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The Many Faces of Alternating-Sign Matrices

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I give a survey of different combinatorial forms of alternating-sign matrices, starting with the original form introduced by Mills, Robbins and Rumsey as well as corner-sum matrices, height-function matrices, three-colorings, monotone triangles, tetrahedral order ideals, square ice, gasket-and-basket tilings and full packings of loops.

Keywords: Alternating-Sign Matrices, Tilings

1 Introduction

An *alternating-sign matrix of order n* is an *n*-by-*n* array of 0's, $+1$'s and -1's with the property that in each row and each column, the non-zero entries alternate in sign, beginning and ending with a $+1$. For example, Figure 1 shows an alternating-sign matrix (ASM for short) of order 4.

$$\begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & -1 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 \end{pmatrix}$$

Figure 1: An alternating-sign matrix of order 4.

Figure 2 exhibits all seven of the ASMs of order 3.

$$\begin{pmatrix} 0 & 0 & +1 \\ 0 & +1 & 0 \\ +1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & +1 \\ +1 & 0 & 0 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ +1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & +1 & 0 \\ +1 & -1 & +1 \\ 0 & +1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

Figure 2: The seven alternating-sign matrices of order 3.

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Matrices satisfying these constraints were first investigated by Mills, Robbins and Rumsey [17]. The matrices arose from their investigation of Dodgson's scheme for computing determinants via "condensation". (see section 10). The number of ASMs of order n, for small values of n, goes like $1, 2, 7, 42, 429, 7436, \ldots$, and it was conjectured by Mills et al. that the number of ASMs of order n is given by the product

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}.$$

However, it took over a decade before this conjecture was proved by Zeilberger [30]. For more details on this history, see the expository article by Robbins [21], the survey article by Bressoud and Propp [7], or the book by Bressoud [6].

Here my concern will be not with the alternating-sign matrix conjecture and its proof by Zeilberger, but with the inherent interest of alternating-sign matrices as combinatorial objects admitting many different representations. I will present here a number of different ways of looking at an ASM. Along the way, I will also mention a few topics related to ASMs in their various guises, such as weighted enumeration formulas and asymptotic shape. Much of what is in this article has appeared elsewhere, but I hope that by gathering these topics together in one place, I will help raise the level of knowledge and interest of the mathematical community concerning these fascinating combinatorial objects.

2 Corner-sum, heights, and colorings

Given an ASM $(a_{i,j})_{i,j=1}^n$ of order n, we can define a *corner-sum matrix* $(c_{i,j})_{i,j=0}^n$ of order n by putting $c_{i,j} = \sum_{i' \le i,\ j' \le j} a_{i,j}$. This definition was introduced in [23]. Figure 3 shows the seven corner-sum matrices of order 3 (note that they are 4-by-4 matrices).

$$\begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&1&2\\0&1&2&3 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&1&1&2\\0&1&2&3 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&0&1&2\\0&1&2&3 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&1&1&2\\0&1&2&3 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&1&2&2\\0&1&2&3 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&1&1&1\\0&1&1&2\\0&1&2&3 \end{pmatrix} \begin{pmatrix} 0&0&0&0\\0&1&1&1\\0&1&2&2\\0&1&2&3 \end{pmatrix}$$

Figure 3: The seven corner-sum matrices of order 3.

Corner-sum matrices, viewed as objects in their own right, have a very simple description: the first row and first column consist of 0's, the last row and last column consist of the numbers from 0 to n (in order), and within each row and column, each entry is either equal to, or one more than, the preceding entry.

Note that the seven corner-sum matrices in Figure 3 correspond respectively to the seven alternating-sign matrices in Figure 2. I will adhere to this pattern throughout, to make it easier for the reader to verify the bijections between the different representations.

Corner-sum matrices can in turn be transformed into a somewhat more symmetrical form. Given a corner-sum matrix $(c_{i,j})_{i,j=0}^n$ define $h_{i,j} = i + j - 2c_{i,j}$. Call the result a *height-function matrix* (see [10]).

Figure 4 shows the seven height-function matrices of order 3 (4-by-4 matrices).

$$\begin{pmatrix} 0&1&2&3\\ 1&2&3&2\\ 2&3&2&1\\ 3&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&3\\ 1&2&3&2\\ 2&1&2&1\\ 3&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&3\\ 1&2&1&2\\ 2&3&2&1\\ 3&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&3\\ 1&2&1&2\\ 2&1&2&1\\ 3&2&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&2&3\\ 1&2&1&2\\ 2&1&0&1\\ 3&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&3\\ 1&0&1&2\\ 2&1&2&1\\ 3&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&3\\ 1&0&1&2\\ 2&1&0&1\\ 3&2&1&0 \end{pmatrix}$$

Figure 4: The seven height-function matrices of order 3.

Height-function matrices have a simple intrinsic description: the first row and first column consist of the numbers from 0 to n (consecutively), the last row and last column consist of the numbers from n to 0 (consecutively), and any two entries that are row-adjacent or column-adjacent differ by 1.

If one reduces a height-function matrix modulo 3, and views the residues 0, 1, and 2 as "colors", one obtains a proper 3-coloring of the $n+1$-by-$n+1$ square grid satisfying specific boundary conditions. Here "proper" means that adjacent sites get distinct colors, and the specific boundary conditions are as follows: colors increase modulo 3 along the first row and first column and decrease modulo 3 along the last row and last column, with the color 0 occurring in the upper left. Figure 5 shows the seven such 3-colorings of the 4-by-4 grid.

$$\begin{pmatrix} 0&1&2&0\\ 1&2&0&2\\ 2&0&2&1\\ 0&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&0\\ 1&2&0&2\\ 2&1&2&1\\ 0&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&0\\ 1&2&1&2\\ 2&0&2&1\\ 0&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&0\\ 1&2&1&2\\ 2&1&2&1\\ 0&2&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&2&0\\ 1&2&1&2\\ 2&1&0&1\\ 0&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&0\\ 1&0&1&2\\ 2&1&2&1\\ 0&2&1&0 \end{pmatrix} \begin{pmatrix} 0&1&2&0\\ 1&0&1&2\\ 2&1&0&1\\ 0&2&1&0 \end{pmatrix}$$

Figure 5: The seven colorings associated with the ASMs of order 3.

