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Discrete Random Walks on One-Sided "Periodic" Graphs

Michael Drmota

Department of Geometry, TU Wien, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria
michael.drmota@tuwien.ac.at

In this paper we consider discrete random walks on infinite graphs that are generated by copying and shifting one finite (strongly connected) graph into one direction and connecting successive copies always in the same way. With help of generating functions it is shown that there are only three types for the asymptotic behaviour of the random walk. It either converges to the stationary distribution or it can be approximated in terms of a reflected Brownian motion or by a Brownian motion. In terms of Markov chains these cases correspond to positive recurrence, to null recurrence, and to non recurrence.

Keywords: discrete random walk, generating functions, singularity analysis

1 Introduction

The purpose of this paper is to consider random walks on infinite graphs G of the following type. Let K and L be a finite strongly connected di-graphs and $K_0, K_1, K_2, \ldots$ copies of K. The set of nodes, $V(G)$, of G is now given by $V(L) \cup V(K_0) \cup V(K_1) \cup \cdots$. The directed edges of G, $E(G)$, consist first of the edges $E(L) \cup E(K_0) \cup E(K_1) \cup \cdots$ and second of edges between L and K_0, between K_0 and K_1, between K_1 and K_2 etc., where the edges from K_j to K_{j+1} are the *same* for all $j = 0, 1, 2, \ldots$. We also assume that every node of K_0 has the same outdegree as K_j for $j = 1, 2, \ldots$, that is, every directed edge from K_{j+1} to K_j (for $j = 0, 1, 2, \ldots$) has a counterpart from K_0 to L.

We consider a discrete random walk X_n (as a Markov chain) on G, where the starting point X_0 is in L. We also assume that the transition probabilities of the corresponding nodes of K_j are the same for all $j = 0, 1, 2, \ldots$.

The simplest case is the one-sided "*linear*" graph, where L and K_j have size 1 (see Figure 1). Of course, the corresponding random walk is just a Markov chain on the non-negative integers with reflection at zero, see Feller (1968, 1971), or a discrete time version of the continuous time Markov chain modeling a M/M/1 queue (see Neuts (1981, 1989); Latouche and Ramaswami (1999)). It is well known that the random walk X_n on G is either positive recurrent, null recurrent, or non recurrent.

The general case corresponds to a discrete time version of a homogeneous quasi-birth-and-death process (see Neuts (1981, 1989); Latouche and Ramaswami (1999)) that is characterised by a Neuts structure given

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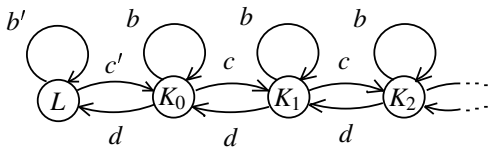

Fig. 1: One-sided "linear" graph (with transition probabilities)

by an infinite matrix of the form

$$\begin{pmatrix} \mathbf{B}' & \mathbf{C}' & \mathbf{0} & \mathbf{0} & \cdots & \\ \mathbf{D}' & \mathbf{B} & \mathbf{C} & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{D} & \mathbf{B} & \mathbf{C} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{D} & \mathbf{B} & \mathbf{C} & \cdots \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where the (finite) matrices $\mathbf{B}', \mathbf{C}', \mathbf{D}', \mathbf{B}, \mathbf{C}, \mathbf{D}$ *collect* the transition probabilities (see Section 3). These kinds of graphs also appear in performance evaluation, for example, compare with Hermanns et al. (2002).

It is also worth mentioning that there are specific problems in combinatorics, where graphs of this type appear, for example, the graph presented in Figure 2[†] is related to a problem of bin-packing (see Prodinger (1985, 1990)).

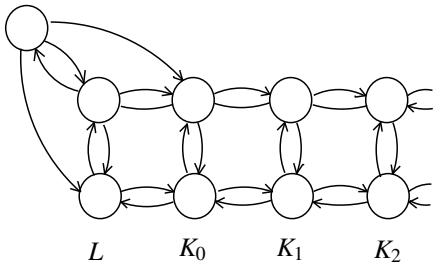

Fig. 2: Graph related to bin-packing

The main purpose of this paper is to indicate that depending on the transition probabilities there are three typical asymptotic behaviours of X_n. It either converges to the stationary distribution or it can be approximated in terms of a reflected Brownian motion or by a Brownian motion.[‡] In terms of Markov chains these cases correspond to positive recurrence, to null recurrence, and to non recurrence.

