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Annihilating random walks and perfect matchings of planar graphs

Massimiliano Mattera

Laboratoire de Mathématique UMR 8628 du CNRS, Bât 425, Université Paris-Sud, 91405 Orsay, France
`Massimiliano.Mattera@math.u-psud.fr`

We study annihilating random walks on $\mathbb{Z}$ using techniques of P.W. Kasteleyn and R. Kenyon on perfect matchings of planar graphs. We obtain the asymptotic of the density of remaining particles and the partition function of the underlying statistical mechanical model.

Keywords: Perfect matchings, Partition fucntion, Interacting particles systems

1 Introduction

Annihilating random walks (ARW) have been studied in the early 70's within the theory of interacting particles systems (see [1],[2],[3] and [9]). The idea was to study a system of particles moving on a graph according to certain laws of attraction. The system we study in this paper is defined as follows: the initial system consists of particles at every site of $2\mathbb{Z}$. Then, each particle simultaneously performs discrete simple random walk on $\mathbb{Z}$, that is, every particle, independently of each other, has probability one half of taking its next step to the left and one half to the right. If two particles are at the same time on the same position they annihilate each other.
Let $x \in \mathbb{Z}$ and $\sigma(x) = 1$ if there is a particle on site x and $\sigma(x) = 0$ if not. Then, for all $T \in \mathbb{N}$, the positions of particles at time T is described by $\sigma_T \in \{0,1\}^{\mathbb{Z}} = \Omega$. ARW is then a discrete time Markov chain with state space Ω.
Many results concerning ARW can be found in the literature but most of them for continuous time systems. We give here some results for ARW on $\mathbb{Z}$ with discrete time and this is obtained by a one-to-one correspondence with a statistical mechanics model: the dimer model on planar graphs.
This model was first studied in the physical literature by P.W. Kasteleyn, H.T emperley and M. Fisher in the 60's ([4],[5]) and then in the mathematical community by R. Kenyon ([6],[7]) in the late 90's and we should note here (see [8]) that this statistical mechanical model has already been useful to understand other discrete random walks, among them loop erased random walks. These links enable one to understand random walks with enumerative combinatorial tools and gives an algebraic aspect to the theory.
Given a graph G the dimer model studies the set $M(G)$ of perfect matchings of G, that is the set of families of edges of G such that every vertex of G lies in exactly one edge of the family (these edges are called dimers). We will describe a graph G such that every configuration of trajectories of ARW correspond to exactly one perfect matching of G. Understanding the global and local statistics of $M(G)$ leads then to results on ARW.

2 Notations and definitions.

Let $G = (E,V)$ be a finite graph and $M(G)$ be the set of perfect matchings of G (which might eventually be the empty set). Let $p : E \to \mathbb{R}^+$ be a weight function. For $m \in M(G)$ let $p(m) = \prod_{e \in m} p(e)$.

The dimer model is the study of the measure μ_p on $M(G)$ given by:

$$\forall m \in M(G) \quad \mu_p(\{m\}) = Z^{-1} p(m)$$

where $Z = \sum_{m \in M(G)} p(m)$ is the partition function of the model. When the graph G is planar, the work of P.W. Kasteleyn and R. Kenyon enables us to compute Z and also local statistics of μ_p. In fact we can define a matrix K indexed by the vertices v_i of G such that:

- (Kasteleyn [4]) $Z = \sqrt{|detK|}$
- (Kenyon [6]) if $E = (v_1v_2, \ldots, v_{2k-1}v_{2k})$ is a set of edges then: $\mu_p(E) = \sqrt{|det(K_E^{-1})|} \prod_{e \in E} p(e)$ where:

$$K_E^{-1} = (K^{-1}(v_i v_j))_{1 \leq i,j \leq 2k}.$$

The link with ARW is then as follows: representing the trajectories of the particles in the upper-plane $\{(x,t), x \in \mathbb{Z}, t \in \mathbb{N}\}$ where (x,t) represents the site x at time t, there are only 6 configurations that are:

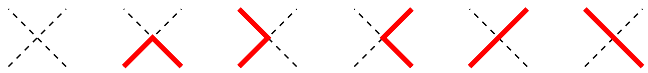

Fig. 1: The 6 configurations

There is then the following correspondence between perfect matchings of a fundamental domain and trajectories: In order to use the techniques mentioned above we need to consider finite graphs and this

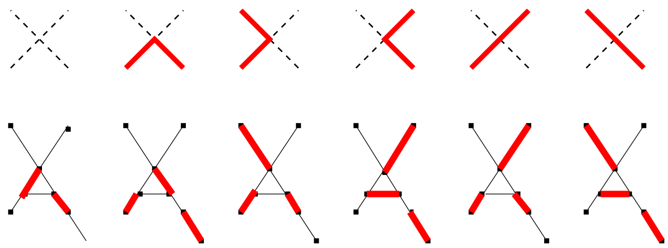

Fig. 2: One-to-one correspondence

leads us to ARW with state space $\Omega_n = \{0,1\}^{\mathbb{Z}/n\mathbb{Z}}$. We also force annihilation of all particles before time N (as a consequence n is an even number). We obtain a finite graph $G_{n,N}$ embedded on a cylinder. Here is

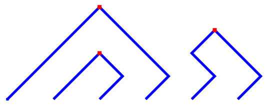

Fig. 3: Paths trajectories

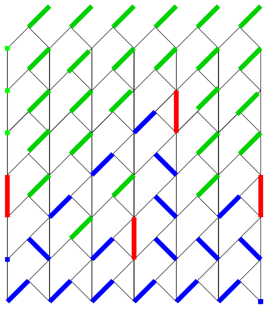

Fig. 4: Perfect matching (cylindrical boundary conditions)

an example given for 6 particles, the last annihilation being at time $T = 4$. Figure 3 shows path trajectories of the particles and figure 4 shows the corresponding perfect matching of $G_{n,N}$. We define the weight function as in Figure 5. In this Figure, b is a non negative real number and corresponds to the weight of an empty site (looking back at the 6 configurations we see that only the empty one uses the edge with the b-weight). Let ARW_b be the process corresponding to μ_b. Let $Z(n,N,b)$ be the partition function for the dimer model on $M(G_{n,N})$ with weight function defined as above. For $z \in \mathbb{C}$ and $j \in \mathbb{N}$, let $H_{b,z}(j)$ be the polynomial: $H_{b,z}(j) = (1+z)^{2j} - (b^2 z)^j$.

3 Results

Using Kasteleyn techniques i.e exact computation of the global statistics of the dimer model using linear algebra, we obtained the following:

Theorem 1 *The partition function of the ARW_b model is:*

$$Z_{n,N}(b) = \sqrt{b^{3n} \prod_{z^n=-1} \frac{(1-z^2)H_{b,z}(N)}{H_{b,z}(1)}} \quad .$$

Using Kenyon's techniques i.e exact computation of the local statistics of the dimer model we obtained the following:

Theorem 2 *Let $p(T)$ be the density of remaining particles after time T for ARW on $\mathbb{Z}$ corresponding to particle performing simple random walk independently of each others. Then when T goes to infinity:*

$$p_(T) \simeq \sqrt{\frac{1}{2T\pi}}.$$

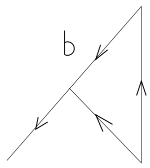

Fig. 5: weights in a fundamental domain

4 Proofs.

$G_{n,N}$, being embedded on a cylinder, is planar and we can use Kasteleyn's method to compute $Z_{n,N}$ as the Pfaffian of its Kasteleyn matrix defined as follows: first, as shown in fig.5 of section 2, we define a weight and an orientation on a fundamental domain of $G_{n,N}$. The choice of this orientation is made to give a "flat orientation" to $G_{n,N}$ so that the number of anticlockwise edges around any face, except the external face, is odd. To have also the external face well oriented we change the orientation of just one type of edge along a vertical strip. We define then, for v and w vertices of $G_{n,N}$, $K(v,w)$ as 0 if v and w are not adjacent vertices, as $p(vw)$ if the edge vw is directed from v to w and as $-p(vw)$ if the edge vw is directed from w to v.
Now, K being antisymmetric, its Pfaffian $Pf(K)$ is such that $Pf(K) = \sqrt{|det(K)|}$.
In order to compute $Z_{n,N}$ we change $G_{n,N}$ into a new graph $G^*_{n,N}$ in the following way:

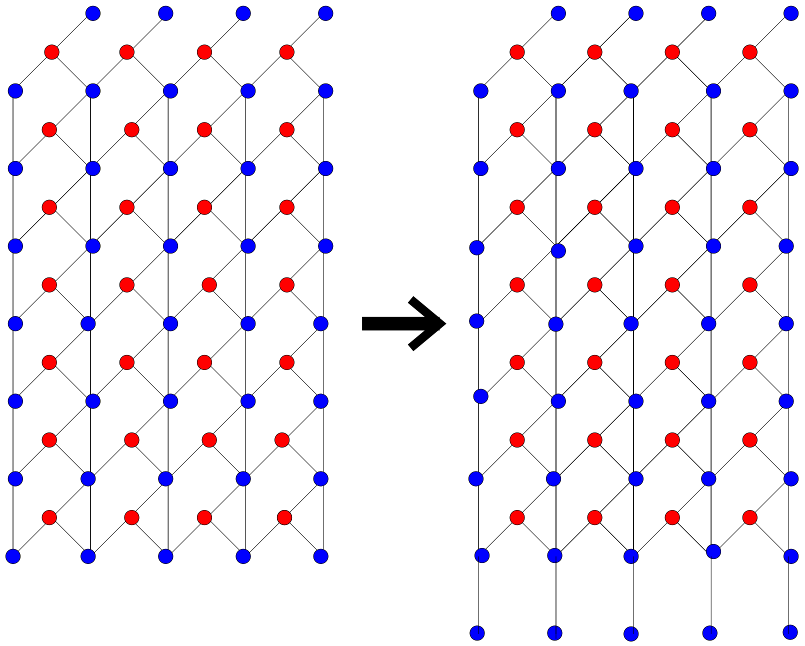

Fig. 6: New graph with only one perfect matching

A vertex of $G^*_{n,N}$ will be designed by a triple $(x,y,\varepsilon) \in \{0,..,n\}\times\{0,...,N+1\}\times\{0,1\}$, where $\varepsilon = 0$ stands for a blue vertex and $\varepsilon = 1$ for a red one, except for the new vertices i.e the ones in $G^*_{n,N}\setminus G_{n,N}$ that we will denote $\mathcal{B} = \{\alpha_0,\alpha_1,..,\alpha_{n-1}\}$. This new graph $G^*_{n,N}$ has the following property: card$M(G^*_{n,N}) = 1$. Let $Z^*_{n,N}$ be the corresponding partition function on $G^*_{n,N}$. We have that $Z^*_{n,N} = b^{n(N+1)}$, since there are $n(N+1)$ b-edges on $G^*_{n,N}$ and the unique perfect matching of $G^*_{n,N}$ uses them all. Let K^* be the Kasteleyn matrix of $G^*_{n,N}$ that is the oriented-weighted adjacency matrix corresponding to weights shown in fig.5. Let V be the set of vertices of $G^*_{n,N}$. Then K^* operates on $\mathbb{C}^V$ in the following way: for $f\in\mathbb{C}^V$ anf $v\in V$ $(K^*f)(v) = \sum_{w\in V} K(v,w)f(w)$.

Let us note $c_{ij} = K^{*-1}(\alpha_i,\alpha_j)$. Since $G_{n,N}$ is obtained from $G^*_{n,N}$ by removing $\{\alpha_0,\alpha_1,..,\alpha_{n-1}\}$ that all lie in the same (external) face we have the following:

Lemma 3

$$\frac{Z_{n,N}}{Z^*_{n,N}} = \sqrt{|det(c_{ij})_{1\le i,j\le n-1}|}$$

We show now how to compute the c_{ij}. Let $z\in\mathbb{C}$ such that $z^n = -1$. We give a function $C_z : V\to\mathbb{C}$ such that: $C_z((j,k,\varepsilon)) = z^j C_z((0,k,\varepsilon))$ and that,

$$\begin{cases} K^*C_z(v) &= 0, \quad v\notin\mathcal{B} \\ K^*C_z(\alpha_i) &= \quad z^i. \end{cases}$$

