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An extremal problem on potentially $K_{p,1,1}$-graphic sequences[†]

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A sequence S is potentially $K_{p,1,1}$ graphical if it has a realization containing a $K_{p,1,1}$ as a subgraph, where $K_{p,1,1}$ is a complete 3-partite graph with partition sizes $p, 1, 1$. Let $\sigma(K_{p,1,1}, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_{p,1,1}, n)$ is potentially $K_{p,1,1}$ graphical. In this paper, we prove that $\sigma(K_{p,1,1}, n) \geq 2[((p+1)(n-1)+2)/2]$ for $n \geq p+2$. We conjecture that equality holds for $n \geq 2p+4$. We prove that this conjecture is true for $p = 3$.

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1 Introduction

If $S = (d_1, d_2, \ldots, d_n)$ is a sequence of non-negative integers, then it is called graphical if there is a simple graph G of order n, whose degree sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is precisely S. If G is such a graph then G is said to realize S or be a realization of S. A graphical sequence S is potentially H graphical if there is a realization of S containing H as a subgraph, while S is forcibly H graphical if every realization of S contains H as a subgraph. Let $\sigma(S) = d(v_1) + d(v_2) + \ldots + d(v_n)$, and $[x]$ denote the largest integer less than or equal to x. We denote $G+H$ as the graph with $V(G+H) = V(G) \bigcup V(H)$ and $E(G+H) = E(G) \bigcup E(H) \bigcup \{xy : x \in V(G), y \in V(H)\}$. Let K_k, and C_k denote a complete graph on k vertices, and a cycle on k vertices, respectively. Let $K_{p,1,1}$ denote a complete 3-partite graph with partition sizes $p, 1, 1$.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdős [3] in 1938 and in general by Turán [12]. In terms of graphic sequences, the number $2ex(n, H)+2$ is the minimum even integer l such that every n-term graphical sequence S with

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$\sigma(S) \geq l$ is forcibly H graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence S with $\sigma(S) \geq l$ is potentially H graphical. We denote this minimum l by $\sigma(H, n)$. Erdős, Jacobson and Lehel [4] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if S does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture is confirmed in [5], [7], [8], [9] and [10].

Gould, Jacobson and Lehel [5] also proved that $\sigma(pK_2, n) = (p-1)(2n-2)+2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. Luo [11] characterized the potentially C_k graphic sequence for $k = 3, 4, 5$. Yin and Li [13] gave sufficient conditions for a graphic sequence being potentially $K_{r,s}$-graphic, and determined $\sigma(K_{r,r}, n)$ for $r = 3, 4$. Lai [6] proved that $\sigma(K_4 - e, n) = 2[\frac{3n-1}{2}]$ for $n \geq 7$. In this paper, we prove that $\sigma(K_{p,1,1}, n) \geq 2[((p+1)(n-1)+2)/2]$ for $n \geq p+2$. We conjecture that equality holds for $n \geq 2p+4$. We prove that this conjecture is true for $p = 3$.

2 Main results.

Theorem 1 $\sigma(K_{p,1,1}, n) \geq 2[((p+1)(n-1)+2)/2]$, *for* $n \geq p+2$.

Proof: If $p = 1$, by Erdős, Jacobson and Lehel [4], $\sigma(K_{1,1,1}, n) \geq 2n$, Theorem 1 is true.

If $p = 2$, by Gould, Jacobson and Lehel [5], $\sigma(K_{2,1,1}, n) = \sigma(K_4 - e, n) \geq \sigma(C_4, n) = 2[(3n-1)/2]$, Theorem 1 is true. Then we can suppose that $p \geq 3$.

We first consider odd p. If n is odd, let $n = 2m+1$, by Theorem 9.7 of [2], K_{2m} is the union of one 1-factor M and $m-1$ spanning cycles $C_1^1, C_2^1, \ldots, C_{m-1}^1$. Let

$$H = C_1^1 \bigcup C_2^1 \bigcup \ldots \bigcup C_{\frac{p-1}{2}}^1 + K_1 \tag{1}$$

Then H is a realization of $((n-1)^1, p^{n-1})$, where the symbol x^y stands for y consecutive terms x. Since $K_{p,1,1}$ contains two vertices of degree $p+1$ while $((n-1)^1, p^{n-1})$ only contains one integer $n-1$ greater than degree p, $((n-1)^1, p^{n-1})$ is not potentially $K_{p,1,1}$ graphic. Thus

$$\sigma(K_{p,1,1}, n) \geq (n-1) + p(n-1) + 2 = 2[((p+1)(n-1)+2)/2]. \tag{2}$$

Next, if n is even, let $n = 2m+2$, by Theorem 9.6 of [2], K_{2m+1} is the union of m spanning cycles $C_1^1, C_2^1, \ldots, C_m^1$. Let

$$H = C_1^1 \bigcup C_2^1 \bigcup \ldots \bigcup C_{\frac{p-1}{2}}^1 + K_1 \tag{3}$$

Then H is a realization of $((n-1)^1, p^{n-1})$, and we are done as before. This completes the discussion for odd p.

