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Number conserving cellular automata: new results on decidability and dynamics

Bruno Durand[1], Enrico Formenti[1], Aristide Grange[2] and Zsuzsanna Róka[2]

[1]*Laboratoire d'Informatique Fondamentale de Marseille (LIF), 39 rue Joliot-Curie, 13453 Marseille Cedex 13,* FRANCE, *{bdurand,eforment}@cmi.univ-mrs.fr*
[2] *Laboratoire d'Informatique Théorique et Appliquée (LITA), Université de Metz, Île du Saulcy, 57045 METZ Cedex 01,* FRANCE, *grange@lita.univ-metz.fr, roka@iut.univ-metz.fr*

This paper is a survey on our recent results about number conserving cellular automata. First, we prove the linear time decidability of the property of number conservation. The sequel focuses on dynamical evolutions of number conserving cellular automata.

Keywords: cellular automata, decidability, discrete dynamical systems, classification

1 Introduction

In the last ten years number conserving cellular automata (NCA) have received greater and greater attention. This interest is confirmed by the hundreds of papers in international conferences or journals since their introduction in [15].

They essentially model particle systems ruled by conservation laws of mass or energy with applications in a wide range of scientific disciplines. For example, they were introduced as a model for car traffic flow on highways.

When building models of particle systems from a (possibly) large number of local interaction rules, it is difficult to be sure that some of the rules do not violate the conservation principles in some special cases. Thus it is very important to have an efficient algorithmic test for number-conservation.

A first solution to this problem was independently found by Boccara *et al.* and by Takesue [2, 16]. They give a formula to decide if a one-dimensional cellular automata is a NCA on rings.

We first present a robust definition of number conservation. Its robustness comes from the equivalence of several definitions based on periodic configurations, on finite configurations, and on density aspects. The existence of such a formula is rather surprising since it allows to design an easy linear time algorithm to decide number conservation, which happens rarely in cellular automata theory where most of the non-trivial properties are undecidable. It is also extremely useful since it allows a "fast" generation of NCA rule tables allowing a more feasible exploration of rule space. For example, in [7], the authors used this algorithm to find out a special NCA in order to prove the main result of their paper, namely, the undecidability of the surjectivity problem for NCA in dimension two (or higher) – a computer-aided proof.

In this paper we present this formula in dimension 2 with 4 neighbors (Proposition 1, for dimension n, see [6]) and we prove that there exists a linear time decision algorithm for the property of number conservation (Theorem 1).

In the second part of the paper we focus on classification problems. In cellular automata theory, the problem of classifying evolutions is one of the oldest, most appealing and still unsolved problem. In this paper we face its restriction to NCA. The strong constraints imposed by number conservation property strongly influence dynamics, simplifying the task.

We give a classification of NCA which is a refinement of two well-known classifications: Kůrka's equicontinuity classification and Cattaneo's one that is based on pattern divergence.

In particular we prove that NCA whose evolutions have bounded divergence are either ultimately periodic or some kind of generalized shift (Theorem 3).

Another interesting result consists of the emergence of the role played by surjectivity. Much as in the case of general cellular automata, it splits each behavioral class into two subclasses: global and ultimate behaviors. Moreover in the case of NCA it helps to exclude regularity (*i.e.* denseness of periodic points) from the basic properties for defining chaotic behavior since all surjective NCA are regular (Theorem 4).

We hope that these results could shed new light on the general classification problem.

2 Definitions and classical results

Cellular automata are formally defined as quadruples (d, S, N, f). The integer d is the *dimension* of the space the cellular automaton will work on. $S = \{0, 1, \ldots, s-1\}$ is called the set of *states*. The *neighborhood* $N = (n_1, \ldots, n_v)$ is a v-tuple of distinct vectors of $\mathbb{Z}^{\mathsf{d}}$. The n_i's are the relative positions of the neighbor cells with respect to the cell, the new state of which is being computed. The states of these neighbors are used to compute the new state of the center cell. The *local function* of the cellular automaton $f : S^v \mapsto S$ gives the local transition rule. A *configuration* is a function from $\mathbb{Z}^{\mathsf{d}}$ to S. The set of all configurations is $\mathcal{C} = S^{\mathbb{Z}^{\mathsf{d}}}$. The *global function* A of the cellular automaton is defined via f as follows:

$$\forall c \in \mathcal{C}, \forall i \in \mathbb{Z}^{\mathsf{d}}, A(c)(i) = f(c(i+n_1), \ldots, c(i+n_v)) \ .$$

2.1 Number-conserving cellular automata

In literature one can find at least two different definitions of number conserving CA. These definitions focus on a particular set of configurations, namely, spatially periodic and finite configurations. A third definition is introduced in [6]. In the same paper, the authors give the proof of the equivalence between all those definitions. Thanks to this equivalence property, we shall use the most convenient one in proofs.