Conversely, every proper 3-coloring of that graph that satisfies the boundary conditions is associated with a unique height-function matrix [5].

3 Monotone triangles and order ideals

Another way to "process" an ASM is to form partial sums of its columns from the top toward the bottom, as shown in Figure 6 for a 4-by-4 ASM. In the resulting square matrix of partial sums, the ith row has i 1's in it and $n-i$ 0's. Hence we may form a triangular array whose ith row consists of precisely those values j for which the i,jth entry of the partial-sum matrix is 1. The result is called a *monotone triangle* [17] (or *Gog triangle* in the terminology of Zeilberger [30]). Figure 7 shows the seven monotone triangles of order 3.

$$\begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & -1 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{array}{ccccccc} & & & 2 & & & \\ & & 1 & & 3 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

Figure 6: Turning an ASM into a monotone triangle.

$$\begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 3 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

$$\begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array}$$

Figure 7: The seven monotone triangles of order 3.

One may intrinsically describe a monotone triangle of order n as a triangular array with n numbers along each side, where the numbers in the bottom row are 1 through n in succession, the numbers in each row are strictly increasing from left to right, and the numbers along diagonals are weakly increasing from left to right. Zeilberger's proof of the ASM conjecture [30] used these Gog triangles and a natural generalization, "Gog trapezoids".

A different geometry comes from looking at the set of ASMs as a distributive lattice. Given two height-function matrices $(h_{i,j})_{i,j=0}^{n}$ and $(h'_{i,j})_{i,j=0}^{n}$, we can define new matrices (called the *join* and *meet*) whose i,jth entries are $\max(h_{i,j}, h'_{i,j})$ and $\min(h_{i,j}, h'_{i,j})$, respectively. These new matrices are themselves height-function matrices, and the operations of join and meet turn the set of ASMs of order n into a distributive lattice L (see [25] for background on finite posets and lattices).

The fundamental theorem of finite distributive lattices tells us that L can be realized as the lattice of order-ideals of a certain poset P, namely, the poset of join-irreducibles of the lattice L. There is a nice geometric description of the ranked poset P. It has $(1)(n-1)$ elements of rank 0, $(2)(n-2)$ elements of rank 1, $(3)(n-3)$ elements of rank 2, etc., up through $(n-1)(1)$ elements of rank $n-1$. These elements are arranged in the fashion of a tetrahedron resting on its edge. A generic element of P, well inside the interior of the tetrahedron, covers 4 elements and is covered by 4 elements.

Using this poset P, we can give a picture of the lattice L that does not require a knowledge of poset-theory (also described in [10]). Picture a tetrahedron that is densely packed with $(1)(n-1)+(2)(n-2)+(3)(n-3)+\ldots+(n-1)(1)$ balls, resting on an edge. Carefully remove the two upper faces of the tetrahedron so as not disturb the balls. One may now start to remove some of the balls, starting from the top, so that removal of a ball does not affect any of the balls below. There are many configurations of this kind, ranging from the full packing to the empty packing. These configurations are in bijective correspondence with the ASMs of order n, and the lattice operations of meet and join correspond to intersection and union.

4 Square ice

Zeilberger's proof of the ASM conjecture was followed in short order by a simpler proof due to Kuperberg. Kuperberg's proof made use of a different representation of ASMs, the "6-vertex model" of statistical mechanics. This model is also called square ice on account of its origins as a two-dimensional surrogate for a more realistic (and still intractable) three-dimensional model of ice proposed by physicists [5]. A square ice state is an orientation of the edges of a square grid or a finite sub-graph thereof with the property that each vertex other than vertices on the boundary has two incoming arrows and two outgoing arrows. Each internal vertex must be of one of the six kinds shown in Figure 8 (hence the name "six-vertex model"). The markings under the six vertex-types can be ignored for the time being.

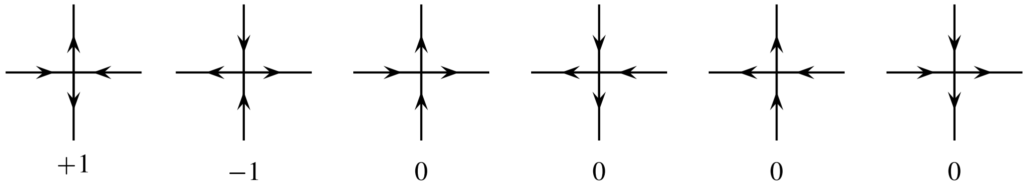

Figure 8: The six vertex-types for the square-ice model.

As our finite subgraph of the square grid, we will take the "generalized tic-tac-toe" graph formed by n horizontal lines and n vertical lines meeting in n^2 intersections of degree 4, with $4n$ vertices of degree 1 at the boundary. We say that an ice state on this graph satisfies *domain-wall boundary conditions* [13] if all the arrows along the left and right flank point inward and all the arrows along the top and bottom point outward.

Figure 9 shows the possibilities when $n = 3$.

