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Some remarks concerning harmonic functions on homogeneous graphs

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We obtain a new result concerning harmonic functions on infinite Cayley graphs X: either every nonconstant harmonic function has infinite radial variation in a certain uniform sense, or there is a nontrivial boundary with hyperbolic properties at infinity of X. In the latter case, relying on a theorem of Woess, it follows that the Dirichlet problem is solvable with respect to this boundary. Certain relations to group cohomology are also discussed.

Keywords: Discrete random walks, Dirichlet problem, radial variation, hyperbolic compactifications

1 Introduction

Let X be a connected, locally finite, infinite, oriented graph with no loops and with countable vertex set $V(X)$ and edge set $E(X)$. Two vertices x and y are *adjacent*, denoted $x \sim y$, if they are connected by an edge, and denote by $\deg(x)$ the number of neighbours of x. The *combinatorial Laplacian* acting on $\{f : V(X) \to \mathbb{R}\}$ is

$$(\Delta f)(x) = \deg(x) f(x) - \sum_{y \sim x} f(y).$$

This yields a notion of *harmonic functions* on X, namely those f satisfying

$$\Delta f \equiv 0,$$

which amounts to a mean value property for f.

The laplacian and corresponding harmonic functions are basic and important objects associated to a graph. As in classical potential theory it is intimately connected with an associated *random walk*, namely, the Markov chain defined by transition probabilities

$$P(x \to y) = \frac{1}{\deg(x)}$$

whenever $y \sim x$ and 0 otherwise. The connection is provided by boundary theory: Suppose that X is compactified with a boundary ∂X and that almost every trajectory of the random walk converges to some

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point in ∂X. Then a *Poisson formula* gives harmonic functions h on X from any boundary value $f \in L^{\infty}(\partial X, \mathbb{R})$:

$$h(x) := \int_{\partial X} f(\xi) d\lambda_x(\xi),$$

where λ_x are the hitting measures from the converging random trajectories starting at x.

We will need the following definition: the differential $df : E(X) \to \mathbb{R}$ of a function $f : V(X) \to \mathbb{R}$ is defined by

$$df([x, xs]) = f(xs) - f(x).$$

The study of harmonic functions on graphs are for example relevant to electrical network theory: Let the edges correspond to 1 Ohm resistances. Then by Kirchoff's laws the *passive currents* are exactly the differentials of harmonic functions f belonging to the space of *Dirichlet finite functions*

$$\{f : V(X) \to \mathbb{R} \text{ such that } df \in l^2(E(X), \mathbb{R})\},$$

(which physically means finiteness of energy). We refer to Soardi's book [So 94] for a nice mathematical exposition on this topic.

The main result in the present note is the following:

Theorem 1. *Let $X = X(\Gamma, S)$ be the Cayley graph of an infinite group Γ generated by a finite set S. Either*

$$\sum_{r>0} \sup_{|e| \geq r} |dh(e)| = \infty$$

for every nonconstant harmonic function h, or else X admits an infinite Floyd boundary $\partial_F X$. In the latter case, the Dirichlet problem is solvable with respect to $\partial_F X$, which moreover together with the hitting measure for the random walk starting at the group identity can be identified with the Poisson boundary.

This theorem is the combination of Theorems 2 and 3 below. Groups for which the first alternative in the theorem necessarily must hold include groups not containing a free noncommutative subgroup, $SL_n(\mathbb{Z})$, $n \geq 3$, as well as most Artin braid groups and mapping class groups. Groups which admit an infinite Floyd boundary include groups with infinitely many ends, nonelementary hyperbolic groups and nonelementary geometrically finite Kleinian groups.

2 Floyd compactifications of infinite graphs

We consider X as a geodesic metric space (X, d) in a standard way: for each edge assign a copy of the unit interval with the standard metric, glue them to their associated vertices and let d denote the induced path metric on all of X. Fix a base point $x_0 \in X$. Let $|x| = d(x_0, x)$ and $|A| = \inf_{a \in A} d(x_0, a)$.

We now describe a compactification of X (basically) following [Fl 80]. Let F be a *Floyd admissible function* by which we here mean a monotonically decreasing function $\mathbb{N} \to \mathbb{R}_{>0}$ which is summable:

$$\sum_{r=0}^{\infty} F(r) < \infty,$$

and for which there is in addition a constant $L > 0$ such that $F(r+1) \leq F(r) \leq LF(r+1)$ for all r.