Let us briefly recall definitions and useful notations, in which $\mathcal{C}_F$ and $\mathcal{C}_P$ denote the sets of finite (*i.e.* of finite support) and periodic configurations, respectively.

Definition 1 (FNC) *Let A be a* d*-dimensional cellular automaton. A is said to be* finite-number-conserving (FNC) *iff*

$$\forall c \in \mathcal{C}_F, \sum_{i \in \mathbb{Z}^{\mathsf{d}}} c(i) = \sum_{i \in \mathbb{Z}^{\mathsf{d}}} A(c)(i) \ .$$

In the following definition, the period of a spatially periodic configuration is denoted by the vector $\pi(c)$. The expression $0 \le k < \pi(c)$, $k \in \mathbb{Z}^{\mathsf{d}}$ means $\forall i, 1 \le i < \mathsf{d},\ 0 \le k_i < \pi(c)_i$ where d is the dimension of the space and k_i is the i-th component of k.

Definition 2 (PNC) *Let A be a d-dimensional cellular automaton. A is said to be* periodic-number-conserving (PNC) *iff*

$$\forall c \in \mathcal{C}_P,\ \sum_{0\le k<\pi(c)} c(k) = \sum_{0\le k<\pi(c)} A(c)(k) \ .$$

Definition 3 (NC) *Let A be a d-dimensional cellular automaton. A* window *is a hypercube of $\mathbb{Z}^{\mathsf{d}}$ centered in $\mathbf{0}$ and determined by its size. Consider the sequence of windows $\{F_n\}$ of size $2n+1$ and denote by $\mu_n(c)$ the sum of states in F_n of a configuration $c \in \mathcal{C}$. Then A is said to be* number-conserving (NC) *iff*

1. $A(\underline{0}) = \underline{0}$;
2. $\forall c \in \mathcal{C} \setminus \{\underline{0}\}, \limsup_{n\to\infty} \dfrac{\mu_n(A(c))}{\mu_n(c)} = \liminf_{n\to\infty} \dfrac{\mu_n(A(c))}{\mu_n(c)} = 1.$

In [13], Moreira proved that the above definitions are robust with respect to relabeling of states, even when the S is an arbitrary finite subset of $\mathbb{Z}$.

For the sake of simplicity, in the sequel we often use cellular automata in dimension $\mathsf{d} = 1$. Most results are easily generalizable to higher dimensions but not all of them. We'll point it out when case occurs.

For any finite configuration c, let $m(c) = \min\{i \in \mathbb{Z} \mid c(i) \neq 0\}$, $M(c) = \max\{i \in \mathbb{Z} \mid c(i) \neq 0\}$ and $\ell(c) = M(c) - m(c)$. Remark that in the sequel, with some abuse of notation, we will use $\ell(w)$ to denote the length of a finite word w. $S^\star$ being the set of finite words on S, define $w_c \in S^\star$ as $w_c = c(m(c))c(m(c)+1)\dots c(M(c))$. For any word $w = w_0 w_1 \dots w_{n-1}$, let $\underline{w}$ be the spatial periodic configuration defined by $\forall i \in \mathbb{Z}$, $(\underline{w})_i = w_{i \bmod n}$.

3 Number conservation: a necessary and sufficient condition

In [2], a necessary and sufficient condition for a one-dimensional cellular automaton to be number-conserving on periodic configurations is given. In order to show the technique of proofs in higher dimensions, we generalize this result to 2-dimensional CA with 4 neighbors as below. The complete generalization to n-dimensional cellular automata is given in [6].

