What is Moonshine?

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This is an informal write up of my talk at the ICM in Berlin. It gives some background to Goddard's talk [Go] about the moonshine conjectures. For other survey talks about similar topics see [B94], [B98], [LZ], [J], [Ge], [Y].

The classification of finite simple groups shows that every finite simple group either fits into one of about 20 infinite families, or is one of 26 exceptions, called sporadic simple groups. The monster simple group is the largest of the sporadic finite simple groups, and was discovered by Fischer and Griess [G]. Its order is

$$\begin{aligned} & 8080,17424,79451,28758,86459,90496,17107,57005,75436,80000,00000 \\ = \; & 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 11^{2}\cdot 13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71 \end{aligned}$$

(which is roughly the number of elementary particles in the earth). The smallest irreducible representations have dimensions $1, 196883, 21296876, \ldots$. The elliptic modular function $j(\tau)$ has the power series expansion

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots,$$

where $q = e^{2\pi i\tau}$, and is in some sense the simplest nonconstant function satisfying the functional equations $j(\tau) = j(\tau+1) = j(-1/\tau)$. John McKay noticed some rather weird relations between coefficients of the elliptic modular function and the representations of the monster as follows:

$$\begin{aligned} 1 &= 1, \\ 196884 &= 196883 + 1, \\ 21493760 &= 21296876 + 196883 + 1, \end{aligned}$$

where the numbers on the left are coefficients of $j(\tau)$ and the numbers on the right are dimensions of irreducible representations of the monster. At the time he discovered these relations, several people thought it so unlikely that there could be a relation between the monster and the elliptic modular function that they politely told McKay that he was talking nonsense. The term "monstrous moonshine" (coined by Conway) refers to various extensions of McKay's observation, and in particular to relations between sporadic simple groups and modular functions.

For the benefit of readers who are not native English speakers, I had better point out that "moonshine" is not a poetic terms referring to light from the moon. It means foolish or crazy ideas. (Quatsch in German.) A typical example of its

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use is the following quotation from E. Rutherford (the discoverer of the nucleus of the atom): "The energy produced by the breaking down of the atom is a very poor kind of thing. Anyone who expects a source of power from the transformations of these atoms is talking moonshine." (Moonshine is also a name for corn whiskey, especially if it has been smuggled or distilled illegally.)

We recall the definition of the elliptic modular function $j(\tau)$. The group $SL_2(\mathbf{Z})$ acts on the upper half plane H by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau+b}{c\tau+d}.$$

A modular function (of level 1) is a function f on H such that $f((a\tau+b)/(c\tau+d)) = f(\tau)$ for all $\left(\begin{smallmatrix} a b \\ c d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$. It is sufficient to assume that f is invariant under the generators $\tau \mapsto \tau+1$ and $\tau \mapsto -1/\tau$ of $SL_2(\mathbf{Z})$. The elliptic modular function j is the simplest nonconstant example, in the sense that any other modular function can be written as a function of j. It can be defined as follows:

$$\begin{aligned} j(\tau) &= \frac{E_4(\tau)^3}{\Delta(\tau)} \\ &= q^{-1} + 744 + 196884q + 21493760q^2 + \cdots, \end{aligned}$$

$$\begin{aligned} E_4(\tau) &= 1 + 240\sum_{n>0}\sigma_3(n)q^n \\ &= 1 + 240q + 2160q^2 + \cdots \\ &\quad (\sigma_3(n) = \sum_{d|n} d^3), \end{aligned}$$

$$\begin{aligned} \Delta(\tau) &= q\prod_{n>0}(1-q^n)^{24} \\ &= q - 24q + 252q^2 + \cdots. \end{aligned}$$

A modular form of weight k is a holomorphic function

$$f(\tau) = \sum_{n\geq 0} c(n)q^n$$

on the upper half plane satisfying the functional equation $f((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f(\tau)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$. The function $E_4(\tau)$ is an Eisenstein series and is a modular form of weight 4, while $\Delta(\tau)$ is a modular form of weight 12.

The function $j(\tau)$ is an isomorphism from the quotient $SL_2(\mathbf{Z})\backslash H$ to $\mathbf{C}$, and is uniquely defined by this up to multiplication by a constant or addition of a constant. In particular any other modular function is a function of j, so j is in some sense the simplest nonconstant modular function.

