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Tilings from some non-irreducible, Pisot substitutions

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A generating method of self-affine tilings for Pisot, unimodular, irreducible substitutions, as well as the fact that the associated substitution dynamical systems are isomorphic to rotations on the torus are established in [AI01]. The aim of this paper is to extend these facts in the case where the characteristic polynomial of a substitution is non-irreducible for a special class of substitutions on five letters. Finally we show that the substitution dynamical systems for this class are isomorphic to induced transformations of rotations on the torus.

Keywords: Substitution, Pisot number, Pisot substitution, atomic surface, tiling, fractal, dynamical system

1 Introduction

In this paper we want to discuss tilings and dynamical systems generated by the following substitutions given by

$$\sigma : \begin{cases} 1 \to \overbrace{11\cdots1}^{K+1\ times}\ 2 \\ 2 \to 3 \\ 3 \to 4 \\ 4 \to \overbrace{1\cdots1}^{K\ times}\ 5 \\ 5 \to 1 \end{cases}, \quad K \geq 0. \tag{1.1}$$

The characteristic polynomial of the incidence matrix L_σ is

$$x^5 - (K+1)x^4 - Kx - 1 = (x^2 - x + 1)(x^3 - Kx^2 - (K+1)x - 1). \tag{1.2}$$

Since it is non-irreducible, its factor $x^3 - Kx^2 - (K+1)x - 1$ is a minimal polynomial of some Pisot number β. Furthermore $|\det(L_\sigma)| = 1$; we thus say that the above substitution σ is of the non-irreducible, Pisot, unimodular type. The aim of this paper is to discuss how we obtain tilings and dynamical systems generated by non-irreducible, Pisot, unimodular substitutions for the special class (1.1) which is coming from Pisot β-expansions.

Let us recall some results in the irreducible, Pisot, unimodular case. (See [AI01].) For example we consider the following substitution σ on 3 letters:

$$\sigma \ : \begin{cases} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1 \end{cases} . \tag{1.3}$$

The substitution σ has the incidence matrix

$$L_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and its characteristic polynomial $x^3 - x^2 - x - 1$. So this substitution is of the irreducible, Pisot, unimodular type.

For the contractive plane $\boldsymbol{P}$ of the matrix L_σ, a stepped surface which is a discrete plane approximation of $\boldsymbol{P}$ is determined. Thus we have a tiling of the plane $\boldsymbol{P}$ with three prototiles, which are parallelograms, by using the projection π defined as the map from $\mathbb{R}^3$ to $\boldsymbol{P}$ along the eigenvector $\boldsymbol{u}$ of L_σ corresponding to the Pisot eigenvalue β.

This substitution σ has a unique fixed point and we denote it by $\omega = s_0 s_1 \cdots s_n \cdots$. Then we obtain the sets X_i $(i = 1, 2, 3)$ given by the closure of $\{\pi \sum_{k=0}^{n-1} e_{s_k} \mid s_n = i,\ n = 1, 2, \cdots\}$ and $X = \cup_{i=1}^{3} X_i$, where $\{\boldsymbol{e}_i\}_{i=1,2,3}$ is the canonical basis of $\mathbb{R}^3$. These sets $X_i,\ X$ are called atomic surfaces of σ.

On the other hand, it is known that the tiling and the atomic surfaces can be generated by the so called tiling substitution $E_1^*(\sigma)$ on the $\mathbb{Z}$-module $\mathcal{G}_1^*$ defined by

$$\mathcal{G}_1^* = \left\{ \sum_{\delta \in \mathbb{Z}^3 \times \{1^*, 2^*, 3^*\}} n_\delta \delta \mid n_\delta \in \mathbb{Z}, {}^{\#}\{\delta \in \mathbb{Z}^3 \times \{1^*, 2^*, 3^*\} \mid n_\delta \neq 0\} < \infty \right\}.$$

Here we identify $(\boldsymbol{x}, i^*) \in \mathbb{Z}^3 \times \{1^*, 2^*, 3^*\}$ with the unit square

$$\{\boldsymbol{x} + s\boldsymbol{e}_j + t\boldsymbol{e}_k \in \mathbb{R}^3 | \{j, k\} = \{1, 2, 3\} \backslash \{i\},\ 0 \leq s \leq 1,\ 0 \leq t \leq 1\},$$

and summation "+" in an element in $\mathcal{G}_1^*$ means the union of these unit squares. More generally we consider a substitution σ denoted by $\sigma(i) = W_0^{(i)} W_1^{(i)} \cdots W_k^{(i)} \cdots W_{l_{(i)}-1}^{(i)}$. By using the canonical homomorphism $\mathbf{f}$ from the free monoid on 3 letters to $\mathbb{Z}^3$ defined by $\mathbf{f}(i) = e_i$ $(i = 1, 2, 3)$ and the notations $P_k^{(i)}$ and $S_k^{(i)}$ stand for respectively the prefix of length k and suffix of length $l_{(i)} - 1 - k$ of $\sigma(i)$, we define the endomorphism $E_1^*(\sigma)$ on $\mathcal{G}_1^*$ as follows:

$$E_1^*(\sigma)(\boldsymbol{x}, i^*) := \sum_{j=1}^{3} \sum_{S_k^{(j)} : W_k^{(j)} = i} (L_\sigma^{-1} \boldsymbol{x} + L_\sigma^{-1} \mathbf{f}(S_k^{(j)}), j^*) .$$

On this setting we can generate the stepped surface of the plane $\boldsymbol{P}$ by $E_1^{*\,n}(\sigma)((\boldsymbol{e}_1, 1^*) + (\boldsymbol{e}_2, 2^*) + (\boldsymbol{e}_3, 3^*))$ $(n \to \infty\ )$ and the atomic surfaces by

$$X_i = -\lim_{n \to \infty} L_\sigma^n \pi E_1^{*\,n}(\sigma)(\boldsymbol{e}_i, i^*)\ (i = 1, 2, 3),$$

where the right side converges in the sense of the Hausdorff metric and we denote $\lim_{n\to\infty} L_\sigma^n \pi E_1^{*\,n}(\sigma)(\boldsymbol{e}_i, i^*)$ by $\widehat{X_i}$. Furthermore, we have the following property:

Property 1.1 *(1) The pieces in the union $\cup_{i=1}^3 \widehat{X_i}$ denoted by $\widehat{X}$ do not overlap up to a set of Lebesgue measure 0,*

(2) the following set equation holds:

$$L_\sigma^{-1}\widehat{X_i} = \bigcup_{j=1}^{3} \bigcup_{P_k^{(j)}:W_k^{(j)}=i} (\widehat{X_j} - L_\sigma^{-1}\pi\mathbf{f}(P_k^{(j)}))\ ,$$

where the sets in the right side of the equation do not overlap up to a set of Lebesgue measure 0, and moreover, the transformation $F : \widehat{X} \to \widehat{X}$ given by

$$F(\boldsymbol{x}) = L_\sigma^{-1}(\boldsymbol{x} + \pi\mathbf{f}(P_k^{(j)})) \quad if\ \boldsymbol{x} \in L_\sigma\widehat{X_j} - \pi\mathbf{f}(P_k^{(j)})$$

is well-defined and it is a Markov transformation with matrix structure L_σ,

(3) the transformation $E : \widehat{X} \to \widehat{X}$, called the domain exchange transformation, given by

$$E(\boldsymbol{x}) = \boldsymbol{x} - \pi\boldsymbol{e}_i \qquad \boldsymbol{x} \in \widehat{X_i}$$

is well-defined and the transformation E is measure-theoretically isomorphic to a rotation on the 2-dimensional torus, and moreover, the orbit of the origin point by E satisfies $E^k(0) \in \widehat{X}_{s_k},\ k = 0, 1, \cdots$.

Property (2) and (3) are not known to hold for any irreducible Pisot substitution. Property (2) holds provided that (1) holds if the substitution σ has the strong coincidence condition (cf. [AI01]). Moreover, for the transformation F to be well-defined, the pieces must not overlap. The isomorphism with a rotation (3) is equivalent with the tiling property.

The aim of the paper is to obtain the analogous property for the non-irreducible, Pisot, unimodular substitutions given by (1.1).

This paper is sketched as follows.

In Section 2, we define a projection map π from $\mathbb{R}^5$ to the contractive 2-dimensional plane of the incidence matrix L_σ of σ. A substitution given by (1.1) has a unique fixed point. Therefore, using the projection we obtain atomic surfaces X and X_i $(i = 1, 2, 3, 4, 5)$ with respect to the letter i in the same way as in the irreducible case. (See Fig. 1.)

In Section 3, 4 and 6, we define the tiling substitution τ^* of a substitution σ according to some modification of the method by [AI01]. Since we deal with the non-irreducible case, we can not use the same method as in the irreducible case. However, we introduce new polygonal tiles instead of parallelograms, a tiling substitution τ^* and the concept of a stepped surface; and try to construct atomic surfaces by using a map τ^* and tilings $\mathcal{T}_{\tau^*}$ with five polygonal prototiles and $\mathcal{T}_{\tau^*}^{\widehat{X}}$ with the atomic surfaces X_i $(i = 1, 2, 3, 4, 5)$. (See Fig. 2 and Fig. 3.)

In Section 5, we introduce two dynamical systems on $\widehat{X} := -X$ associated with non-irreducible substitutions, that is, a Markov transformation and a domain exchange transformation related to Property 1.1

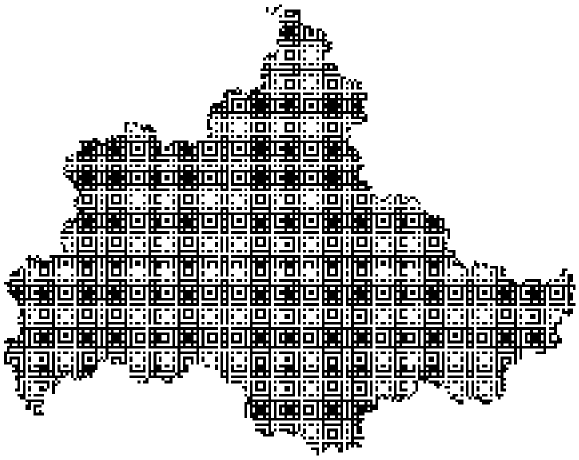

Fig. 1: The atomic surface X of the substitution σ in the case of $K = 0$

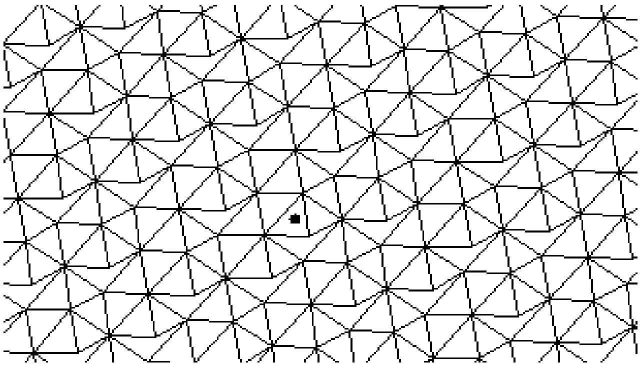

Fig. 2: The tiling $\mathcal{T}_{\tau^*}$ in the case of $K = 0$

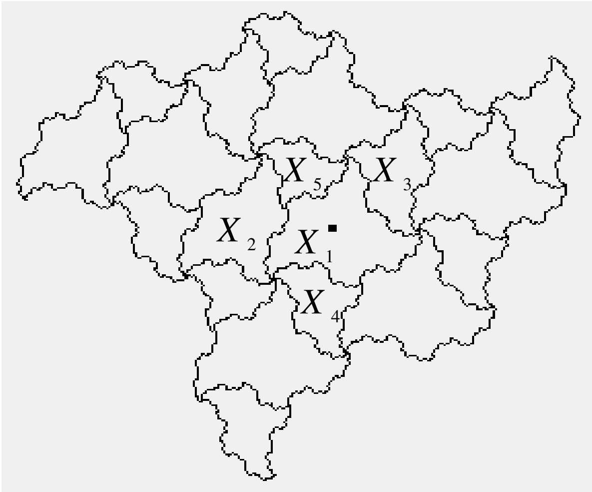

Fig. 3: The tiling $\mathcal{T}_{\tau^*}^{\widehat{X}}$ in the case of $K = 0$

(2) and (3). The main theorems in this paper are Theorem 5.2 and Theorem 5.3, and these theorems say the following:

The domain exchange transformation $E : \widehat{X} \to \widehat{X}$ is defined by

$$E(\boldsymbol{x}) = \boldsymbol{x} - \pi \boldsymbol{e}_i \qquad \boldsymbol{x} \in \widehat{X_i}$$

and the orbit of the origin point by the transformation E satisfies $E^k(0) \in \widehat{X}_{s_k},\ k = 0, 1, \cdots$. The transformation E is not measure-theoretically isomorphic to a rotation on the 2-dimensional torus, but isomorphic to the *induced* transformation of a rotation on the torus.

2 Substitutions and atomic surfaces

2.1 General results on atomic surfaces

Let $\mathcal{A}$ be an alphabet consisting of d letters $\{1, 2, \cdots, d\}$. The free monoid on the alphabet $\mathcal{A}$ with the empty word ϵ is denoted by $\mathcal{A}^* = \cup_{n=0}^{\infty} \mathcal{A}^n$ and $\mathcal{A}^{\mathbb{N}}$ denotes the collection of all right infinite sequences of symbols from $\mathcal{A}$. Let σ be an endomorphism on $\mathcal{A}^*$ such that $\sigma(i) \in \mathcal{A}^* \setminus \{\epsilon\}$ for all $i \in \mathcal{A}$, which is called a *substitution*. By defining $\sigma(UV) = \sigma(U)\sigma(V)$ for a concatenation UV of words, the substitution σ is extended to an endomorphism on $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}}$. Put

$$\begin{aligned} \sigma(i) &= W_0^{(i)} W_1^{(i)} \cdots W_{l_{(i)}-1}^{(i)} \\ &= P_k^{(i)} W_k^{(i)} S_k^{(i)} \end{aligned}$$

and

$$\begin{aligned} \sigma^n(i) &= W_0^{(n,i)} W_1^{(n,i)} \cdots W_{l_{(n,i)}-1}^{(n,i)} \\ &= P_k^{(n,i)} W_k^{(n,i)} S_k^{(n,i)} \qquad , \end{aligned}$$

where $P_0^{(i)} = P_0^{(n,i)} = \epsilon$ for any $i \in \mathcal{A}$ and any positive integer n. We call $P_k^{(i)}$ (resp., $S_k^{(i)}$) the *k-prefix* (resp., the *k-suffix*) of a word $\sigma(i)$. We define the canonical homomorphism $\mathbf{f} : \mathcal{A}^* \to \mathbb{Z}^d$ by $\mathbf{f}(\epsilon) = 0$ and $\mathbf{f}(i) = \boldsymbol{e}_i$ $(i \in \mathcal{A})$, where $\{\boldsymbol{e}_i\}_{i=1,\cdots d}$ is the canonical basis of $\mathbb{R}^d$. This is naturally extended to a map on $\mathcal{A}^*$ by defining $\mathbf{f}(UV) = \mathbf{f}(U) + \mathbf{f}(V)$ for words $U,\ V$ in $\mathcal{A}^*$. There is a matrix L_σ on $\mathbb{Z}^d$ for a substitution σ satisfying the following commutative diagram:

$$\begin{array}{ccccc} & \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* & \\ \mathbf{f} & \downarrow & & \downarrow & \mathbf{f} \\ & \mathbb{Z}^d & \xrightarrow{L_\sigma} & \mathbb{Z}^d & . \end{array}$$

The matrix L_σ is called the *incidence matrix* of the substitution and its entry $L_\sigma(i,j)$ is equal to the number of occurrences of the letter i in $\sigma(j)$.

Recall that a *Pisot number* is an algebraic integer greater than 1 and whose conjugates have modulus strictly less than 1. Before discussing atomic surfaces of substitutions given by (1.1), we will define atomic surfaces on a general setting with the following assumption:

Assumption *Throughout this paper, we assume that for a substitution σ*

(1) $W_0^{(1)} = 1$, that is, $\sigma(1)$ begins with 1,

(2) σ is unimodular, that is, $\mid \det(L_\sigma) \mid = 1$,

(3) the characteristic polynomial of L_σ is not irreducible and is decomposed as $f(x)g(x)$ such that $f(x)$ is a minimal polynomial of some Pisot number β, and the roots of $g(x)$ have modulus 1.

A substitution with Assumption (2) and (3) is referred to be of *unimodular, non-irreducible, Pisot* type. From Assumption (1), there exists a fixed point ω of the substitution σ:

$$\omega = \lim_{n\to\infty} \sigma^n(1) = s_0 s_1 \cdots s_n \cdots \ .$$

From Assumption (2) and (3), we define the expanding subspace $\mathcal{L}(\boldsymbol{u})$ spanned by the eigenvector $\boldsymbol{u}$ corresponding to the eigenvalue β and the contractive subspace $\boldsymbol{P} \neq \{0\} (\subset \mathbb{R}^d)$ corresponding to the other conjugate eigenvalues of β. Then we have a direct sum $\mathbb{R}^d = \boldsymbol{P} \oplus \mathcal{L}(\boldsymbol{u}) \oplus \boldsymbol{P}'$, where $\boldsymbol{P}'$ corresponds to the other eigenvectors coming from $g(x)$. Let us define the projection $\pi : \mathbb{R}^d \to \boldsymbol{P}$ by

$$\pi(\boldsymbol{p} + \boldsymbol{x} + \boldsymbol{p}') = \boldsymbol{p},$$

where $\boldsymbol{p} \in \boldsymbol{P},\ \boldsymbol{x} + \boldsymbol{p}' \in \mathcal{L}(\boldsymbol{u}) \oplus \boldsymbol{P}'$.

Definition 2.1 *From sets $Z_i,\ Z_i'$ of prefixes of a fixed point ω:*

$$\begin{aligned} Z_i &:= \{s_0 s_1 \cdots s_{k-1} \mid s_k = i, k = 1, 2, \cdots\} \\ Z_i' &:= \{s_0 s_1 \cdots s_k \mid s_k = i, k = 0, 1, \cdots\}, \end{aligned}$$

we define sets $Y_i,\ Y_i'$ in $\boldsymbol{P}$ as follows:

$$\begin{aligned} Y_i &:= \pi\mathbf{f}(Z_i) = \{\pi\mathbf{f}(s_0 s_1 \cdots s_{k-1}) \mid s_k = i, k = 1, 2, \cdots\} \\ Y_i' &:= \pi\mathbf{f}(Z_i') = \{\pi\mathbf{f}(s_0 s_1 \cdots s_k) \mid s_k = i, k = 0, 1, \cdots\}\ . \end{aligned}$$

The sets $X,\ X_i,\ X'_i$ in $\boldsymbol{P}$ are defined by

$$\begin{aligned} X &:= cl.(\cup_{i=1}^{d} Y_i) = cl.(\cup_{i=1}^{d} Y'_i) \\ X_i &:= cl.(Y_i) \\ X'_i &:= cl.(Y'_i)\ , \end{aligned}$$

where $cl.S$ means the closure of a set S.

We call the sets $X,\ X_i\ (i \in \mathcal{A})$ *atomic surfaces* of σ. Note that the equality $X'_i = X_i + \pi \boldsymbol{e}_i$ holds from $Y'_i = Y_i + \pi \boldsymbol{e}_i$, where $S + \boldsymbol{a} = \{\boldsymbol{x} + \boldsymbol{a} \mid \boldsymbol{x} \in S\}$.