[†] This example was in fact the motivating example for writing this paper.

[‡] We only present one dimensional distributional results. However, with help of the same proof techniques we easily obtain corresponding functional versions. One has to show finite-dimensional distributional results and tightness. Both properties can be shown with help of analytic methods applied to corresponding multivariate generating functions, for related problems and methods see Drmota and Gittenberger (1997) or Drmota et al. (2001).

In what follows we present a unified approach to these kinds of problems that is bases on generating functions and on analytic methods (singularity analysis, saddle point techniques) for obtaining asymptotic relations for the coefficients via Cauchy's formula. It seems that this kind of method has not been used in this context in this generality.

It also seems that the precise statements given below (in particular the second and third part of Theorem 2) are new in this generality. The case of positive recurrence (of Theorem 2) has been discussed in detail (see Latouche and Ramaswami (1999)). Also, it is well known that Dyck paths and Motzkin paths can be approximated by a reflected Brownian motion. Further, the paper of Lalley (2001) deals with random walks on regular languages – it seems that our case may be viewed as special cases – but the results there concern only asymptotic expansions for the probabilities $\mathbf{Pr}\{X_0 = v, X_n = w\}$, where $v, w \in V(G)$ are fixed and n tends to infinity, compare also with Lalley (1995).

It would be interesting, too, to extend the present results to graphs G with specific infinite graphs L and K_j. This would cover one-sided versions of the random walk on the d-dimensional grid. (One can either try to use the method by Lalley (2002) for infinite systems of functional equations or the Fourier analytic methods by Guivarc'h (1984) and Krámli and Szász (1984), compare also with (Woess, 2000, Section 13).)

In section 2 we first consider the simplest case of a *one-sided "linear" graph* (see Figure 1) that is related to the classical random walk on the non-negative integers (Dyck paths, Motzkin paths etc.). The general case will then treated in section 3.

2 The one-sided "linear" graph

2.1 Statement of the Result

In this section we will describe in detail the asymptotic behaviour of X_n with $X_0 = L$ for the one-sided "linear" graph G, where b, c, d and b', c' denote the corresponding transition probabilities (compare with Figure 1).[§]

Theorem 1. *Suppose that b, c, d and b', c' are positive numbers with $b + c + d = b' + c' = 1$; and let X_n be the random walk on the one-sided "linear" graph G with $X_0 = L$.*

1. *If $c < d$ then we have*

$$\lim_{n\to\infty} \mathbf{Pr}\{X_n = L\} = \frac{d-c}{d-c+c'} \quad \text{and} \quad \lim_{n\to\infty} \mathbf{Pr}\{X_n = K_\ell\} = \frac{c'(d-c)}{d(d-c+c')} \left(\frac{c}{d}\right)^\ell \quad (\ell \geq 0).$$

   that is, X_n is positive recurrent. The distribution of X_n converges to the stationary distribution.

2. *If $c = d$ then $X_n/\sqrt{cn}$ is null recurrent and converges weakly to the absolute normal distribution. In particular, we have, as $n \to \infty$,*

$$\mathbf{Pr}\{X_n = L\} = \frac{1}{c'}\sqrt{\frac{2c}{n\pi}} + O\left(\frac{1}{n}\right).$$

[§] Theorem 1 is surely not new, but it seems that the proof method is. Furthermore, it plays the rôle of a *prototype* for the general case covered by Theorem 2.

and (uniformly for all $\ell \geq 0$)

$$\mathbf{Pr}\{X_n = K_\ell\} = \sqrt{\frac{2}{nc\pi}} \exp\left(-\frac{\ell^2}{2cn}\right) + O\left(\frac{1}{n}\right).$$

3. *If $c > d$ then X_n is non recurrent and $(X_n - (c-d)n)/\sqrt{(c+d-(c-d)^2)n}$ converges weakly to the standard normal distribution. We also have, as $n \to \infty$ and uniformly for all $\ell \geq 0$,*

$$\mathbf{Pr}\{X_n = K_\ell\} = \frac{1}{\sqrt{2\pi(c+d-(c-d)^2)n}} \exp\left(-\frac{(\ell-(c-d)n)^2}{2(c+d-(c-d)^2)n}\right) + O\left(\frac{1}{n}\right).$$

Note that the assumption that $b > 0$ and $b' > 0$ are not that restrictive. In particular if one of them is zero then the result remains true as it is. Only if both are zero then $X_n \notin K_\ell$ (and consequently $\mathbf{Pr}\{X_n \in K_\ell\} = 0$) if n and ℓ have the same parity. However, if $n \not\equiv \ell \bmod 2$ then we get qualitatively the same result (and the proofs are a little bit more technical).

