y = -(\cot\theta)x$. Let $\mathcal{R}$ be a distribution on the positive half line $\mathbf{R}_+ = \{x \geq 0\}$, and $R_0, \ldots, R_{n-1}$ be independent $\mathcal{R}$-distributed random variables. Let $\Theta_0, \ldots, \Theta_{n-1}$ be i.i.d. random variables, uniformly distributed on $[0, 2\pi)$. For $0 \leq i \leq n-1$, denote by L_i the line $L(R_i, \Theta_i)$. Let $y = a_i x + b_i$ be the equation of L_i. (Note that, with probability 1, none of the L_i's is parallel to the y-axis.) Clearly, $a_i = -\cot\Theta_i$ and, with probability 1, all a_i's are distinct. For each pair $(i, j), 0 \leq i < j \leq n-1$, denote by V_{ij} the intersection point of the lines L_i and L_j. Put $V(n) = \{V_{ij} : 0 \leq i, j \leq n-1\}$. Let $N_{\text{ch}}^{(n)}$ be the number of points in the convex hull of $V(n)$ and $N_{\text{ol}}^{(n)}$ the number of points in the outer layer of $V(n)$. Suppose the L_i's are sorted by their slopes. Atallah [1] showed that $CH(V(n)) = CH(\{V_{i,i+1} : i = 0, \ldots, n-1\})$. (Here and elsewhere in the paper, the addition of indices is modulo n.)

As mentioned above, the reason Devroye and Toussaint's algorithm works in linear expected time is

Theorem A [4, Theorem 1] *If $\mathcal{R}$ has a finite mean, then there exists a universal constant γ such that, uniformly over all n,*

$$EN_{\text{ch}}^{(n)} \leq EN_{\text{ol}}^{(n)} \leq \gamma < \infty. \tag{1}$$

The constant γ does not depend on the distribution $\mathcal{R}$.

Going over the proof of the theorem, one realizes that the authors prove the existence of a constant γ such that, given a distribution $\mathcal{R}$, we have $E(O_n) \leq \gamma$ for sufficiently large n. The following proposition implies that, no matter how large we choose γ to be in Theorem A, the initial n for which 1 holds may be arbitrary large as we change $\mathcal{R}$.

Proposition 2.1 *There exists a sequence of distributions $(\mathcal{R}_n)_{n=2}^{\infty}$ of the radius such that each $\mathcal{R}_n$ has mean 1 and the expected number of outer layer points in $V(n)$ under the distribution $\mathcal{R}_n$ exceeds $\frac{1}{2e}n$.*

What happens if the condition in Theorem A, whereby the distribution $\mathcal{R}$ has a finite expectation, is dropped? Our next result is that then Theorem A is inapplicable.

Proposition 2.2 *There exists a distribution $\mathcal{R}$ with an infinite mean such that $EN_{\text{ol}}^{(n)} \geq \frac{1}{2e}n$ for each $n \geq 2$.*

Note that Propositions 2.1 and 2.2 do not refer to the expected size of the convex hull, which may be much smaller than that of the outer layer. If this size is indeed uniformly bounded, the algorithm of Devroye and Toussaint [4] does work in linear expected time independently of $\mathcal{R}$. However, this does not follow from their approach.

We present here an alternative algorithm for finding the convex hull of the set of intersection points.

Require: The input lines $L_i = L(R_i, \Theta_i),\ 0 \le i \le n-1$, are random and selected according to [4] (i.e., all variables are independent, the R_i's are $\mathcal{R}$-distributed, and the Θ_i's are $U[0, 2\pi)$).
Ensure: H is the convex hull of the set $V(n)$ of intersection points of the lines $L(R_i, \Theta_i),\ 0 \le i \le n-1$.