For words $U = u_0 u_1 \cdots u_l$, $V = v_0 v_1 \cdots v_m \in \mathcal{A}^*$, $U \prec V$ denotes that $l < m$ and $v_0 v_1 \cdots v_l = U$. The following proposition can be found in [AI01][FFIW00].

Proposition 2.1 *The following set equations hold:*

(1)
$$L_\sigma^{-1} X_i = \bigcup_{j=1}^{d} \bigcup_{P_k^{(j)} : W_k^{(j)} = i} (X_j + L_\sigma^{-1} \pi \mathbf{f}(P_k^{(j)}))$$

(2)
$$L_\sigma^{-1} X'_i = \bigcup_{j=1}^{d} \bigcup_{S_k^{(j)} : W_k^{(j)} = i} (X'_j - L_\sigma^{-1} \pi \mathbf{f}(S_k^{(j)}))\ .$$

Remark 1 *We will see the following property in Theorem 4.1 (4):*
the sets $X_j + L_\sigma^{-1}\pi\mathbf{f}(P_k^{(j)})$ (resp., $X'_j - L_\sigma^{-1}\pi\mathbf{f}(S_k^{(j)})$) with $W_k^{(j)} = i$ in the equation (1) (resp., (2)) do not overlap up to a set of Lebesgue measure 0.

Proof: From the property

$$cl.(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} cl.(A_i),$$

it is enough to show that

$$\mathbf{f}(Z_i) = \bigcup_{j=1}^{d} \bigcup_{P_k^{(j)} : W_k^{(j)} = i} (L_\sigma \mathbf{f}(Z_j) + \mathbf{f}(P_k^{(j)})). \tag{2.1}$$

Take $s_0 s_1 \cdots s_n \in Z_i$ with $s_{n+1} = i$, then there exists an integer $m\ (m < n)$ such that

$$\sigma(s_0 s_1 \cdots s_{m-1}) \prec s_0 s_1 \cdots s_n i \preceq \sigma(s_0 s_1 \cdots s_m).$$

Consequently there exists an integer t such that

$$s_0 s_1 \cdots s_n i = \sigma(s_0 s_1 \cdots s_{m-1}) P_t^{(s_m)} W_t^{(s_m)},$$

where $W_t^{(s_m)} = i$. Let $\mathbf{f}$ act on both sides of the above equality,

$$\mathbf{f}(s_0 s_1 \cdots s_n i) = \mathbf{f}(\sigma(s_0 s_1 \cdots s_{m-1}) P_t^{(s_m)} W_t^{(s_m)}).$$

Hence, by $\mathbf{f} \circ \sigma = L_\sigma \circ \mathbf{f}$, we have

$$\mathbf{f}(s_0 s_1 \cdots s_n) = L_\sigma \mathbf{f}(s_0 s_1 \cdots s_{m-1}) + \mathbf{f}(P_t^{(s_m)}) \ .$$

Thus $\mathbf{f}(s_0 s_1 \cdots s_n) \in L_\sigma \mathbf{f}(Z_{s_m}) + \mathbf{f}(P_t^{(s_m)})$ with $W_t^{(s_m)} = i$. This shows $\subseteq$ is true for the equality (2.1) by choosing $j = s_m$ and $k = t$.

On the other hand, for any $s_0 s_1 \cdots s_m \in Z_j$ with $s_{m+1} = j$ and $W_k^{(j)} = i$, we have

$$\begin{aligned} L_\sigma \mathbf{f}(s_0 s_1 \cdots s_m) + \mathbf{f}(P_k^{(j)}) &= \mathbf{f}(\sigma(s_0 s_1 \cdots s_m)) + \mathbf{f}(P_k^{(j)}) \\ &= \mathbf{f}(\sigma(s_0 s_1 \cdots s_m) P_k^{(j)}) \ . \end{aligned}$$

Since $s_{m+1} = j$, then it is clear that

$$\sigma(s_0 s_1 \cdots s_m) P_k^{(j)} i \preceq \sigma(s_0 s_1 \cdots s_m s_{m+1}) \ .$$

Therefore, $\sigma(s_0 s_1 \cdots s_m) P_k^{(j)} \in Z_i$. This leads to the other direction of the equality (2.1). The second set equation is shown by (1) and $X_i = X_i' - \pi \boldsymbol{e}_i$. $\square$

2.2 Atomic surfaces generated by the labeled graph G^*

To obtain a numerical representation of X or X_i as we will see in Theorem 2.1, we introduce the new alphabet $\mathcal{B}$ and the subset $\mathcal{B}_\sigma^*$ of the monoid $\mathcal{B}^*$ by using the prefix automaton as follows:

$$\mathcal{B} := \left\{ \begin{pmatrix} i \\ t_i \end{pmatrix} \mid \begin{array}{l} i \in \mathcal{A} \\ t_i \in \{0, 1, \cdots, |\sigma(i)| - 1\} \end{array} \right\},$$

$$\mathcal{B}_\sigma^* := \left\{ \begin{pmatrix} i_0 \\ k_0 \end{pmatrix} \begin{pmatrix} i_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} i_N \\ k_N \end{pmatrix} \in \mathcal{B}^* \mid W_{k_n}^{(i_n)} = i_{n-1} \ (n = 1, 2, \cdots, N), \ N = 0, 1, \cdots \right\},$$

where $|U|$ means the length of a word U. (See Fig. 4.)

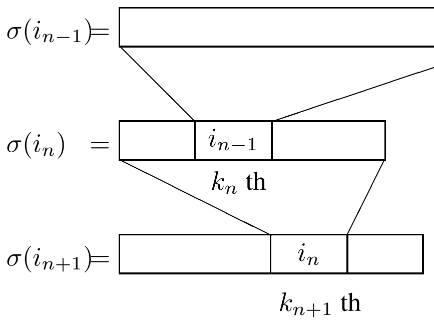

Fig. 4: A subword $\binom{i_n}{k_n}\binom{i_{n+1}}{k_{n+1}}$ of a word in $\mathcal{B}_\sigma^*$

We define a labeled graph G^* such that the set of vertices is $\mathcal{V} = \{1, 2, \cdots, d\}$ and the set of edges is $\mathcal{E} = \{0, 1, \cdots, |\sigma(i_0)| - 1\}$ with the largest $|\sigma(i_0)|$ for $i_0 \in \mathcal{A}$. If $W_k^{(j)} = i$, that is, the letter i occurs in $\sigma(j)$ as k-th letter, then one edge from the vertex i to the vertex j named k is drawn. (See Fig. 5. cf. [CS01])

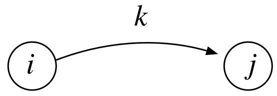

Fig. 5: The edge k from the vertex i to the vertex j ($W_k^{(j)} = i$)

From the definition of the labeled graph G^*, the set $\mathcal{B}_\sigma^*$ is given by all finite paths of $\begin{pmatrix} vertex \\ edge \end{pmatrix}$ in G^* (cf. Fig 6). Define the subset $\mathcal{B}_\sigma^{*(i)}$ of $\mathcal{B}_\sigma^*$:

$$\mathcal{B}_\sigma^{*(i)} := \left\{ \begin{pmatrix} i_0 \\ k_0 \end{pmatrix} \begin{pmatrix} i_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} i_N \\ k_N \end{pmatrix} \in \mathcal{B}_\sigma^* \mid W_{k_0}^{(i_0)} = i,\ N = 0, 1, \cdots \right\} .$$

The set $\mathcal{B}_\sigma^{*(i)}$ is the set of all finite paths whose initial vertex is i in G^*.

Theorem 2.1 *For any substitution σ satisfying Assumption, the following equalities hold:*

$$\begin{aligned} X_i &= cl. \left\{ \sum_{n=0}^{N} L_\sigma^n \pi \mathbf{f}(P_{k_n}^{(i_n)}) \mid \begin{pmatrix} i_0 \\ k_0 \end{pmatrix} \begin{pmatrix} i_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} i_N \\ k_N \end{pmatrix} \in \mathcal{B}_\sigma^{*(i)},\ P_{k_N}^{(i_N)} W_{k_N}^{(i_N)} \prec \omega,\ N = 0, 1, \cdots \right\} \\ X &= cl. \left\{ \sum_{n=0}^{N} L_\sigma^n \pi \mathbf{f}(P_{k_n}^{(i_n)}) \mid \begin{pmatrix} i_0 \\ k_0 \end{pmatrix} \begin{pmatrix} i_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} i_N \\ k_N \end{pmatrix} \in \mathcal{B}_\sigma^*,\ P_{k_N}^{(i_N)} W_{k_N}^{(i_N)} \prec \omega,\ N = 0, 1, \cdots \right\} . \end{aligned}$$

Proof: We show

$$Y_i = \left\{ \sum_{n=0}^{N} L_\sigma^n \mathbf{f}(P_{k_n}^{(i_n)}) \mid \begin{pmatrix} i_0 \\ k_0 \end{pmatrix} \begin{pmatrix} i_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} i_N \\ k_N \end{pmatrix} \in \mathcal{B}_\sigma^{*(i)},\ P_{k_N}^{(i_N)} W_{k_N}^{(i_N)} \prec \omega,\ N = 0, 1, \cdots \right\} .$$

By the proof in Proposition 2.1, for $s_0 s_1 \cdots s_m \in Z_i$, there exist positive integers i_0, k_0, m_0 such that

$$\mathbf{f}(s_0 s_1 \cdots s_m) = L_\sigma \mathbf{f}(s_0 s_1 \cdots s_{m_0}) + \mathbf{f}(P_{k_0}^{(i_0)})$$

and

$$W_{k_0}^{(i_0)} = i, \ \ s_0 s_1 \cdots s_{m_0} \in Z_{i_0} .$$

For $s_0s_1\cdots s_{m_0}\in Z_{i_0}$, let us continue the same procedure. Because $|s_0s_1\cdots s_{m_l}|$ is monotone decreasing with respect to l, at last we have the following equality:

$$\begin{aligned}\mathbf{f}(s_0s_1\cdots s_m) &= L_\sigma\mathbf{f}(s_0s_1\cdots s_{m_0})+\mathbf{f}(P_{k_0}^{(i_0)})\\ &= L_\sigma(L_\sigma\mathbf{f}(s_0s_1\cdots s_{m_1})+\mathbf{f}(P_{k_1}^{(i_1)}))+\mathbf{f}(P_{k_0}^{(i_0)})\\ &= \cdots\\ &= L_\sigma^N\mathbf{f}(P_{k_N}^{(i_N)})+L_\sigma^{N-1}\mathbf{f}(P_{k_{N-1}}^{(i_{N-1})})+\cdots+\mathbf{f}(P_{k_0}^{(i_0)})\ ,\end{aligned}$$

where $\binom{i_0}{k_0}\binom{i_1}{k_1}\cdots\binom{i_N}{k_N}\in\mathcal{B}_\sigma^{*(i)}$. This shows

$$Y_i\subseteq\left\{\sum_{n=0}^{N}L_\sigma^n\mathbf{f}(P_{k_n}^{(i_n)})\mid\begin{pmatrix}i_0\\k_0\end{pmatrix}\begin{pmatrix}i_1\\k_1\end{pmatrix}\cdots\begin{pmatrix}i_N\\k_N\end{pmatrix}\in\mathcal{B}_\sigma^{*(i)},\ P_{k_N}^{(i_N)}W_{k_N}^{(i_N)}\prec\omega,\ N=0,1,\cdots\right\}.$$

We obtain the converse inclusion as in the proof of Proposition 2.1. Thus the first equality is proved. Analogously we prove the second one. □

From the above decomposition of $\mathbf{f}(s_0s_1\cdots s_m)$, we have the following corollary:

Corollary 2.1 *For any substitution which satisfies Assumption, the atomic surface X is bounded.*

2.3 Atomic surfaces corresponding to a substitution given by (1.1)

From now on, let us deal with the non-irreducible, Pisot, unimodular substitutions σ given by (1.1):

$$\sigma\ :\begin{cases}1\ \rightarrow\ \overbrace{11\cdots1}^{K+1\ times}\ 2\\ 2\ \rightarrow\ 3\\ 3\ \rightarrow\ 4\\ 4\ \rightarrow\ \overbrace{1\cdots1}^{K\ times}\ 5\\ 5\ \rightarrow\ 1\end{cases}.$$

In this case,

$$L_\sigma=\begin{pmatrix}K+1&0&0&K&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}.$$

We see that the characteristic polynomial of the incidence matrix L_σ is given by (1.2) and it is also the characteristic polynomial of β (cf. [PAR60]). It is easy to check that the equation $x^3-Kx^2-(K+1)x-1=0$ gives one real root β and two imaginary roots $\beta^{(2)},\overline{\beta^{(2)}}$ if $K\leq2$, or three real roots $\beta,\beta^{(2)},\beta^{(3)}$ if $K\geq3$. The five roots of $x^5-(K+1)x^4-Kx-1=0$ are denoted by λ_i $(i=1,2,3,4,5)$, where $\lambda_1=\beta^{(2)},\ \ \lambda_2=\overline{\beta^{(2)}},\ \lambda_3=\beta$ if $K\leq2$, or $\lambda_1=\beta^{(2)},\ \ \lambda_2=\beta^{(3)},\ \lambda_3=\beta$ if $K\geq3$, and λ_4,λ_5 are the imaginary roots of $x^2-x+1=0$. Choose any eigenvector $\boldsymbol{u}_i$ associated with the eigenvalue

λ_i $(i = 1, 2, 3, 4, 5)$ such that $\boldsymbol{u}_2 = \overline{\boldsymbol{u}_1}$ if $K \le 2$ and $\boldsymbol{u}_5 = \overline{\boldsymbol{u}_4}$. Put

$$\begin{array}{llllll} \boldsymbol{v}_1 = \frac{1}{2}(\boldsymbol{u}_2 + \boldsymbol{u}_1), & \boldsymbol{v}_2 = \frac{1}{2i}(\boldsymbol{u}_2 - \boldsymbol{u}_1), & \boldsymbol{v}_3 = \boldsymbol{u}_3, & \boldsymbol{v}_4 = \frac{1}{2}(\boldsymbol{u}_5 + \boldsymbol{u}_4), & \boldsymbol{v}_5 = \frac{1}{2i}(\boldsymbol{u}_5 - \boldsymbol{u}_4) & if\ K \le 2\,, \\ \boldsymbol{v}_1 = \boldsymbol{u}_1, & \boldsymbol{v}_2 = \boldsymbol{u}_2, & \boldsymbol{v}_3 = \boldsymbol{u}_3, & \boldsymbol{v}_4 = \frac{1}{2}(\boldsymbol{u}_5 + \boldsymbol{u}_4), & \boldsymbol{v}_5 = \frac{1}{2i}(\boldsymbol{u}_5 - \boldsymbol{u}_4) & if\ K \ge 3\,, \end{array}$$

and let us define the 5×5 matrix V by

$$V := (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5).$$

The real vectors $\boldsymbol{v}_i$ $(i = 1, 2, 3, 4, 5)$ and the real matrix V satisfy the following relation:

$$L_\sigma V = V \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{R}} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \Re[\lambda_4] & -\Im[\lambda_4] \\ 0 & 0 & 0 & \Im[\lambda_4] & \Re[\lambda_4] \end{pmatrix},$$

where $\Re[a]$ (resp., $\Im[a]$) means the real (resp., imaginary) part of a and

$$R = \begin{cases} \begin{pmatrix} \Re[\beta^{(2)}] & -\Im[\beta^{(2)}] \\ \Im[\beta^{(2)}] & \Re[\beta^{(2)}] \end{pmatrix} & if\ K \le 2 \\ \begin{pmatrix} \beta^{(2)} & 0 \\ 0 & \beta^{(3)} \end{pmatrix} & if\ K \ge 3 \end{cases} .$$

The space $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$ spanned by vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ is an invariant contractive space of the linear transformation L_σ. More precisely, we know

$$L_\sigma \boldsymbol{x} = (\boldsymbol{v}_1, \boldsymbol{v}_2) R \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

where $\boldsymbol{x} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 \in \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$, $x_1, x_2 \in \mathbb{R}$. We have the direct sum $\mathbb{R}^5 = \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>} \oplus \mathcal{L}(\boldsymbol{v}_3) \oplus \boldsymbol{P}'_{<\boldsymbol{v}_4,\boldsymbol{v}_5>}$, where $\boldsymbol{P}'_{<\boldsymbol{v}_4,\boldsymbol{v}_5>}$ is spanned by $\boldsymbol{v}_4, \boldsymbol{v}_5$; and define the projection map $\pi : \mathbb{R}^5 \to \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$ by

$$\pi(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 + x_4\boldsymbol{v}_4 + x_5\boldsymbol{v}_5) = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2\ ,\ \ x_i \in \mathbb{R}\ (i = 1, 2, 3, 4, 5).$$

It is easy to see that

$$\pi\boldsymbol{e}_i = v'_{1i}\boldsymbol{v}_1 + v'_{2i}\boldsymbol{v}_2\ ,$$

where $V^{-1} = (v'_{ij})_{\substack{1 \le i \le 5 \\ 1 \le j \le 5}}$.