Now we consider even p. If n is odd, let $n = 2m+1$, by Theorem 9.7 of [2], K_{2m} is the union of one 1-factor M and $m-1$ spanning cycles $C_1^1, C_2^1, \ldots, C_{m-1}^1$. Let

$$H = M \bigcup C_1^1 \bigcup C_2^1 \bigcup \ldots \bigcup C_{\frac{p-2}{2}}^1 + K_1 \tag{4}$$

Then H is a realization of $((n-1)^1, p^{n-1})$, and we are done as before.

Next, if n is even, let $n = 2m + 2$, by Theorem 9.6 of [2], K_{2m+1} is the union of m spanning cycles $C_1^1, C_2^1, \ldots, C_m^1$. Let

$$\begin{aligned} C_1^1 &= x_1x_2\ldots x_{2m+1}x_1 \\ H &= (C_1^1 \bigcup C_2^1 \bigcup \cdots \bigcup C_{\frac{p}{2}}^1 + K_1) - \{x_1x_2, x_3x_4, \ldots, x_{2m-1}x_{2m}, x_{2m+1}x_1\} \end{aligned}$$

Then H is a realization of $((n-1)^1, p^{n-2}, (p-1)^1)$. It is easy to see that $((n-1)^1, p^{n-2}, (p-1)^1)$ is not potentially $K_{p,1,1}$ graphic. Thus

$$\begin{aligned} \sigma(K_{p,1,1}, n) &\geq (n-1) + p(n-2) + p - 1 + 2 \\ &= 2[((p+1)(n-1)+2)/2]. \end{aligned}$$

This completes the discussion for even p, and so finishes the proof of Theorem 1 □

Theorem 2 *For $n = 5$ and $n \geq 7$,*

$$\sigma(K_{3,1,1}, n) = 4n - 2.$$

For $n = 6$, if S is a 6-term graphical sequence with $\sigma(S) \geq 22$, then either there is a realization of S containing $K_{3,1,1}$ or $S = (4^6)$. (Thus $\sigma(K_{3,1,1}, 6) = 26$.)

Proof: By Theorem 1, for $n \geq 5$, $\sigma(K_{3,1,1}, n) \geq 2[((3+1)(n-1)+2)/2] = 4n - 2$. We need to show that if S is an n-term graphical sequence with $\sigma(S) \geq 4n - 2$, then there is a realization of S containing a $K_{3,1,1}$ (unless $S = (4^6)$). Let $d_1 \geq d_2 \geq \cdots \geq d_n$, and let G be a realization of S.

Case $n = 5$: If a graph has size $q \geq 9$, then clearly it contains a $K_{3,1,1}$, so that $\sigma(K_{3,1,1}, 5) \leq 4n - 2$.

Case $n = 6$: If $\sigma(S) = 22$, we first consider $d_6 \leq 2$. Let S' be the degree sequence of $G - v_6$, so $\sigma(S') \geq 22 - 2 \times 2 = 18$. Then S' has a realization containing a $K_{3,1,1}$. Therefore S has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 3$. It is easy to see that S is one of $(5^2, 3^4)$, $(5^1, 4^2, 3^3)$ or $(4^4, 3^2)$. Obviously, all of them are potentially $K_{3,1,1}$-graphic. Next, if $\sigma(S) = 24$, we first consider $d_6 \leq 3$. Let S' be the degree sequence of $G - v_6$, so $\sigma(S') \geq 24 - 3 \times 2 = 18$. Then S' has a realization containing a $K_{3,1,1}$. Therefore S has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 4$. It is easy to see that $S = (4^6)$. Obviously, (4^6) is graphical and (4^6) is not potentially $K_{3,1,1}$ graphic. Finally, suppose that $\sigma(S) \geq 26$. We first consider $d_6 \leq 4$. Let S' be the degree sequence of $G - v_6$, so $\sigma(S') \geq 26 - 2 \times 4 = 18$. Then S' has a realization containing a $K_{3,1,1}$. Therefore S has a realization containing a $K_{3,1,1}$. Now we consider $d_6 \geq 5$. It is easy to see that $S = (5^6)$. Obviously, (5^6) is potentially $K_{3,1,1}$-graphic.