- Sort the lines according to the θ-coordinates of the points (R_i, Θ_i) defining them, using bucket sort.
- For the sorted lines, find the index $k, 0 \le k \le n-1$, such that $\Theta_i < \pi$ if $i < k$. Put $\mathcal{L}_{<\pi} = \{L_i : i < k\}$ and $\mathcal{L}_{\ge\pi} = \{L_i : i \ge k\}$. /* Each of the two sets is sorted by slope. */
- Merge $\mathcal{L}_{\ge\pi}$ and $\mathcal{L}_{<\pi}$ into a single list, sorted by slope.
- Calculate the intersection points of the (sorted) consecutive lines $\{L_i\}$: $V_i = L_i \cap L_{i+1},\ 0 \le i \le n-1$.
- Let (h_i, ψ_i) be the polar coordinates of V_i. Sort the V_i's by their ψ-coordinates using bucket sort.
- Calculate $d_i = \psi_{i+1} - \psi_i, 0 \le i < n-1$ and $d_{n-1} = \psi_0 + 2\pi - \psi_{n-1}$ (for the newly sorted V_i's). Check if there exists $0 \le k \le n-1$ such that $d_k \ge \pi$.
- **if** such k exists **then**
    - For each $j, 0 \le j \le n-1$, denote by U_j the point V_{j+k} if $0 \le j \le n-k$ and V_{j+k-n} otherwise. The set of points U_j determines a simple polyline. Use Melkman's algorithm [9] to find $H = CH(\{U_j : 0 \le j \le n-1\}) = CH(V(n))$.
- **else**
    - The origin is an internal point of $CH(V(n))$. Also, the V_i's are in sorted order around the origin. Use Graham's scan [11] with the origin as internal point to construct the convex hull H.
- **end if**

Algorithm 1: CONVEX HULL FOR INTERSECTIONS OF RANDOM LINES

The next theorem is our main result.

Theorem 2.3 *Algorithm* 1 *provides a construction of the convex hull of* $V(n)$*, consuming linear space and expected linear time.*

3 Proofs of the Negative Results

Proof of Proposition 2.1. Let $n \ge 3$ be arbitrarily fixed. Denote:

$$W_{ij} = L(1, \Theta_i) \cap L(1, \Theta_j), \qquad 0 \le i < j \le n-2.$$

Consider the random variable $M = \max\{\|W_{ij}\| : 0 \le i < j \le n-2\}$. Let a be the median of M: $P(M < a) = \frac{1}{2}$. Obviously, $a > 1$. Denote by $\mathcal{R}_n$ the discrete distribution taking the two values 1 and a with probabilities $1 - \frac{1}{n}$ and $\frac{1}{n}$, respectively. Let $R_0, \ldots, R_{n-1}$ be independent $\mathcal{R}_n$-distributed random variables, and

$$V_{ij} = L(R_i, \Theta_i) \cap L(R_j, \Theta_j), \qquad 0 \le i < j \le n-1.$$

Consider the random variables

$$M_k = \max\{\|V_{ij}\| : 0 \le i < j \le n-1,\ i \ne k,\ j \ne k\}, \qquad k = 0, \ldots, n-1.$$

Consider the events: $A = \{N_{\text{ol}}^{(n)} \ge n-1\}$, $B_k = \{R_k = a, R_j = 1\ \forall j \ne k\}$, $C_k = \{M_k < a\}$ for $k = 0, \ldots, n-1$. The events $B_0 \cap C_0, \ldots, B_{n-1} \cap C_{n-1}$ are equi-probable and pairwise disjoint. If $B_k \cap C_k$ occurs for some k, then all points $V_{ij},\ i, j \ne k$, are at a distance smaller than a from the origin, while the points $V_{ij}, i \ne k$, are at a distance at least a. Hence in this case all points $V_{ik}, i \ne k$, are outer layer points; see Figure 1. (In fact, some of the L_i's may be parallel to L_k, in which case we miss the corresponding intersection points, but the probability for this is 0.) Hence $\bigcup_{k=0}^{n-1}(B_k \cap C_k) \subseteq A$.

Hence:

$$P(A) \ge P\left(\bigcup_{k=0}^{n-1}(B_k \cap C_k)\right) = nP(B_0 \cap C_0) = nP(B_0)P(C_0|B_0).$$

Further we obtain

$$\begin{aligned} P(C_0|B_0) &= P(M_0 < a | R_0 = a, R_1 = 1, \ldots, R_{n-1} = 1) \\ &= P(M_0 < a | R_1 = 1, \ldots, R_{n-1} = 1) = P(M < a) = \tfrac{1}{2}. \end{aligned}$$