Proposition 1 *Let $A = (2, N, Q, f)$ be a 2D cellular automaton with $N = \begin{pmatrix} (0,0) & (1,0) \\ (0,-1) & (1,-1) \end{pmatrix}$ and $f : Q^4 \to Q$. A is number-conserving if and only if*

$$\begin{aligned} f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a &+ \left[f\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} - f\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \right] \\ &+ \left[f\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} - f\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \right] \\ &+ \left[f\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} - f\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \right] \\ &+ \left[f\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} - f\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \right]. \end{aligned}$$

Sketch of the proof.

1. The condition is necessary *i.e.* if A is number-conserving, then the formula holds.

   We use the following notations:

   - $c^{s_1 s_2 s_3 s_4}$ is the finite configuration obtained by padding with zeros the square $\begin{matrix} s_1 s_2 \\ s_3 s_4 \end{matrix}$ and with $c^{s_1 s_2 s_3 s_4}(0,0) = s_1$;
   - $\sharp c$ denotes the sum of the states in a finite configuration c of A;
   - $\vec{c}_j = (c(j), c(j+(0,1)), c(j+(0,-1)), c(j+(1,-1)))$.

   Since A is in particular FNC, $\sharp c^{s_1 s_2 s_3 s_4} = \sharp A(c^{s_1 s_2 s_3 s_4}) = \sum_{i=1}^{4} s_i$ and hence

   $$\sum_{i=1}^{4} s_i \quad = \quad \sum_{j \in \mathbb{Z}^2} f(\vec{c}_j^{\,s_1 s_2 s_3 s_4}) \tag{1}$$

   where the number of non-zero f-terms is finite.

   We establish Equation (1) for $s_1 s_2 s_3 s_4 = abcd, 0bcd, 00cd$ and $0b0d$. Then, with appropriate subtractions and substitutions, one obtains the formula for $f(a,b,c,d)$.

2. The condition is sufficient *i.e.* if the formula holds, then A is number-conserving.

   Let us prove that the condition implies PNC. For any periodic configuration consider the sum of terms mentioned in the condition over the period. The terms of type $[f(\dots) - f(\dots)]$ cancel giving exactly the condition for PNC. Since PNC $\Leftrightarrow$ FNC $\Leftrightarrow$ NC, we have the thesis.

$\square$

Theorem 1 *The "number-conserving" property is decidable in linear time for cellular automata of any dimension on the size s of the transition table. Linear time depends on the chosen computational model, and here we assume a constant access cost to the table. If in the model the access cost is logarithmic then the decision time is $O(s \log s)$.*

Remark that the complexity $O(s \log s)$ does not depend on the dimension of the space. More precisely, the multiplicative constant of the O depends neither on the dimension nor on the number of states or neighborhood of the cellular automaton. All these quantities are subsumed in s.

Proof. The number of terms of our condition in Proposition 3 (and also in formulas for higher dimensions) is proportional to the number of neighbors. As a consequence, the size of the set of conditions to be checked is proportional to the size of the local transition rule of the cellular automata, which is upper bounded by the size of the cellular automaton in the chosen representation. Access to the table costs at most $O(\log s)$. $\square$

Remark that if cellular automata are given by a program computing their rule and not by the rule explicitly, then Theorem 1 does not hold. This drawback is often present in cellular automata theory. However if we assume that access to the transition table is performed in constant time, then the algorithm is linear.

4 Dynamics

This section surveys results in [8]. Most of the proofs are just sketched; details or technical difficulties are hidden in order to focus on the ideas.

We are interested in the study of dynamical systems in symbolic spaces. For this reason, $\mathcal{C}$ is endowed with the product topology (also called Cantor topology) of a countable product of discrete spaces on S. It is easy to see that the following metric induces such a product topology on $\mathcal{C}$.

For $w \in S^\star$ and $i \in \mathbb{Z}$, the sets

$$[w]_i = \left\{x \in \mathcal{C} \mid \forall j, 0 \leq j < \ell(w), x(i+j) = w_j\right\}$$

are called *cylinders* and form a basis for the product topology on $\mathcal{C}$. For any finite configuration x, we call $[w_x]_{m(x)}$ the cylinder *induced* by x.

The *metric* d on $\mathcal{C}$ is defined by $\forall x, y \in \mathcal{C}$,

1. $d(x,y) = 0$ if $x = y$;
2. $d(x,y) = 2^{-n}$ if $x \neq y$, where $n = \inf\{i \geq 0, x(i) \neq y(i) \text{ or } x(-i) \neq y(-i)\}$ is the smallest absolute coordinate where x and y differ.