An amusing property of j (which so far seems to have no relation with moonshine) is that $j(\tau)$ is an algebraic integer whenever τ is an imaginary quadratic irrational number. A well known consequence of this is that

$$\exp(\pi\sqrt{163}) = 262537412640768743.99999999999925\ldots$$

is very nearly an integer. The explanation of this is that $j((1+i\sqrt{163})/2)$ is exactly the integer

$$-262537412640768000 = -2^{18}3^3 5^3 23^3 29^3\,,$$

and

$$\begin{aligned} j((1+i\sqrt{163})/2) &= q^{-1} + 744 + 196884q + \cdots \\ &= -e^{\pi\sqrt{163}} + 744 + (\text{something very small}). \end{aligned}$$

McKay and Thompson suggested that there should be a graded representation $V = \oplus_{n\in\mathbf{Z}} V_n$ of the monster, such that $\dim(V_n) = c(n-1)$, where $j(\tau) - 744 = \sum_n c(n)q^n = q^{-1} + 196884q + \cdots$. Obviously this is a vacuous statement if interpreted literally, as we could for example just take each V_n to be a trivial representation. To characterize V, Thompson suggested looking at the McKay-Thompson series

$$T_g(\tau) = \sum_n Tr(g|V_n)q^{n-1}$$

for each element g of the monster. For example, $T_1(\tau)$ should be the elliptic modular function. Conway and Norton [C-N] calculated the first few terms of each McKay-Thomson series by making a reasonable guess for the decomposition of the first few V_n's into irreducible representations of the monster. They discovered the astonishing fact that all the McKay-Thomson series appeared to be Hauptmoduls for certain genus 0 subgroups of $SL_2(\mathbf{R})$. (A Hauptmodul for a subgroup Γ is an isomorphism from $\Gamma\backslash H$ to $\mathbf{C}$, normalized so that its Fourier series expansion starts off $q^{-1} + O(1)$.)

As an example of some Hauptmoduls of elements of the monster, we will look at the elements of order 2. There are 2 conjugacy classes of elements of order 2, usually called the elements of types $2A$ and $2B$. The corresponding McKay-Thompson series start off

$$\begin{aligned} T_{2B}(\tau) &= q^{-1} + 276q - 2048q^2 + \cdots \quad \text{Hauptmodul for } \Gamma_0(2) \\ T_{2A}(\tau) &= q^{-1} + 4372q + 96256q^2 + \cdots \quad \text{Hauptmodul for } \Gamma_0(2)+ \end{aligned}$$

The group $\Gamma_0(2)$ is $\{\binom{ab}{cd} \in SL_2(\mathbf{Z}) | c \text{ is even}\}$, and the group $\Gamma_0(2)+$ is the normalizer of $\Gamma_0(2)$ in $SL_2(\mathbf{R})$. Ogg had earlier commented on the fact that the full normalizer $\Gamma_0(p)+$ of $\Gamma_0(p)$ for p prime is a genus 0 group if and only if p is one of the primes 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, or 71 dividing the order of the monster.

Conway and Norton's conjectures were soon proved by A. O. L. Atkin, P. Fong, and S. D. Smith. The point is that to prove something is a virtual character of a finite group it is only necessary to prove a finite number of congruences. In the case of the moonshine module V, proving the existence of an infinite dimensional representation of the monster whose McKay-Thompson series are give

Hauptmoduls requires checking a finite number of congruences and positivity conditions for modular functions, which can be done by computer.

This does not give an explicit construction of V, or an explanation about why the conjectures are true. Frenkel, Lepowsky, and Meurman managed to find an explicit construction of a monster representation $V = \oplus V_n$, such that $\dim(V_n) = c(n-1)$, and this module had the advantage that it came with some extra algebraic structure preserved by the monster. However it was not obvious that V satisfied the Conway-Norton conjectures. So the main problem in moonshine was to show that the monster modules constructed by Frenkel, Lepowsky and Meurman on the one hand, and by Atkin, Fong, and Smith on the other hand, were in fact the same representation of the monster.

Peter Goddard [Go] has given a description of the proof of this in his talk in this volume, so I will only give a quick sketch of this. The main steps of the proof are as follows:

- 1. The module V constructed by Frenkel, Lepowsky, and Meurman has an algebraic structure making it into a "vertex algebra". A detailed proof of this is given in [F-L-M].

- 2. Use the vertex algebra structure on V and the Goddard-Thorn no-ghost theorem [G-T] from string theory to construct a Lie algebra acted on by the monster, called the monster Lie algebra.

- 3. The monster Lie algebra is a "generalized Kac-Moody algebra" ([K90]); use the (twisted) Weyl-Kac denominator formula to show that $T_g(\tau)$ is a "completely replicable function".