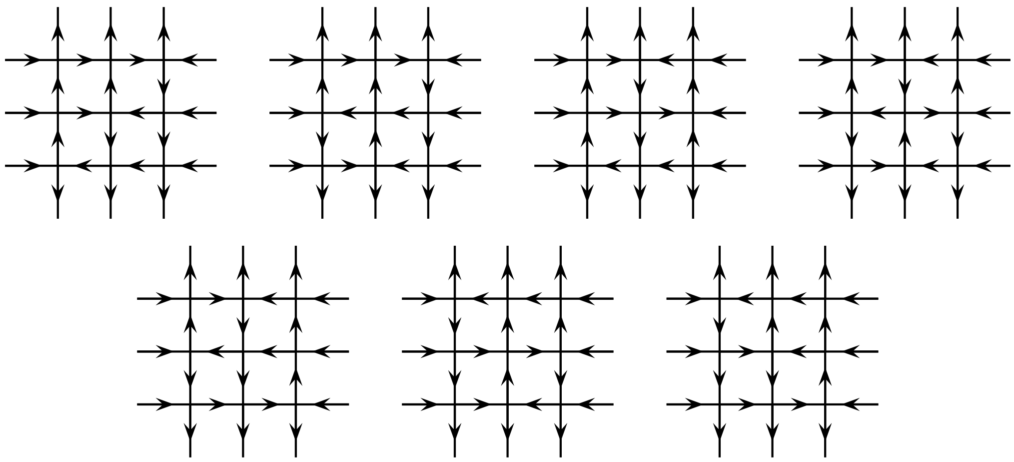

Figure 9: The seven square-ice states for $n = 3$ with domain-wall boundary conditions.

These states of the square-ice model are in bijective correspondence with ASMs. To turn a state of the square-ice model on an n-by-n grid with domain-wall boundary conditions into an alternating-sign matrix of order n, replace each vertex by $+1$, -1, or 0 according to the marking given in Figure 8. Kuperberg was able to give a simplified proof of the ASM conjecture by making use of results about the square-ice model in the mathematical physics literature [13].

An amusing variant of the square-ice model is a tiling model in which the tiles are deformed versions of squares that the physicist Joshua Burton has dubbed "gaskets" and "baskets", depicted in Figure 10. (To see why the basket deserves its name, you might want to rotate the page by 45 degrees, so that the "handle" of the basket is pointing up.)

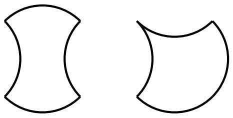

Figure 10: A gasket and a basket.

The gasket and basket correspond respectively to the first and last vertex-types shown in Figure 8; the other five vertex-types correspond to tiles obtained by rotating the gasket by 90 degrees or by rotating the basket by 90, 180, or 270 degrees. The directions of the bulges of the four sides of a tile correspond to the orientations of the four edges incident with a vertex. Thus, the seven ASMs of order 3 correspond to the seven distinct ways of tiling the region shown in Figure 11 (a "supergasket" of order 3) with gaskets and baskets.

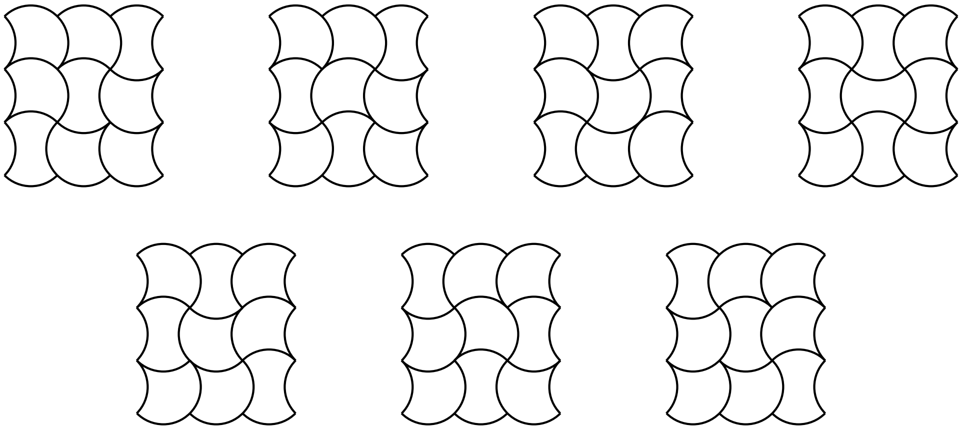

Figure 11: The seven tilings of an order-3 supergasket with gaskets and baskets.

For another fanciful embodiment of ASMs as tilings, see the cover of [6].

5 Symmetric ASMs and partial ASMs

Some ASMs are more symmetrical than others. More precisely, the eight-element dihedral group D_4 acts on ASMs, and for every subgroup G of D_4 there are ASMs that are invariant under the action of every element of G. In [22], Robbins gave some conjectures for the number of ASMs of order n that are invariant under particular groups G; for most (but not all) of the subgroups G of D_4, numerical evidence suggested specific product-formula. Since then, Kuperberg [15] has proved some of these, but others remain conjectural.

At the same time, one may also look at halves (or even quarters or eighths) of ASMs — the fundamental regions under the action of the aforementioned groups G — and look at them in their own right, asking, How many partial ASMs are there if one limits attention to such a region? There are some interesting phenomena here.

For instance, for c_1,c_2,c_3 each equal to $+1$ or -1, define $N(c_1,c_2,c_3)$ as the number of 4-by-7 partial height-function matrices of the form shown in Figure 12.

$$\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & ? & ? & ? & ? & ? & 5 \\
2 & ? & ? & ? & ? & ? & 4 \\
3 & 3+c_1 & 3 & 3+c_2 & 3 & 3+c_3 & 3
\end{array}$$

Figure 12: Half of a height-function matrix of order 6.