4.1 Background

The definitions relative to dynamical systems are well known in the general case. We present them adapted to the model of CA. In all the following definitions, we will assume that the structure $(\mathcal{C}, A)$ consists of a metric space $\mathcal{C}$ (the set of all configurations) and a continuous self-map A (the global function of the cellular automaton). A^t denotes the t-fold composition of A with itself.

A configuration x is an *equicontinuity* point for A if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in \mathcal{C}$, $d(x,y) \leq \delta$ implies that for all $t \in \mathbb{N}$, $d(A^t(x), A^t(y)) < \varepsilon$. A CA A is *sensitive to initial conditions* if there exists $\varepsilon > 0$ such that for any $x \in \mathcal{C}$ and any $\delta > 0$, there exists $y \in \mathcal{C}$ such that $d(x,y) < \delta$ implies that there exists $t \in \mathbb{N}$ such that $d(A^t(x), A^t(y)) > \varepsilon$. Since $\mathcal{C}$ is perfect (*i.e.* it has no isolated points), A has no equicontinuity points iff it is sensitive to initial conditions. A is *expansive* if there exists $\varepsilon > 0$ such that for any $x, y \in \mathcal{C}$, $d(x,y) > 0$ implies that $\exists t \in \mathbb{N}$ such that $d(A^t(x), A^t(y)) \geq \varepsilon$. If for any pair of non-empty open sets of configurations X, Y there exists $t \in \mathbb{N}$ such that $A^t(X) \cap Y \neq \emptyset$ then A is *transitive*. A configuration x is called *ultimately periodic* if there exist $t_0, p \in \mathbb{N}$ $(p > 0)$ such that $A^{t_0+kp+i}(x) = A^{t_0+i}(x)$ for $k, i \in \mathbb{N}$. The minimal t_0, p with such a property are called, respectively, the *preperiod* and the *period* of x. If $t_0 = 0$ then x is called *periodic*. A is *regular* if periodic configurations are dense in $\mathcal{C}$. Note that regular systems are necessarily surjective. The *shift map* σ is often used as a paradigmatic example of chaotic symbolic system. It is defined as $\forall c \in \mathcal{C}, \forall i \in \mathbb{Z},\ \sigma(c)(i) = c(i+1)$.

4.2 Classification of CA dynamical behavior

A well-known classification of cellular automata with quiescent state is given in [3, 4]. In these papers, cellular automata are classified according to their pattern divergence. Three classes of behavior are defined:

C_1 : (vanishing) $\forall c \in \mathcal{C}_F\ \lim_{t\to\infty} \ell(A^t(c)) = 0$;

C_2 : (bounded) $\forall c \in \mathcal{C}_F\ \sup_{t\in\mathbb{N}} \ell(A^t(c)) < \infty$;

C_3 : (growing) $\exists c \in \mathcal{C}_F\ \sup_{t\in\mathbb{N}} \ell(A^t(c)) = \infty$.

In [10], Kůrka proposed a classification based on local behavior of cellular automata and increasing degrees of chaos. Kůrka devised the following classes:

K_1: equicontinuous cellular automata;

K_2: cellular automata with equicontinuity points but not equicontinuous;

K_3: cellular automata sensitive to initial conditions but not expansive;

K_4: expansive cellular automata.

Kůrka's classes are defined by properties on the set of all configurations. Using Proposition 3, one can express the properties defining some of Kůrka's classes in terms of behavior on finite patterns. This will be useful when comparing Kůrka's classes with pattern growth classification.

We close this section with some results which will be very useful in the sequel.

A CA A is *surjective* (resp. *injective*) if its global rule is surjective (resp. injective). Denote A_F the restriction of A to $\mathcal{C}$.

Theorem 2 (Moore-Myhill's theorem) *Let A be a cellular automaton in which $\underline{0}$ is a quiescent configuration. Then A is surjective if and only if A_F is injective.*

Proof. [12, 14] □

In the NCA case, Moore-Myhill's theorem can be strengthened as follows.