By Proposition 2.1, in this case we obtain equations in concrete terms as stated in the following corollary:

Corollary 2.2 *For the substitution σ given by (1.1), the sets $\{X_i\}_{i=1,2,3,4,5}$ given in Theorem 2.1 satisfy the following set equations:*

(1)

$$\begin{aligned}
L_\sigma^{-1}X_1 &= \begin{cases} X_1 \cup X_5 & if\ K=0 \\ \bigcup_{n=0}^{K}(X_1+nL_\sigma^{-1}\pi\boldsymbol{e}_1) \cup \bigcup_{n=0}^{K-1}(X_4+nL_\sigma^{-1}\pi\boldsymbol{e}_1) \cup X_5 & if\ K \geq 1\ , \end{cases} \\
L_\sigma^{-1}X_2 &= X_1+(K+1)L_\sigma^{-1}\pi\boldsymbol{e}_1, \\
L_\sigma^{-1}X_3 &= X_2, \\
L_\sigma^{-1}X_4 &= X_3, \\
L_\sigma^{-1}X_5 &= X_4+KL_\sigma^{-1}\pi\boldsymbol{e}_1,
\end{aligned}$$

(2)

$$\begin{aligned}
L_\sigma^{-1}X_1' &= \begin{cases} (X_1'-L_\sigma^{-1}\pi\boldsymbol{e}_2) \cup X_5' & if\ K=0 \\ \bigcup_{n=0}^{K}(X_1'-L_\sigma^{-1}\pi(\boldsymbol{e}_2+n\boldsymbol{e}_1)) \cup \bigcup_{n=0}^{K-1}(X_4'-L_\sigma^{-1}\pi(\boldsymbol{e}_5+n\boldsymbol{e}_1)) \cup X_5' & if\ K \geq 1\ , \end{cases} \\
L_\sigma^{-1}X_2' &= X_1', \\
L_\sigma^{-1}X_3' &= X_2', \\
L_\sigma^{-1}X_4' &= X_3', \\
L_\sigma^{-1}X_5' &= X_4'\ .
\end{aligned}$$

For the substitution σ, the alphabet $\mathcal{B}$ and the graph G^* are as follows:

$$\mathcal{B}=\left\{\binom{1}{0},\binom{1}{1},\cdots,\binom{1}{K+1},\binom{2}{0},\binom{3}{0},\binom{4}{0},\binom{4}{1},\cdots,\binom{4}{K},\binom{5}{0}\right\}\ .$$

From the definition of σ given by (1.1), we have $\mathbf{f}(P_k^{(j)})=k\boldsymbol{e}_1$ for any $j\in\{1,2,3,4,5\}$. Hence it is enough to take only the path $(k_0,k_1,\cdots,k_N)$ of the edges in $\binom{i_0}{k_0}\binom{i_1}{k_1}\cdots\binom{i_N}{k_N}\in\mathcal{B}^*$. Then the set X is written as follows:

$$\begin{aligned}
X &= cl.\left\{\sum_{n=0}^{N}L_\sigma^n\pi\mathbf{f}(P_{k_n}^{(i_n)})\mid\binom{i_0}{k_0}\binom{i_1}{k_1}\cdots,\binom{i_N}{k_N}\in\mathcal{B}_\sigma^*,\ P_{k_N}^{(i_N)}W_{k_N}^{(i_N)}\prec\omega,\ N=0,1,\cdots\right\} \\
&= cl.\left\{\sum_{n=0}^{N}k_nL_\sigma^n\pi\boldsymbol{e}_1\mid t(k_n)=i(k_{n+1})\ in\ G^*,\ N=0,1,\cdots\right\}\ ,
\end{aligned}$$

where $t(e)\in\mathcal{V}$ (resp., $i(e)$) means the terminal (resp., the initial) vertex of an edge e. The condition $P_{k_N}^{(i_N)}W_{k_N}^{(i_N)}\prec\omega$ in the first line of the above equality means $\binom{i_N}{k_N}\in\mathcal{B}\setminus\{\binom{2}{0},\binom{3}{0},\binom{4}{K}\}$ by the definition of σ, but we can omit this condition for the following reason: Let us consider the case $\binom{i_N}{k_N}=\binom{4}{K}$, that is, the path in $\mathcal{B}_\sigma^*$ is written as $\binom{i_0}{k_0}\binom{i_1}{k_1}\cdots\binom{i_{N-1}}{k_{N-1}}\binom{4}{K}$. From the graph G^* in Fig. 6, the path is determined as $\binom{i_0}{k_0}\cdots\binom{1}{k_{N-2}}\binom{5}{0}\binom{4}{K}$, and it provides the same summation $\sum_{n=0}^{N}L_\sigma^n\pi\mathbf{f}(P_{k_n}^{(i_n)})$ as the one for the path

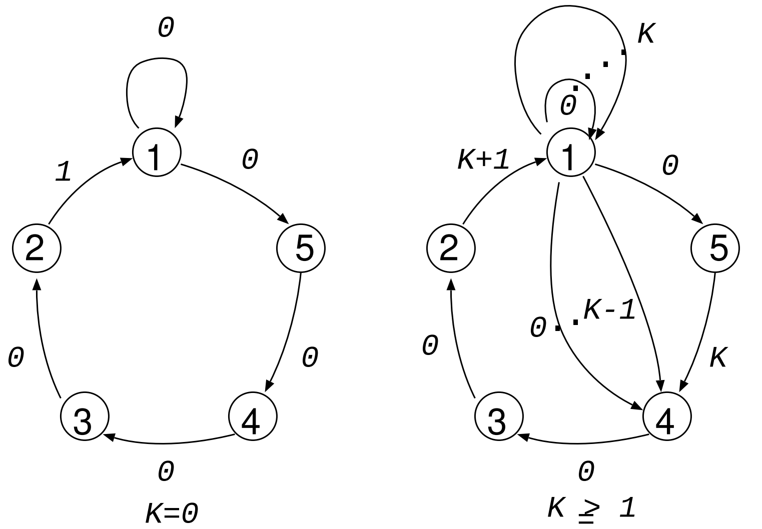

Fig. 6: The graph G^*

$\binom{i_0}{k_0} \cdots \binom{1}{k_{N-2}} \binom{1}{0} \binom{1}{K}$. Therefore we can allow $\binom{i_N}{k_N} = \binom{4}{K}$. The other cases $\binom{i_N}{k_N} = \binom{2}{0}$ or $\binom{3}{0}$ can be shown analogously.

Moreover, from the labeled graph G^*, the sets $\{X_i\}_{i=1,2,3,4,5}$ are given by

$$X_i = cl. \left\{ \sum_{n=0}^{N} k_n L_\sigma^n \pi \boldsymbol{e}_1 \mid \begin{array}{l} i(k_0) = i,\ t(k_n) = i(k_{n+1})\ in\ G^* \\ N = 0, 1, \cdots \end{array} \right\} .$$

This leads to the following corollary:

Corollary 2.3 *An element in the atomic surface X has an infinite expansion, that is,*

$$X = \left\{ \sum_{n=0}^{\infty} k_n L_\sigma^n \pi \boldsymbol{e}_1 \mid t(k_n) = i(k_{n+1})\ \ in\ G^* \right\} .$$

Example 2.1 *Let us consider the case of $K = 0$. The substitution σ is*

$$\sigma\ :\ \left\{ \begin{array}{lcl} 1 & \to & 12 \\ 2 & \to & 3 \\ 3 & \to & 4 \\ 4 & \to & 5 \\ 5 & \to & 1 \end{array} \right. .$$

By the labeled graph G^* in Fig. 6, the following admissible conditions hold for the sequence of digits $\{k_n\}$:

$$\begin{aligned} X &= \left\{ \sum_{n=0}^{\infty} k_n L_\sigma^n \pi \boldsymbol{e}_1 \mid t(k_n) = i(k_{n+1}) \ in \ G^* \right\} \\ &= \left\{ \sum_{n=0}^{\infty} k_n L_\sigma^n \pi \boldsymbol{e}_1 \mid k_n = 1 \Rightarrow k_{n+1} = k_{n+2} = k_{n+3} = k_{n+4} = 0 \right\} . \end{aligned}$$

3 A tiling with polygonal tiles of the plane $P_{<v_1,v_2>}$

3.1 A tiling substitution τ^*

In this subsection we introduce a tiling substitution τ^* associated with σ according to [AI01]. From now on let σ be a substitution given by (1.1) in Section 1. Then we have the direct sum $\mathbb{R}^5 = \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>} \oplus \mathcal{L}(\boldsymbol{v}_3) \oplus \boldsymbol{P}'_{<\boldsymbol{v}_4,\boldsymbol{v}_5>}$, and the map π is a projection from $\mathbb{R}^5$ to the 2-dimensional plane $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$.

Lemma 3.1 *The following relations hold:*

$$\pi \boldsymbol{e}_3 + \pi \boldsymbol{e}_4 = \pi \boldsymbol{e}_1 \ , \ \ \pi \boldsymbol{e}_4 + \pi \boldsymbol{e}_5 = \pi \boldsymbol{e}_1 + \pi \boldsymbol{e}_2 \ , \ \ \pi \boldsymbol{e}_5 = \pi \boldsymbol{e}_2 + \pi \boldsymbol{e}_3 \ .$$

Proof: We check that we can choose the vector ${}^t(-1,0,1,1,0) = -\boldsymbol{e}_1 + \boldsymbol{e}_3 + \boldsymbol{e}_4$ as the vector $\boldsymbol{v}_5$. Since $\boldsymbol{v}_5$ is mapped to $\mathbf{0}$ by the projection π, we have

$$-\pi \boldsymbol{e}_1 + \pi \boldsymbol{e}_3 + \pi \boldsymbol{e}_4 = \mathbf{0}.$$

From the fact that $L_\sigma \circ \pi = \pi \circ L_\sigma$, we have the second and third equalities. □

We introduce $\mathbb{Z}$-modules $\mathcal{F}$ and $\mathcal{F}^*$ using finite integer combinations of elements of $\mathbb{Z}^5 \times \{1,2,3,4,5\}$ and $\mathbb{Z}^5 \times \{1^*,2^*,3^*,4^*,5^*\}$ as follows:

$$\mathcal{F}_u := \left\{ \sum_{\delta \in \mathbb{Z}^5 \times \{1,2,3,4,5\}} n_\delta \delta \mid n_\delta \in \mathbb{Z}, {}^\#\{\delta \in \mathbb{Z}^5 \times \{1,2,3,4,5\} \mid n_\delta \neq 0\} < \infty \right\},$$

$$\mathcal{F}_u^* := \left\{ \sum_{\delta \in \mathbb{Z}^5 \times \{1^*,2^*,3^*,4^*,5^*\}} n_\delta \delta \mid n_\delta \in \mathbb{Z}, {}^\#\{\delta \in \mathbb{Z}^5 \times \{1^*,2^*,3^*,4^*,5^*\} \mid n_\delta \neq 0\} < \infty \right\}.$$

From Lemma 3.1 we can introduce the equivalence relation $\sim$ on $\mathcal{F}_u$ (resp., $\mathcal{F}_u^*$) defined by $\sum_{k=1}^N (\boldsymbol{x}_k, i_k) \sim \sum_{k=1}^N (\boldsymbol{y}_k, i_k)$ if $\pi \boldsymbol{x}_k = \pi \boldsymbol{y}_k$ for all k (resp., $\sum_{k=1}^N (\boldsymbol{x}_k, i_k^*) \sim \sum_{k=1}^N (\boldsymbol{y}_k, i_k^*)$ if $\pi \boldsymbol{x}_k = \pi \boldsymbol{y}_k$ for all k) and we set $\mathcal{F} := \mathcal{F}_u / \sim$ (resp., $\mathcal{F}^* := \mathcal{F}_u^* / \sim$). $\mathcal{F}$ will be used in Section 4 mainly.

To give a geometrical meaning of $(\boldsymbol{x}, i^*)$, first we define the map $\pi_1 : \mathcal{F} \to \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$, which gives a one dimensional geometric representation of the symbolic object $(\boldsymbol{x}, i)$, by

$$\pi_1(\boldsymbol{x}, i) = \{\pi \boldsymbol{x} + t \pi \boldsymbol{e}_i \mid 0 \leq t \leq 1\} ,$$

where for $\sum_{k=1}^{N} n_k(\boldsymbol{x}_k, i_k) \in \mathcal{F}$ with $n_k \in \mathbb{Z} - \{0\}$

$$\pi_1\left(\sum_{k=1}^{N} n_k(\boldsymbol{x}_k, i_k)\right) = \bigcup_{k=1}^{N} \pi_1(\boldsymbol{x}_k, i_k) \, .$$

(See Fig. 7.)

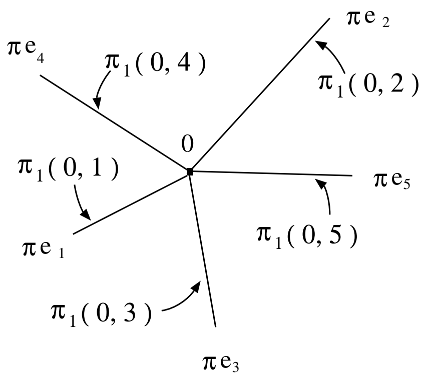

Fig. 7: Representation of $\pi_1(0, i)$ $(i = 1, 2, 3, 4, 5)$

A set consisting of three vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^2)$ is called a *hexa-generator* if the domain

$$\{t_1\boldsymbol{a} + s_1\boldsymbol{b}, t_2\boldsymbol{a} + s_2\boldsymbol{c}, t_3\boldsymbol{b} + s_3\boldsymbol{c} \mid 0 \le t_i \le 1, 0 \le s_i \le 1, i = 1, 2, 3\}$$

is a hexagon. (See Fig. 8.)

Lemma 3.2 $\{\pi\boldsymbol{e}_2, \pi\boldsymbol{e}_3, \pi\boldsymbol{e}_4\}$ *and* $\{-\pi\boldsymbol{e}_1, -\pi\boldsymbol{e}_5, -\pi\boldsymbol{e}_1 - \pi\boldsymbol{e}_2\}$ *are hexa-generators.*

Proof: We show $\{\pi\boldsymbol{e}_2, \pi\boldsymbol{e}_3, \pi\boldsymbol{e}_4\}$ is a hexa-generator. The other part can be shown analogously.

For $\boldsymbol{x} = \binom{x_1}{x_2} \in \mathbb{R}^2$, $\mathbf{n}(\boldsymbol{x})$ denotes $\binom{-x_2}{x_1}$, that is, a normal vector of $\boldsymbol{x}$. We can calculate the coordinates of $\pi\boldsymbol{e}_i$ $(i = 1, 2, 3, 4, 5)$ and easily check that every $\mathbf{n}(\pi(\boldsymbol{e}_2)) \cdot \pi(\boldsymbol{e}_4)$, $\mathbf{n}(\pi(\boldsymbol{e}_4)) \cdot \pi(\boldsymbol{e}_3)$ and $\mathbf{n}(\pi(\boldsymbol{e}_3)) \cdot \pi(\boldsymbol{e}_2)$ has the same signature, where $\boldsymbol{a} \cdot \boldsymbol{b}$ means the inner product for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$. Therefore $\{\pi\boldsymbol{e}_2, \pi\boldsymbol{e}_3, \pi\boldsymbol{e}_4\}$ is a hexa-generator. (See Fig. 8.) □

From Lemma 3.1 and Lemma 3.2, we can consider the map $\pi_2 : \mathcal{F}^* \to \boldsymbol{P}_{<\boldsymbol{v}_1, \boldsymbol{v}_2>}$, which gives a two-dimensional geometric representation of the symbolic object $(\boldsymbol{x}, i^*)$, by

$$\begin{aligned}
\pi_2(\mathbf{0}, 1^*) &= [\pi_1(\mathbf{0}, 2), \pi_1(\mathbf{0}, 5), \pi_1(\boldsymbol{e}_2, 3)] \\
\pi_2(\mathbf{0}, 2^*) &= [\pi_1(\mathbf{0}, 1), \pi_1(\mathbf{0}, 3), \pi_1(\boldsymbol{e}_3, 4)] \\
\pi_2(\mathbf{0}, 3^*) &= [\pi_1(\mathbf{0}, 2), \pi_1(\mathbf{0}, 4), \pi_1(\boldsymbol{e}_2, 1), \pi_1(\boldsymbol{e}_4, 5)] \\
\pi_2(\mathbf{0}, 4^*) &= [\pi_1(\mathbf{0}, 3), \pi_1(\mathbf{0}, 5), \pi_1(\boldsymbol{e}_3, 2)] \\
\pi_2(\mathbf{0}, 5^*) &= [\pi_1(\mathbf{0}, 1), \pi_1(\mathbf{0}, 4), \pi_1(\boldsymbol{e}_4, 3)] \\
\pi_2(\boldsymbol{x}, i^*) &= \pi_2(\mathbf{0}, i^*) + \pi\boldsymbol{x} \, ,
\end{aligned}$$

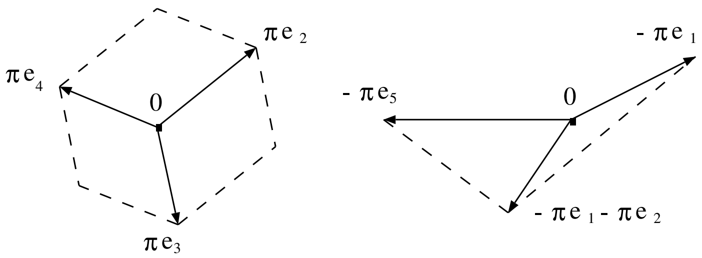

Fig. 8: Hexa-generators $\{\pi e_2, \pi e_3, \pi e_4\}$ and $\{-\pi e_1, -\pi e_5, -\pi e_1 - \pi e_2\}$

where $[x, y, \cdots, z]$ is the convex hull of the segments $x, y, \cdots, z$ and for $\sum_{k=1}^{N} n_k(\boldsymbol{x}_k, {i_k}^*) \in \mathcal{F}^*$ with $n_k \in \mathbb{Z} - \{0\}$

$$\pi_2\left(\sum_{k=1}^{N} n_k(\boldsymbol{x}_k, {i_k}^*)\right) = \bigcup_{k=1}^{N} \pi_2(\boldsymbol{x}_k, {i_k}^*) .$$

(See. Fig. 9.)

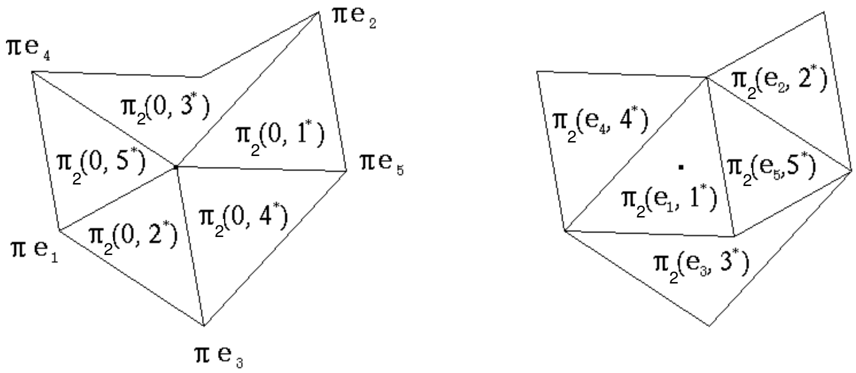

Fig. 9: Representation of $(0, i^*), (e_i, i^*)$ and $\mathcal{U}', \mathcal{U}$

By analogy with the definition of $E_1^*(\sigma)$ in [AI01], let us define the endomorphism τ^* on $\mathcal{F}^*$ called a *tiling substitution* associated with σ, by

$$\tau^*(\boldsymbol{x}, i^*) := \sum_{j=1}^{5} \sum_{S_k^{(j)}:W_k^{(j)}=i} (L_\sigma^{-1}\boldsymbol{x} + L_\sigma^{-1}\mathbf{f}(S_k^{(j)}), j^*) \ . \tag{3.1}$$

We remark that the sign is plus here in this definition with respect to the sign minus in Formula (2) in Proposition 2.1. (See Remark 4 in Section 4.)

Remark 2 *For the substitution σ given by (1.1), the tiling substitution τ^* is defined explicitly by*

$$
\begin{aligned}
\tau^*(\boldsymbol{x},1^*) &= \begin{cases} (L_\sigma^{-1}\boldsymbol{x}+\boldsymbol{e}_1-\boldsymbol{e}_5,1^*)+(L_\sigma^{-1}\boldsymbol{x},5^*) & if\ K=0 \\ \sum_{n=1}^{K+1}(L_\sigma^{-1}\boldsymbol{x}+\boldsymbol{e}_1-n\boldsymbol{e}_5,1^*)+\sum_{n=1}^{K}(L_\sigma^{-1}\boldsymbol{x}+\boldsymbol{e}_4-n\boldsymbol{e}_5,4^*)+(L_\sigma^{-1}\boldsymbol{x},5^*) & if\ K\geq 1 \end{cases}, \\
\tau^*(\boldsymbol{x},2^*) &= (L_\sigma^{-1}\boldsymbol{x},1^*)\ , \\
\tau^*(\boldsymbol{x},3^*) &= (L_\sigma^{-1}\boldsymbol{x},2^*)\ , \\
\tau^*(\boldsymbol{x},4^*) &= (L_\sigma^{-1}\boldsymbol{x},3^*)\ , \\
\tau^*(\boldsymbol{x},5^*) &= (L_\sigma^{-1}\boldsymbol{x},4^*)\ .
\end{aligned}
$$

(See Fig. 10.)