Note also that the probabilities $\mathbf{Pr}\{X_n = L\}$ have been discussed in Lalley (2001) for the case $b = 0$ and $b' > 0$.

Finally, as mentioned above, with a little bit more effort it can be shown that in the case $c = d$ the *normalized* discrete processes

$$\left(\frac{X_{\lfloor tn \rfloor}}{\sqrt{cn}}, t \geq 0\right)_{n \geq 1}$$

converges weakly to a reflected Brownian motion as $n \to \infty$; and for $c < d$ the processes

$$\left(\frac{X_{\lfloor tn \rfloor} - t(c-d)n}{\sqrt{(c+d-(c-d)^2)n}}, t \geq 0\right)_{n \geq 1}$$

converges weakly to the standard Brownian motion.

2.2 Generating Functions

We start with a property of one-sided paths on the integers (see Figure 3).

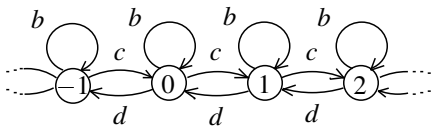

Fig. 3: Random walk on the integers

Lemma 1. *Let Y_n denote the random walk on the integers (see Figure 3) with $Y_0 = 0$. Then the generating function of one-sided return probabilities*

$$M(x) = \sum_{n \geq 0} \mathbf{Pr}\{Y_1 \geq 0,\ Y_2 \geq 0, \ldots,\ Y_{n-1} \geq 0,\ Y_n = 0\} \cdot x^n \tag{1}$$

satisfies the functional equation

$$M(x) = 1 + bxM(x) + cdx^2M(x)^2 \tag{2}$$

and is thus explicitly given by

$$M(x) = \frac{1 - bx - \sqrt{(1-bx)^2 - 4cdx^2}}{2cdx^2}.$$

The radius of convergence x_0 is given by

$$x_0 = \frac{1}{b + 2\sqrt{cd}} = \frac{1}{1 - (\sqrt{c} - \sqrt{d})^2}.$$

If $b > 0$ then x_0 is also the only singularity on the circle of convergence $|x| = x_0$. Furthermore, $M(x)$ has a local expansion of the form

$$M(x) = \frac{b + 2\sqrt{cd}}{\sqrt{cd}} - \frac{1}{\sqrt{2}} \left(\frac{b + 2\sqrt{cd}}{\sqrt{cd}} \right)^{3/2} \cdot \sqrt{1 - (b + 2\sqrt{cd})x} + O\left(1 - (b + 2\sqrt{cd})x\right) \tag{3}$$

around its singularity $x = x_0$.

Note that the generating function $M(x)$ is closely related to the generating functions $U(x)$, $G(x)$, and $R(x)$ presented in (Latouche and Ramaswami, 1999, p. 96). We have $M(x) = 1/(1 - U(x))$, $G(x) = M(x) \cdot dx$, and $R(x) = cx \cdot M(x)$.

Proof. The functional equation (2) is immediately clear by writing it in the following way:

$$M(x) = 1 + bx \cdot M(x) + cx \cdot M(x) \cdot dx \cdot M(x).$$

If the first step is the loop (with probability b) then the remaining part is just a non-negative path from 0 to 0 and the contribution is $bx \cdot M(x)$. If the first step goes to the right (with probability c) then we decompose the path into four parts: first the step from 0 to the right, then we consider the part from 1 to 1 that is followed by the first step back from 1 to 0, the third part is this step back, and the last part is again a non-negative path from 0 to 0. Hence, in terms of generating functions this case contributed $cx \cdot M(x) \cdot dx \cdot M(x)$. This proves (2).

The remaining properties follow directly from (2). □

Next consider the *original* one-sided "linear" graph.