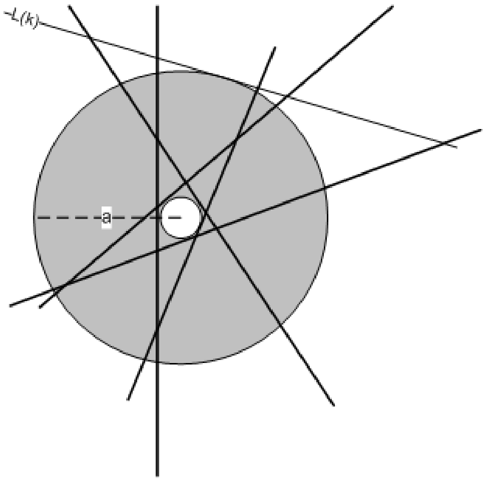

Fig. 1: Outer layer points

Therefore,

$$P(A) \geq n\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\frac{1}{2} > \frac{1}{2e}.$$

Consequently,

$$EN_{\text{ol}}^{(n)} \geq P(A)\cdot(n-1) + (1-P(A))\cdot 1 \geq \frac{1}{2e}n.$$

Since for $n=2$ we have $EN_{\text{ol}}^{(2)} = 1$, the inequality holds in this case as well.

$\square$

Proof of Proposition 2.2

Define a sequence $(a_m)_{m=1}^{\infty}$ inductively. Put $a_1 = 1$. Let $m \geq 1$ be arbitrary and fixed, and suppose $a_1, a_2, \dots, a_m$ have been defined. Let $(\Theta_i)_{i=0}^{m-1}$ be independent $U[0,2\pi)$-distributed random variables. Denote

$$\begin{aligned} W_{ij}^{st} &= L(a_s,\Theta_i)\cap L(a_t,\Theta_j), \qquad 1\leq s,t\leq m,\ 0\leq i<j\leq m-1, \\ M_m &= \max\{\|W_{ij}^{st}\| : \qquad 1\leq s,t\leq m,\ 0\leq i<j\leq m-1\}. \end{aligned}$$

Let b_{m+1} be the median of M_m such that $P(M_m < b_{m+1}) = \frac{1}{2}$, and put $a_{m+1} = \max(b_{m+1}, m)$. Define a distribution $\mathcal{R}$ by: $R \sim \mathcal{R}$ if

$$P(R = a_m) = \frac{1}{m(m+1)}, \qquad m = 1,2,\dots.$$

Clearly, $E(R) = \infty$. We shall prove that $\mathcal{R}$ satisfies the required condition. Let $R_0,\dots,R_{n-1}$ be independent $\mathcal{R}$-distributed random variables, and denote

$$\begin{aligned} V_{ij} &= L(R_i,\Theta_i)\cap L(R_j,\Theta_j), \qquad 0\leq i<j\leq n-1, \\ H_k &= \max\{\|V_{ij}\| : 0\leq i<j\leq n-1,\ i\neq k,\ j\neq k\}, \qquad k=0,\dots,n-1. \end{aligned}$$

Consider the events $A = \{N_{\text{ol}}^{(n)} \geq n-1\},\ \ B_k = \{R_k \geq a_n, R_j \leq a_{n-1} \forall j\neq k\},\ \ C_k = \{H_k < a_n\}$ for $k = 0,\dots,n-1$. According to the definition of $\mathcal{R}$,

$$P(R_0 \geq a_n) = \sum_{m=n}^{\infty}\frac{1}{m(m+1)} = \frac{1}{n},$$

which implies

$$P(B_0) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}.$$

The events $B_0 \cap C_0, \ldots, B_{n-1} \cap C_{n-1}$ are equi-probable and pairwise disjoint and $\bigcup_{k=0}^{n-1}(B_k \cap C_k) \subseteq A$. Hence

$$P(A) \geq P\left(\bigcup_{k=0}^{n-1}(B_k \cap C_k)\right) = nP(B_0 \cap C_0) = nP(B_0)P(C_0|B_0).$$

Further, we obtain

$$\begin{aligned} P(C_0|B_0) &= P(H_0 < a_n | R_0 \geq a_n, R_1 < a_n, \ldots, R_{n-1} < a_n) \\ &= P(H_0 < a_n | R_1 < a_n, \ldots, R_{n-1} < a_n) \geq P(M_{n-1} < a_n) \geq \tfrac{1}{2}. \end{aligned}$$

Therefore,

$$P(A) \geq n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} \cdot \frac{1}{2} > \frac{1}{2e}.$$

Consequently, $EN_{\mathrm{ol}}^{(n)} \geq P(A) \cdot (n-1) + (1 - P(A)) \cdot 1 \geq \frac{1}{2e}n$.