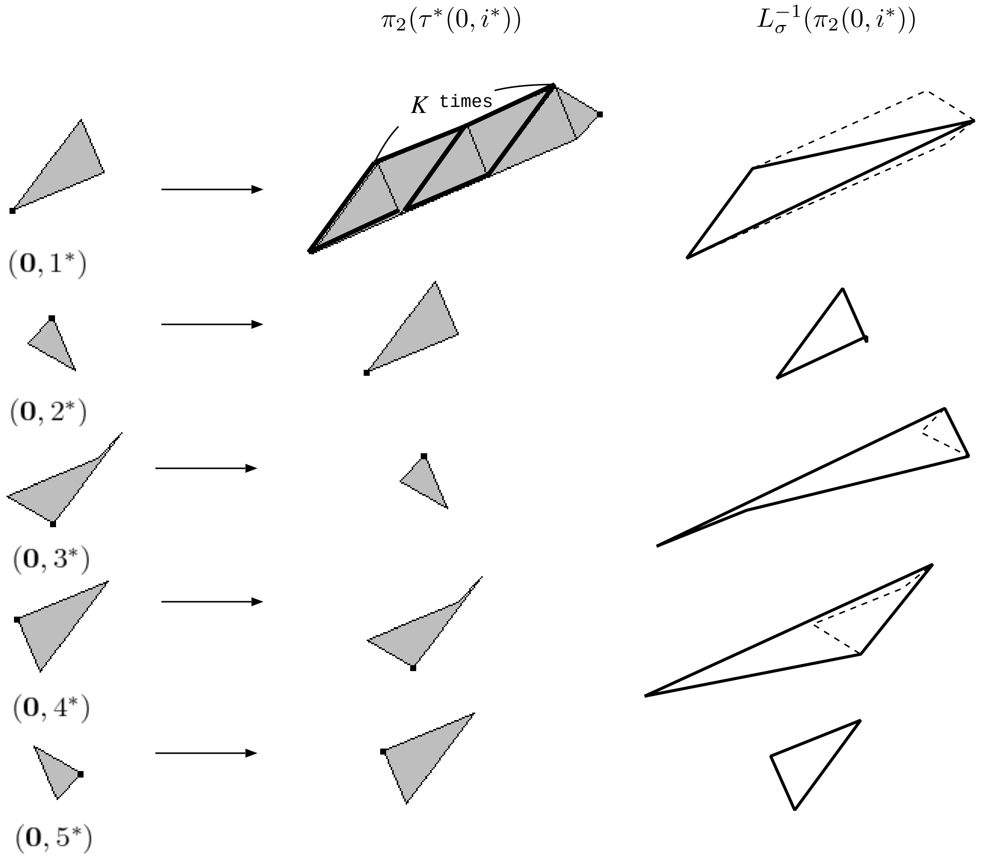

Fig. 10: The tiling substitution τ^*

Remark 3 *The formula of the tiling substitution τ^* can be found in [AI01] under the notation $E_1^*(\sigma)$. But the geometrical meaning of $(\boldsymbol{x},i^*)$ $(i=1,2,3,4,5)$ is different. In [AI01] we mean by $(\boldsymbol{x},i^*)$ a unit* *cube of dimension four in $\mathbb{R}^5$ and a cube $(\boldsymbol{x}, i^*)$ projected by π is a good prototile for the tiling of $\mathbb{R}^4$ for an irreducible substitution, however, for a non-irreducible substitution which we deal with now, a cube $(\boldsymbol{x}, i^*)$ projected by π does not work as prototile of a tiling of the contractive space $\boldsymbol{P}_{<\boldsymbol{v}_1, \boldsymbol{v}_2>}$. In this section we will see that the tiles $\pi_2(\boldsymbol{x}, i^*)$ given by Fig. 9 work well.*

We define $\mathcal{U} = \sum_{i=1}^{5} (\boldsymbol{e}_i, i^*)$, $\mathcal{U}' = \sum_{i=1}^{5} (\boldsymbol{0}, i^*)$ as the union of these five elements in $\mathcal{F}^*$. (See Fig. 9.)

Proposition 3.1 *The following relations hold:*

$$\tau^*(\mathcal{U}) \supset \mathcal{U}, \ \ \tau^*(\mathcal{U}') \supset \mathcal{U}' \ ,$$

where for $\mathcal{V}$, $\mathcal{W} \in \mathcal{F}^*$, $\mathcal{V} \supset \mathcal{W}$ means that $\pi_2(\mathcal{V}) \supset \pi_2(\mathcal{W})$, and that there exists $\mathcal{W}' = \sum_{k=1}^{N} n_k(x_k, i_i^*) \in \mathcal{F}^*$ with $n_i > 0$ for all i such that $\mathcal{V} = \mathcal{W} + \mathcal{W}'$; moreover,

$$\tau^*(\mathcal{U}) - \tau^*(\mathcal{U}') = \mathcal{U} - \mathcal{U}'$$

and for any positive integer n

$$\tau^{*\,n}(\mathcal{U}) - \tau^{*\,n}(\mathcal{U}') = \mathcal{U} - \mathcal{U}'.$$

Proof: From the definition of τ^*, we see

$$\tau^*(\mathcal{U}) = \begin{cases} \mathcal{U} + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 1^*) & if\ K = 0 \\ \mathcal{U} + \displaystyle\sum_{n=1}^{K+1} (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_1 - n\boldsymbol{e}_5, 1^*) + \sum_{n=1}^{K} (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_4 - n\boldsymbol{e}_5, 4^*) & if\ K \geq 1 \end{cases}$$

$$\tau^*(\mathcal{U}') = \begin{cases} \mathcal{U}' + (\boldsymbol{e}_1 - \boldsymbol{e}_5, 1^*) & if\ K = 0 \\ \mathcal{U}' + \displaystyle\sum_{n=1}^{K+1} (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_1 - n\boldsymbol{e}_5, 1^*) + \sum_{n=1}^{K} (L_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e}_4 - n\boldsymbol{e}_5, 4^*) & if\ K \geq 1\ . \end{cases}$$

Therefore, we know $\tau^*(\mathcal{U}) - \tau^*(\mathcal{U}') = \mathcal{U} - \mathcal{U}'$. □

The replacing and re-dividing method

Observe the two domains $\pi_2(\tau^*(\boldsymbol{x}, i^*))$ and $L_\sigma^{-1}(\pi_2(\boldsymbol{x}, i^*))$, then we have the following three cases (See Fig. 10.):

(1) $\pi_2(\tau^*(\boldsymbol{x}, i^*)) = L_\sigma^{-1}(\pi_2(\boldsymbol{x}, i^*))$ $(i = 2, 5)$

(2) $\pi_2(\tau^*(\boldsymbol{x}, i^*)) \subset L_\sigma^{-1}(\pi_2(\boldsymbol{x}, i^*))$ $(i = 3, 4)$

(3) $\pi_2(\tau^*(\boldsymbol{x}, i^*)) \supset L_\sigma^{-1}(\pi_2(\boldsymbol{x}, i^*))$ $(i = 1)$

In the case where $i = 1, 3, 4$, each of these domains $\pi_2(\boldsymbol{x}, i^*)$ $(i = 1, 3, 4)$ contains at least one edge of the form $\pi_1(\boldsymbol{y}, 2)$ and one edge of the form $\pi_1(\boldsymbol{y}', 5)$, so we introduce the following “replacing and re-dividing” method to get the domain $\pi_2(\tau^*(\boldsymbol{x}, i^*))$ from the domain $L_\sigma^{-1}(\pi_2(\boldsymbol{x}, i^*))$.

First, we "replace" every edge $L_\sigma^{-1}\pi_1(\boldsymbol{y}, 2)$ on $L_\sigma^{-1}(\pi_2(\boldsymbol{x}, i^*))$ $(i = 1, 3, 4)$ with

$$\pi_1\{(L_\sigma^{-1}\boldsymbol{y}, 1) + \sum_{n=1}^{K+1}(L_\sigma^{-1}\boldsymbol{y} + \boldsymbol{e}_1 - n\boldsymbol{e}_5, 5)\},$$

what is more, in the case of $K \geq 1$, replace every edge $L_\sigma^{-1}\pi_1(\boldsymbol{y}', 5)$ on $L_\sigma^{-1}(\pi_2(\boldsymbol{x}, i^*))$ $(i = 1, 3, 4)$ with

$$\pi_1\{(L_\sigma^{-1}\boldsymbol{y}', 4) + \sum_{n=1}^{K}(L_\sigma^{-1}\boldsymbol{y}' + \boldsymbol{e}_4 - n\boldsymbol{e}_5, 5)\}.$$

By this procedure, we have $\pi_2(\tau^*(\boldsymbol{x}, i^*))$ in the case of $i = 3, 4$.

Secondly, we "re-divide" the domain in the case of $i = 1$ (i.e. $\pi_2(\tau^*(\boldsymbol{x}, 1^*))$). Then, we have $\pi_2(\tau^*(\boldsymbol{x}, i^*))$. (See Fig. 11.)

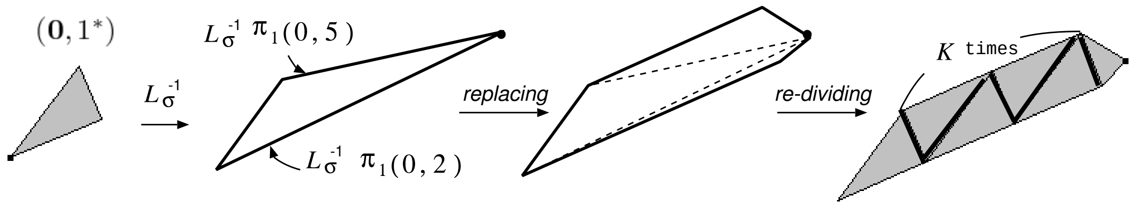

Fig. 11: The replacing and re-dividing method for the domain $L_\sigma^{-1}(\pi_2(x, 1^*))$

3.2 A tiling of the plane $P_{<v_1,v_2>}$ generated by τ^*

We construct a first quasi-periodic tiling of $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$ with five polygonal prototiles generated by the tiling substitution τ^* associated to σ. (See Fig. 2 in Section 1.) This tiling corresponds to the projection of a discrete plane approximation (stepped surface) in the Pisot case. First we will see the property of non-overlap.

Proposition 3.2 *The sets $\pi_2(\tau^{*\ n}(\mathcal{U}))$ (resp., $\pi_2(\tau^{*\ n}(\mathcal{U}'))$) consisting of five prototiles of the form $\pi_2(\boldsymbol{x}, i^*)$ $(i = 1, 2, 3, 4, 5)$ do not overlap for any positive integer n.*

Proof: Suppose that the pieces of $\pi_2(\tau^{*\ n}(\mathcal{U}))$ do not overlap, then the pieces of $L_\sigma^{-1}\pi_2(\tau^{*\ n}(\mathcal{U}))$ do not overlap. By the replacing and re-dividing method, we obtain $\pi_2(\tau^{*\ n+1}(\mathcal{U}))$ from $L_\sigma^{-1}\pi_2(\tau^{*\ n}(\mathcal{U}))$. Therefore, to show the pieces of $\pi_2(\tau^{*\ n+1}(\mathcal{U}))$ do not overlap, it is enough to show that replaced edges on $L_\sigma^{-1}\pi_2(\tau^{*\ n}(\mathcal{U}))$ do not cause overlap. From (3) in the replacing and re-dividing method, it is possible that overlaps occur by replacing edges on $\pi_2(\boldsymbol{x}, 1^*)$. We list the pairs of tiles which are just touching with the segment $\pi_1(\boldsymbol{x}, 2)$ or $\pi_1(\boldsymbol{x}, 5)$ on $\pi_2(\boldsymbol{x}, 1^*)$, which are given by

$$\{(\boldsymbol{x}, 1^*), (\boldsymbol{x} - \boldsymbol{e}_3, 4^*)\}, \{(\boldsymbol{x}, 1^*), (\boldsymbol{x}, 3^*)\}, \{(\boldsymbol{x}, 1^*), (\boldsymbol{x}, 4^*)\}, \{(\boldsymbol{x}, 1^*), (\boldsymbol{x} - \boldsymbol{e}_4, 3^*)\}\ .$$

We deal with the pair $\{(\boldsymbol{x}, 1^*), (\boldsymbol{x} - \boldsymbol{e}_3, 4^*)\}$. The edge which substitutes for $L_\sigma^{-1}\pi_1(\boldsymbol{x}, 2)$ on $L_\sigma^{-1}\pi_2(\boldsymbol{x}, 1^*)$ is not included in the domain $L_\sigma^{-1}\pi_2(\boldsymbol{x}, 1^*)$, but in the adjoining domain $L_\sigma^{-1}\pi_2(\boldsymbol{x} - \boldsymbol{e}_3, 4^*)$. Moreover, the edge does not cross the other edges. Thus there is no overlap by replacing edges. We see other cases analogously. (See Fig. 12.)

In the case where $\pi_1(\boldsymbol{x}, i)$ $(i = 2, 5)$ in $\pi_2(\boldsymbol{x}, 1^*)$ is a part of the boundary of $\pi_2(\tau^{*\ n}(\mathcal{U}))$, it is easy to see that replacing edges do not cause overlap. □

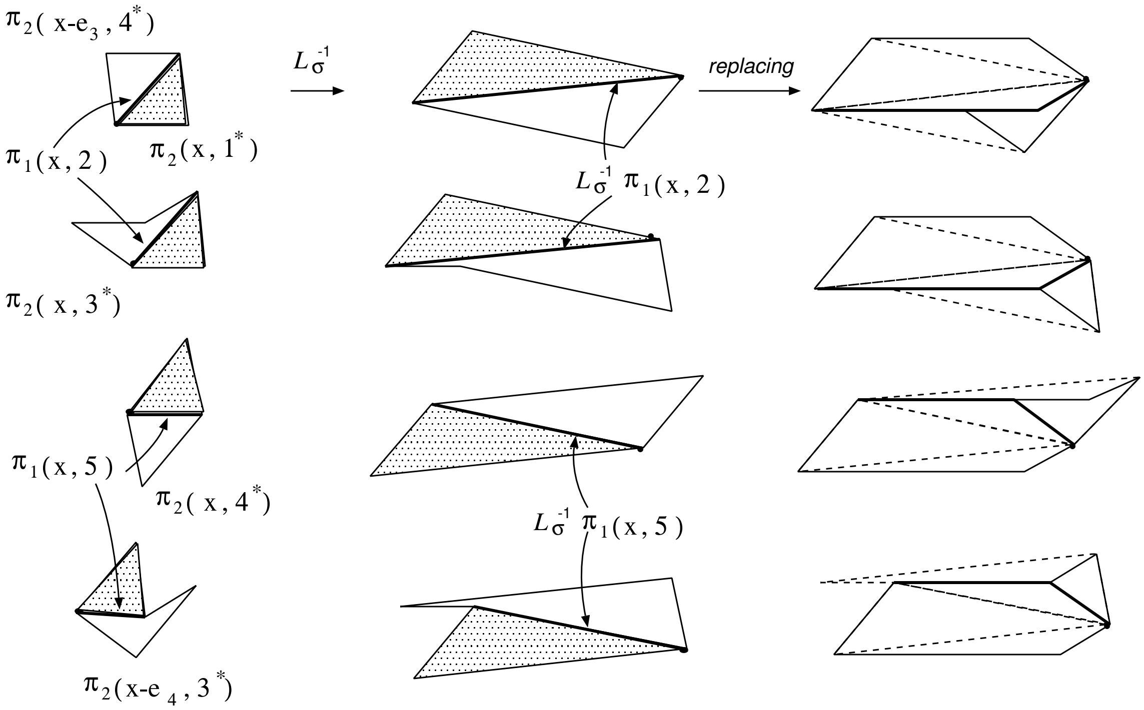

Fig. 12: Four pairs including $(x, 1^*)$

Secondly we consider the covering of the plane $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$ by $\pi_2(\tau^{*\ n}(\mathcal{U}))$.

Proposition 3.3 $\cup_{n=1}^{\infty}\pi_2(\tau^{*\ n}(\mathcal{U})) = \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$*, that is,* $\pi_2(\tau^{*\ n}(\mathcal{U}))$ *covers the plane as* n *goes to* ∞*.*

The proof of this proposition is long and not easy. So the detail of this proof is put in Section 6. In the irreducible case, we can prove the property in Proposition 3.3 by using the notion of stepped surface of a substitution σ (cf. [AI01]), but here we must prove it without such a notion. That is the reason why the proof is difficult.

The following proposition is deduced from Proposition 3.2 and Proposition 3.3:

Proposition 3.4 *The sets* $\tau^{*\ n}(\mathcal{U})$ *generate a tiling of the plane* $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$*, that is,*

$$\mathcal{T}_{\tau^*} := \{\pi_2(\boldsymbol{x}, i^*) \mid (\boldsymbol{x}, i^*) \subset \tau^{*\ n}(\mathcal{U})\ for\ some\ n\}$$

is a tiling of $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$*. (See Fig. 2 in Section 1.)*

Finally we discuss the periodicity of the tiling $\mathcal{T}_{\tau^*}$. Let us introduce the following notation:

$$\mathcal{G}^* := \{(\boldsymbol{x}, i^*) \in \mathcal{F}^* \mid (\boldsymbol{x}, i^*) \subset {\tau^*}^n(\mathcal{U}) \ for\ some\ n\},$$

that is, $\mathcal{T}_{\tau^*} = \{\pi_2(\boldsymbol{x}, i^*) \mid (\boldsymbol{x}, i^*) \in \mathcal{G}^*\}$, and we also introduce the new prototiles $(\boldsymbol{x}, \widetilde{i^*})$ based on $\pi\boldsymbol{x}$ and the tiling associated to $\widetilde{\mathcal{G}^*}$ defined by

$$(\boldsymbol{x}, \widetilde{i^*}) := \tau^*(L_\sigma \boldsymbol{x}, i^*) \ \text{ for } (L_\sigma \boldsymbol{x}, i^*) \in \mathcal{G}^* ,$$

$$\widetilde{\mathcal{G}^*} := \{(\boldsymbol{x}, \widetilde{i^*}) \mid (L_\sigma \boldsymbol{x}, i^*) \in \mathcal{G}^*\} .$$

So we have the ordinary tiling $\mathcal{T}_{\tau^*}$ by dividing the prototiles in $\widetilde{\mathcal{G}^*}$ following the method in Subsection 3.1. We say $\mathcal{G}^*$ (or $\mathcal{T}_{\tau^*}$) is *periodic* if there exists at least one non-zero period $\boldsymbol{p} \in \mathbb{R}^5$ such that $(\boldsymbol{x}, i^*) \in \mathcal{G}^*$ implies $(\boldsymbol{x} + \boldsymbol{p}, i^*) \in \mathcal{G}^*$.