This is done by noticing that the values of C_z at $(0,k,0),(0,k,1),(0,k+1,0)$ give the values of C_z at $(0,k+1,0),(0,k+1,1),(0,k+2,0)$. In fact we have that

$$K^*C_z(1,k+1,0) = bC_z(0,k,1) - zC_z(0,k,0) - (1+z)C_z(0,k+1,1 + zC_z(0,k+2,0) = 0,$$

$$\text{and} \quad K^*C_z(0,k+1,0) = (1+z)C_z(0,k+1,0) - bzC_z(0,k+2,0) = 0.$$

So that if

$$W_k = \begin{pmatrix} \underline{C_z(0,k,0)} \\ C_z(0,k,1) \\ C_z(0,k+1,0) \end{pmatrix}$$

we have that

$$W_{k+1} = MW_k$$

where

$$M = \begin{pmatrix} 0 & 0 & 1 \\ \frac{-z}{1+z} & \frac{b}{1+z} & \frac{1+z}{bz} \\ 0 & 0 & \frac{1+z}{bz} \end{pmatrix}$$

and then $W_j = M^jW_0$ where M^j is given by:

$$M^j = \begin{pmatrix} 0 & 0 & (\frac{1+z}{bz})^{j-1} \\ \frac{-z}{b}(\frac{b}{1+z})^j & (\frac{b}{1+z})^j & \frac{z(z^2-1)}{H(1)}(\frac{b}{1+z})^j + \frac{(1+z)^2-b^2z^2}{bH(1)}(\frac{1+z}{bz})^{j-1} \\ 0 & 0 & (\frac{1+z}{bz})^j \end{pmatrix}$$

Moreover, $K^{*-1}(0,0,0) = 1$ since $\mathrm{card}M(G^*_{n,N}) = 1$ and that $K^{*-1}(0,0,0)$ is the probability of having the edge between $(0,0,0)$ and α_0. We may then take $C_z(0,0,0)$ equal to 1 and then since

$$\begin{cases} K^*C_z(1,0,0) &= -(1+z)C_z(0,0,1) - zC_z(\alpha_0) + zC_z(1,0,0) &= 0 \\ K^*C_z(0,0,1) &= (1+z)C_z(0,0,0) - bzC_z(0,1,0) &= 0 \end{cases}$$

we have that $W_0 = \begin{pmatrix} 1 \\ \frac{1}{b} - \frac{z}{1+z}C_z(\alpha_0) \\ \frac{1+z}{bz} \end{pmatrix}$

Now, looking at the upper-boundaries conditions we must have:

$$bC_z(N-1,0,1) = zC_Z(N-1,0,0).$$

Plugging in together all these relations gives us that:

$$C_z(\alpha_0) = \frac{b(1-z^2)}{(b^2z)^N}\frac{H_z(N)}{H_z(1)}.$$

Let us note $c_i = c_{0i}$.
Using the fact that $\forall j \in \{1,..,n-1\} \sum_{z^n=-1} z^j = 0$, we have the following:

Proposition 4

$$c_j = \frac{b}{n}\sum_{z^n=-1}\frac{(1-z^2)}{(b^2z)^N}\frac{H_z(N)}{H_z(1)}z^j.$$

Using basic linear algebra we also have that :

Proposition 5

$$det(c_{ij})_{1\le i,j\le n} = det\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ -c_1 & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{n-1} & -c_{n-2} & \cdots & \cdots & c_0 \end{pmatrix} = \prod_{z^n=-1}(c_0 + c_1 z + .. + c_{n-1}z^{n-1}).$$