The new length of an edge connecting x and y is

$$F(|\{x,y\}|)$$

(instead of 1). A *path* is a sequence of vertices $\{x_i\}$ such that every x_i and x_{i+1} are adjacent. We define the d_F-length L_F of a path $\alpha = \{x_i\}$ in the graph:

$$L_F(\alpha) = \sum_i F(|\{x_i, x_{i+1}\}|)$$

and the new distance is

$$d_F(z,w) := \inf_\alpha L_F(\alpha),$$

where the infimum is taken over all paths α connecting z and w. It is straightforward to verify that d_F satisfies the axioms of a metric. Note also that X has finite d_F-diameter because F is summable.

We now define $\overline{X}^F$ to be the completion of (X, d_F) in the sense of metric spaces and the boundary is $\partial_F X = \overline{X}^F \setminus X$, which we refer to as a *Floyd boundary of* X. It is simple to see that this completion is compact, see [Fl 80].

The Cayley(–Dehn) graph of a group. Let Γ be an infinite, finitely generated group. Choose a finite generating set S and fix the corresponding Cayley graph $X = X(\Gamma, S)$: the vertex set is Γ and the oriented edges are $[x, xs]$ where $s \in S$ and $x \in \Gamma$.

Stone–Čech compactifications. Different but related compactifications can be constructed as follows: Let $\mathcal{F}$ be the family of functions $f : X \to [0,1]$ such that

$$\sum_{r>0} \sup_{|e|\geq r} |df(e)| < \infty.$$

The corresponding Stone–Čech compactification is obtained by embedding X via the evaluation map to the Cartesian product $[0,1]^{\mathcal{F}}$ and taking the closure. Compare with [R91] or [Gr 93, Ch. 8].

3 The Dirichlet problem

Given a compactification $\overline{X}$, one says that the Dirichlet problem is solvable if every continuous function on ∂X has a continuous extension to $\overline{X}$ which is harmonic on X. By the maximum principle (see Proposition 1 below) any such extension is unique.

Let X be as in the introduction and assume that $|\partial_F X| > 2$ for some Floyd-admissible F. It is established in [K 03b] that the convergence and projectivity axioms of Woess are satisfied and that the action of $Aut(X)$ extends continuously to an action by homeomorphism of $\overline{X}^F$. In other words, $\overline{X}^F$ is a contractive $Aut(X)$-compactification in the sense of [Wo 00].

Assume now that X is a Cayley graph as in section 2 of an infinite, finitely generated group Γ. It is clear that Γ acts transitively on itself by translation on the left as a group of graph automorphisms and moreover it is proved in [K 03b] that if $|\partial_F X| > 2$ then $|\partial_F X| = \infty$ and that Γ does not have a global fixpoint in $\partial_F X$. Therefore, we can invoke Woess' Theorem 20.13 in [Wo 00] and [K 03a] (which uses works of Kaimanovich) to obtain the following:

Theorem 2. *Let $X = X(\Gamma, S)$ be the Cayley graph of a group Γ generated by a finite set S and assume that $\partial_F X$ is a Floyd boundary containing more than 2 points. Then the Dirichlet problem with respect to $\overline{X}^F$ is solvable and $\partial_F X$ with the induced hitting measure is isomorphic to the Poisson boundary.*

This generalizes some previous results on the space of ends and the standard boundary of Gromov hyperbolic graphs, see the notes and references in [Wo 00] and [KW 92].

Convergence of random walks. In order to give some intuition behind this theorem we give a simple argument in a special case to illustrate how random trajectories converge: Let $F(r) = 1/(1+r^2)$ for simplicity and suppose y_n is a sequence of points in X such that $d(y_n, y_{n+1}) = 1$ and $d(x_0, y_n) \geq An$ for some $A > 0$. (The latter condition is satisfied for almost every trajectory of the random walk on a nonamenable group.) Then, since it is readily checked that

$$d_F(y_n, y_m) \leq \sum_{n \leq k \leq m} \frac{1}{1+(Ak)^2},$$

it follows that y_n is a d_F-Cauchy sequence and hence converges to a point in $\partial_F X$.

Nonhomogeneous graphs. For graphs which are not necessarily a Cayley graph (or homogeneous) but which satisfy a strong isoperimetric inequality and infinite $\partial_F X$ for $F << r^{-3}$, it seems that the results and methods in the paper of Kaimanovich and Woess ([KW 92]) can be applied and establish the solvability of the corresponding Dirichlet problem. (Instead of their ρ_a and θ_a we have d_F, and the estimate $d_F(x,y) \leq G((x|y)) << r^{-2}$ from Lemma 1 in [K 03b] can be used in the proof of Theorem 5 in [KW 92].)