Lemma 3.3 *If $\boldsymbol{p}$ is a non-zero period of $\mathcal{G}^*$, then $\widetilde{\mathcal{G}^*}$ also has the period $\boldsymbol{p}$.*

Proof: Assume $\mathcal{G}^*$ has a non-zero period $\boldsymbol{p}$. At first we consider the periodicity of the tiles of the form $(\boldsymbol{x}, \widetilde{1^*})$ in $\widetilde{\mathcal{G}^*}$. Note that the only $\tau^*(\boldsymbol{x}, 1^*)$ in the images by τ^* includes a tile of the form $(\boldsymbol{y}, 5^*)$, that is, $(\boldsymbol{x}, \widetilde{1^*}) \in \widetilde{\mathcal{G}^*}$ if and only if $(\boldsymbol{x}, 5^*) \in \mathcal{G}^*$. Suppose $(\boldsymbol{x}, \widetilde{1^*}) \in \widetilde{\mathcal{G}^*}$, then $(\boldsymbol{x}, 5^*) \in \mathcal{G}^*$, and so $(\boldsymbol{x} + \boldsymbol{p}, 5^*) \in \mathcal{G}^*$ by the assumption, and finally we have $(\boldsymbol{x} + \boldsymbol{p}, \widetilde{1^*}) \in \widetilde{\mathcal{G}^*}$. It means that $\boldsymbol{p}$ is also a period for the tiles of the form $(\boldsymbol{x}, \widetilde{1^*})$. Now we want to observe the periodicity for the tiles in $\widetilde{\mathcal{G}^*}$ except the tiles $(\boldsymbol{x}, \widetilde{1^*})$. Put

$$\begin{aligned} \mathcal{D} &:= \{(\boldsymbol{y}, j^*) \in \mathcal{G}^* \mid (\boldsymbol{y}, j^*) \subset (\boldsymbol{x}, \widetilde{1^*}) \ for\ some\ (\boldsymbol{x}, \widetilde{1^*}) \in \widetilde{\mathcal{G}^*}\} \\ \widetilde{\mathcal{D}} &:= \{(\boldsymbol{x}, \widetilde{1^*}) \mid (\boldsymbol{x}, \widetilde{1^*}) \in \widetilde{\mathcal{G}^*}\} . \end{aligned}$$

From the above discussion we know that $\widetilde{\mathcal{D}}$ has a period $\boldsymbol{p}$. This means that $\mathcal{D}$ is closed for the translation by $\boldsymbol{p}$, and has the same period $\boldsymbol{p}$. Thus $\mathcal{G}^* - \mathcal{D}$ also has a period $\boldsymbol{p}$ by the assumption. After projection by π_2, $\mathcal{G}^* - \mathcal{D}$ and $\widetilde{\mathcal{G}^*} - \widetilde{\mathcal{D}}$ provide the same covering of $P_{<\boldsymbol{v}_1, \boldsymbol{v}_2>}$ with many holes by the equality

$$(\boldsymbol{x}, \widetilde{i^*}) = (\boldsymbol{x}, (i-1)^*) \ \ (i = 2, 3, 4, 5) .$$

So $\widetilde{\mathcal{G}^*} - \widetilde{\mathcal{D}}$ has a period $\boldsymbol{p}$. Therefore, $\widetilde{\mathcal{G}^*}$ has a period $\boldsymbol{p}$. □

Theorem 3.1 *The tiling $\mathcal{T}_{\tau^*}$ is not periodic.*

Proof: Suppose the tiling $\mathcal{T}_{\tau^*}$ is periodic, that is, $\mathcal{G}^*$ is periodic. Since $\mathcal{G}^*$ is a discrete set, there exists a non-zero and minimum period $\boldsymbol{p}$ of $\mathcal{G}^*$, where a minimum period is a period whose norm $\| \pi\boldsymbol{p} \|$ is minimum. From Lemma 3.3, $\boldsymbol{p}$ is also a period of $\widetilde{\mathcal{G}^*}$. Define the map $\iota^* : \widetilde{\mathcal{G}^*} \to \mathcal{G}^*$ by

$$\iota^*(\boldsymbol{x}, \widetilde{i^*}) = (L_\sigma \boldsymbol{x}, i^*) \ \ (i = 1, 2, 3, 4, 5) \ .$$

By the definition of $\widetilde{\mathcal{G}}^*$, $\iota^*(\boldsymbol{x}, \widetilde{i^*})$ is in $\mathcal{G}^*$, that is, ι^* is well-defined. Moreover, it is a bijection and the inverse is given by

$$\iota^{*-1}(\boldsymbol{y}, i^*) = (L_\sigma^{-1} \boldsymbol{y}, \widetilde{i^*}) \ \text{ for } (\boldsymbol{y}, i^*) \in \mathcal{G}^* .$$

Hence, $(\boldsymbol{x}, \widetilde{j^*}), (\boldsymbol{x}+\boldsymbol{p}, \widetilde{j^*}) \in \widetilde{\mathcal{G}^*}$ implies $\iota^*(\boldsymbol{x}, \widetilde{j^*}) = (L_\sigma \boldsymbol{x}, j^*) \in \mathcal{G}^*$, $\iota^*(\boldsymbol{x}+\boldsymbol{p}, \widetilde{j^*}) = (L_\sigma \boldsymbol{x} + L_\sigma \boldsymbol{p}, j^*) \in \mathcal{G}^*$. This means $\pi(L_\sigma \boldsymbol{p})$ is a period of the tiling $\mathcal{T}_{\tau^*}$. On the other hand, L_σ is contractive on $\boldsymbol{P}_{<\boldsymbol{v}_1, \boldsymbol{v}_2>}$. Therefore, it contradicts the minimality of the period $\boldsymbol{p}$. □

Definition 3.1 *A tiling* $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda, T_\lambda$ *is a tile on* $\boldsymbol{P}\}$ *of the space* $\boldsymbol{P}$ *is called a quasi-periodic tiling if for any* $r > 0$ *there exists* $R > 0$ *such that any patch* $\gamma = \bigcup_{\substack{\lambda' \in \Lambda' \subset \Lambda \\ \#\Lambda' < \infty}} T_{\lambda'}$ *whose diameter is smaller than r occurs somewhere in a neighbourhood of radius R of any point.*

For $\gamma, \delta \in \mathcal{F}^*$, $\gamma \succ \delta$ denotes that there exists $\boldsymbol{z} \in \mathbb{Z}^5$ such that $M_{\boldsymbol{z}}\gamma \supset \delta$, where $M_{\boldsymbol{z}}$ is the translation map given by

$$M_{\boldsymbol{z}}(\sum_{k=1}^{N} n_k(\boldsymbol{x}_k, {i_k}^*)) = \sum_{k=1}^{N} n_k(\boldsymbol{x}_k + \boldsymbol{z}, {i_k}^*)$$

for $\sum_{k=1}^{N} n_k(\boldsymbol{x}_k, {i_k}^*) \in \mathcal{F}^*$.

Theorem 3.2 *The tiling* $\mathcal{T}_{\tau^*}$ *is quasi-periodic.*

Proof: Take any $r > 0$. There exists a positive integer N such that ${\tau^*}^N(\mathcal{U}) \succ \gamma$, for any $\gamma \in \mathcal{G}^*$ satisfying the diameter of $\pi_2(\gamma)$ is smaller than r, because the number of such γ's is finite. From ${\tau^*}^8(\boldsymbol{e}_i, i^*) \supset {\tau^*}^4(\boldsymbol{e}_1, 1^*) \supset \mathcal{U}$ for any $i = 1, 2, 3, 4, 5$, putting M=N+8, we have

$${\tau^*}^M(\boldsymbol{e}_i, i^*) \succ \gamma \qquad (i = 1, 2, 3, 4, 5).$$

By the definition of $\mathcal{G}^*$, for any $(\boldsymbol{x}, i^*) \in \mathcal{G}^*$ there exists $(\boldsymbol{y}, j^*) \in \mathcal{G}^*$ such that

$$(\boldsymbol{x}, i^*) \subset {\tau^*}^M(\boldsymbol{y}, j^*).$$

Therefore, we have

$$U_R(\boldsymbol{x}) \supset \pi_2({\tau^*}^M(\boldsymbol{y}, j^*)) \,,$$

where $R = \max_{i=1,2,3,4,5} diam.(\pi_2({\tau^*}^M(\boldsymbol{e}_i, i^*)))$ and $U_R(\boldsymbol{x})$ means the neighbourhood of $\boldsymbol{x}$ with the radius R. Thus, $U_R(\boldsymbol{x})$ contains any configuration of $\pi_2(\gamma)$ whose diameter is smaller than r. □

4 Atomic surfaces given by τ^* and a second tiling

In Section 2 we constructed atomic surfaces X, X_i $(i = 1, 2, 3, 4, 5)$ from the fixed point of a substitution and the projection map π. In Subsection 4.1 we generate the atomic surfaces by using the tiling substitution τ^*; and by the virtue of this construction, we can observe the boundaries of atomic surfaces in Subsection 4.2; and in Subsection 4.3 we obtain a second tiling with atomic surfaces by replacing the polygonal tiles on the first tiling $\mathcal{T}_{\tau^*}$ by atomic surfaces.

4.1 Atomic surfaces given by τ^*

Definition 4.1 *Define the domains $D_n,\ D_n^{(i)}$, (resp., ${D_n}',\ {D_n^{(i)}}'$) as follows:*

$$\begin{aligned} D_n &:= \pi_2(\tau^{*\,n}(\mathcal{U})) \\ D_n^{(i)} &:= \pi_2(\tau^{*\,n}(\boldsymbol{e}_i, i^*)) \\ {D_n}' &:= \pi_2(\tau^{*\,n}(\mathcal{U}')) \\ {D_n^{(i)}}' &:= \pi_2(\tau^{*\,n}(\mathbf{0}, i^*)). \end{aligned}$$

Theorem 4.1 *We take a renormalization of the domains $D_n,\ D_n^{(i)}$, then*

(1) the following limit sets exist in the sense of the Hausdorff metric:

$$\begin{aligned} \widehat{X_i} &:= \lim_{n\to\infty} L_\sigma^n D_n^{(i)} \\ \widehat{X_i'} &:= \lim_{n\to\infty} L_\sigma^n {D_n^{(i)}}' \end{aligned}$$

and they satisfy the relations:

$$\widehat{X_i} = \widehat{X_i'} + \pi\boldsymbol{e}_i\ ,\ (\widehat{X} :=) \cup_{i=1}^5 \widehat{X_i} = \cup_{i=1}^5 \widehat{X_i'}\ ,$$

(2) the following inequality holds:

$$L_\sigma \begin{pmatrix} \mu(\widehat{X_1}) \\ \mu(\widehat{X_2}) \\ \vdots \\ \mu(\widehat{X_5}) \end{pmatrix} \geq \beta \begin{pmatrix} \mu(\widehat{X_1}) \\ \mu(\widehat{X_2}) \\ \vdots \\ \mu(\widehat{X_5}) \end{pmatrix} ;$$

moreover, the vector of volumes ${}^t(\mu(\widehat{X_1}), \mu(\widehat{X_2}), \cdots, \mu(\widehat{X_5}))$ is an eigenvector of L_σ with respect to the maximum eigenvalue β, where μ is the Lebesgue measure,

(3) the following set equations hold:

$$\begin{aligned} L_\sigma^{-1}\widehat{X_i} &= \bigcup_{j=1}^{5} \bigcup_{P_k^{(j)}:W_k^{(j)}=i} (\widehat{X_j} - L_\sigma^{-1}\pi\mathbf{f}(P_k^{(j)})) \\ L_\sigma^{-1}\widehat{X_i'} &= \bigcup_{j=1}^{5} \bigcup_{S_k^{(j)}:W_k^{(j)}=i} (\widehat{X_j'} + L_\sigma^{-1}\pi\mathbf{f}(S_k^{(j)}))\ , \end{aligned}$$

(4) the sets in the right side of the equation in (3) do not overlap up to a set of Lebesgue measure 0.

The proof of the theorem can be obtained by a quite similar way as in [AI01] following Lemma 11, Lemma 12 and Corollary 2.

Remark 4 *We can find a relationship between $\widehat{X_i}$ and the atomic surfaces X_i. By Proposition 2.1, we have the following equation for the atomic surfaces X_i:*

$$L_\sigma^{-1}(-X_i) = \bigcup_{j=1}^{5} \bigcup_{P_k^{(j)}:W_k^{(j)}=i} ((-X_j) - L_\sigma^{-1}\mathbf{f}(P_k^{(j)})).$$

This means $-X_i$ and $\widehat{X_i}$ $(i = 1,2,3,4,5)$ satisfy the same set equations. Since L_σ is a contractive transformation on the plane $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$ and from the uniqueness of self-affine sets (See Theorem 1 in [MW88] for the uniqueness.), we have the following relation

$$-X_i \ = \widehat{X_i} \qquad (i = 1,2,3,4,5).$$

Remark 5 *We are interested in the disjointness of the partitions $\widehat{X_i}$ $(i = 1,2,3,4,5)$ of $\widehat{X}$. From the property $\tau^{*\,4}(\boldsymbol{e}_1, 1^*) \supset \mathcal{U}$, any $\widehat{X_i}$ is included in the right side of the equation*

$$L_\sigma^{-4}\widehat{X_1} = \bigcup_{j=1}^{5} \bigcup_{P_k^{(4,j)}:W_k^{(4,j)}=1} (\widehat{X_j} - L_\sigma^{-4}\pi\mathbf{f}(P_k^{(4,j)}))\ ,$$

thus by Theorem 4.1 (4), for any $i,\ j$ $(i \neq j)$

$$\mu(\widehat{X_i} \cap \widehat{X_j}) = 0\ .$$

This property of disjointness holds for substitutions satisfying the strong coincidence property in [AI01].

4.2 Boundaries of atomic surfaces

In this subsection we will observe the boundaries of atomic surfaces $\widehat{X},\ \widehat{X_i}$. One of our aims here is to obtain Proposition 4.1, which says that the origin is an inner point of the domain $\widehat{X_1}$. For that we will show that the distance between the origin point and the boundary is positive by studying the boundary.

We introduce an endomorphism $\overline{\tau}$ on $\mathcal{F}$ as follows (See Fig. 13.):

$$\begin{aligned}
\overline{\tau}(\boldsymbol{x},1) &= (L_\sigma^{-1}\boldsymbol{x},5) \\
\overline{\tau}(\boldsymbol{x},2) &= (L_\sigma^{-1}\boldsymbol{x},1) - \sum_{n=1}^{K+1}(L_\sigma^{-1}\boldsymbol{x}+\boldsymbol{e}_1 - n\boldsymbol{e}_5,5) \\
\overline{\tau}(\boldsymbol{x},3) &= (L_\sigma^{-1}\boldsymbol{x},2) \\
\overline{\tau}(\boldsymbol{x},4) &= (L_\sigma^{-1}\boldsymbol{x},3) \\
\overline{\tau}(\boldsymbol{x},5) &= \begin{cases} (L_\sigma^{-1}\boldsymbol{x},4) & if\ K=0 \\ (L_\sigma^{-1}\boldsymbol{x},4) - \sum_{n=1}^{K}(L_\sigma^{-1}\boldsymbol{x}+\boldsymbol{e}_4 - n\boldsymbol{e}_5,5) & if\ K \geq 1 \end{cases}
\end{aligned}$$

Define the boundary map $\partial_{\mathcal{F}^*} : \mathcal{F}^* \to \mathcal{F}$ as follows:

$$\begin{aligned}
\partial_{\mathcal{F}^*}(\boldsymbol{x},1^*) &= -(\boldsymbol{x},2) + (\boldsymbol{x},5) - (\boldsymbol{x}+\boldsymbol{e}_2,3) \\
\partial_{\mathcal{F}^*}(\boldsymbol{x},2^*) &= (\boldsymbol{x},1) - (\boldsymbol{x},3) - (\boldsymbol{x}+\boldsymbol{e}_3,4) \\
\partial_{\mathcal{F}^*}(\boldsymbol{x},3^*) &= (\boldsymbol{x},2) - (\boldsymbol{x},4) + (\boldsymbol{x}+\boldsymbol{e}_2,1) - (\boldsymbol{x}+\boldsymbol{e}_4,5) \\
\partial_{\mathcal{F}^*}(\boldsymbol{x},4^*) &= (\boldsymbol{x},3) - (\boldsymbol{x},5) + (\boldsymbol{x}+\boldsymbol{e}_3,2) \\
\partial_{\mathcal{F}^*}(\boldsymbol{x},5^*) &= -(\boldsymbol{x},1) + (\boldsymbol{x},4) + (\boldsymbol{x}+\boldsymbol{e}_4,3)\ .
\end{aligned}$$

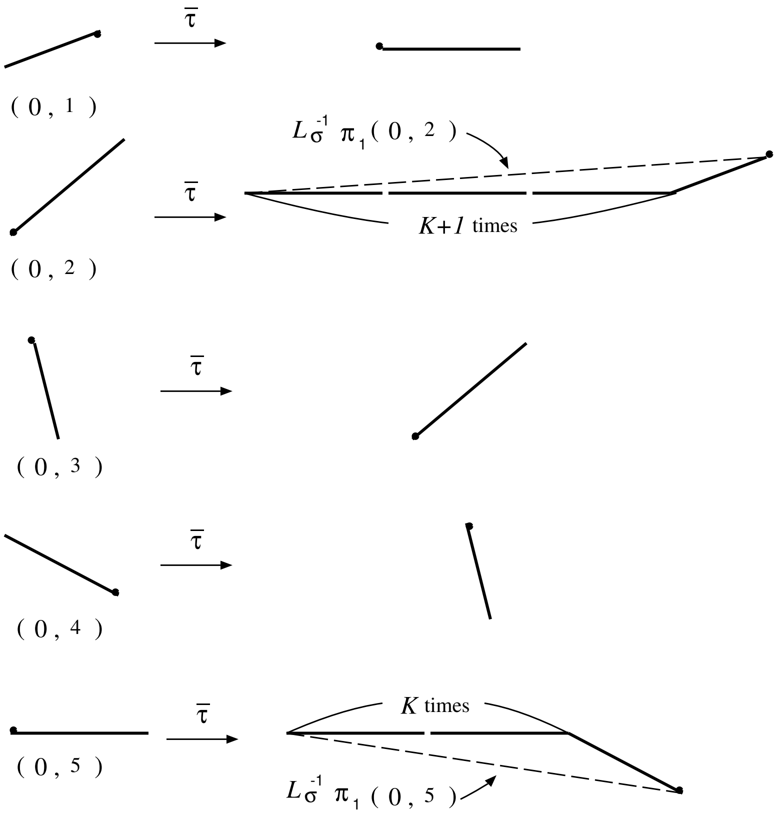

Fig. 13: Geometrical meaning of $\overline{\tau}$

The following diagram is commutative:

$$\begin{array}{ccccc} & \mathcal{F}^* & \overset{\tau^*}{\rightarrow} & \mathcal{F}^* & \\ \partial_{\mathcal{F}^*} & \downarrow & & \downarrow & \partial_{\mathcal{F}^*} \\ & \mathcal{F} & \overset{\overline{\tau}}{\rightarrow} & \mathcal{F} & \end{array},$$

and from the definition of the maps π_1, π_2, we have

$$\begin{array}{ccccc} & \mathcal{F}^* & \overset{\pi_2}{\rightarrow} & \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>} & \\ \partial_{\mathcal{F}^*} & \downarrow & & \downarrow & \partial \\ & \mathcal{F} & \overset{\pi_1}{\rightarrow} & \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>} & \end{array},$$

where ∂D denotes the boundary of the domain D.

From the two diagrams, we deduce that

$$\begin{aligned}\partial(L_\sigma^n D_n^{(i)}) &= \partial(L_\sigma^n \pi_2(\tau^{*\ n}(\boldsymbol{e}_i, i^*))) \\ &= L_\sigma^n(\partial\ \pi_2(\tau^{*\ n}(\boldsymbol{e}_i, i^*))) \\ &= L_\sigma^n \pi_1(\overline{\tau}^n(\partial_{\mathcal{F}^*}(\boldsymbol{e}_i, i^*)))\ .\end{aligned}$$

Let $B_n^{(i)}$ denote the set of vertices on $\partial(L_\sigma^n D_n^{(i)})$.