Lemma 2. *Let X_n denote the random walk on the graph represented by Figure 1 and set*

$$M_L(x) = \sum_{n \geq 0} \mathbf{Pr}\{X_n = L\} \cdot x^n \quad \textit{and} \quad M_\ell(x) = \sum_{n \geq 0} \mathbf{Pr}\{X_n = K_\ell\} \cdot x^n \quad (\ell \geq 0).$$

Then

$$M_L(x) = \frac{1}{1 - b'x - c'dx^2M(x)} \quad \textit{and} \quad M_\ell(x) = \frac{c'}{c} \frac{(cxM(x))^{\ell+1}}{1 - b'x - c'dx^2M(x)} \quad (\ell \geq 0).$$

Proof. With help of the same reasoning as in the proof of Lemma 1 one gets the relation

$$M_L(x) = 1 + b'xM_L(x) + c'xM(x)dxM_L(x)$$

that proves the proposed representation for $M_L(x)$.

Next we have $M_0(x) = M_L(x)c'xM(x)$. Here we have to divide all paths from L to K_0 into three parts. The first part is just the path from L to L that is followed by the last step from L to K_0. This step is the second part, and the third part is a *non-negative* path from K_0 to K_0. In a similar way we also obtain the recurrence $M_{\ell+1}(x) = M_\ell(x)cxM(x)$. This completes the proof of Lemma 2. □

2.3 Analytic Methods

We now use the above explicit representations for $M_L(x)$ and $M_\ell(x)$ $(\ell \geq 0)$ and Cauchy's formula to *extract* the coefficients, e.g.

$$\mathbf{Pr}\{X_n = K_\ell\} = \frac{1}{2\pi i}\int_{|x|=r} \frac{M_\ell(x)}{x^{n+1}}\,dx,$$

where r is smaller that the radius of convergence of $M_\ell(x)$. By shifting the path of integration suitably in the analyticity region of $M_\ell(x)$ and evaluating asymptotically the integral we will thus obtain asymptotic expansions for $\mathbf{Pr}\{X_n = K_\ell\}$. In particular we have to deal with three different cases, first with a polar singularity, second with a square-root singularity, and third we have to apply saddle point techniques. These kinds of techniques are very well established in the literature. Therefore we will not work out all the details but refer to proper references (e.g. to Drmota (1994)).

We start with the case $c < d$.

Lemma 3. *Suppose that $c < d$. Then the radius of convergence of $M_L(x)$ and $M_\ell(x)$ $(\ell \geq 0)$ is $x_1 = 1$ that is also a polar singularity of order* 1. *Furthermore, we have*

$$\lim_{n\to\infty} \mathbf{Pr}\{X_n = L\} = \frac{d-c}{d-c+c'} \quad \textit{and} \quad \lim_{n\to\infty} \mathbf{Pr}\{X_n = K_\ell\} = \frac{c'(d-c)}{d(d-c+c')}\left(\frac{c}{d}\right)^\ell \quad (\ell \geq 0). \tag{4}$$

Proof. First note that (for $c < d$) we have $M(1) = 1/d$ and $M'(1) = (1-d+c)/(d(d-c))$. Thus,

$$1 - b'x - c'dx^2M(x) = \frac{d-c+c'}{d-c}(1-x) + O((1-x)^2).$$

and consequently

$$M_L(x) = \frac{d-c}{d-c+c'}\frac{1}{1-x} + \text{analytic function}$$

and

$$M_\ell(x) = \frac{c'(d-c)}{d(d-c+c')}\left(\frac{c}{d}\right)^\ell \frac{1}{1-x} + \text{analytic function}$$

for $|x| < 1/(b+2\sqrt{cd})$. (Note that $1/(b+2\sqrt{cd}) > 1$.) Of course, this directly implies (4). □

The most interesting case is the case $c = d$.

Lemma 4. *Suppose that $c = d$. Then the radius of convergence of $M_L(x)$ and $M_\ell(x)$ $(\ell \geq 0)$ is $x_1 = 1$ that is an algebraic singularity. Here we get, as $n \to \infty$,*

$$\mathbf{Pr}\{X_n = L\} = \frac{1}{c'}\sqrt{\frac{2c}{n\pi}} + O\left(\frac{1}{n}\right). \tag{5}$$

and (uniformly for all $\ell \geq 0$)

$$\mathbf{Pr}\{X_n = K_\ell\} = \sqrt{\frac{2}{nc\pi}}\exp\left(-\frac{\ell^2}{2cn}\right) + O\left(\frac{1}{n}\right). \tag{6}$$