Not surprisingly, the eight values of $N(c_1,c_2,c_3)$ as c_1,c_2,c_3 vary are not all equal to one another. But it is surprising that the four numbers

$$N(1,1,1),$$

$$(N(1,1,-1)+N(1,-1,1)+N(-1,1,1))/3,$$

$$(N(1,-1,-1)+N(-1,1,-1)+N(-1,-1,1))/3,$$

and

$$N(-1,-1,-1)$$

are all equal. More generally, consider $(n+1)$-by-$(2n+1)$ partial height-function matrices of the following form:

$$\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & \ldots & 2n-1 & 2n \\
1 & ? & ? & ? & ? & ? & \ldots & ? & 2n-1 \\
2 & ? & ? & ? & ? & ? & \ldots & ? & 2n-2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n-1 & ? & ? & ? & ? & ? & \ldots & ? & n+1 \\
n & n+c_1 & n & n+c_2 & n & n+c_3 & \ldots & n+c_n & n
\end{array}.$$

Figure 13: Half of a height-function matrix of order $2n$.

Here each c_i $(1 \leq i \leq n)$ is either $+1$ or -1. Let $N(c_1,c_2,\ldots,c_n)$ be the number of such partial height-function matrices. Then one finds empirically that for every k in $-n,-(n-2),-(n-4),\ldots,n-4,n-2,n$, the average of $N(c_1,c_2,\ldots,c_n)$ over all vectors $(c_1,\ldots,c_n)$ satisfying $c_1+\ldots+c_n = k$ depends only on n, not on k. Kuperberg has found an algebraic proof of this using the Tsuchiya determinant formula [26] invented for the study of the square ice model, but there ought to be a purely combinatorial proof of this simple relation.

6 Weighted enumeration

There are some interesting results in the literature on weighted enumeration of ASMs. Here one assigns to each ASM of order n some weight, and tries to compute the sum of the weights of all the ASMs of order n. A priori it might be unclear why this would be interesting, but with certain weighting scheme one gets beautiful (and mysterious) formulas, which are their own justification.

For instance, following [17], one can assign weight x^k to every ASM that contains exactly k entries equal to -1. What is the sum of the weights of the ASMs of order n? When $x = 1$, this is nothing other than ordinary enumeration of ASMs. When $x = 2$, there is a very nice answer [17]: the sum of the weights is exactly $2^{n(n-1)/2}$. When $x = 3$, the answer is more complicated, but it is roughly similar in type to the formula for the case $x = 1$, and roughly similar in difficulty; the "3-enumeration" formula was first conjectured by Mills, Robbins and Rumsey [17] and was eventually proved by Kuperberg [15] (with corrections provided by Robin Chapman). No other positive integer x seems to give nice answers. One can also assign weight x^k to every ASM that contains exactly k entries equal to $+1$, but this is essentially the same weighting scheme, since in any ASM of order n, the number of $+1$'s is always n plus the number of -1's.

More interestingly, one can also use a hybrid weighting scheme in which the exponent of x is equal to the number of entries $a_{i,j}$ such that *either* $i+j$ is even and $a_{i,j} = -1$ *or* $i+j$ is odd and $a_{i,j} = +1$. When $x = 2$, this too leads to an interesting result: the sum of the weights is always a power of 2 times a power of 5! [29]

One can come up with many open problems by combining the ideas of this section and the previous section. Here is one example: Each way of filling in Figure 13 (with the c_i's now permitted to vary freely) gives rise to a "half-ASM". For instance, the filling

$$\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 4 & 5 \\
2 & 3 & 4 & 3 & 4 & 3 & 4 \\
3 & 2 & 3 & 2 & 3 & 4 & 3
\end{array}$$

of Figure 12 gives rise to the half-ASM

$$\begin{array}{rrrrrr}
0 & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & +1 & 0 & 0 & 0 \\
+1 & 0 & 0 & 0 & -1 & +1
\end{array}$$

which has a single -1. If we assign each rectangular array that arises in this way a weight equal to 2 to the power of the number of -1's, we apparently get 2^{n^2}. Robin Chapman has found a nice proof of this. On the other hand, suppose we now permit every entry in the bottom row of Figure 13 to vary freely (aside from the n on the left and the n on the right). When we 2-enumerate half-ASMs of this sort, as a function of n, we get the following sequence of numbers: 2, $20 = 2^2 \cdot 5$, $896 = 2^7 \cdot 7$, $177408 = 2^8 \cdot 3^2 \cdot 7 \cdot 11$, $154632192 = 2^{15} \cdot 3 \cdot 11^2 \cdot 13$, $592344383488 = 2^{17} \cdot 11^2 \cdot 13^3 \cdot 17, \ldots$. Clearly the absence of larger prime factors indicates that there is some nice product formula governing these numbers. Can someone find the right conjecture? Can someone prove it? (For more data of this kind, see `http://www.math.wisc.edu/~propp/half-asm`. Late-breaking news: Theresia Eisenkölbl has made progress with the data-set and proved a number of theorems about half-ASMs.)

7 Full packings of loops

Given an ice state of order n, we can form a subgraph of the underlying tic-tac-toe graph by selecting precisely those edges that are oriented so as to point from an odd vertex to an even vertex, where we have assigned parities to vertices so that each odd vertex has only even vertices as neighbors and vice versa. Then one gets a subgraph of the tic-tac-toe graph such that each of the n^2 internal vertices lies on exactly 2 of the selected edges, and the $4n$ external vertices, taken in cyclical order, alternate between lying on a selected edge and not lying on a selected edge. Moreover, every such subgraph arises from an ice state in this way.

Let us say that the leftmost vertex in the top row of external vertices is even. Figure 14 shows the seven subgraphs that result from applying the transformation to the seven ice states of order 3.