Lemma 4.1 *The sequences of sets $\{\partial(L_\sigma^n D_n^{(i)})\}$ and $\{B_n^{(i)}\}$ converge towards the same set as n goes to ∞ in the sense of the Hausdorff metric.*

Proof:
Existence of the limit set of $\{\partial(L_\sigma^n D_n^{(i)})\}_{n=1}^\infty$. It is enough to show that the following limit set exists:

$$I_j = \lim_{n\to\infty} L_\sigma^n \pi_1(\overline{\tau}^n(\mathbf{0}, j)) \text{ for any } j = 1, 2, 3, 4, 5\ .$$

Put

$$c_0 := \max_{i=1,2,3,4,5} d_H(\ L_\sigma \pi_1(\overline{\tau}(\mathbf{0}, i)),\ \pi_1(\mathbf{0}, i)\ ),$$

where d_H is the Hausdorff metric. In general the following property holds:

$$d_H(A \cup B, C \cup D) \le \max(d_H(A, C), d_H(B, D))\ ,$$

for sets A, B, C, D. It is easy to check that $\overline{\tau}^n(\boldsymbol{x}, i)$ does not have cancellation for any $(\boldsymbol{x}, i)$ and any positive integer n. Therefore we see

$$d_H(\ L_\sigma \pi_1(\overline{\tau}^{n+1}(\mathbf{0}, i)),\ \pi_1(\overline{\tau}^n(\mathbf{0}, i))\ ) \le c_0\ .$$

Hence,

$$d_H(\ L_\sigma^{n+1} \pi_1(\overline{\tau}^{n+1}(\mathbf{0}, i)),\ L_\sigma^n \pi_1(\overline{\tau}^n(\mathbf{0}, i))\ ) \le c_0 {\beta_0}^n\ , \tag{4.1}$$

where $\beta_0 = \begin{cases} \frac{1}{\sqrt{\beta}} & K \le 2 \\ \max\{|\beta^{(2)}|, |\beta^{(3)}|\} & K \ge 3 \end{cases}$.

This means the sequence $\{L_\sigma^n \pi_1(\overline{\tau}^n(\mathbf{0}, i))\}_{n=1}^\infty$ is a Cauchy sequence and it has a limit set in the sense of the Hausdorff metric.

Analogously we see that the sequence $B_n^{(i)}$ converges.

By the construction of $B_n^{(i)}$ and a simple approximation argument, we see these limit sets are equal.

□

Let $B^{(i)}$ (resp., B) denote the limit set $\lim_{n\to\infty} \partial(L_\sigma^n D_n^{(i)})(= \lim_{n\to\infty} B_n^{(i)})$ (resp., $\lim_{n\to\infty} \partial(L_\sigma^n D_n)$).

Lemma 4.2 $\mu(\widehat{X_i} \cap U_r(\boldsymbol{x})) > 0$ $(i = 1, 2, 3, 4, 5)$ *for any $\boldsymbol{x} \in \widehat{X_i}$ and any $r > 0$.*

Proof: From Theorem 4.1 (3),

$$\widehat{X_i} = \bigcup_{j=1}^5 \bigcup_{P_k^{(n,j)}: W_k^{(n,j)} = i} (L_\sigma^n \widehat{X_j} - \pi\mathbf{f}(P_k^{(n,j)}))\ .$$

By the boundedness of $\widehat{X_i}$ (See Corollary 2.1.), for any $\boldsymbol{x} \in \widehat{X_i}$ and any $r > 0$, there exist positive integers n, j, k such that

$$\boldsymbol{x} \in L_\sigma^n \widehat{X_j} - \pi \mathbf{f}(P_k^{(n,j)}) \subset U_r(\boldsymbol{x}) \ .$$

Thus we have

$$\mu(\widehat{X_i} \cap U_r(\boldsymbol{x})) \geq \mu(\widehat{X_i} \cap (L_\sigma^n \widehat{X_j} - \pi \mathbf{f}(P_k^{(n,j)}))) = \mu(L_\sigma^n \widehat{X_j}) > 0 \ .$$

□

Lemma 4.3 *We have* $\partial \widehat{X_i} = B^{(i)}$ *for* $i = 1, 2, 3, 4, 5$.

Lemma 4.3 says that the boundary of $\widehat{X_i}$ is constructed by the map $\overline{\tau}$. **Proof:** We show that $\partial \widehat{X_1} = B^{(1)}$ and the other cases are shown analogously. To see $B^{(1)} \subset \partial \widehat{X_1}$, it is sufficient to see $B_n^{(1)} \subset \partial \widehat{X_1}$ for any n. Let $\mathcal{N}$ be the collection of tiles which consists of $(\boldsymbol{e}_1, 1^*)$ and its neighbour tiles:

$$\mathcal{N} := (\boldsymbol{e}_1, 1^*) + (\boldsymbol{e}_3, 3^*) + (\boldsymbol{e}_4, 4^*) + (\boldsymbol{e}_5, 5^*) \ .$$

By $\mathcal{N} \subset {\tau^*}^4(\boldsymbol{e}_1, 1^*)$ and Theorem 4.1 (3),

$$\mu(\widehat{X_1} \cap \widehat{X_i}) = 0 \ \text{ for } \ (\boldsymbol{e}_i, i^*) \subset \mathcal{N}, i \neq 1 \ .$$

Take $\boldsymbol{x} \in B_n^{(1)}$, then there exists $(\boldsymbol{e}_j, j^*) \subset \mathcal{N}, j \neq 1$ such that $\boldsymbol{x} \in \widehat{X_1}$ and $\boldsymbol{x} \in \widehat{X_j}$. Suppose that $\boldsymbol{x}$ is an inner point of $\widehat{X_1}$, that is, there is $r > 0$ such that $U_r(\boldsymbol{x}) \subset \widehat{X_1}$. By $\boldsymbol{x} \in \widehat{X_j}$ and Lemma 4.2, we have

$$\mu(\widehat{X_j} \cap U_r(\boldsymbol{x})) > 0,$$

therefore, for $(\boldsymbol{e}_i, i^*) \in \mathcal{N}$

$$\mu(\widehat{X_1} \cap \widehat{X_j}) \geq \mu(U_r(\boldsymbol{x}) \cap \widehat{X_j}) > 0 \ ,$$

which leads to a contradiction and it implies $B_n^{(1)} \subset \partial \widehat{X_1}$ for any n.

Conversely, suppose that $\boldsymbol{x} \in \partial \widehat{X_1} \subset \widehat{X_1}$. Then there are sequences $\{\boldsymbol{x}_n\}_{n=1}^\infty$ with $\boldsymbol{x}_n \in L_\sigma^n D_n^{(1)}$ and $\{\boldsymbol{y}_n\}_{n=1}^\infty$ with $\boldsymbol{y}_n \notin \widehat{X_1}$ such that $d(\boldsymbol{x}, \boldsymbol{x}_n) < \frac{1}{n}$ and $d(\boldsymbol{x}, \boldsymbol{y}_n) < \frac{1}{n}$ for any positive integer n, where d is the usual Euclidean distance on the plane $\boldsymbol{P}_{<\boldsymbol{v}_1, \boldsymbol{v}_2>}$. By $\boldsymbol{y}_n \notin \widehat{X_1}$, we have sequences $\{\boldsymbol{y}_{n,m}\}_{m=1}^\infty$:

$$\lim_{m \to \infty} \boldsymbol{y}_{n,m} = \boldsymbol{y}_n,$$

and for any m there exists $M \geq m$ such that $\boldsymbol{y}_{n,M} \notin L_\sigma^M D_M^{(1)}$. Therefore, we can choose $\boldsymbol{y}_{n,k_n} \notin L_\sigma^{k_n} D_{k_n}^{(1)}$ so that $d(\boldsymbol{y}_{n,k_n}, \boldsymbol{y}_n) < \frac{1}{n}$ and $k_1 < k_2 < \cdots$. Thus the sequence $\{\boldsymbol{y}_{n,k_n}\}_{n=1}^\infty$ satisfies

$$d(\boldsymbol{y}_{n,k_n}, \boldsymbol{x}) \leq d(\boldsymbol{y}_{n,k_n}, \boldsymbol{y}_n) + d(\boldsymbol{y}_n, \boldsymbol{x}) = \frac{2}{n} \ ,$$

and this means $\lim_{n \to \infty} \boldsymbol{y}_{n,k_n} = \boldsymbol{x}$. On the segment between $\boldsymbol{x}_{k_n} \in L_\sigma^{k_n} D_{k_n}^{(1)}$ and $\boldsymbol{y}_{n,k_n} \notin L_\sigma^{k_n} D_{k_n}^{(1)}$, there exists $\boldsymbol{c}_{k_n} \in \partial L_\sigma^{k_n} D_{k_n}^{(1)}$, and $\lim_{n \to \infty} \boldsymbol{c}_{k_n} = \boldsymbol{x}$. This implies

$$\boldsymbol{x} \in \lim_{n \to \infty} \partial L_\sigma^{k_n} D_{k_n}^{(1)} = \lim_{n \to \infty} \partial L_\sigma^n D_n^{(1)}$$

and $\partial \widehat{X_1} \subset B^{(1)}$. □

Proposition 4.1 *The origin is an inner point of $\widehat{X_1}$.*

Proof: By Lemma 4.3, we see

$$L_\sigma^{-N}\partial(\widehat{X_i}) = \lim_{n\to\infty} L_\sigma^n \pi_1(\,\overline{\tau}^n(\partial_{\mathcal{F}^*}(\tau^{*\,N}(\boldsymbol{e}_i, i^*)))\,)\,,$$

for any positive integer N. This means that $L_\sigma^{-N}\partial(\widehat{X_i})$ is constructed by replacing each edge $\pi_1(\boldsymbol{x}, j)$ $(j = 1,2,3,4,5)$ on $\partial D_N^{(i)}$ with $I_j + \boldsymbol{x}$, where $I_j = \lim_{n\to\infty} L_\sigma^n \pi_1(\overline{\tau}^n(\boldsymbol{0}, j))$.

By the inequality (4.1), we have the following inequality

$$d_H\left(L_\sigma^n \pi_1(\overline{\tau}^n(\boldsymbol{0}, i)), \pi_1(\boldsymbol{0}, i)\right) \le \sum_{j=0}^{n-1} c_0 {\beta_0}^j < c \quad \text{for any } n,$$

where c is some positive number. Therefore,

$$d_H(I_i, \pi_1(\boldsymbol{0}, i)) = d_H\left(\lim_{n\to\infty} L_\sigma^n \pi_1(\overline{\tau}^n(\boldsymbol{0}, i)), \pi_1(\boldsymbol{0}, i)\right) \le c\,. \tag{4.2}$$

By the fact that $\pi_2(\tau^{*\,n}(\mathcal{U}))$ is covering the plane $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$ as n goes to ∞ (See Proposition 3.3 and Section 6.), there exists a positive integer N such that

$$\inf\left\{d(\boldsymbol{0}, \boldsymbol{x}) \mid \boldsymbol{x} \in \partial\pi_2(\tau^{*\,N}(\boldsymbol{e}_1, 1^*))\right\} > 2c\,.$$

From the construction of $L_\sigma^{-N}\partial(\widehat{X_1})$ and (4.2), we have

$$\inf\left\{d(\boldsymbol{0}, \boldsymbol{x}) \mid \boldsymbol{x} \in L_\sigma^{-N}\partial(\widehat{X_1})\right\} > c\,.$$

Thus we see that for some positive number c'

$$\inf\left\{d(\boldsymbol{0}, \boldsymbol{x}) \mid \boldsymbol{x} \in \partial(\widehat{X_1})\right\} > c' > 0\,.$$

This implies the origin is an inner point of $\widehat{X_1}$. $\square$

Proposition 4.2 *The Hausdorff dimension of the limit set B of the boundaries $\lim_{n\to\infty} \partial(L_\sigma^n D_n)$ satisfies*

$$(\dim_H \partial\widehat{X} =) \dim_H B \le \frac{2\log\lambda_\theta}{\log\beta}\,,$$

where $M_\theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & K+1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & K \end{pmatrix}$ *and λ_θ is the maximum eigenvalue of the matrix M_θ, that is, the maximum solution of the equation $x^5 - Kx^4 - (K+1)x - 1 = 0$.*

Proof: We define the endomorphism θ on the free group $< 1^{\pm 1}, 2^{\pm 1}, 3^{\pm 1}, 4^{\pm 1}, 5^{\pm 1} >$ associated with $\overline{\tau}$:

$$\theta\ :\ \begin{cases} 1 \ \rightarrow\ 5 \\ 2 \ \rightarrow\ 1\,\overbrace{5^{-1}5^{-1}\cdots 5^{-1}}^{K+1\ times} \\ 3 \ \rightarrow\ 2 \\ 4 \ \rightarrow\ 3 \\ 5 \ \rightarrow\ 4\,\overbrace{5^{-1}\cdots 5^{-1}}^{K\ times} \end{cases} .$$

We see that $\theta^n(15^{-1}34^{-1}23^{-1})$ has no cancellation for all n. Therefore, from the method of [BED86], [DEK82], [IO93], [IK91], we have this result. □

Corollary 4.1 *For any $K \geq 0$ the Hausdorff dimension of B satisfies the inequality:*

$$(\dim_H \partial\widehat{X} =) \dim_H B < 2.$$

In the case of $K = 0$, $\dim_H \partial\widehat{X} = 1.10026\cdots$.

Proof: By Proposition 4.2, it is enough to show the inequality $\lambda_\theta < \beta$. We can assume $K \geq 1$. Put

$$\begin{aligned} f(x) &= x^3 - Kx^2 - (K+1)x - 1 \\ g(x) &= x^5 - Kx^4 - (K+1)x - 1. \end{aligned}$$

By a simple computation, one gets that

$$f(K+1) < 0\ ,\ f(K+1+\frac{1}{3K+2}) > 0,$$
$$g(\frac{K+2}{2}) < 0\ ,\ g(K+1) > 0\ .$$

This means $K+1 < \beta < K+1+\frac{1}{3K+2}$ and there is some solution λ' of the equation $g(x) = 0$ such that $\frac{K+2}{2} < \lambda' < K+1$. From $g'(K+1) > 0$ and $g''(x) > 0$ with $x \geq K+1$, we know there is no solution of the equation $g(x) = 0$ with $x \geq K+1$, that is, $\lambda' = \lambda_\theta$ and $\frac{K+2}{2} < \lambda_\theta < K+1$. Using the above two inequalities we have the conclusion. □

4.3 A second tiling given by τ^* with atomic surfaces

In Section 3 we introduce a first tiling $\mathcal{T}_{\tau^*}$ with five polygonal prototiles $\pi_2(\boldsymbol{x}, i^*)$ (i=1,2,3,4,5). Here we consider a second tiling with five prototiles $\widehat{X_i}$ with fractal boundary by replacing each prototile $\pi_2(\boldsymbol{x}+\boldsymbol{e}_i, i^*)$ in $\mathcal{T}_{\tau^*}$ with $\widehat{X_i} + \pi\boldsymbol{x}$. Put

$$\mathcal{T}_{\tau^*}^{\widehat{X}} := \{\widehat{X_i} + \pi\boldsymbol{x} \mid \pi_2(\boldsymbol{x}+\boldsymbol{e}_i, i^*) \in \mathcal{T}_{\tau^*}\}\ .$$

Theorem 4.2 *The family of tiles $\{\widehat{X_i} + \pi\boldsymbol{x} \mid \pi_2(\boldsymbol{x}+\boldsymbol{e}_i, i^*) \in \mathcal{T}_{\tau^*}\}$ is a quasi periodic tiling with five prototiles $\widehat{X_i}$ $(i = 1,2,3,4,5)$ of the plane $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$.*

Proof: At first we show that $\mathcal{T}_{\tau^*}^{\widehat{X}}$ is a tiling of $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$. From Proposition 4.1, there exists a positive number δ such that

$$U_\delta(\mathbf{0}) \subset \widehat{X_1}\ .$$

Notice that for any positive integer n

$$U_{\beta_0{}^{-n}\delta}(\mathbf{0}) \subset L_\sigma^{-n} U_\delta(\mathbf{0}).$$

By Theorem 4.1 (3)

$$U_{\beta_0{}^{-n}\delta}(\mathbf{0}) \subset L_\sigma^{-n}\widehat{X_1} = \bigcup_{j=1}^{5} \bigcup_{P_k^{(n,j)}:W_k^{(n,j)}=1} (\widehat{X_j} - L_\sigma^{-n}\pi\mathbf{f}(P_k^{(n,j)}))\ ,$$

where the tiles $(\widehat{X_j} - L_\sigma^{-n}\pi\mathbf{f}(P_k^{(n,j)}))$ do not overlap for any k, j. From

$$\mathcal{T}_{\tau^*} = \cup_{n=0}^{\infty}\pi_2(\tau^{*\ n}(\mathcal{U})) = \cup_{n=0}^{\infty}\pi_2(\tau^{*\ n}(\boldsymbol{e}_1, 1^*)),$$

the union of tiles $\bigcup_{j=1}^{5}\bigcup_{P_k^{(n,j)}:W_k^{(n,j)}=1}(\widehat{X_j} - L_\sigma^{-n}\pi\mathbf{f}(P_k^{(n,j)}))$ is a part of $\mathcal{T}_{\tau^*}^{\widehat{X}}$. This means that the pieces of $\mathcal{T}_{\tau^*}^{\widehat{X}}$ do not overlap and cover $U_{\beta_0{}^{-n}\delta}(\mathbf{0})$ for any n. Therefore $\mathcal{T}_{\tau^*}^{\widehat{X}}$ is a tiling of $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$ by taking $n \to \infty$. On the other hand, from the quasi periodicity of the tiling $\mathcal{T}_{\tau^*}$, $\mathcal{T}_{\tau^*}^{\widehat{X}}$ is also quasi periodic. $\square$

At the beginning of this paper, we started from the substitution σ given by (1.1). And we obtained atomic surfaces $\{X_i\}_{i=1,2,3,4,5} (= \{-\widehat{X_i}\}_{i=1,2,3,4,5})$ and the tiling $\mathcal{T}_{\tau^*}^{\widehat{X}}$. We also get tiles $\{T_i\}_{i=1,2,3,4,5}$ and a tiling $\mathcal{T}_\beta$ by using the numeration system related to a Pisot number β as in [AKI99], [THU89]. We plan to make the relation between $\{X_i\}_{i=1,2,3,4,5}$ (resp. $\mathcal{T}_{\tau^*}^{\widehat{X}}$) and $\{T_i\}_{i=1,2,3,4,5}$ (resp. $\mathcal{T}_\beta$) explicit with the subdivision rule in [EIR02].

5 Dynamical systems

We introduce two types of measured dynamical systems on $\widehat{X}$ a Markov transformation and a domain exchange transformation with σ-structure.

From Theorem 4.1 (3), $\boldsymbol{x} \in \widehat{X_i}$ implies that there exist integers j, k such that

$$L_\sigma^{-1}\boldsymbol{x} \in \widehat{X_j} - L_\sigma^{-1}\pi\mathbf{f}(P_k^{(j)}).$$

Therefore we get the division of $\widehat{X_i}$:

$$\widehat{X_i} = \bigcup_{\binom{j}{k}:W_k^{(j)}=i} \widehat{X_i}^{\binom{j}{k}},$$

where $\widehat{X_i}^{\binom{j}{k}} := \{\boldsymbol{x} \in \widehat{X_i} \mid L_\sigma^{-1}\boldsymbol{x} \in \widehat{X_j} - L_\sigma^{-1}\pi\mathbf{f}(P_k^{(j)})\}$. Here we have the following theorem which provides a Markov transformation.