Proposition 2 *A NCA A is surjective iff A_F is bijective.*

Sketch of the proof. Use FNC condition for the direct implication, and Moore-Myhill theorem for the converse. □

Proposition 3 ([9]) *Let A be a cellular automaton. Then A_F is transitive (resp. sensitive to initial conditions) iff A is transitive (resp. sensitive to initial conditions).*

A similar result holds if in Proposition 3 we consider the set of spatial periodic configurations instead of finite configurations.

4.3 New classification results

Before starting to review the new results we introduce an interesting new definition which will be very useful to simplify most of the following proofs. We think that it is a "fine tool" which could be fruitfully reused to simplify other results and, of course, to make new ones.

Definition 4 (spaced enough) *Consider a CA A and three finite configurations c, x, y. We say that the couple (x, y) is* spaced enough *in c iff* $\forall t \in \mathbb{N}, \exists k_t \in \mathbb{N} : w_{A^t(c)} = w_{A^t(x)} 0^{k_t} w_{A^t(y)}$.

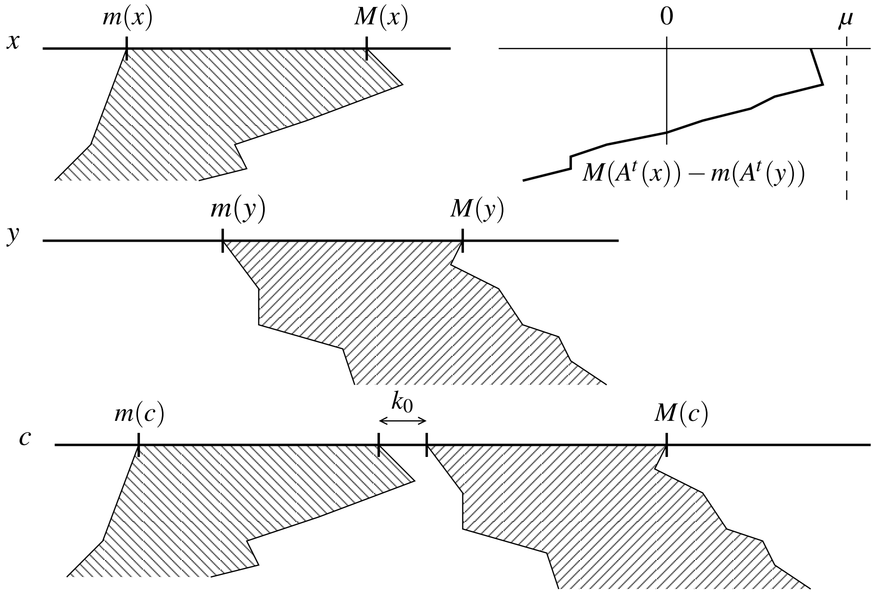

Figure 1: *x and y are spaced enough in c (*cf. *Definition 4).*

Proposition 4 *Consider a CA A. Then $A \in C_2$ iff for any $x, y \in \mathcal{C}_F$, there exists $c \in \mathcal{C}_F$ such that (x, y) is spaced enough in c.*

Sketch of the proof. First we proved that, for a CA A and any two finite configurations x, y, there exists a finite configuration in which (x, y) is spaced enough iff $\exists \mu \in \mathbb{N}, \forall t > 0, M(A^t(x)) - m(A^t(y)) \leq \mu$ (see Figure 4 for a visual interpretation of this counter-intuitive inequality). Next, we prove that if $A \in C_2$, the sequence $(M(A^t(x)) - m(A^t(y)))_{t \in \mathbb{N}}$ is ultimately periodic. The direct implication is the consequence of these two results. The converse requires only the first one, but applied twice, to both (x, y) and (y, x). $\square$

4.3.1 Ultimate behaviors

Preliminaries: generalized subshifts. Generalized shifts were introduced in [11] for giving a model "in which complexity in finite-dimensional systems can be discussed in a precise manner". Here we have a fairly weak adaptation to CA of such a model that is called a *generalized subshift*. Roughly speaking, we study models which behave like "a kind of shift" on some (preferably large) invariant set of configurations. The following definition is essentially taken from [5].