□

4 Cyclic Functions

Our work depends on numerous computations involving angles. It will be convenient to view angles as points of the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. As representatives of the cosets we shall take points in $[0, 2\pi)$ (with the quotient topology of $\mathbb{R}/2\pi\mathbb{Z}$). Addition modulo 2π will be denoted by $\oplus$ and subtraction by $\ominus$. Let $X_0, ..., X_{n-1} \sim U[0, 2\pi)$ be independent random variables.

Definition 4.1 A function $f : \mathbb{T} \times \mathbb{T} \mapsto \mathbb{T}$ is *cyclic* if $f(x \oplus \alpha, y \oplus \alpha) = f(x, y) \oplus \alpha, \qquad x, y, \alpha \in \mathbb{T}.$

Let ρ be the random permutation of $\{0 \ldots n-1\}$ which determines the index of the successor of each X_i when the values are ordered on $\mathbb{T}$. More accurately, $\rho(i), i = 0, \ldots, n-1$, is the index $j, 0 \leq j \leq n-1$, such that X_j is the nearest point to X_i on the unit circle (in the counterclockwise direction):

$$X_{\rho(i)} \ominus X_i = \min_{0 \leq k \leq n-1, k \neq i}(X_k \ominus X_i)$$

In the following two lemmas, we shall deal with functions $f : \mathbb{T}^2 \times \mathbb{R}^2 \mapsto \mathbb{T}$ whose restriction $f(\cdot, \cdot, r_1, r_2)$ is cyclic for arbitrary fixed r_1, r_2. By $R_0, \ldots, R_{n-1}$ we shall denote i.i.d. random variables independent of $X_0, \ldots, X_{n-1}$.

Denote $I = [0, \frac{2\pi}{n})$ and $M = \sum_{i=0}^{n-1} \mathbf{1}_I(f(X_i, X_{\rho(i)}, R_i, R_{\rho(i)}))$. The proofs of the next two lemmas are omitted due to their length. They can be found in the full version of the paper.

Lemma 4.2 $E(M) = 1$.

Lemma 4.3 $E(M^2) \leq 3 + \frac{n(n-1)}{(n-2)(n-3)}$.

5 Proof of the Main Result

This section is devoted to the proof of Theorem 2.3. For a pair of points $\theta, \eta \in [0, 2\pi)$, let $A(\theta, \eta)$ be the arc from θ to η, taken counterclockwise:

$$A(\theta, \eta) = \{\nu \in [0, 2\pi) : (\eta \ominus \nu) + (\nu \ominus \theta) = \eta \ominus \theta\}.$$

The following lemmas describe some properties of A. The proofs are simple, and will be omitted.

Lemma 5.1 *Let $\alpha, \beta, \gamma, \eta, \theta \in \mathbb{T}$ be points such that $\eta \in A(\alpha, \beta)$, $\theta \in A(\beta, \gamma)$ and $\beta \in A(\alpha, \gamma)$. Then $\beta \in A(\eta, \theta)$.*

Lemma 5.2 *Let $\alpha, \beta, \gamma, \eta, \theta, \zeta$ be points such that $\theta \in A(\alpha, \beta)$, $\zeta \in A(\gamma, \eta)$, $\beta \in A(\alpha, \gamma)$ and the two sets $A(\alpha, \beta)$ and $A(\gamma, \eta)$ intersect in at most the single point γ. Then $A(\beta, \gamma) \subseteq A(\theta, \zeta) \subseteq A(\alpha, \eta)$.*

Lemma 5.3 *Let $\alpha, \beta, \gamma, \eta$ be points such that $\beta \ominus \alpha < \pi$ and $\eta, \gamma \in A(\alpha, \beta)$. Then $A(\gamma, \eta) \subseteq A(\alpha, \beta)$ if and only if $\eta \ominus \gamma < \pi$.*

For $0 \leq \theta < 2\pi$, denote $\underline{\theta} = \max\{\theta, \theta \oplus \pi\}$ and $\overline{\theta} = \min\{\theta, \theta \oplus \pi\}$. Note that $0 \leq \overline{\theta} < \pi \leq \underline{\theta} = \overline{\theta} + \pi < 2\pi$. Thus, $\overline{\theta}$ is always at the top half circle and $\underline{\theta}$ at the bottom half circle (which is the reason for our notation).