Theorem 5.1 *Let us define the map $F : \widehat{X} \to \widehat{X}$ by*

$$F(\boldsymbol{x}) = L_\sigma^{-1}\boldsymbol{x} + L_\sigma^{-1}\pi\mathbf{f}(P_k^{(j)}) \text{ if } \boldsymbol{x} \in \widehat{X_i}^{\binom{j}{k}},$$

then the map F is well-defined and

$$F(\widehat{X_i}^{\binom{j}{k}}) = \widehat{X_j}.$$

The transformation F is well-defined because of Theorem 4.1 (4), and it is called a Markov transformation with matrix structure L_σ with respect to partitions $\{\widehat{X_i}\}_{i=1,2,3,4,5}$.

From now on we consider a domain exchange transformation with σ-structure.

Definition 5.1 *Let (X, T, μ) be a measured dynamical system, σ a substitution over the alphabet $\mathcal{A}$ such that*

$$\sigma(i) = W_0^{(i)}W_1^{(i)} \cdots W_{l_{(i)}-1}^{(i)} ,$$

and consider a measurable partition $\{X^{(i)} \mid i \in \mathcal{A}\}$ of X, a subset A of X and a measurable partition $\{A^{(i)} \mid i \in \mathcal{A}\}$ of A.

We say that the transformation T has σ-structure with respect to the pair of partitions $\{X^{(i)}\}, \{A^{(i)}\}$ if T satisfies the following condition:

$$\begin{array}{lll} T^k A^{(i)} \subset X^{(W_k^{(i)})} & for\ all\ \ i \in \mathcal{A}\ ,\ k = 0, 1, \cdots, l_{(i)} - 1 & \\ T^{l_{(i)}} A^{(i)} \subset A & for\ all\ i \in \mathcal{A} & . \\ X = \bigcup_{i\in\mathcal{A}} \bigcup_{0\le k\le l_{(i)}-1} T^k A^{(i)} & (non-overlapping) & \end{array}$$

For the transformation T with σ-structure, the induced transformation $T|_A$ on A is defined by

$$T|_A(\boldsymbol{x}) = T^{l_{(i)}}(\boldsymbol{x})\ \ for\ \boldsymbol{x} \in A^{(i)}\ .$$

Proposition 5.1 *Let σ be a substitution satisfying (1.1) and put*

$$L_\sigma^{-n} = \left(\boldsymbol{f}_1^{(n)}, \boldsymbol{f}_2^{(n)}, \boldsymbol{f}_3^{(n)}, \boldsymbol{f}_4^{(n)}, \boldsymbol{f}_5^{(n)}\right)\ .$$

(1) The transformation $E_n : D_n \to D_n$ given by

$$E_n(\boldsymbol{x}) = \boldsymbol{x} - \pi\boldsymbol{f}_i^{(n)} \quad if\ \boldsymbol{x} \in D_n^{(i)}\ (i = 1, 2, 3, 4, 5)$$

is well-defined and preserves the Lebesgue measure μ. (See Fig. 14.)

(2) The transformation $E_1 : D_1 \to D_1$ has σ-structure with respect to the pair of partitions $\{D_1^{(i)}\}$, $\{D_0^{(i)}\}$ and the induced transformation satisfies

$$E_1|_{D_0} = E_0\ .$$

Moreover, for any positive integer k the transformation $E_k : D_k \to D_k$ has σ-structure with respect to the pair of partitions $\{D_k^{(i)}\}$, $\{D_{k-1}^{(i)}\}$ and the induced transformation satisfies

$$E_k|_{D_{k-1}} = E_{k-1}\ .$$

(3) *For any positive integer k the transformation $E_k : D_k \to D_k$ has σ^k-structure with respect to the pair of partitions $\{D_k^{(i)}\}$, $\{D_0^{(i)}\}$ and the induced transformation satisfies*

$$E_k|_{D_0} = E_0 \ .$$

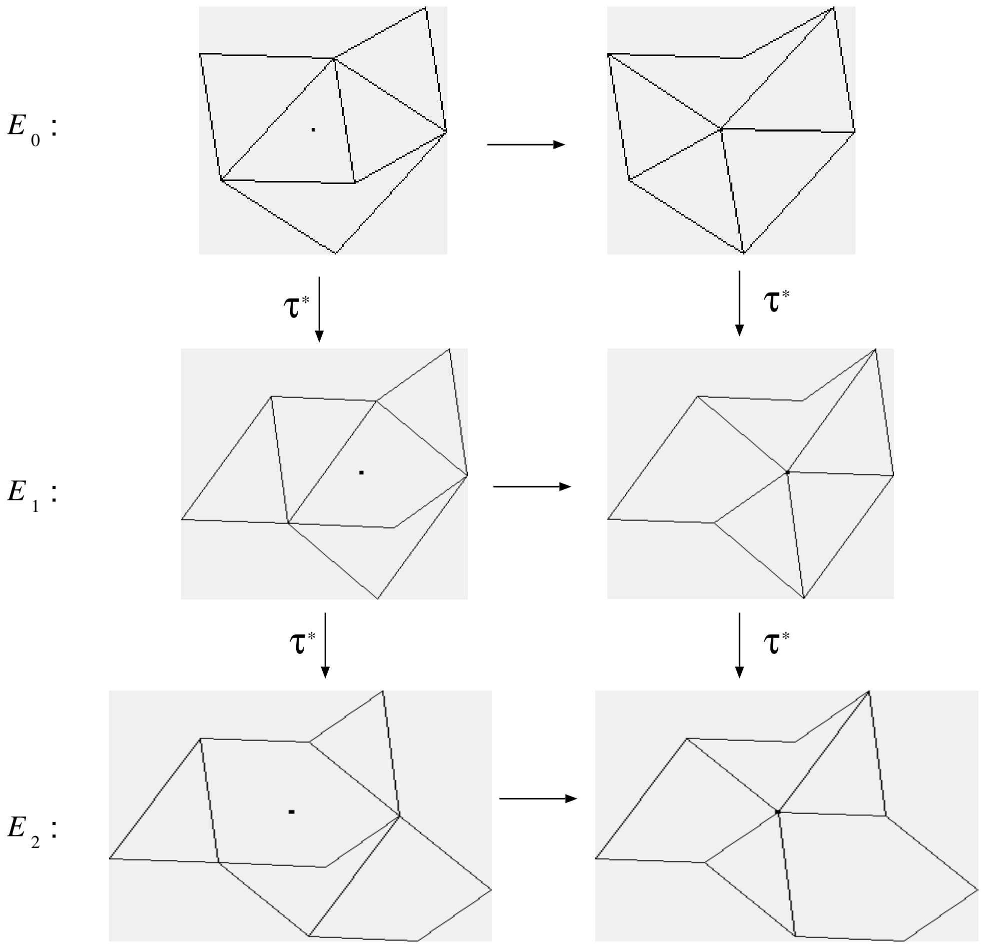

Fig. 14: Domain exchange transformations E_i on D_i

The transformations E_k are called *domain exchange transformations* on D_k.

Proof: From the equation

$$\tau^{*\,n}(\boldsymbol{e}_i, i^*) = M_{\boldsymbol{f}_i^{(n)}}(\tau^{*\,n}(\mathbf{0}, i^*))$$

and $D_n = {D_n}'$ by Proposition 3.1, we see the transformation E_n is well-defined.

First statement of (2) is obtained from Fig. 14. The second and third statements are proved inductively. □

This proposition leads to the following:

Theorem 5.2 *Define the transformation $E : \widehat{X} \to \widehat{X}$ by*

$$E(\boldsymbol{x}) = \boldsymbol{x} - \pi \boldsymbol{e}_i \qquad \boldsymbol{x} \in \widehat{X_i} \ .$$

The transformation E, which preserves the Lebesgue measure μ, is well-defined on $\widehat{X}$. And E has σ-structure with respect to the pair of partitions $\{\widehat{X_i}\},\ \{L_\sigma \widehat{X_i}\}$, moreover, E has σ^n-structure with respect to the pair of partitions $\{\widehat{X_i}\},\ \{L_\sigma^n \widehat{X_i}\}$ for all $n \in \mathbb{N}$.

Proof: From Theorem 4.1 (1), the transformation E is well-defined. From the equation given by Theorem 4.1 (3) and (4), we have

$$\widehat{X_i} = \bigcup_{j=1}^{5} \bigcup_{P_k^{(j)}:W_k^{(j)}=i} (L_\sigma \widehat{X_j} - \pi \mathbf{f}(P_k^{(j)})) \ ,$$

$$L_\sigma \widehat{X_i} \subset \widehat{X}_{W_0^{(i)}} \ .$$

Hence,

$$E(L_\sigma \widehat{X_i}) = L_\sigma \widehat{X_i} - \pi \mathbf{f}(W_0^{(i)}) \subset \widehat{X}_{W_1^{(i)}} \ .$$

Analogously we say

$$E^k(L_\sigma \widehat{X_i}) = L_\sigma \widehat{X_i} - \pi \mathbf{f}(P_k^{(i)}) \subset \widehat{X}_{W_k^{(i)}}$$

for $k = 0, 1, \cdots, l_{(i)-1}$ and

$$\begin{aligned} E^{l_{(i)}}(L_\sigma \widehat{X_i}) &= L_\sigma \widehat{X_i} - \pi \mathbf{f}(\sigma(i)) \\ &= L_\sigma (\widehat{X_i} - \pi \boldsymbol{e}_i) \\ &= L_\sigma \widehat{X_i}' \ . \end{aligned}$$

This means E has σ-structure with respect to the pair of partitions $\{\widehat{X_i}\},\ \{L_\sigma \widehat{X_i}\}$. By induction, we can show the second statement. □

This transformation $E : \widehat{X} \to \widehat{X}$ is also called the domain exchange transformation associated with a substitution σ. From this theorem and Proposition 4.1 we have the following corollaries:

Corollary 5.1 *For $k = 0, 1, \cdots$, we have*

$$E^k(\mathbf{0}) \in \widehat{X}_{s_k} \ ,$$

where $\omega = \lim_{n\to\infty} \sigma^n(1) = s_0 s_1 \cdots s_n \cdots$.

From Corollary 5.1 we have the following corollary: (See Lemma 6 in [AI01] and [BFMS02].)

Corollary 5.2 *Let (Ω_σ, S) be the substitution dynamical system generated by a substitution σ given by (1.1). The dynamical system $(\widehat{X}, E)$ is a realization of (Ω_σ, S), and the realization map ϕ from Ω_σ to $\widehat{X}$ is given by using $\phi(S^k(s_0 s_1 \cdots)) = E^k(0)$ for all positive integers k.*

Finally to observe ergodic property of the domain exchange transformation E, we define new domains $\widetilde{D_n}$ (resp., $\widetilde{\widehat{X}}$) and domain exchange transformations $\widetilde{E_n}$ (resp., $\widetilde{E}$) on the domains which are measure-theoretically isomorphic to a rotation on the 2-dimensional torus.

Since the domain $\pi_2(\mathcal{U})$ is not a 2-dimensional fundamental domain, we introduce the symmetrical image of $\mathcal{U}$ denoted by $\overline{\mathcal{U}}$ and the union of $\mathcal{U}$ and $\overline{\mathcal{U}}$ denoted by $\widetilde{\mathcal{U}}$ (cf. Fig. 15 and Fig. 16), which will be a 2-dimensional fundamental domain as follows:

$$
\begin{aligned}
\overline{\mathcal{U}} &:= \{(\boldsymbol{e}_4, 1^*) + (\boldsymbol{e}_4, 3^*) + (\boldsymbol{e}_4, 5^*) + (2\boldsymbol{e}_4, 1^*) + (\boldsymbol{e}_2 + \boldsymbol{e}_4, 5^*)\},\\
\widetilde{\mathcal{U}} &:= \mathcal{U} + \overline{\mathcal{U}},\\
\mathbb{L} &:= \{m(\boldsymbol{e}_2 - \boldsymbol{e}_4) + n(\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_3) \mid m, n \in \mathbb{Z}\},\\
\widetilde{D_0} &:= \pi_2(\widetilde{\mathcal{U}}) = D_0 \cup \{-D_0 + (\pi\boldsymbol{e}_1 + \pi\boldsymbol{e}_2 + \pi\boldsymbol{e}_4)\},\\
\pi(\mathbb{L}) &:= \{\pi\boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{L}\}.
\end{aligned}
$$

Then we have the following lemma:

Lemma 5.1 *The domains $L_\sigma^n \pi_2(\tau^{*\,n}(\widetilde{\mathcal{U}}))$ are 2-dimensional fundamental domains for the lattice $\pi(\mathbb{L})$, that is, $L_\sigma^n \pi_2(\tau^{*\,n}(\widetilde{\mathcal{U}})) = \mathbb{R}^2/\mathbb{L}$ for any non negative integer n. In other words, we have periodic tilings $\{L_\sigma^n \pi_2(\boldsymbol{y}, j^*) \mid (\boldsymbol{x}, i^*) \subset M_{\boldsymbol{z}}(\widetilde{\mathcal{U}})$ for some $\boldsymbol{z} \in \mathbb{L},\ (\boldsymbol{y}, j^*) \subset \tau^{*\,n}(\boldsymbol{x}, i^*)\}$ for any non negative integer n.*

Proof: Replace the segment $\{t\pi(\boldsymbol{e}_2 - \boldsymbol{e}_4) \mid 0 \le t \le 1\}$, which is an edge of the fundamental domain $\{s\pi(\boldsymbol{e}_2 - \boldsymbol{e}_4) + t\pi(\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_3) \mid 0 \le s \le 1,\ 0 \le t \le 1\}$ for the lattice $\pi(\mathbb{L})$, with $\pi_1((-\boldsymbol{e}_4, 4) + (-\boldsymbol{e}_4, 2))$; and the segment $\{t\pi(\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_3) \mid 0 \le t \le 1\}$ with $\pi_1((-\boldsymbol{e}_3, 3) + (-\boldsymbol{e}_3, 1) + (\boldsymbol{e}_1 - 2\boldsymbol{e}_3, 3) + (\boldsymbol{e}_1 - 2\boldsymbol{e}_3, 2))$. We replace the other sides of these edges analogously. Then we have the domain $\pi_2(M_{-\boldsymbol{e}_1}(\widetilde{\mathcal{U}}))$. Thus $\pi_2(\widetilde{\mathcal{U}})$ is a 2-dimensional fundamental domain. By replacing every edge $\pi_1(\boldsymbol{x}, i)$ on $\pi_2(\widetilde{\mathcal{U}})$ with $L_\sigma^n \pi_1(\overline{\tau}^n(\boldsymbol{x}, i))$, we can also say that $L_\sigma^n \pi_2(\tau^{*\,n}(\widetilde{\mathcal{U}}))$ is a 2-dimensional fundamental domain. (See Fig. 15.) □

Let us define the map $\widetilde{E_0} : \widetilde{D_0} \to \widetilde{D_0}$ by

$$\widetilde{E_0}(\boldsymbol{x}) = \boldsymbol{x} - \pi\boldsymbol{e}_2 \pmod{\pi(\mathbb{L})}\ .$$

For example, the domain $\pi_2(\boldsymbol{e}_1, 1^*)$ is mapped by $\widetilde{E_0}$ as follows:

$$\pi_2(\boldsymbol{e}_1, 1^*) \xrightarrow{\widetilde{E_0}} \pi_2(2\boldsymbol{e}_4, 1^*) \xrightarrow{\widetilde{E_0}} \pi_2(\boldsymbol{e}_4, 1^*) \xrightarrow{\widetilde{E_0}} \pi_2(\boldsymbol{0}, 1^*).$$

Therefore, $\widetilde{E_0}$ is well-defined and measure-theoretically isomorphic to a rotation on the 2-dimensional torus, moreover, we have

$$\widetilde{E_0}|_{D_0} = E_0\ .$$

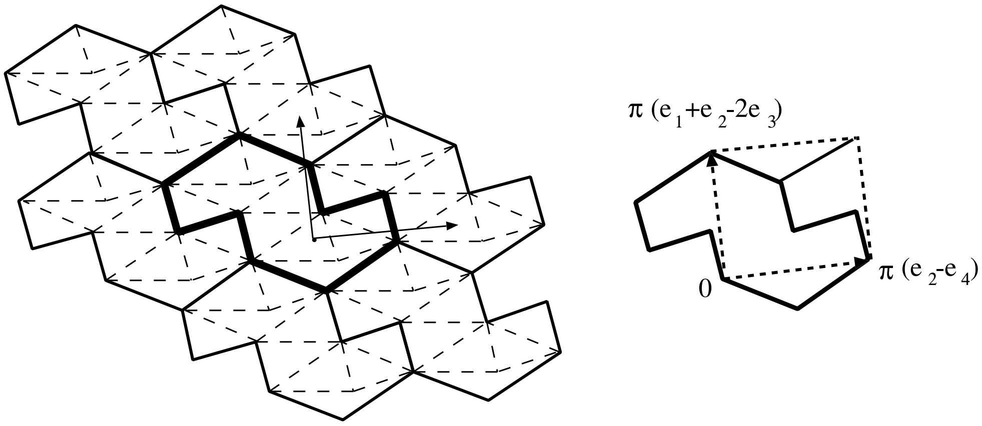

Fig. 15: The periodic tiling $\{\pi_2(x,i^*) \mid (x,i^*) \subset M_z(\widetilde{\mathcal{U}}) \text{ for some } z \in \mathbb{L}\}$

In Lemma 5.1 we obtained 2-dimensional fundamental domains and they were constructed by replacing every edge $\pi_1(\boldsymbol{x},i)$ on $\pi_2(\widetilde{\mathcal{U}})$ with $L_\sigma^n \pi_1(\overline{\tau}^n(\boldsymbol{x},i))$. Moreover, in the same way as in the proof of Proposition 4.1, we also obtain a new 2-dimensional fundamental domain by replacing every edge $\pi_1(\boldsymbol{x},i)$ on the boundary of the domains D_0 and $\{-D_0+(\pi\boldsymbol{e}_1+\pi\boldsymbol{e}_2+\pi\boldsymbol{e}_5)\}$ with $I_i+\pi\boldsymbol{x}$. Then we have the following theorem, which says the domain exchange transformation E is measure-theoretically isomorphic to the induced transformation of a rotation on the 2-dimensional torus:

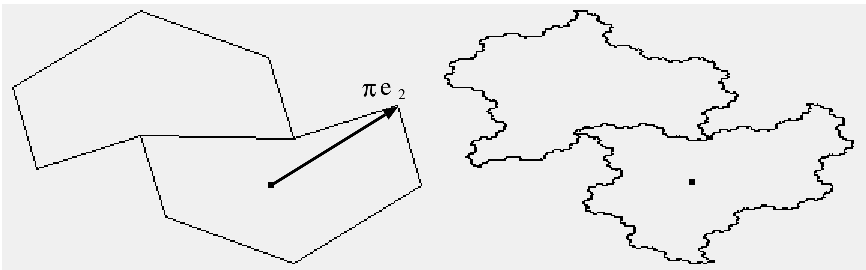

Fig. 16: The domains $\widetilde{D_0}$ and $\widetilde{\widehat{X}}$

Theorem 5.3 *The domain given by*

$$\widetilde{\widehat{X}} := \widehat{X} \cup \{-\widehat{X} + (\pi\boldsymbol{e}_1 + \pi\boldsymbol{e}_2 + \pi\boldsymbol{e}_5)\}$$

is a 2-dimensional fundamental domain.