- 4. Y. Martin [M], C. Cummins, and T. Gannon [C-G] proved several theorems showing that completely replicable functions were modular functions of Hauptmoduls for genus 0 groups. By using these theorems it follows that T_g is a Hauptmodul for a genus 0 subgroup of $SL_2(\mathbf{Z})$, and hence V satisfies the moonshine conjectures. (The original proof used an earlier result by Koike [Ko] showing that the appropriate Hauptmoduls were completely replicable, together with a boring case by case check and the fact that a completely replicable function is characterized by its first few coefficients.)

We will now give a brief description of some of the terms above, starting with vertex algebras. The best reference for finding out more about vertex algebras is Kac's book [K]. In this paragraph we give a rather vague description. Suppose that V is a commutative ring acted on by a group G. We can form expressions like

$$u(x)v(y)w(z)\,,$$

where $u, v, w \in V$ and $x, y, z \in G$, and the action of $x \in G$ on $u \in V$ is denoted rather confusingly by $u(x)$. (This is not a misprint for $x(u)$; the reason for this strange notation is to make the formulas compatible with those in quantum field theory, where u would be a quantum field and x a point of space-time.) For each fixed $u, v, \ldots \in V$, we can think of $u(x)v(y)\cdots$ as a function from G^n to V. We can rewrite the axioms for a commutative ring acted on by G in terms of these functions. We can now think of a vertex algebra roughly as follows: we are given lots of functions from G^n to V satisfying the axioms mentioned above, with the difference that these functions are allowed to have certain sorts of singularities. In other words a vertex algebra is a sort of commutative ring acted on by a group G, except that the multiplication is not defined everywhere but has singularities. In particular we cannot recover an underlying ring by defining the product of u and v to be $u(0)v(0)$, because the function $u(x)v(y)$ might happen to have a singularity at $u = v = 0$.

It is easy to write down examples of vertex algebras: any commutative ring acted on by a group G is an example. (Actually this is not quite correct: for technical reasons we should use a formal group G instead of a group G.) Conversely any vertex algebra "without singularities" can be constructed in this way. Unfortunately there are no easy examples of vertex algebras that are not really commutative rings. One reason for this is that nontrivial vertex algebras must be infinite dimensional; the point is that if a vertex algebra has a nontrivial singularity, then by differentiating it we can make the singularity worse and worse, so we must have an infinite dimensional space of singularities. This is only possible if the vertex algebra is infinite dimensional. However there are plenty of important infinite dimensional examples; see for example Kac's book for a construction of the most important examples, and [FLM] for a construction of the monster vertex algebra.

Next we give a brief description of generalized Kac-Moody algebras. The best way to think of these is as infinite dimensional Lie algebras which have most of the good properties of finite dimensional reductive Lie algebras. Consider a typical finite dimensional reductive Lie algebra G, (for example the Lie algebra $G = M_n(\mathbf{R})$ of $n \times n$ real matrices). This has the following properties:

- 1. G has an invariant symmetric bilinear form $(,)$ (for example $(a, b) = -Tr(a, b)$).
- 2. G has a (Cartan) involution ω (for example, $\omega(a) = -a^t$).
- 3. G is graded as $G = \oplus_{n \in \mathbf{Z}} G_n$ with G_n finite dimensional and with ω acting as -1 on the "Cartan subalgebra" G_0. (For example, we could put the basis element $e_{i,j}$ of $M_n(\mathbf{R})$ in G_{i-j}.)
- 4. $(a, \omega(a)) > 0$ if $g \in G_n$, $g \neq 0$.

Conversely any Lie algebra satisfying the conditions above is essentially a sum of finite dimensional and affine Lie algebras. Generalized Kac-Moody algebras are defined by the same conditions with one small change: we replace condition 4 by

- 4'. $(a, \omega(a)) > 0$ if $g \in G_n$, $g \neq 0$ and $n \neq 0$.

This has the effect of allowing an enormous number of new examples, such as all Kac-Moody algebras and the Heisenberg Lie algebra (which behaves like a sort of degenerate affine Lie algebra). Generalized Kac-Moody algebras have many of the properties of finite dimensional semisimple Lie algebras, and in particular they have an analogue of the Weyl character formula for some of their representations, and an analogue of the Weyl denominator formula. An example of the Weyl-Kac denominator formula for the algebra $G = SL_2[z, z^{-1}]$ is

$$\prod_{n>0}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1}) = \sum_{n\in\mathbf{Z}}(-1)^n q^{n^2} z^n.$$

This is the Jacobi triple product identity, and is also the Macdonald identity for the affine Lie root system corresponding to A_1.