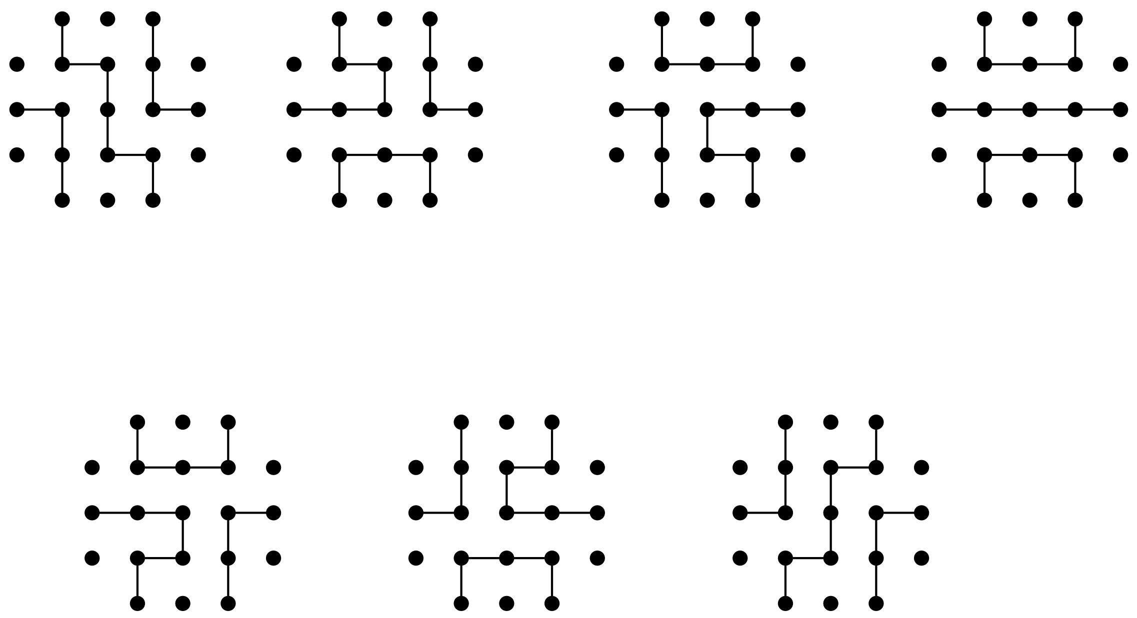

Figure 14: The seven FPL states of order 3.

Leaving aside the behavior at the boundary, these are states of what physicists call the fully packed loop (FPL) model on the square grid (see e.g. [3]). I sometimes prefer to call such states "near 2-factors" since nearly all of the vertices in these subgraphs have degree 2; only the external vertices can have smaller degree.

If one starts from an external vertex, there is a unique path that one can follow using edges in the subgraph; this path must eventually lead to one of the other external vertices. In addition to these paths ("open loops"), the edges of the subgraph can also form closed loops (see Figure 15, for example).

Note that these loops (both open and closed) cannot cross one another. In particular, the open loops must join up the $2n$ external edges in some non-crossing fashion. If one numbers the vertices of degree 1 in cyclic order from 1 to $2n$, the FPL state yields a pairing of odd-indexed external vertices with even-indexed external vertices. For instance, the FPL state shown in Figure 15 links 1 with 12, 2 with 11, 3 with 4, 5 with 6, 7 with 8, and 9 with 10.

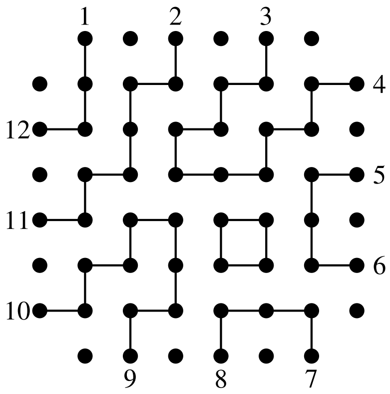

Figure 15: A fully packed loop state of order 6.

It has been conjectured, on the strength of numerical evidence, that the number of ASMs of order n in which the open paths link 1 with 2, 3 with 4, ..., and $2n-1$ with $2n$ is exactly the total number of ASMs of order $n-1$.

We do not have a proof of this, but curiously, we have a proof of something else: that the number of FPL states of order n in which the open paths link 1 with 2, 3 with 4, ..., and $2n-1$ with $2n$ is equal to the number of FPL states of order n in which the open paths link 1 with $2n$, 2 with 3, 4 with 5, ..., and $2n-2$ with $2n-1$. Note that when n is divisible by 4, the geometries of the two linking-patterns is different, with respect to the tic-tac-toe graph. Yet the number of FPL states is the same.

This is a special case of a far more general fact proved by Wieland [27]. For any two non-crossing pairings π and π' of the numbers 1 through $2n$ (viewed as equally spaced points on a circle), if π and π' are conjugate via a rotation or reflection, then (if we now treat the numbers 1 through $2n$ as the labels of vertices of degree 1 in the tic-tac-toe graph of order n) the number of FPL states with linking-pattern π equals the number of FPL states with linking-pattern π'. It is as if the tic-tac-toe graph, in some mystical sense, had an automorphism sending 1 to 2, 2 to 3, etc.

One might also ask for the number of FPL states of order n in which 1 is linked with 2 (ignoring all the other linking going on). Here too we have a conjectural answer, due to David Wilson: the number of such FPL states is just the total number of FPL states of order n multiplied by

$$\frac{3}{2}\frac{n^2+1}{4n^2-1}.$$

This would imply, in particular, that as n goes to infinity, the probability that a randomly chosen FPL state links 1 with 2 is asymptotically $3/8$.

For a very recent discussion of the FPL model, see [20]. In this article, Razumov and Stroganov point out that the FPL model is closely related to a seemingly quite different lattice model.