Lemma 5.4 *Let α, β be angles such that $-\cot\alpha < -\cot\beta$. Then $\overline{\alpha} < \overline{\beta}$, $\underline{\alpha} < \underline{\beta}$ and $\underline{\beta} \ominus \overline{\alpha} > \pi$.*

Lemma 5.5 *Let $l = L(r, \theta), l' = L(r', \theta')$ be two intersecting lines. Denote by (ψ, h) the polar representation of the intersection point $l \cap l'$. If $\theta' \ominus \theta > \pi$, then $\psi \in A\left(\theta \ominus \frac{\pi}{2}, \theta' \oplus \frac{\pi}{2}\right)$.*

Lemma 5.6 *Let $r_1, r_2 \geq 0$ be two real numbers and θ_1, θ_2 be two angles such that $-\cot\theta_1 < -\cot\theta_2$. Denote $\hat{V} = L(r_1, \overline{\theta}_1) \cap L(r_2, \underline{\theta}_2)$. Let $(\hat{r}, \hat{\theta})$ be the polar representation of $\hat{V}$. Then $\hat{\theta} \in A\left(\underline{\theta}_1 \oplus \frac{\pi}{2}, \underline{\theta}_2 \oplus \frac{\pi}{2}\right)$.*

Proof Due to Lemma 5.4, $\underline{\theta}_2 \ominus \overline{\theta}_1 > \pi$. Using Lemma 5.5 and the fact that $\overline{\theta} \ominus \frac{\pi}{2} = \underline{\theta} \oplus \frac{\pi}{2}$ for any $\theta \in [0, 2\pi)$, we have

$$\hat{\theta} \in A\left(\overline{\theta}_1 \ominus \frac{\pi}{2}, \underline{\theta}_2 \oplus \frac{\pi}{2}\right) = A\left(\underline{\theta}_1 \oplus \frac{\pi}{2}, \underline{\theta}_2 \oplus \frac{\pi}{2}\right).$$

□

Lemma 5.7 *Let $J = A(\theta, \eta)$ be an arc such that $\eta \ominus \theta < \pi$. Let $\alpha, \beta, \gamma, \zeta \in \mathbb{T}$ be angles such that $-\cot\alpha < -\cot\beta < -\cot\gamma$ and $\underline{\alpha} \oplus \zeta, \underline{\gamma} \oplus \zeta \in J$. Then $\underline{\beta} \oplus \zeta \in J$.*

Proof of Lemma 5.7 Using Lemma 5.4, we obtain $\underline{\gamma} \ominus \underline{\alpha} < \pi$. Therefore we can apply Lemma 5.3 to conclude that $A(\underline{\alpha} \oplus \zeta, \underline{\gamma} \oplus \zeta) \subseteq J$. According to Lemma 5.4, $\underline{\alpha} < \underline{\beta} < \underline{\gamma}$ and $\underline{\alpha}, \underline{\beta}, \underline{\gamma} \in [\pi, 2\pi)$. Thus $\underline{\beta} \in A(\underline{\alpha}, \underline{\gamma})$ and $\underline{\beta} \oplus \zeta \in A(\underline{\alpha} \oplus \zeta, \underline{\gamma} \oplus \zeta) \subseteq J$. □

We now turn to prove Theorem 2.3. To this end, we have to deal with various stages of Algorithm 1. Recall that, after step 4 of Algorithm 1, the lines $L_i = L(R_i, \Theta_i), 0 \leq i \leq n-1$, are sorted by slope. Denote

$$\hat{V}_i = L(R_i, \overline{\Theta}_i) \cap L(R_{i+1}, \underline{\Theta}_{i+1}), \qquad 0 \leq i \leq n-1,$$

and let $\hat{\psi}_i$ be the angle between the positive x-axis and the line segment connecting 0 with $\hat{V}_i$.