Proof. The essential difference between the present case and that of Lemma 3 is that $M(x)$ is not regular at $x = 1$. We have to use the singular expansion (3) of Lemma 1 and obtain (around $x = 1$)

$$1 - b'x - c'dx^2M(x) = \frac{c'}{\sqrt{2c}}\sqrt{1-x} + O(1-x).$$

Furthermore

$$(cxM(x))^\ell = \exp\left(-\frac{\ell}{\sqrt{2c}}\sqrt{1-x} + O(\ell(1-x))\right).$$

Thus, the dominant behaviour of $M_\ell(x)$ around $x_0 = 1$ is of the form

$$\sqrt{\frac{2}{c}} \cdot \frac{\exp\left(-\frac{\ell}{\sqrt{2c}}\sqrt{1-x}\right)}{\sqrt{1-x}}.$$

We can now proceed as in the proof of Theorem 4 of Drmota (1994). We just have to use the formula

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-\lambda\sqrt{-t}-t}}{\sqrt{-t}}\,dt = \frac{1}{\sqrt{\pi}}e^{-\lambda^2},$$

where γ denotes a Hankel contour. This directly leads to (5) and (6). □

The analysis of the final case $c > d$ is a little bit different from the previous ones. In the first two cases the singular behaviour of $M_L(x)$ and $M_\ell(x)$ around the point $x_0 = 1$ has governed the asymptotic behaviour of the coefficients. In the third case we will again work around the *critical* point $x_0 = 1$ but now with help of a saddle point method. The radius of convergence is larger than 1.

Lemma 5. *Suppose that $c > d$. Then X_n satisfies a central limit theorem with mean value $\mathbf{E}X_n \sim (c-d)n$ and $\mathbf{Var}X_n \sim (c+d-(c-d)^2)n$. In particular we have the following local limit theorem as $n \to \infty$ and uniformly for all $\ell \geq 0$:*

$$\mathbf{Pr}\{X_n = K_\ell\} = \frac{1}{\sqrt{2\pi(c+d-(c-d)^2)n}}\exp\left(-\frac{(\ell-(c-d)n)^2}{2(c+d-(c-d)^2)n}\right) + O\left(\frac{1}{n}\right). \tag{7}$$

Proof. Note first that $M_\ell(x) = \frac{c'}{c}M_L(x)(cxM(x))^{\ell+1}$ and that $x_1 = 1$ is a regular point of $M_\ell(x)$. Thus, $M_\ell(x)$ is (despite of an analytic factor) a power of the function $cxM(x)$. Consequently, we can directly apply the (saddle point) methods of Drmota (1994) and obtain the result. □

3 The general case

3.1 Matrices of Generating Functions

We are now going to consider the general situation. We will denote by $\mathbf{B}, \mathbf{C}, \mathbf{D}$ the corresponding matrices containing the transition probabilities inside K_j, from K_j to K_{j+1}, from K_{j+1} to K_j and by and $\mathbf{B}', \mathbf{C}', \mathbf{D}'$ the transition probabilities inside L, from L to K_0 and from K_0 to L. (Note that in contrast to the "linear" case $\mathbf{D}$ and $\mathbf{D}'$ are different in general.)

We now assume that the random walk X_n starts at a vertex w in L.

The first (and easy) step is to generalize the above relations for generating functions. Let $\mathbf{M}_L(x) = (M_{L;w,w'}(x))_{w,w' \in V(L)}$ denote the matrix of the generating functions

$$M_{L;w,w'}(x) = \sum_{n \geq 0} \mathbf{Pr}\{X_0 = w,\ X_n = w'\} \cdot x^n$$

and $\mathbf{M}_\ell(x) = (M_{\ell;w,v}(x))_{w \in V(L), v \in K_\ell}$ the matrix of functions

$$M_{\ell;w,v}(x) = \sum_{n \geq 0} \mathbf{Pr}\{X_0 = w,\ X_n = v\} \cdot x^n.$$

Lemma 6. *Let $\mathbf{M}(x) = (M_{v,v'}(x))_{v,v' \in K}$ denote the (analytic) solution with $\mathbf{M}(0) = \mathbf{I}$ of the matrix equation*

$$\mathbf{M}(x) = \mathbf{I} + x\mathbf{B}\,\mathbf{M}(x) + x^2\mathbf{C}\,\mathbf{M}(x)\,\mathbf{D}\,\mathbf{M}(x). \tag{8}$$