It is worth pointing out that these conjectures are truly native to the FPL incarnation of ASMs; it is hard to see how they could have arisen from one of the other models. So, even though the transformation between ASMs and FPL states is fairly shallow mathematically, some interesting questions can arise from it that might not otherwise have been noticed. Likewise, the passage between square-ice states and ASMs is not conceptually deep, but it made possible the shortest known solution of the ASM problem by putting the problem into a form where known methods from the physics literature could be applied. Hence a good subtitle for this paper might have been “The non-trivial power of trivial transformations”.

8 Numerology

As far as I know, the first manifestation of the sequence 1,2,7,42,429,7436,... occurred in connection with combinatorial objects called descending plane partitions or DPPs [1]. Another manifestation was totally symmetric self-complementary plane-partitions (TSSCPPs) [2]. Andrews discovered the proofs of both these formulas. Indeed, it was Andrews' proof of the 1,2,7,42,... formula for TSSCPPs that galvanized Zeilberger into tackling the ASM conjecture. Zeilberger showed that ASMs are equinumerous with TSSCPPs. However, his proof was not bijective, and to this day nobody knows of a good bijection between ASMs of order n and TSSCPPs of order n.

Another context in which these numbers arise is the study of the XXZ model in statistical mechanics [19] [4].

A different context in which numbers related to ASMs have occurred is certain “number walls” investigated by Somos [24]. (Number walls are arrays of Hankel determinants, arranged so as to faciliate calculations of successively larger ones; see [9] for details.) In Somos' examples it is not the sequence 1,2,7,42,... that crops up, but sequences that enumerate various sorts of symmetric ASMs. Xin [28] has found a Hankel determinant theorem that involves the sequence $1,2,7,42,\ldots$ itself: he has shown that for all n, the n-by-n matrix whose i,jth entry is equal to the coefficient of x^{i+j-2} in the Taylor expansion of the generalized Catalan generating function $\frac{1-(1-9x)^{1/3}}{3x}$ [16] is equal to

$$3^{\binom{n}{2}}$$

times the number of n-by-n alternating-sign matrices.

It would be desirable to have some sort of understanding of why the number of ASMs of order n (with or without symmetry-constraints) turns up in these seemingly disparate situations.

9 Large random ASMs

Another sort of phenomenon associated with ASMs of order n is their typical “shape” when n is large. I remarked above that the ASMs of order n form a distributive lattice; consequently, the method of “coupling from the past” can be applied [18]. Figure 16 shows a random ASM of order 40, represented as a gaskets-and-baskets tiling (where an attempt has been made to give each of the six tile-types its own distinctive shading).

Note that the gaskets (which correspond to the non-zero entries of the ASM) stay away from the corners. Computer experiments strongly indicate that this is typical behavior: the probability of finding a non-zero entry close to one of the corners appears to be quite small. Another way of expressing this is in terms of the entries $h_{i,j}$ of the height-function matrix. Say that a location (i,j) in a particular height-function

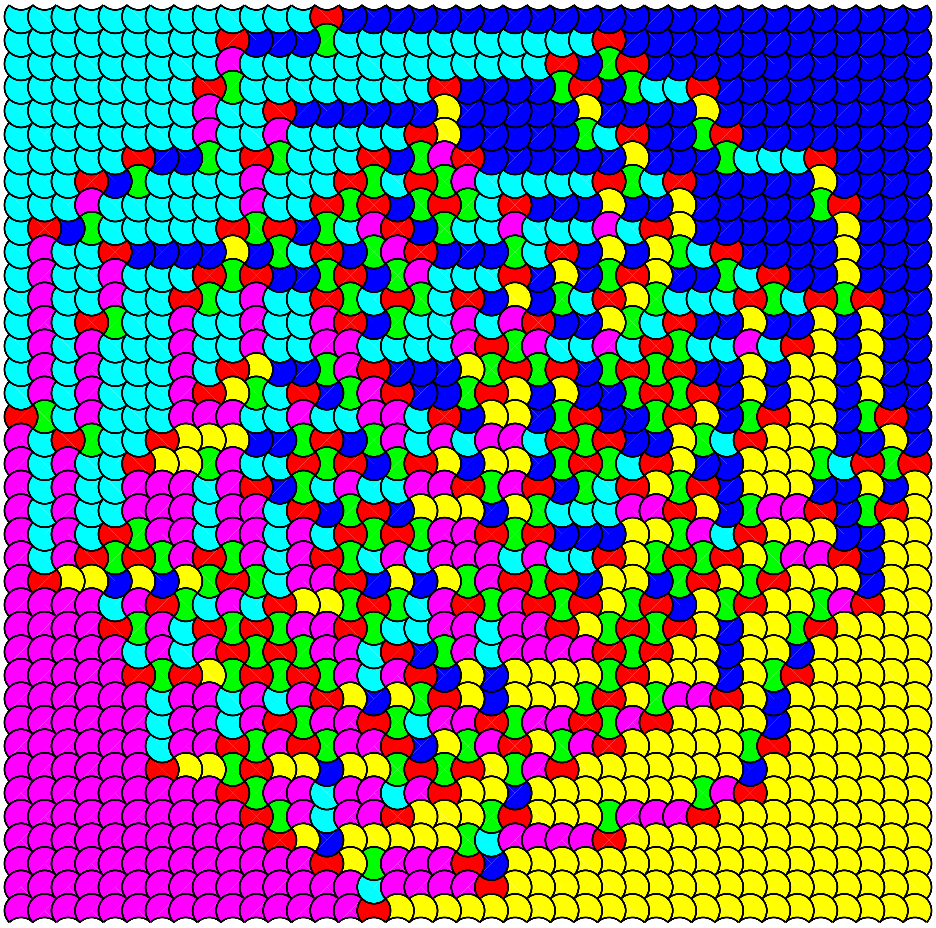

Figure 16: A random gaskets-and-baskets tiling of order 40.

matrix of order n is *frozen* if the height there is equal to either the maximum possible height that any height-function matrix of order n can exhibit at that location or the minimum possible height that any height-function matrix of order n can exhibit at that location. Then the claim is that a significant portion of the height-function matrix, concentrated in the four corners, tends to be frozen.