More general random walks. For simplicity we have in this paper restricted our discussion to the case of standard simple, nearest neighbor random walk, but it would be possible to treat more general classes of random walks defined by a measure on the group (and corresponding notion of harmonic functions) such as for example the ones considered in [KW 92], [Wo 00], or [K 03b].

4 Constructing Floyd compactifications using harmonic functions

Let X be as in the introduction. We want to construct a compactification of X as in the previous section starting from a harmonic function of a certain kind. We begin by recalling the maximum principle:

Proposition 1. *Suppose that h is a harmonic function such that $h(x) \to 0$ as $x \to \infty$. Then $h \equiv 0$.*

Proof. Since h tends to 0 at infinity we may assume it has a maximum (or minimum) at some point x. But then by the mean value property ($\Delta h \equiv 0$) it holds for all neighbours y of x that $h(x) = h(y)$. As X is connected this equation propagates everywhere and we get that h must be constant. ☐

Note that the above proof shows that on a connected finite graph every harmonic function is constant.

Suppose that h is a function on X such that

$$\sum_{r>0} \sup_{|e| \geq r} |dh(e)| < \infty.$$

(As usual, e denotes the elements of $E(X)$ and $|e|$ is the distance from the endpoints of the edge e to x_0.) Note that it follows that h is bounded and

$$\lim_{n \to \infty} h(\gamma(n))$$

exists for any geodesic ray γ. We assume moreover that h is nonconstant and harmonic.

The preliminary candidate for a Floyd admissible function is

$$H(r) := \sup_{|e|\geq r} |dh(e)|,$$

since this function is clearly monotonically decreasing, positive and summable. It remains to prove the existence of $L > 0$ such that $H(r+1) \leq H(r) \leq LH(r+1)$ for all r. This technical property (which might not be necessary) is used in [K 03b] to show that the isometric action of Γ on the Cayley graph extends to an action by homeomorphisms of the completion $\overline{X}^H$. To guarantee this property we replace H by a function F which the following technical lemma provides:

Lemma 1. *There is a summable function F such that $F(r) \geq H(r)$ and $F(r+1) \leq F(r) \leq 2F(r+1)$ for every r.*

Proof. Let $[a_i, b_i]$ be all the disjoint maximal intervals for which it holds that

$$H(r) \leq 2^{-(r-a_i)} H(a_i),$$

for all $r \in [a_i, b_i]$. For r outside these intervals we let $F(r) = H(r)$ and for $r \in [a_i, b_i]$ we let

$$F(r) = 2^{-(r-a_i)} H(a_i).$$

By construction $F(r) \geq H(r)$ and $F(r+1) \leq F(r) \leq 2F(r+1)$ for all r. To see the summability, note that

$$\sum_{r>0} F(r) \leq \sum_{r>0} H(r) + \sum_{i>0} \sum_{k=1}^{b_i-a_i} 2^{-k} H(a_i).$$

We therefore should estimate the double-sum. For that, first note that $H(a_{i+1}) \leq 2^{-(b_i-a_i)} H(a_i) \leq 2^{-1} H(a_i)$ because of the monotonicity of H. Hence $H(a_i) \leq 2^{-i} H(0)$ and the double-sum is dominated by

$$H(0) \sum_{i\geq 0} 2^{-i}.$$

The summability of F is now established. □

Now we show that $|\partial_F X| \geq 2$. To see this note that for any path $\{x_i\}$ connecting two points $x, y \in X$ we have

$$\begin{aligned} |h(x) - h(y)| &= |h(x) - h(x_1) + h(x_1) - h(x_2) + \ldots - h(y)| \\ &\leq \sum_i |h(x_i) - h(x_{i+1})| \leq \sum_i H(|\{x_i, x_{i+1}\}|) \\ &\leq L_F(\{x_i\}). \end{aligned}$$

Since this estimate holds for any path connecting x and y we get that

$$|h(x) - h(y)| \leq d_F(x, y).$$

Therefore, as h is a nonconstant harmonic function it has at least two distinct limiting values (as follows from Proposition 1), there are two geodesic rays which converge to (end at) two different points in $\partial_F\Gamma$. Thus the boundary $\partial_F\Gamma$ we constructed from h is nontrivial in the sense that it contains at least two points.