Definition 5 *A cellular automaton $(1, S, N, f)$ is said to be a* generalized subshift *on the set $U \subseteq \mathcal{C}$ iff*

1. *U is non-trivial,* i.e. *$U \neq \{\underline{0}\}$;*
2. *U is A-positively invariant,* i.e. *$A(U) \subseteq U$;*
3. *there exist mappings $T : \mathcal{C} \to \mathbb{N} \setminus \{0\}$ and $T' : \mathcal{C} \to \mathbb{Z} \setminus \{0\}$ such that $\forall c \in U$, $A^{T(c)}(c) = \sigma^{T'(c)}(c)$.*

Proposition 5 presents peculiar properties of the shifting behavior of generalized subshifts on $\mathcal{C}_F$. Roughly speaking, the proposition states that in generalized shifts all finite patterns move in the same direction with identical speed.

Proposition 5 *Consider A a generalized subshift on $\mathcal{C}_F$ and let T, T' as in Definition 5. Then $\forall x, y \in \mathcal{C}_F$, $T(x) = T(y)$ and $T'(x) = T'(y)$.*

Sketch of the proof. By contradiction using Proposition 4. □

4.3.2 Partitioning C_2 according to ultimate behaviors

Definition 6 (Limit sets Ω and Ω_F) *Let A be a CA. Then, $\Omega = \cap_{t\in\mathbb{N}} A^t(\mathcal{C})$ (resp., $\Omega_F = \cap_{t\in\mathbb{N}} A^t(\mathcal{C}_F)$) is called the* limit set *of A (resp., of A_F).*

Ω (resp. Ω_F) can be viewed as the set of configurations (resp., finite configurations) which has an infinite number of predecessors (resp., predecessors of finite type).

Definition 7 *A CA A is said to be* ultimately periodic *(resp.,* an ultimately generalized subshift*) on $U \subseteq \mathcal{C}$ if A is periodic (resp., a generalized subshift) on U and for all $x \in \mathcal{C}$, there exists $t_x \in \mathbb{N}$ such that $A^{t_x}(x) \in U$.*

Proposition 6 *Consider a CA A in class C_2. Then A is either ultimately periodic on Ω_F or an ultimately generalized subshift on Ω_F.*

Sketch of the proof. In the proof of Proposition 4, we have already used the fact that each finite configuration is ultimately periodic or ultimately shifted by A. It remains to prove that A behaves in the same way on all finite configurations. If we assume that it is not the case for two configurations x and y (e.g., x is shifted and y periodic), it is quite easy to place them in such a way that the length of the resulting configuration grows unboundedly during the evolution. □

4.3.3 Partitioning C_2 along the presence of equicontinuity points

Definition 8 (Blocking word) *A word $V \in S^\star$ is called a* blocking *word if there exists an infinite sequence of words $(v_t)_{t\in\mathbb{N}}$ such that*

1. *for any $t \in \mathbb{N}$, $\ell(v_t)$ is finite, odd and greater or equal to r;*
2. *for any $c \in [V]_{-(\ell(V)-1)/2}$ and any $t \in \mathbb{N}$, $A^t(c) \in [v_t]_{-(\ell(v_t)-1)/2}$.*

In other words V partitions the evolution of A into two disconnected parts: perturbations made in one side are completely “blocked” by V.

Proposition 7 ([1]) *Any equicontinuity point has an occurrence of a blocking word. Conversely, if there exist blocking words, then any point with infinitely many occurrences of a blocking word to the left and to the right (of 0) is an equicontinuity point.*

Proposition 8 *There are no expansive CA in class C_2,* i.e. $C_2 \cap K_4 = \emptyset$.

Sketch of the proof. A non-zero expansivity constant is incompatible with ultimate periodicity. □

Theorem 3 *Consider a CA A in class C_2. It satisfies exactly one of the following conditions:*

1. *A is ultimately periodic on Ω_F and belongs to $K_1 \cup K_2$.*
2. *A is an ultimately generalized subshift on Ω_F and belongs to K_3.*

Sketch of the proof. In the first case, A has a blocking word V. We construct a finite configuration c from $x = \dots 00V00\dots$ and some y on which A behaves like an ultimately generalized subshift. At this point we remark that the length of $A^n(c)$ grows unboundedly during evolutions, contradicting the hypothesis $A \in C_2$.