Lemma 5.8 *Let $J = A(\tau, \sigma)$ be an arc such that $\sigma \ominus \tau < \pi$. For any n*

$$\sum_{i=0}^{n-1} \mathbf{1}_J\left(\hat{\psi}_i\right) \leq \sum_{i=0}^{n-1} \mathbf{1}_J\left(\underline{\Theta}_i \oplus \frac{\pi}{2}\right) + 2.$$

Proof If the set $\{i : \hat{\psi}_i \in J\}$ contains at most two elements, then the required inequality is trivial. Suppose that $|\{i : \hat{\psi}_i \in J\}| > 2$. Denote $l = \min_{0 \leq i \leq n-2}\{i : \hat{\psi}_i \in J\}, k = \max_{0 \leq i \leq n-2}\{i : \hat{\psi}_i \in J\}$. Putting $\hat{V}_l = (\hat{\psi}_l, \hat{r}_l)$ and applying Lemma 5.6 with $\theta_1 = \Theta_l,\ \theta_2 = \Theta_{l+1},\ \hat{V} = \hat{V}_l,\ r_1 = R_l,\ r_2 = R_{l+1}$, we conclude that $\hat{\psi}_l \in A(\underline{\Theta}_l \oplus \frac{\pi}{2}, \underline{\Theta}_{l+1} \oplus \frac{\pi}{2})$.

Similarly, applying Lemma 5.6 with $\theta_1 = \Theta_k,\ \theta_2 = \Theta_{k+1},\ \hat{V} = \hat{V}_k,\ r_1 = R_l,\ r_2 = R_{l+1}$, we conclude that $\hat{\psi}_k \in A(\underline{\Theta}_k \oplus \frac{\pi}{2}, \underline{\Theta}_{k+1} \oplus \frac{\pi}{2})$.

Since $\underline{\Theta}_l \leq \underline{\Theta}_{l+1} \leq \underline{\Theta}_k$, we have $\underline{\Theta}_{l+1} \oplus \frac{\pi}{2} \in A(\underline{\Theta}_l \oplus \frac{\pi}{2}, \underline{\Theta}_k \oplus \frac{\pi}{2})$. Applying Lemma 5.2 with

$$\alpha = \underline{\Theta}_l \oplus \frac{\pi}{2},\ \beta = \underline{\Theta}_{l+1} \oplus \frac{\pi}{2},\ \gamma = \underline{\Theta}_k \oplus \frac{\pi}{2},\ \eta = \underline{\Theta}_{k+1} \oplus \frac{\pi}{2},\ \theta = \hat{\psi}_l,\ \zeta = \hat{\psi}_k,$$

we obtain

$$\underline{\Theta}_{l+1} \oplus \frac{\pi}{2},\ \underline{\Theta}_k \oplus \frac{\pi}{2} \in A(\hat{\psi}_l, \hat{\psi}_k)$$

and

$$\hat{\psi}_k \ominus \hat{\psi}_l = \zeta \ominus \theta \leq \eta \ominus \alpha = \left(\underline{\Theta}_{k+1} \oplus \frac{\pi}{2}\right) \ominus \left(\underline{\Theta}_l \oplus \frac{\pi}{2}\right).$$

Thus $\hat{\psi}_k \ominus \hat{\psi}_l \leq \underline{\Theta}_{k+1} \ominus \underline{\Theta}_l \leq \pi$. Applying Lemma 5.3 with $\alpha = \tau,\ \beta = \sigma,\ \gamma = \hat{\psi}_l,\ \eta = \hat{\psi}_k$, we get:

$$A(\hat{\psi}_l, \hat{\psi}_k) = A(\gamma, \eta) \subseteq A(\alpha, \beta) = A(\tau, \sigma) = J.$$

Taking into account that

$$\underline{\Theta}_{l+1} \oplus \frac{\pi}{2}, \underline{\Theta}_k \oplus \frac{\pi}{2} \in A(\hat{\psi}_l, \hat{\psi}_k),$$

we therefore have

$$\underline{\Theta}_{l+1} \oplus \frac{\pi}{2}, \underline{\Theta}_k \oplus \frac{\pi}{2} \in J.$$