This is analogous to phenomena that have been observed for other sorts of combinatorial models. Indeed, if one adopt a non-uniform distribution on the set of ASMs of order n, where the probability associated with an ASM containing exactly k entries equal to -1 is proportional to 2^k, then it is rigorously known that the frozen region tend in probability to a perfect circular disk [12] [8]. However, it is not rigorously known that this holds when the uniform distribution on ASMs is used.

10 Back to Dodgson

I conclude this article by coming full circle and returning the context from which ASMs first came to light: the study of Dodgson's condensation algorithm and its variants. I will not discuss Dodgson's algorithm per se, but rather a variation of it invented by Robbins and Rumsey [23]. This modified form of Dodgson condensation is an algebraic recurrence relation

$$f_{i,j,k+1} = (f_{i+1,j,k}f_{i-1,j,k} + f_{i,j+1,k}f - i, j-1, k)/f_{i,j,k-1}$$

satisfied by certain functions $f : \mathbf{Z}^3 \to \mathbf{R}$. (Note that this equation can be written slightly more symmetrically as

$$f_{i,j,k+1}f_{i,j,k-1} - f_{i+1,j,k}f_{i-1,j,k} - f_{i,j+1,k}f_{i,j-1,k} = 0.$$

Physicists and researchers in the field of integrable systems call this a discrete Hirota equation, and have developed a great deal of theory associated with it; however, the observations I make here seem to be currently unknown outside of a small circle of algebraic combinatorialists.) If we let $f_{i,j,-1} = x_{i,j}$ and $f_{i,j,0} = y_{i,j}$ for formal indeterminates $x_{i,j}, y_{i,j}$ (with (i, j) ranging over $\mathbf{Z}^2$), then the recurrence relation lets us express all the $f'_{i,j,k}s$ in terms of the $x_{i,j}$'s and $y_{i,j}$'s, at least formally. A priori, one expects each $f_{i,j,k}$ to be a rational function of the x and y variables; the surprise (the first of several surprises, in fact) is that these rational functions are actually Laurent polynomials (that is, they are polynomials functions of the x and y variables along with their reciprocals).

This observation seems to have first been made by Mills, Robbins and Rumsey, who actually considered a more general recurrence

$$f_{i,j,k+1} = (f_{i+1,j,k}f_{i-1,j,k} + \lambda f_{i,j+1,k}f - i, j-1, k)/f_{i,j,k-1}.$$

The case $\lambda = -1$ corresponds to the original Dodgson algorithm, but Mills et. al noticed that the same surprising cancellations occur for more general values of λ, including the especially nice case $\lambda = 1$, and that one always obtains Laurent polynomials.

It was from studying these Laurent polynomials that Mills, Robbins and Rumsey were led to discover alternating-sign matrices. Every term in one of these Laurent polynomials has a coefficient equal to 1 (that is the second surprise), and is a product of powers of a finite number of x and y variables. The third surprise is that all the exponents of the variables are $+1$, -1, and 0. The fourth and final surprise is that these patterns of exponents encode ASMs. More specifically, the exponents of the x variables (after a global sign flip) encode one ASM, and the exponents of the y variables encode another. These two ASMs satisfy a combinatorial relationship that the researchers dubbed “compatibility”. They showed that the number of compatible pairs of ASMs is exactly $2^{n(n+1)/2}$.

As it happens, this formula is not hard to verify as a consequence of the other claims I have made. If indeed all the coefficients in the Laurent polynomial equal 1, then one can count the terms (and thereby count the compatible pairs of ASMs) just by setting all the x and y variables equal to 1. But in this case, $f_{i,j,k}$ depends only on k (call it F_k), and the three-dimensional recurrence boils down to the one-dimensional recurrence

$$F_{k+1} = (F_k F_k + F_k F_k)/F_{k-1}$$

with initial conditions $F_0 = F_1 = 1$, which is readily solved.

The terms of these Laurent polynomials were originally understood in terms of compatible pairs of ASMs. A few years after this work was done, it turned out that compatible pairs of ASMs admit a much more geometrical representation, namely, as tilings of regions called Aztec diamonds by means of tiles called dominos (1-by-2 and 2-by-1 rectangles). See [10] for more details.

I will close by pointing out that a kindred recurrence relation cries out to be studied, namely

$$f_{i,j,k} = (f_{i-1,j,k} f_{i,j-1,k-1} + f_{i,j-1,k} f_{i-1,j,k-1} + f_{i,j,k-1} f_{i-1,j-1,k})/f_{i-1,j-1,k-1},$$

with initial conditions

$$f_{i,j,k} = \begin{cases} x_{i,j,k} & \text{if } i+j+k = -1, \\ y_{i,j,k} & \text{if } i+j+k = 0, \\ z_{i,j,k} & \text{if } i+j+k = +1. \end{cases}$$