Then $\mathbf{M}_L(x)$ and $\mathbf{M}_\ell(x)$ are given by

$$\mathbf{M}_L(x) = \left(\mathbf{I} - x\mathbf{B}' - x^2\mathbf{C}'\,\mathbf{M}(x)\,\mathbf{D}'\right)^{-1} \tag{9}$$

and (for $\ell \geq 0$)

$$\mathbf{M}_\ell(x) = x^{\ell+1}\left(\mathbf{I} - x\mathbf{B}' - x^2\mathbf{C}'\,\mathbf{M}(x)\,\mathbf{D}'\right)^{-1}\mathbf{C}'\,\mathbf{M}(x)\,(\mathbf{C}\,\mathbf{M}(x))^\ell. \tag{10}$$

Proof. The proof is exactly the same as that of Lemma 1 and 2 and already appears (for the case of Figure 2) in Prodinger (1990), compare also with Kuich and Urbanek (1983). For $x = 1$ the matrix $\mathbf{M} = \mathbf{M}(1)$ is also related to the matrices $\mathbf{U}$, $\mathbf{G}$, and $\mathbf{R}$ of (Latouche and Ramaswami, 1999, p. 137), in particular, $\mathbf{M} = (\mathbf{I} - \mathbf{U})^{-1}$, $\mathbf{G} = \mathbf{MD}$, and $\mathbf{R} = \mathbf{CM}$. □

The main difference to the "linear" case is that we are now not able to solve the above system (8)–(10) explicitly. Nevertheless, from an analytic point of view they behave in a similar way. Let us start with $\mathbf{M}(x)$, the solution of (8).

Lemma 7. *Suppose that $\mathbf{B}$ is a primitive irreducible matrix and let $\mathbf{M}(x) = (M_{v,v'}(x))_{v,v' \in V(K)}$ denote the solution of (8). Then all functions $M_{v,v'}(x)$ have a common radius of convergence $x_0 \geq 1$. Furthermore, x_0 is the only singularity on the circle of convergence $|x| = x_0$ and there is a local expansion of the form*

$$\mathbf{M}(x) = \tilde{\mathbf{M}}_1 - \tilde{\mathbf{M}}_2\sqrt{1 - \frac{x}{x_0}} + O\left(1 - \frac{x}{x_0}\right) \tag{11}$$

around its singularity $x = x_0$, where $\tilde{\mathbf{M}}_1$ and $\tilde{\mathbf{M}}_2$ are matrices with positive elements.

Proof. The relation (8) is a system of $|V(K)|^2$ algebraic equation for the functions $M_{v,v'}(x)$ that can be written in the form $\mathbf{Q}(x) = \mathbf{F}(x, \mathbf{Q}(x))$, where $\mathbf{Q}(x)$ is just the vector of functions $M_{w,w'}(x)$ and $\mathbf{F}(x, \mathbf{y})$ is a proper (non-linear) polynomial vector function with non-negative coefficients. By assumption **B** is irreducible (and non-negative). Thus, the so-called *dependency graph* (compare with Drmota (1997)) of this system is strongly connected, that is, it is impossible to solve a subsystem before solving the whole system. Consequently, all (algebraic) functions $M_{v,v'}(x)$ have the same finite radius of convergence and by Lalley (2001) (compare also with Drmota (1997)) they have a square-root singularity at $x = x_0$ of the form (11), where all entries of $\tilde{\mathbf{M}}_1$ and $\tilde{\mathbf{M}}_2$ are positive.

The assumption that **B** is primitive implies that all (sufficiently large) coefficients of the power series $M_{v,v'}(x)$ are positive. This property shows that $x = x_0$ is the only singularity on the circle of convergence $|x| = x_0$ (compare with Drmota (1997), where this property is called *of simple type*).

Finally, we surely have $x_0 \geq 1$. For, if $x_0 < 1$ then the coefficients of $M_{v,v'}(x)$ are unbounded. However, the coefficients of $M_{v,v'}(x)$ are probabilities (compare also with (1)) and thus bounded. This completes the proof of the lemma. □

This lemma also shows that all entries of the matrix function

$$\mathbf{M}_L(x) = \left(\mathbf{I} - x\mathbf{B}' - x^2\mathbf{C}'\,\mathbf{M}(x)\,\mathbf{D}'\right)^{-1}$$

have a finite radius of convergence x_1 that satisfies

$$1 \leq x_1 \leq x_0.$$

(Note that x_1 cannot be smaller than 1 since the coefficients are probabilities and thus bounded.)