By Lemma 5.7, for any $l+1 \le h \le k$ we have $\underline{\Theta}_h \oplus \frac{\pi}{2} \in J$. Finally,

$$\sum_{i=0}^{n-1} \mathbf{1}_J\left(\hat{\psi}_i\right) \le \sum_{i=0}^{n-2} \mathbf{1}_J\left(\hat{\psi}_i\right) + 1 \le k - l + 2 \le \sum_{i=0}^{n-1} \mathbf{1}_J\left(\underline{\Theta}_i \oplus \frac{\pi}{2}\right) + 2.$$

□

Similarly, let $\check{V}_i = L(R_i, \underline{\Theta}_i) \cap L(R_{i+1}, \overline{\Theta}_{i+1})$. Let $\left(\check{r}_i, \check{\psi}_i\right)$ be the polar representation of $\check{V}_i$. Using the same observations as in Lemma 5.8, we obtain

Lemma 5.9 *Let $J = A(\tau, \sigma)$ be an arc such that $\sigma \ominus \tau < \pi$. For any n*

$$\sum_{i=0}^{n-1} \mathbf{1}_J\left(\check{\psi}_i\right) \le \sum_{i=0}^{n-1} \mathbf{1}_J\left(\overline{\Theta}_i \oplus \frac{\pi}{2}\right) + 2.$$

We shall refer to V_i as "far" if $\Theta_{i+1} \ominus \Theta_i > \pi$ and as "near" otherwise. Let F be the set of indices $i, 0 \le i \le n-1$, for which V_i is far, and N the set of those for which it is near. Put $I = [0, \frac{2\pi}{n})$.

Lemma 5.10 *For any n:*

$$E\left(\sum_{i \in F} \mathbf{1}_I\left(\psi_i\right)\right)^2 < 40.$$

Proof Clearly, if V_i is a "far" point, then $V_i \in \{\hat{V}_i, \check{V}_i\}$, which implies

$$E\left(\sum_{i \in F} \mathbf{1}_I(\psi_i)\right)^2 \le E\left(\sum_{i=1}^{n} \mathbf{1}_I\left(\hat{\psi}_i\right) + \mathbf{1}_I\left(\check{\psi}_i\right)\right)^2.$$

Using Lemmas 5.8, 5.9 and the fact that $\underline{\Theta}_i \in \{\Theta_i, \Theta_i + \pi\}$, we obtain that

$$E\left(\sum_{i=1}^{n} \mathbf{1}_I\left(\hat{\psi}_i\right) + \mathbf{1}_I\left(\check{\psi}_i\right)\right) \le \sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \tfrac{\pi}{2}\right) + \sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \tfrac{3\pi}{2}\right) + 4.$$

Therefore,

$$\begin{aligned} E\left(\sum_{i \in F} \mathbf{1}_I(\psi_i)\right)^2 \le \quad & E\left(\sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \tfrac{\pi}{2}\right) + \sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \tfrac{3\pi}{2}\right)\right)^2 \\ & + 16E\left(\sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \tfrac{\pi}{2}\right)\right) + 16. \end{aligned}$$

Now, $\sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \frac{\pi}{2}\right)$ and $\sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \frac{3\pi}{2}\right)$ are both $B(n, \frac{1}{n})$-distributed, and hence

$$E\left(\sum_{i=0}^{n-1} \mathbf{1}_I\left(\hat{\psi}_i\right)\right)^2 \le 2E\left(\sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \frac{\pi}{2}\right)\right)^2 + 2E\left(\sum_{i=0}^{n-1} \mathbf{1}_I\left(\Theta_i \oplus \frac{3\pi}{2}\right)\right)^2 + 32 \le 40.$$

□

Lemma 5.11 *Let α, β be angles such that $\beta \ominus \alpha \le \pi$ and $-\cot\alpha < -\cot\beta$. Then α and β belong to the same semicircle i.e., either $\alpha, \beta \in [0, \pi)$ or $\alpha, \beta \in [\pi, 2\pi)$.*

Proof Suppose, say, that $\alpha \in [0, \pi)$ and $\beta \in [\pi, 2\pi)$. Then $\alpha + \pi \in [\pi, 2\pi)$ and $\alpha + \pi \ge \beta$. Therefore $-\cot\alpha = -\cot(\alpha + \pi) \ge -\cot\beta$, which contradicts the conditions. □