For the second case, in a similar manner one can prove that if the hypothesis is not true then one can build an equicontinuity point by a suitable "spaced enough construction". □

4.4 The role of surjectivity

4.4.1 Cases where surjectivity is equivalent to regularity

Proposition 9 *Consider a surjective CA A in class C_2. Then A is either periodic or a generalized subshift on $\mathcal{C}_F$.*

Sketch of the proof. Use Proposition 6 and Moore-Myhill's theorem. □

Proposition 10 *In class C_2, surjectivity is equivalent to regularity.*

Sketch of the proof. A being in class C_2, the delicate matter consists of proving the regularity of CAs of class C_2 which are generalized subshifts on $\mathcal{C}_F$. Since $\mathcal{C}_F$ is dense in $\mathcal{C}$, given any configuration $x \in \mathcal{C}$, we can construct a sequence of spatial periodic configurations which converges to x. The fact that regularity implies surjectivity, and the result in Proposition 9, close the proof. □

Theorem 4 *Surjective NCA are regular.*

Sketch of the proof. For the direct case, consider a surjective NCA A. We claim that there is at least one periodic point in every cylinder. The proof of the claim requires the construction of a quite complex spatial periodic configuration. The idea is to make it sparse enough to ensure the existence of holes between which one can reason like in the finite case. The converse is obvious since regularity implies surjectivity. □

Remark that the above result holds both for surjective NCA in C_2 and in C_3, although the C_2 case is already a consequence of Proposition 10.

4.4.2 Case where surjectivity is equivalent to transitivity

Proposition 11 *Consider a surjective CA A in class C_2. Then A is transitive iff it is a generalized subshift on $\mathcal{C}_F$.*

Sketch of the proof. The direct case is immediate. For the converse, being given two arbitrary finite configurations x and y, construct a finite configuration c in which x and y are spaced enough. They are placed in c in such a way that c belongs to the cylinder induced by x and $\exists t > 0 : A^t(c)$ belongs to the cylinder induced by y. Then, by Proposition 3, A is transitive. □

Corollary 1 *For generalized subshifts on $\mathcal{C}_F$ in class C_2, surjectivity is equivalent to transitivity.*

4.5 Empty classes

The property of number conservation imposes strong constraints on dynamical behavior. In this section we prove that some classes are empty.

Proposition 12 *There are no CA in class $K_1 \cap C_3$.*

Sketch of the proof. Consider a CA $A \in C_3$. For any $\delta > 0$, there exists $c \in \mathcal{C}_F$ such that $d(\underline{0}, c) < \delta$ and $\sup_{t\in\mathbb{N}} \ell(A^t(c)) = \infty$. Place beside c a copy c' of c such that $d(\underline{0}, c') \leq d(\underline{0}, c) < \delta$ and, for some t, $d(A^t(\underline{0}), A^t(c')) \geq 1$. Hence $\underline{0}$ is not an equicontinuity point. □

The following propositions follow immediately from the definition of FNC.

Proposition 13 *There are no NCA in class C_1.*

Proposition 14 *There are no expansive NCA,* i.e. *no NCA is in class K_4.*

Sketch of the proof. Similar to the proof of Proposition 8. □

Table 1 summarizes the situation. We provide examples for each remaining case in [8].

	C_1	C_2	C_3
K_1	$\emptyset$ (Prop. 13)	$\exists$	$\emptyset$ (Prop. 12)
K_2	$\emptyset$ (Prop. 13)	$\exists$	$\exists$
K_3	$\emptyset$(Prop. 13)	$\exists$	$\exists$
K_4	$\emptyset$ (Prop. 14)	$\emptyset$ (Prop. 14)	$\emptyset$ (Prop. 14)

Table 1: *NCA in Kůrka's and pattern growth classifications.*

5 Conclusion and open problems

In this paper we reviewed recent results on NCA. This work can have many future developments. For example, it could be interesting to find formulas for different types of conserved laws, namely non linear laws, dissipative laws and so on.

Another direction of research could be the generalization of Takesue's approach ([16]) to find out "all" invariants along with their support.

The second part of the paper deserves further study too. In particular, we have pointed out the role played by surjectivity in dynamics. We have also remarked how each behavioral class is sharply split into two parts by such a property. A closer analysis of the content of the surjective subclasses lets one immediately see that in some cases they are "almost" empty. For instance, the only known surjective NCA in class $K_1 \cap C_2$ is the identity rule. Moreover, in the case of surjective NCA in $K_2 \cap C_3$ we found no example at all.

We think that a further study of those subclasses can clarify the relations between surjectivity and the strong combinatorial properties that it imposes on dynamics.

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