3.2 A General Theorem

As in the "linear" case there are three kinds of asymptotic behaviours for X_n, where we assume that $X_0 = w_0$ with a given node $w_0 \in V(L)$.

Theorem 2. *Suppose that the matrices* **B** *and* $\mathbf{B}'$ *are primitive irreducible, that no row of* **C** *is zero, and that the matrices* $\mathbf{D}, \mathbf{C}', \mathbf{D}'$ *are non-zero. Let* X_n *denote the random walk on* G *with* $X_0 = w_0 \in V(L)$ *and let* x_0 *and* x_1 *denote the radius of convergence of the entries of* $\mathbf{M}(x)$ *and* $\mathbf{M}_L(x)$.

1. *If* $x_0 > 1$ *and* $x_1 = 1$ *then* X_n *is positive recurrent and for all* $v \in V(G) = V(L) \cup V(K_0) \cup V(K_1) \cup \cdots$ *we have*

$$\lim_{n\to\infty} \mathbf{Pr}\{X_n = v\} = p_v,$$

where $(p_v)_{v\in V(G)}$ *is the (unique) stationary distribution on* G. *Furthermore, there exists a non-negative matrix* **R** *(where all eigenvalues have moduli* < 1*) such that*

$$\mathbf{p}_{\ell+1} = \mathbf{p}_\ell \mathbf{R}, \tag{12}$$

in which $\mathbf{p}_\ell = (p_v)_{v\in K_\ell}$.

2. *If* $x_0 = x_1 = 1$ *then* X_n *is null recurrent and there exist* $\rho_{v'} > 0$ $(v' \in V(K))$, $\rho'_w > 0$ $(w \in V(L))$ *and* $\eta > 0$ *such that, as* $n \to \infty$,

$$\mathbf{Pr}\{X_n = w\} = \rho'_w \sqrt{\frac{1}{n\pi}} + O\left(\frac{1}{n}\right) \quad (w \in V(L)). \tag{13}$$

and (uniformly for all $\ell \geq 0$)

$$\mathbf{Pr}\{X_n = v\} = \rho_{\tilde{v}} \sqrt{\frac{1}{n\pi}} \exp\left(-\frac{\ell^2}{2\eta n}\right) + O\left(\frac{1}{n}\right) \quad (v \in V(K_\ell)), \tag{14}$$

where $\tilde{v}$ (for $v \in K_\ell$) denotes the corresponding node in K.

3. *If $x_1 > 1$ then X_n is non recurrent and there exist $\tau_{v'} > 0$ ($v' \in V(K)$), $\mu > 0$ and $\sigma > 0$ such that, as $n \to \infty$ and uniform for all $\ell \geq 0$,*

$$\mathbf{Pr}\{X_n = v\} = \frac{\tau_{\tilde{v}}}{\sqrt{n}} \exp\left(-\frac{(\ell - \mu n)^2}{2\sigma^2 n}\right) + O\left(\frac{1}{n}\right) \quad (v \in V(K_\ell)), \tag{15}$$

where $\tilde{v}$ (for $v \in K_\ell$) denotes the corresponding node in K.

Theorem 2 is, of course, a direct generalization of Theorem 1. As above the second and the third case can be generalized to functional limit theorems in the following sense. For $v \in V(L)$ let $\hat{v} := -1$ and for $v \in V(K_\ell)$ set $\hat{v} := \ell$. Then $\hat{X}_n$ is a process on the integers ≥ -1 and after a proper scaling $\hat{X}_n$ can be approximated by a reflected Brownian motion or by a Brownian motion. Note further that the matrix $\mathbf{R}$ in the first part of Theorem 2 is the classical $\mathbf{R}$-matrix for positive recurrent quasi-birth-and-death processes, it is given by $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}(1)$ and satisfies the equation $\mathbf{R} = \mathbf{C} + \mathbf{R}\mathbf{B} + \mathbf{R}^2\mathbf{D}$, compare with (Latouche and Ramaswami, 1999, Theorem 6.2.1).