Let ρ be the permutation of $\{0, 1, \ldots, n-1\}$ such that $\Theta_{\rho(i)}$ is the nearest to Θ_i in the counterclockwise direction among all Θ_k's:

$$\Theta_{\rho(i)} \ominus \Theta_i = \min_{0 \le k \le n-1, k \ne i} (\Theta_k \ominus \Theta_i), \quad i = 0, 1, \ldots, n-1.$$

Lemma 5.12 *For all $i \in N$ we have $\rho(i) = i + 1$.*

Proof We may assume that $\Theta_i \neq \Theta_{i+1}$, for any $0 \leq i \leq n-1$, . According to Lemma 5.11, the angles Θ_i, Θ_{i+1} belong to the same semicircle. The function $-\cot$ is increasing on $(0,\pi)$ and $(\pi, 2\pi)$, therefore $\Theta_{i+1} > \Theta_i$. Suppose that there exists $0 \leq k \leq n-1$ such that $\Theta_i < \Theta_k < \Theta_{i+1}$. It is easy to see that Θ_k belongs to the same semicircle as Θ_i, Θ_{i+1}, and therefore $a_i < a_k < a_{i+1}$. This contradicts the increasing order of the slopes a_i of L_i's. □

Lemma 5.13 $E\left(\sum_{i\in N} \mathbf{1}_I\left(\psi_i\right)\right)^2 < 3 + \frac{n(n-1)}{(n-2)(n-3)}$.

Proof Let (ψ_i', r_i') be the polar representation of the point $V_{i,\rho(i)}$. We have, according to Lemma 5.12:

$$\sum_{i\in N} \mathbf{1}_I(\psi_i) \leq \sum_{i=0}^{n-1} \mathbf{1}_I\left(\psi_i'\right).$$

It is easy to verify that

$$\psi_i' = \arctan\left(\frac{R_i \cos\Theta_{\rho(i)} - R_{\rho(i)}\cos\Theta_i}{R_{\rho(i)}\sin\Theta_i - R_i \sin\Theta_{\rho(i)}}\right).$$

Denote

$$f(\theta_1, \theta_2, r_1, r_2) = \arctan\left(\frac{r_1\cos\theta_2 - r_2\cos\theta_1}{r_2\sin\theta_1 - r_1\sin\theta_2}\right).$$

The function is cyclic according to Definition 4.1. Therefore, by Lemma 4.3:

$$E\left(\sum_{i=0}^{n-1} \mathbf{1}_I\left(\psi_i'\right)\right)^2 = E\left(\sum_{i=0}^{n-1} \mathbf{1}_I\left(f(\Theta_i, \Theta_{\rho(i)}, R_i, R_{\rho(i)})\right)\right)^2 \leq 3 + \frac{n(n-1)}{(n-2)(n-3)}.$$

□

Lemma 5.14 $E\left(\sum_{i=0}^{n-1} \mathbf{1}_I(\psi_i)\right)^2 \leq 86 + \frac{n(n-1)}{(n-2)(n-3)}$.

Proof According to lemmas 5.10, 5.13,

$$E\left(\sum_{i=0}^{n-1} \mathbf{1}_I(\psi_i)\right)^2 \leq 2E\left(\sum_{i\in F} \mathbf{1}_I(\psi_i)\right)^2 + 2E\left(\sum_{i\in N} \mathbf{1}_I(\psi_i)\right)^2 \leq 86 + \frac{2n(n-1)}{(n-2)(n-3)}.$$

□

Lemma 5.15 *Step* 5 *of algorithm 1 sorts the* ψ_i*'s in expected linear time.*

Proof To show that bucket sort works in expected linear time on the ψ_i's , it is sufficient to show that there exists a constant C, such that $E\left(\sum_{i=0}^{n-1} \mathbf{1}_I(\psi_i)\right)^2 < C$. According to Lemma 5.14, this is in fact true. □

Proof of Theorem 2.3

The algorithm indeed calculates the required convex hull due to [1], [9] and [10]. The algorithm works in expected linear time due to Lemma 5.15.

□

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