5 Radial variation of harmonic functions

In classical analysis, the radial variation of harmonic functions f on the disk

$$V(f,\theta)=\int_0^1 |f'(re^{i\theta}|dr$$

has been studied by Fatou, Zygmund, Rudin, Bourgain and others. Some analogous results for harmonic functions on trees have been obtained, see [Ca 72] for an early paper and see [CFPR 01] and [KW 00] for two recent contributions. Here we establish that certain graphs have no nonconstant harmonic functions of uniformly finite (in the sense specified in the theorem) radial variation.

Theorem 3. *Assume X is a Cayley graph of an infinite, finitely generated group not admitting an infinite Floyd boundary. If h is a harmonic function on X such that*

$$\sum_{r>0}\sup_{|e|\geq r}|dh(e)|<\infty,$$

then h is constant.

Proof. It is well known that if $|\partial_F X|=2$ then Γ is virtually $\mathbb{Z}$ and it does not admit any nonconstant bounded harmonic functions. In view of section 4 the theorem now follows. □

The above result can be viewed as an anti-Fatou-type or Liouville-property statement.

In [K 03b], it is proved that if Γ does not contain a free nonabelian subgroup then every $\partial_F\Gamma$ contains at most two points. Moreover groups, such as $SL_n(\mathbb{Z})$, for $n\geq 3$, most Artin braid groups, mapping class groups and automorphism groups of free groups, are shown in [KN 02] not to admit any nontrivial Floyd boundary. The same is true if Γ has an infinite, amenable (or, more generally, does not contain a nonabelian free subgroup) normal subgroup, see [K 03b]. In [Fl 80] it is proved that any Floyd boundary of the direct product of two infinite groups is trivial.

It is known that any nonamenable group admits many nonconstant bounded harmonic functions. Note also that it is easy to write down harmonic functions h on a d-regular tree for which $dh(e)$ decays as fast as $(d-1)^{-|e|}$. It therefore makes sense to formulate:

Theorem 4. *Let Γ be $SL_n(\mathbb{Z})$ for $n\geq 3$, or a group containing an infinite normal subgroup not containing a nonabelian free group, or a product of two finitely generated infinite groups. Choose some finite generating set S. Then for any nonconstant harmonic function h it holds that*

$$\sum_{r>0}\sup_{|e|\geq r}|dh(e)|=\infty.$$

6 Some group cohomology

The present section is included for at least two reasons. First, it points out that the sections 2 and 4 are rather cohomological in nature. Second, it outlines a conceivable strategy to study the question of the triviality of the first L^2-Betti number, or in a different language, the existence of nonconstant Dirichlet finite harmonic functions (see [BV 97]).

It is possible to relate Floyd admissible functions which give nontrivial boundaries to a certain reduced first cohomology group: Let L^1C_0 denote the functions on Γ whose supremum outside balls is integrable as in section 4. Define moreover

$$\overline{L^1C_0(X)} = \{f \in C_0(X) : df \in L^1C_0(E(X)\}$$

and finally

$$\overline{H}^1(\Gamma, L^1C_0) \cong \{f : \Gamma \to \mathbb{R} \text{ such that } df \in L^1C_0(E_\Gamma)\}/(\mathbb{R}+\overline{L^1C_0(\Gamma)}).$$

(This is just one possible way of defining these objects.)

Theorem 5. *Let Γ be an infinite, finitely generated group. The maximal number of points in a Floyd boundary of Γ equals*

$$1 + \dim \overline{H}^1(\Gamma, L^1C_0).$$

(Here, as well as in the next theorem, by "equal" is meant that, if both numbers are finite then they are equal, otherwise both are infinite.) This is in analogy with a standard result essentially due to Specker [Sp 49]:

Theorem 6. *Let Γ be an infinite, finitely generated group. The number of ends equals*

$$1 + \dim H^1(\Gamma, C_c).$$

(Here C_c denotes the finitely supported real valued functions on a finitely generated group Γ, also called the group algebra and denoted $\mathbb{R}\Gamma$.) As expected,

$$H^1(\Gamma, C_c) \rightarrowtail \overline{H}^1(\Gamma, L^1C_0)$$

in a natural way. These statements can be proved following the paper of Bekka and Valette [BV 97] with suitable modifications.

Since L^2-cohomology is of particular importance (for example to electrical network theory), it make sense to raise the following question (which partly came up in some discussions with Marc Bourdon):

Question. Given a finitely generated infinite group Γ with nontrivial first L^2-Betti number, what properties does the action of Γ on the associated compactification of Stone–Čech type have?

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