3.3 Proof of the Theorem

Proof. First, let us consider the case $x_0 > 1$ and $x_1 = 1$. By assumption, $x = 1$ is a regular point of $\mathbf{M}(x)$ and, thus, the function

$$f(x) = \det\left(\mathbf{I} - x\mathbf{B}' - x^2\mathbf{C}'\mathbf{M}(x)\,\mathbf{D}'\right)$$

is regular at $x = 1$ and satisfies $f(1) = 0$. Equivalently, 1 is an eigenvalue of the matrix $\mathbf{B}' + \mathbf{C}'\mathbf{M}(1)\,\mathbf{D}'$. Since the matrix $\mathbf{B}' + \mathbf{C}'\mathbf{M}(1)\,\mathbf{D}'$ is primitive irreducible, 1 is a simple eigenvalue. Consequently $x = 1$ is a simple zero of $f(x)$ (and there are no further zeros on the circle $|x| = 1$). Hence all functions of the inverse matrix $\left(\mathbf{I} - x\mathbf{B}' - x^2\mathbf{C}'\mathbf{M}(x)\,\mathbf{D}'\right)^{-1}$ have a simple pole at $x = 1$ (and no other singularities on the circle $|x| = 1$). Thus, it follows as in the proof of Lemma 3 that the limits

$$\lim_{n\to\infty} \mathbf{Pr}\{X_n = w\}$$

exist for $w \in V(L)$. Similarly we get the existence of the corresponding limits for $v \in K_\ell$ and (12) with $\mathbf{R} = \mathbf{C}\mathbf{M}(1)$. Since $\sum_{v \in V(G)} p_v = 1$ the moduli of all eigenvalues of $\mathbf{R}$ have to be smaller than 1.

Next, suppose that $x_0 = x_1 = 1$. Now $\mathbf{M}(x)$ is singular at $x = 1$ and behaves like (11). We also have $f(1) = 0$ (with $f(x)$ from above) and by using the definition of the determinant it also follows that $f(x)$ has a square-root singularity of the form

$$f(x) = c\sqrt{1 - x} + O(1 - x),$$

where $c \neq 0$. (If we consider $s = \sqrt{1 - x}$ as a new variable then it follows as in the first part of the proof that $f(x) = \tilde{f}(s)$ has a simple zero in s. Thus, $c \neq 0$.)

Next, consider the powers $(x\mathbf{CM}(x))^\ell$. By assumption $x\mathbf{CM}(x)$ has just positive entries (for real x with $0 < x \leq 1$). Hence, there exists a unique positive eigenvalue $\lambda(x)$ of $x\mathbf{CM}(x)$ such that the moduli of all other eigenvalues are smaller than $\lambda(x)$. By continuity this is also true in a neighborhood of the real axis. Thus,

$$(x\mathbf{CM}(x))^\ell = \lambda(x)^\ell \mathbf{Q} + O\left(\lambda(x)^{(1-\eta)\ell}\right)$$

for some matrix $\mathbf{Q}$ and some $\eta > 0$. Since $\mathbf{M}(x)$ has a square-root singularity at $x = 1$, the eigenvalue $\lambda(x)$ has the same property:

$$\lambda(x) = c_1 - c_2\sqrt{1-x} + O(1-x).$$

Hence, we are in a similar situation as in Lemma 4 and (13) and (14) follow with the only difference that an additional factor $c_1^\ell = \lambda(1)^\ell$ appears. However, if $c_1 < 1$ then the probabilities do not sum up to 1 but the sum is bounded by $O(1/\sqrt{n})$. On the other hand, if $c_1 > 1$ then the sum of the probabilities does not converge. This provides $c_1 = 1$ and completes the proof of the second part of Theorem 2.

Finally, suppose that $x_1 > 1$. Then we also have $x_0 > 1$. Thus, if we consider $\mathbf{M}_\ell(x)$ in a neighborhood of $x = 1$ then all components of $\mathbf{M}_\ell(x)$ behave (almost) as powers of $\lambda(x)$ (the largest eigenvalue of $x\mathbf{CM}(x)$) that is now analytic at $x = 1$. Thus, we can again use the (saddle point) methods of Drmota (1994) and obtain (15), however, again with a factor $\lambda(1)^\ell$. As above it follows that $\lambda(1) = 1$ and we are done. Note that $\mu = 1/\lambda'(1)$. □

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