Case $n = 7$: First we assume that $\sigma(S) = 26$. Suppose $d_7 \leq 2$ and let S' be the degree sequence of $G - v_7$, so $\sigma(S') \geq 26 - 2 \times 2 = 22$. Then S' has a realization containing a $K_{3,1,1}$ or $S' = (4^6)$. Therefore S has a realization containing a $K_{3,1,1}$ or $S = (5^1, 4^5, 1^1)$. Obviously, $(5^1, 4^5, 1^1)$ is potentially $K_{3,1,1}$ -graphic. In either event, S has a realization containing a $K_{3,1,1}$. Now we assume that $d_7 \geq 3$. It is easy to see that S is one of $(6^1, 5^1, 3^5)$, $(6^1, 4^2, 3^4)$, $(5^2, 4^1, 3^4)$, $(5^1, 4^3, 3^3)$ or $(4^5, 3^2)$. Obviously, all of them are potentially $K_{3,1,1}$-graphic. Next, if $\sigma(S) = 28$, Suppose $d_7 \leq 3$. Let S' be the degree sequence of $G - v_7$, so $\sigma(S') \geq 28 - 3 \times 2 = 22$. Then

S' has a realization containing a $K_{3,1,1}$ or $S' = (4^6)$. Therefore S has a realization containing a $K_{3,1,1}$ or $S = (5^2, 4^4, 2^1)$. Obviously, $(5^2, 4^2, 2^1)$ is potentially $K_{3,1,1}$-graphic. In either event, S has a realization containing a $K_{3,1,1}$. Now we assume that $d_7 \geq 4$, then $S = (4^7)$. Clearly, (4^7) has a realization containing a $K_{3,1,1}$. Finally, suppose that $\sigma(S) \geq 30$. If $d_7 \leq 4$. Let S' be the degree sequence of $G - v_7$, so $\sigma(S') \geq 30 - 2 \times 4 = 22$. Then S' has a realization containing a $K_{3,1,1}$ or $S' = (4^6)$. Therefore S has a realization containing a $K_{3,1,1}$ or $S = (5^3, 4^3, 3^1)$. Clearly, $(5^3, 4^3, 3^1)$ has a realization containing a $K_{3,1,1}$. In either event, S has a realization containing a $K_{3,1,1}$. Now we consider $d_7 \geq 5$. It is easy to see that $\sigma(S) \geq 5 \times 7 = 35$. Obviously $\sigma(S) \geq 36$. Clearly, S has a realization containing a $K_{3,1,1}$.

We proceed by induction on n. Take $n \geq 8$ and make the inductive assumption that for $7 \leq t < n$, whenever S_1 is a t-term graphical sequence such that

$$\sigma(S_1) \geq 4t - 2 \tag{5}$$

then S_1 has a realization containing a $K_{3,1,1}$. Let S be an n-term graphical sequence with $\sigma(S) \geq 4n - 2$. If $d_n \leq 2$, let S' be the degree sequence of $G - v_n$. Then $\sigma(S') \geq 4n - 2 - 2 \times 2 = 4(n-1) - 2$. By induction, S' has a realization containing a $K_{3,1,1}$. Therefore S has a realization containing a $K_{3,1,1}$. Hence, we may assume that $d_n \geq 3$. By Proposition 2 and Theorem 4 of [5] (or Theorem 3.3 of [7]) S has a realization containing a K_4. By Lemma 1 of [5], there is a realization G of S with v_1, v_2, v_3, v_4, the four vertices of highest degree containing a K_4. If $d(v_2) = 3$, then $4n - 2 \leq \sigma(S) \leq n - 1 + 3(n-1) = 4n - 4$. This is a contradiction. Hence, we may assume that $d(v_2) \geq 4$. Let v_1 be adjacent to v_2, v_3, v_4, y_1. If y_1 is adjacent to one of v_2, v_3, v_4, then G contains a $K_{3,1,1}$. Hence, we may assume that y_1 is not adjacent to v_2, v_3, v_4. Let v_2 be adjacent to v_1, v_3, v_4, y_2. If y_2 is adjacent to one of v_1, v_3, v_4, then G contains a $K_{3,1,1}$. Hence, we may assume that y_2 is not adjacent to v_1, v_3, v_4. Since $d(y_1) \geq d_n \geq 3$, there is a new vertex y_3, such that $y_1 y_3 \in E(G)$.

Case 1: Suppose $y_3 v_3 \in E(G)$. If $y_3 v_4 \in E(G)$, then G contains a $K_{3,1,1}$. Hence, we may assume that $y_3 v_4 \notin E(G)$. Then the edge interchange that removes the edges $y_1 y_3, v_3 v_4$ and $v_2 y_2$ and inserts the edges $y_1 v_2, y_3 v_4$ and $y_2 v_3$ produces a realization G' of S containing a $K_{3,1,1}$.

Case 2: Suppose $y_3 v_3 \notin E(G)$. Then the edge interchange that removes the edges $y_1 y_3, v_3 v_4$ and $v_2 y_2$ and inserts the edges $y_1 v_2, y_3 v_3$ and $y_2 v_4$ produces a realization G' of S containing a $K_{3,1,1}$.

This finishes the inductive step, and thus Theorem 2 is established. □

We make the following conjecture:

Conjecture 1 $\sigma(K_{p,1,1}, n) = 2[((p+1)(n-1)+2)/2]$, *for* $n \geq 2p+4$.

This conjecture is true for $p = 1$, by Theorem 3.5 of [4], for $p = 2$, by Theorem 1 of [6], and for $p = 3$, by the above Theorem 2.

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