Let us define the map $\widetilde{E} : \widetilde{\widehat{X}} \to \widetilde{\widehat{X}}$ by

$$\widetilde{E}(\boldsymbol{x}) = \boldsymbol{x} - \pi\boldsymbol{e}_2 \pmod{\pi(\mathbb{L})} ,$$

then it is well-defined and measure-theoretically isomorphic to a rotation on the 2-dimensional torus, moreover, we have

$$\widetilde{E}|_{\widehat{X}} = E \ .$$

6 Appendix: The proof of Proposition 3.3

In this section we prove the following proposition stated in Section 3:

Proposition 3.3 $\cup_{n=1}^{\infty}\pi_2(\tau^{*\ n}(\mathcal{U})) = \boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>}$, *that is,* $\pi_2(\tau^{*\ n}(\mathcal{U}))$ *covers the plane as* n *goes to* ∞.

First we introduce the notion of $\mathcal{C}$-covered property of a set $\Delta \in \mathcal{F}^*$ associated with the connectivity of the domain $\pi_2(\Delta)$ as in [IO93]. Define the subset $\mathcal{C}_0$ of $\mathcal{F}^*$ and $\mathcal{C}$ consisting of translated elements of $\mathcal{C}_0$ by

$$\begin{aligned}\mathcal{C}_0 \ := \ & \{(\boldsymbol{e}_1, 1^*) + (\boldsymbol{e}_4, 4^*), (\boldsymbol{e}_1, 1^*) + (\boldsymbol{e}_3, 3^*), (\boldsymbol{e}_2, 2^*) + (\boldsymbol{e}_5, 5^*), (\boldsymbol{e}_1, 1^*) + (\boldsymbol{e}_5, 5^*), (\boldsymbol{0}, 1^*) + (\boldsymbol{e}_2, 4^*), \\ & (\boldsymbol{0}, 1^*) + (\boldsymbol{0}, 3^*), (\boldsymbol{0}, 2^*) + (\boldsymbol{0}, 5^*), (\boldsymbol{0}, 1^*) + (\boldsymbol{0}, 4^*), (\boldsymbol{0}, 3^*) + (\boldsymbol{0}, 5^*), (\boldsymbol{0}, 2^*) + (\boldsymbol{0}, 4^*)\} \ ,\end{aligned}$$

$$\mathcal{C} := \{M_{\boldsymbol{z}}\xi \in \mathcal{F}^* \mid \xi \in \mathcal{C}_0, \ \boldsymbol{z} \in \mathbb{Z}^5\}.$$

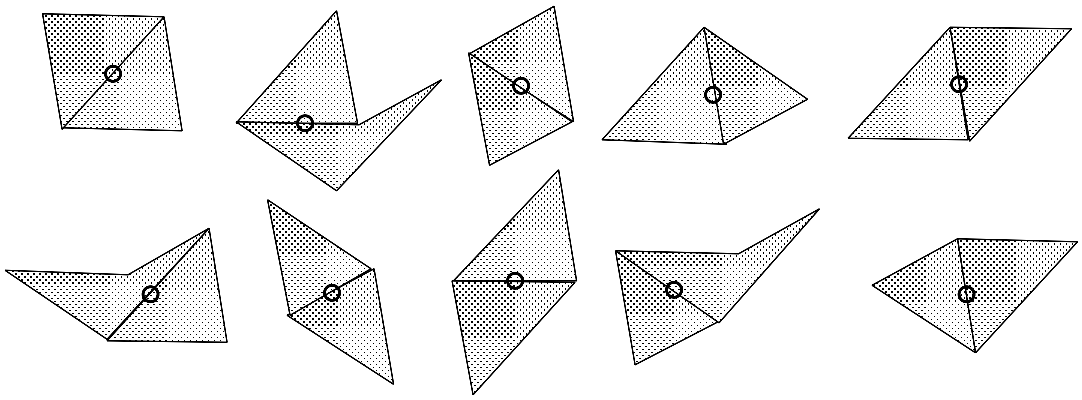

Fig. 17: The elements of $\mathcal{C}_0$

Definition 6.1 *An element* $\Delta \in \mathcal{F}^*$ *is* $\mathcal{C}$*-covered if there exists a finite subset* $\Gamma = \{\gamma_i \in \mathcal{C} \mid i = 1, 2, \cdots, N\}$ *of* $\mathcal{C}$ *such that*

(1) for any $(\boldsymbol{x}, i^*), (\boldsymbol{y}, j^*) \subset \Delta$, *there exists a subset* $\{\gamma_{s_t}\}_{t=1}^{k}$ *of* Γ *such that*

$$(\boldsymbol{x}, i^*) \subset \gamma_{s_1}, \gamma_{s_t} \cap \gamma_{s_{t+1}} \neq \emptyset \ (t = 1, 2, \cdots, k-1), (\boldsymbol{y}, j^*) \subset \gamma_{s_k} \ ,$$

where for $\gamma, \gamma' \in \mathcal{F}^*$, $\gamma \cap \gamma' = \emptyset$ *means* $\mu(\pi_2(\gamma) \cap \pi_2(\gamma')) = 0$, *(2)* $\pi_2(\Gamma) = \pi_2(\Delta)$*, where* $\pi_2(\Gamma) = \cup_{i=1}^{N} \pi_2(\gamma_i)$*.*

The subset Γ *is called a* $\mathcal{C}$*-cover of* Δ*.*

Lemma 6.1 *If* $\Delta \in \mathcal{F}^*$ *is* $\mathcal{C}$*-covered, then* $\tau^*(\Delta)$ *is* $\mathcal{C}$*-covered.*

Proof: It is enough and easy to check that the image of every element of $\mathcal{C}_0$ by τ^* is also $\mathcal{C}$-covered. (See Fig. 18.) □

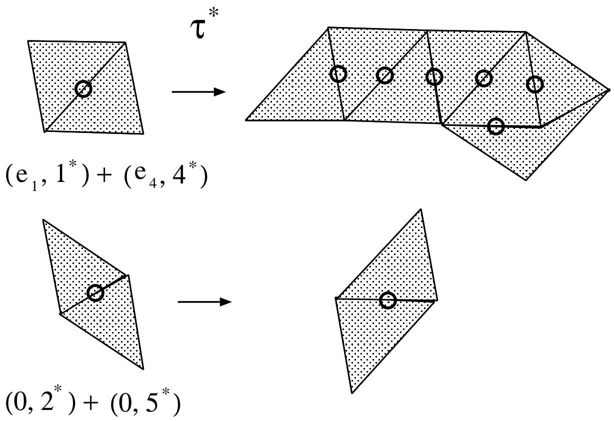

Fig. 18: The images of $(e_1, 1^*) + (e_4, 4^*)$ and $(0, 2^*) + (0, 5^*)$ by τ^*

Definition 6.2 $\mathcal{C}$*-covered* $\Delta \in \mathcal{F}^*$ *is called a* $\mathcal{C}$*-covered cell if* $\pi_2(\Delta)$ *is a topological cell.*

Lemma 6.2 *For any* n*,* $\tau^{*\,n}(\mathcal{U})$ *is a* $\mathcal{C}$*-covered cell.*

Proof: $\mathcal{U}$ is a $\mathcal{C}$-covered cell. Suppose that $\tau^{*\,n}(\mathcal{U})$ is a $\mathcal{C}$-covered cell but $\tau^{*\,n+1}(\mathcal{U})$ is not, that is, $\boldsymbol{P}_{<\boldsymbol{v}_1,\boldsymbol{v}_2>} - \pi_2(\tau^{*\,n+1}(\mathcal{U}))$ has a bounded component $\boldsymbol{D}_1$ and an unbounded component $\boldsymbol{D}_2$. Recall that from Lemma 5.1 $\sum_{\boldsymbol{z}\in\mathbb{L}} M_{L_\sigma^{-n}\boldsymbol{z}}(\tau^{*\,n}(\widetilde{\mathcal{U}}))$ give periodic tilings. Then, there exist $(\boldsymbol{x}, i^*)$ and $(\boldsymbol{y}, j^*) \subset \sum_{\boldsymbol{z}\in\mathbb{L}} M_{L_\sigma^{-(n+1)}\boldsymbol{z}}(\tau^{*\,n+1}(\widetilde{\mathcal{U}}))$ such that

$$\pi_2(\boldsymbol{x}, i^*) \subset \boldsymbol{D}_1 \text{ and } \pi_2(\boldsymbol{y}, j^*) \subset \boldsymbol{D}_2 .$$

And for $(\boldsymbol{x}, i^*)$, $(\boldsymbol{y}, j^*)$ there exist $(\boldsymbol{x}', i'^*)$, $(\boldsymbol{y}', j'^*) \subset \sum_{\boldsymbol{z}\in\mathbb{L}} M_{L_\sigma^{-n}\boldsymbol{z}}(\tau^{*\,n}(\widetilde{\mathcal{U}}))$ such that

$$(\boldsymbol{x}, i^*) \subset \tau^*(\boldsymbol{x}', i'^*) \text{ and } (\boldsymbol{y}, j^*) \subset \tau^*(\boldsymbol{y}', j'^*).$$

From Fig. 15 we have the following properties:

(1) $\overline{\mathcal{U}}, \widetilde{\mathcal{U}}$ are $\mathcal{C}$-covered,

(2) if $\boldsymbol{z}, \boldsymbol{z}' \in \mathbb{L}$ $(\boldsymbol{z} \neq \boldsymbol{z}')$ satisfy $\pi_2(M_{\boldsymbol{z}}(\widetilde{\mathcal{U}})) \cap \pi_2(M_{\boldsymbol{z}'}(\widetilde{\mathcal{U}})) \neq \emptyset$, then $M_{\boldsymbol{z}}(\widetilde{\mathcal{U}}) + M_{\boldsymbol{z}'}(\widetilde{\mathcal{U}})$ is $\mathcal{C}$-covered,

(3) if $\boldsymbol{z} \in \mathbb{L}$ ($\boldsymbol{z} \neq \mathbf{0}$) satisfies $\pi_2(\overline{\mathcal{U}}) \cap \pi_2(M_{\boldsymbol{z}}(\widetilde{\mathcal{U}})) \neq \emptyset$, then $\overline{\mathcal{U}} + M_{\boldsymbol{z}}(\widetilde{\mathcal{U}})$ is $\mathcal{C}$-covered.

Hence, there exists a finite subset $\Gamma = \{\gamma_t \in \mathcal{C} \mid t = 1, 2, \cdots, N\}$ of $\mathcal{C}$ such that

$$(\boldsymbol{x}', {i'}^*) \subset \gamma_1, \gamma_t \cap \gamma_{t+1} \neq \emptyset\ (t = 1, 2, \cdots, N-1), (\boldsymbol{y}', {j'}^*) \subset \gamma_N,$$

$$\gamma_t \cap \tau^{*\ n}(\mathcal{U}) = \emptyset\ (t = 1, \cdots, N).$$

The second condition implies

$$\sum_{t=1}^{N} \tau^*(\gamma_t) \cap \tau^{*\ n+1}(\mathcal{U}) = \emptyset\ .$$

On the other hand, $\tau^*(\Gamma) := \{\tau^*(\gamma_t) \mid t = 1, 2, \cdots, N\}$ has the following property:

$$\begin{array}{rcl} (\boldsymbol{x}, i^*) & \subset & \tau^*(\gamma_1), \\ \tau^*(\gamma_t) & \cap & \tau^*(\gamma_{t+1}) \neq \emptyset\ (t = 1, \cdots, N-1), \\ (\boldsymbol{y}, j^*) & \subset & \tau^*(\gamma_N), \end{array}$$

and $\tau^*(\gamma_t)$ is $\mathcal{C}$-covered by Lemma 6.1 for any t, that is, $\pi_2(\boldsymbol{x}, i^*)$ and $\pi_2(\boldsymbol{y}, j^*)$ are connected by using elements of $\mathcal{C}$. This provides $\sum_{t=1}^{N} \tau^*(\gamma_t) \cap \tau^{*\ n+1}(\mathcal{U}) \neq \emptyset$, which leads to a contradiction. □

Finally we want to show that $\pi_2(\tau^{*\ n}(\mathcal{U}))$ is expanding on $\boldsymbol{P}_{<\boldsymbol{v}_1, \boldsymbol{v}_2>}$ as n goes to ∞.

Definition 6.3 *An element $\Delta \in \mathcal{F}^*$ including $\tau^{*\ n}(\mathcal{U})$ has an n-th $\mathcal{C}$-belt if there exists a finite subset $\Gamma = \{\gamma_t \in \mathcal{C} \mid t = 1, 2, \cdots N\}$ of $\mathcal{C}$ such that*

(1) $\pi_2(\Delta) \supset \pi_2(\Gamma)$,

(2) $\gamma_t \cap \tau^{*\ n}(\mathcal{U}) = \emptyset$ *for any* t $(t = 1, 2, \cdots, N)$,

(3) there exists $(\boldsymbol{x}, i^*) \subset \Delta$ *satisfying*
$(\boldsymbol{x}, i^*) \subset \gamma_1,\ \gamma_t \cap \gamma_{t+1} \neq \emptyset\ (t = 1, 2, \cdots, N-1),\ (\boldsymbol{x}, i^*) \subset \gamma_N$,

(4) $\pi_2(\Gamma)$ *is an annulus and* $\pi_2(\Gamma)^{bd} \supset \pi_2(\tau^{*\ n}(\mathcal{U}))$*, where* A^{bd} *means the bounded domain of the complement of an annulus* A.

Γ *is called an n-th $\mathcal{C}$-belt of* Δ .

Lemma 6.3 *If $\Delta \in \mathcal{F}^*$ including $\tau^{*\ n}(\mathcal{U})$ has an n-th $\mathcal{C}$-belt Γ, then $\tau^*(\Delta)$ has an $(n+1)$-th $\mathcal{C}$-belt.*

Proof: Assume that $\Gamma = \{\gamma_t \in \mathcal{C} \mid t = 1, 2, \cdots, N\}$ is an n-th $\mathcal{C}$-belt of Δ. Since the subset $\Gamma' = \{\gamma' \in \mathcal{C} \mid \gamma' \subset \tau^*(\gamma_t),\ t = 1, \cdots, N\}$ of $\mathcal{C}$ related to $\tau^*(\Delta)$ satisfies (1), (2), (3) of Definition 6.3, it is enough to show that $\pi_2(\Gamma')$ is an annulus and $\pi_2(\Gamma')^{bd} \supset \pi_2(\tau^{*\ n+1}(\mathcal{U}))$. Let us consider two domains $L_\sigma^{-1}\pi_2(\Gamma)$ and $L_\sigma^{-1}\pi_2(\tau^{*\ n}(\mathcal{U}))$, and replace every edge on these domains by using the replacing and re-dividing method. Then we have $\pi_2(\Gamma')$ and $\pi_2(\tau^{*\ n+1}(\mathcal{U}))$. By the property (3) of Definition 6.3, $\pi_2(\Gamma')$ is an annulus. From the relation $L_\sigma^{-1}\pi_2(\Gamma)^{bd} \supset L_\sigma^{-1}\pi_2(\tau^{*\ n}(\mathcal{U}))$ by the assumption, we see the relation of inclusion.

□

Proof of Proposition 3.3. In the case of $K = 0$, we can take a first $\mathcal{C}$-belt Γ_1 of $\tau^{*\,8}(\mathcal{U})$ concretely. (See Fig 19.) In the case of $K \geq 1$, take a first $\mathcal{C}$-belt Γ_1 as follows:

$$\begin{aligned}\Gamma_1 \;\; = \;\; & \{\gamma \in \mathcal{C} \mid \gamma \supset (\boldsymbol{x}, i^*), \\ & (\boldsymbol{x}, i^*) \in \{(\boldsymbol{e}_1 - \boldsymbol{e}_5, 1^*), (\boldsymbol{e}_4 - \boldsymbol{e}_5, 4^*), (\boldsymbol{e}_1 - \boldsymbol{e}_4, 1^*), (\boldsymbol{e}_5, 4^*), (\boldsymbol{e}_5 - \boldsymbol{e}_3, 5^*), (\boldsymbol{e}_1 - \boldsymbol{e}_3, 1^*), \\ & (\boldsymbol{e}_4 - \boldsymbol{e}_3, 4^*), (\boldsymbol{e}_1 - \boldsymbol{e}_3 - \boldsymbol{e}_5, 1^*), (\boldsymbol{e}_1 - \boldsymbol{e}_2, 1^*), (\boldsymbol{e}_5 - \boldsymbol{e}_2, 4^*), (\boldsymbol{e}_3 - \boldsymbol{e}_2, 3^*), (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_4, 1^*), \\ & (\boldsymbol{e}_4 - \boldsymbol{e}_2, 4^*), (\boldsymbol{e}_4 + \boldsymbol{e}_5 - \boldsymbol{e}_1, 4^*), (\boldsymbol{e}_5, 1^*), (\boldsymbol{e}_5 - \boldsymbol{e}_3, 1^*), (\boldsymbol{e}_4 + \boldsymbol{e}_5 - \boldsymbol{e}_1, 3^*)\}\;\}.\end{aligned}$$

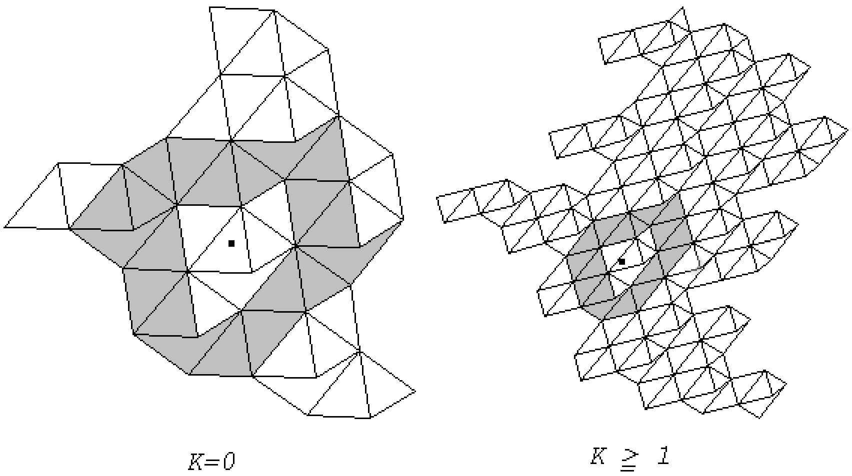

Fig. 19: The figure of $\pi_2(\tau^{*\,8}(\mathcal{U}))$ $(K = 0)$ and $\pi_2(\tau^{*\,5}(\mathcal{U}))$ $(K \geq 1)$

From Lemma 6.3, $\tau^{*\,8n}(\mathcal{U})$ has an $8(n-1)$-th $\mathcal{C}$-belt $\Gamma_{8(n-1)}$ for any n such that

$$\pi_2(\tau^{*\,8n}(\mathcal{U}) - \tau^{*\,8(n-1)}(\mathcal{U})) \supset \pi_2(\Gamma_{8(n-1)}).$$

Thus the distance of the boundary of $\pi_2(\tau^{*\,n}(\mathcal{U}))$ from the origin tends to ∞; and from Lemma 6.2, $\pi_2(\tau^{*\,n}(\mathcal{U}))$ is a topological cell for any n. Therefore, $\pi_2(\tau^{*\,n}(\mathcal{U}))$ is covering the plane $\boldsymbol{P}_{<\boldsymbol{v}_1, \boldsymbol{v}_2>}$ as n goes to ∞. $\square$

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