Here (just as in the Mills, Robbins, and Rumsey recurrence) one finds empirically that each value of $f_{i,j,k}$ is expressible as a Laurent polynomial in the x, y, and z variables (in fact, shortly before this article went to press, this Laurent property was proved by Fomin and Zelevinsky[11]); here too one finds empirically that each coefficient in these Laurent polynomials is equal to 1; and here too one finds that the exponents of the x, y and z variables that occur in the Laurent monomials are universally bounded (in this case between -1 and $+4$ rather than between -1 and $+1$). If all this is true, then the exponent-patterns that arise are some sort of analogue of compatible pairs of ASMs, and moreover, we know exactly how many there are: $3^{\lfloor n^2/4 \rfloor}$. (This comes from reducing the original three-dimensional recurrence to a one-dimensional recurrence, as we did before.) So, assuming that our empirical observations are not leading us astray, there is some new kind of combinatorial gadget that governs these Laurent polynomials (or vice versa!), and we know exactly how many gadgets of order n there are. And it is easy to generate these Laurent polynomials (and with them the gadgets) using MAPLE, e.g. with the following short program:

```
f := proc (i,j,k)
        if   (i+j+k < 3) then x(i,j,k) else
        simplify(
        ( f(i-1,j,k)*f(i,j-1,k-1)+
          f(i,j-1,k)*f(i-1,j,k-1)+
          f(i,j,k-1)*f(i-1,j-1,k) )
        /f(i-1,j-1,k-1));
        fi; end;
```

Nonetheless, we do not know what these gadgets are, combinatorially! They are analogous to pairs of compatible ASMs, which in turn are equivalent to domino tilings of Aztec diamonds, so one hopes that the gadgets have some intuitive geometric meaning. For more information about the properties of these gadgets, see `http://www.math.wisc.edu/~propp/cube-recur`.

References

[1] G. Andrews, Andrews, George E. Plane partitions III: The weak Macdonald conjecture, *Inventiones Math.* **53** (1979), 193–225.

[2] G. Andrews, Plane partitions V: The T.S.S.C.P. conjecture. *J. Combin. Theory Ser. A* **66** (1994), 28–39.

[3] M.T. Batchelor, H.W.J. Blote, B. Nienhuis, and C.M. Yung, Critical behavior of the fully packed loop model on the square lattice, *J. Phys. A* **29** (1996), L399-L404.

[4] M.T. Batchelor, J. de Gier, and B. Nienhuis, The quantum symmetric XXZ chain at Delta=–1/2, alternating sign matrices and plane partitions; arXiv:`cond-mat/0101385`.

[5] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics.* Academic Press, London, 1982.

[6] D.M. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture.* Cambridge University, Cambridge, 1999.

[7] D. Bressoud and J. Propp, How the alternating sign matrix conjecture was solved, *Notices of the AMS* **46** (1999), 637–646; `www.ams.org/notices/199906/fea-bressoud.pdf`.

[8] H. Cohn, N. Elkies, and J. Propp, Local statistics for random domino tilings of the Aztec diamond Duke Mathematical Journal **85** (1996), 117–166; arXiv:`math.CO/0008243`.

[9] J. Conway and R. Guy, *The Book of Numbers.* Copernicus, 1996.

[10] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, Alternating-sign matrices and domino tilings, *J. Algebraic Combin.* **1** (1992), 111–132; `www.math.wisc.edu/~propp/aztec.ps.gz`.

[11] S. Fomin and A. Zelevinsky, The Laurent phenomenon; arXiv:`math.CO/0104241`.

[12] W. Jockusch, J. Propp, and P. Shor, Random domino tilings and the arctic circle theorem; arXiv:`math.CO/9801068`.

[13] V.E. Korepin, N.M. Bogoliubov, and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions.* Cambridge University Press, New York, 1993.

[14] G. Kuperberg, Another proof of the alternating sign matrix conjecture, *Inter. Math. Res. Notes* **1996** (1996), 139–150; math.CO/9712207.

[15] G. Kuperberg, Symmetry classes of alternating-sign matrices under one roof; arXiv:`math.CO/0008184`.

[16] W. Lang, On generalizations of the Stirling number triangles, *J. Integer Sequences* **3** (2000), 00.2.4.

[17] W.H. Mills, D.P. Robbins, and H. Rumsey, Alternating-sign matrices and descending plane partitions, *J. Combin. Theory Ser. A* **34** (1983), 340–359.

[18] J. Propp, Generating random elements of a finite distributive lattice, *Electron. J. Combinat.* **4**:2 (1997), R15.

[19] A.V. Razumov and Yu.G. Stroganov, Spin chains and combinatorics; arXiv:`cond-mat/0012141`.

[20] A.V. Razumov and Yu.G. Stroganov, Combinatorial nature of ground state vector of $O(1)$ loop model; arXiv:`math.CO/0104216`.

[21] D.P. Robbins, The story of 1,2,7,42,429,7436,..., *Math. Intelligencer* **13** no. 2 (1991), 12–19.

[22] D.P. Robbins, Symmetry classes of alternating sign matrices; arXiv:`math.CO/0008045`.

[23] D.P. Robbins and H. Rumsey, Determinants and alternating-sign matrices, *Advances in Math.* **62** (1986), 169–184.

[24] M. Somos, `grail.cba.csuohio.edu/~somos/nwic.html`.

[25] R. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986, xi + 306 pages; second printing, Cambridge University Press, Cambridge, 1996.

[26] O. Tsuchiya, Determinant formula for the six-vertex model with reflecting end; arXiv:`solv-int/9804010`.

[27] B. Wieland, A large dihedral symmetry of the set of alternating sign matrices; `www.combinatorics.org/Volume_7/Abstracts/v7i1r37.html`, arXiv:`math.CO/0006234`.

[28] G. Xin, private communication.

[29] B.-Y. Yang, Three enumeration problems concerning Aztec diamonds, Ph.D. thesis, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1991.

[30] D. Zeilberger, Proof of the alternating sign matrix conjecture, *Electronic J. Comb.* **3** (1996), R13; arXiv:`math.CO/9407211`.