Let $c_j = < P(z)z^j >$ where $P(z) = \frac{(1-z^2)}{(b^2z)^N}\frac{H_z(N)}{H_z(1)}$ and $< Q(z) > = \frac{b}{n}\sum_{z^n=-1} Q(z)$ for any expression $Q(z)$.
We then have that:

$$\prod_{w^n=-1}(c_0 + c_1 w + .. + c_{n-1}w^{n-1}) = \prod_{z^n=-1}\sum_{k=0}^{n-1} < P(z)z^k > w^k.$$

And, for fixed w:

$$\sum_{k=0}^{n-1} < P(z)z^k > w^k = \sum_{k=0}^{n-1}\frac{b}{n}\sum_{z^n=-1}P(z)(zw)^k = \frac{b}{n}\sum_{z^n=-1}P(z)\sum_{k=0}^{n-1}(zw)^k = bP(w^{-1}),$$

so that:

$$\prod_{z^n=-1}(c_0+c_1z+..+c_{n-1}z^{n-1}) = \prod_{w^n=-1} bP(w^{-1}).$$

Using Proposition 6 we get the expression for the partition function $Z_{n,N}$ and this completes the proof of Theorem 1.

We now prove Theorem 2. Using the same ideas of the one in the proof of theorem 1 we get that:

Proposition 6

$$K^{-1}((k,0,1),(k+1,0,0)) =$$

$$\frac{1}{n}\sum_{z^n=-1} z^j\left(\frac{1+z}{bz}\right)^{k+1}\frac{(bz(1+z))^k}{(1-z^2)H_{b,z}(N)}\big(H_{b,z}(N-k) - z(1+z)^2H_{b,z}(N-k-1)\big)-\frac{z^{j-1}}{b}$$

Now, according to [6], $b|K^{-1}((k,0,1),(k+1,0,0))|$ is the probability of having the edge between $(k,0,1)$ and $(k+1,0,0)$ on a random matching we have that the density $p_b(T)$ of particles after time T is given by $1-b|K^{-1}((k,0,1),(k+1,0,0))|$.
Moreover, it is easily seen that when $b=2$, ARW describes particles moving independently of each other, that is each particle performs simple random walk. Using proposition 6 we can compute the asymptotic of $p_2(T)$ and we get the asymptotic density given in theorem 2. We may notice that a similar result was found by D.Balding in [10] in the context of ARW where particles perform continuous time symmetric random walk where particles move with parameter-1 exponential waiting time.

Other directions may be explored using these techniques and for instance the previous computations show that remaining particles are distributed according to a "Pfaffian point process" that we aim to study in a future work.

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References

[1] R. Arratia, Limiting point processes for rescalings of coalescing and annihilating random walks on Z^d.*Ann. Probab.* 9 (1981), no. 6, 909–936.

[2] R. Arratia, Site recurrence for annihilating random walks on Z_d.*Ann. Probab.* 11 (1983), no. 3, 706–713.

[3] M.Bramson, D.Griffeath, Clustering and dispersion rates for some interacting particle systems on *Z*. *Ann. Probab.* 8 (1980), no. 2, 183–213.

[4] P.W. Kasteleyn, The statistics of dimers on a lattice I, The number of dimer arrangements on a quadratic lattice, *Physica* 27 (1961), 1209-1225.

[5] H. Temperley, M. Fisher, The dimer model in statistical mechanics-an exact result. *Phil Mag.*, 6 (1961), 1061-1063.

[6] R. Kenyon, Local statistics of lattice dimers, *Ann.Inst.H.Poicaré, Probabilités*, 33 (1997), 591-618.

[7] R. Kenyon,The planar dimer model with boudary: a survey, Directions in mathematical quasicrystals, 307-328, CRM Monogr. Ser., 13,*Amer. Math. Soc.*, Providence, RI, 2000. A survey of recent mathematical work on the planar dimer model.

[8] R. Kenyon, Long-Range properties of spanning trees in $\mathbb{Z}^2$,*J. Math. Phys.* 41 (2000) 1338-1363.

[9] T.M. Liggett, Interacting Particle Systems, *Grund.der Math.Wiss.* 276 (1985), Spinger-Verlag.

[10] D. Balding, Diffusion-Reaction in one dimension, *J.Appl.Prob.* 25, 733-743 (1988).