Dyson described Macdonald's discovery of the Macdonald identities in [D]. Dyson found identities for $\eta(\tau)^m = q^{m/24}\prod_{n>0}(1-q^n)^m$ for the following values of m:

$$3, 8, 10, 14, 15, 21, 24, 26, 28, \ldots$$

and wondered where this strange sequence of numbers came from. (The case $m = 3$ is just the Jacobi triple product identity with $z = 1$.) Macdonald found his identities corresponding to affine root systems, which gave an explanation for the sequence above: with one exception, the numbers are the dimensions of simple finite dimensional complex Lie algebras. The exception is the number 26 (found by Atkin), which as far as I know has not been explained in terms of Lie algebras. It seems possible that it is somehow related to the fake monster Lie algebra and the special dimension 26 in string theory.

Next we give a quick explanation of "completely replicable" functions. A function is called completely replicable if its coefficients satisfy certain relations. As an example of a completely replicable function, we will look at the elliptic modular function $j(\tau) - 744 = \sum c(n)q^n$. This satisfies the identity

$$j(\sigma) - j(\tau) = p^{-1}\prod_{\substack{m>0\\ n\in\mathbf{Z}}}(1-p^m q^n)^{c(mn)},$$

where $p = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$. (This formula was proved independently in the 80's by Koike, Norton, and Zagier, none of whom seem to have published their proofs.) Comparing coefficients of $p^m q^n$ on both sides gives many relations between the coefficients of j whenever we have a solution of $m_1 n_1 = m_2 n_2$ in positive integers, which are more or less the relations needed to show that j is completely replicable. For example, from the relation $2 \times 2 = 1 \times 4$ we get the relation

$$c(4) = c(3) + \frac{c(1)^2 - c(1)}{2},$$

or equivalently

$$20245856256 = 864299970 + \frac{196884^2 - 196884}{2}.$$

In the rest of this paper we will discuss various extensions of the original moonshine conjectures, some of which are still unproved. The first are Norton's "generalized moonshine" conjectures [N]. If we look at the Hauptmodul $T_{2A}(\tau) = q^{-1} + 4372q + \ldots$ we notice that one of the coefficients is almost the same as the dimension 4371 of the smallest non-trivial irreducible representation of the baby monster simple group, and the centralizer of an element of type 2A in the monster is a double cover of the baby monster. Similar things happen for other elements of the monster, suggesting that for each element g of the monster there should be some sort of graded moonshine module $V_g = \oplus_n V_{g,n}$ acted on by a central extension of the centralizer $Z_M(g)$. In particular we would get series $T_{g,h}(\tau) = \sum_n Tr(h|V_{g,n})q^n$ satisfying certain conditions. Some progress has been made on this by Dong, Li, and Mason [D-L-M], who proved the generalized moonshine conjectures in the case when g and h generate a cyclic group by reducing to the case when $g = 1$ (the ordinary moonshine conjectures). G. Höhn [H] has made some progress in the harder case when g and h do not generate a cyclic group by constructing the required modules for the baby monster (when g is of type $2A$). It seems likely that his methods would also work for the Fischer group Fi_{24}, but it is not clear how to go further than this. There might be some relation to elliptic cohomology (see [Hi]for more discussion of this), as this also involves pairs of commuting elements in a finite group and modular forms.

The space V_g mentioned above does not always have an invariant vertex algebra structure on it. Ryba discovered that a vertex algebra structure sometimes magically reappears when we reduce V_g modulo the prime p equal to the order of g. In fact V_g/pV_g can often be described as the Tate cohomology group $\hat{H}^0(g, V)$ for a suitable integral form V of the monster vertex algebra. This gives natural examples of vertex algebras over finite fields which do not lift naturally to characteristic 0. (Note that most books and papers on vertex algebras make the assumption that we work over a field of characteristic 0; this assumption is often unnecessary and excludes many interesting examples such as the one above.)

We will finish by describing some more of McKay's observations about the monster, which so far are completely unexplained. The monster has 9 conjugacy classes of elements that can be written as the product of two involutions of type $2A$, and their orders are 1, 2, 3, 4, 5, 6, 2, 3, 4. McKay pointed out that these are exactly the numbers appearing on an affine E_8 Dynkin diagram giving the linear relation between the simple roots. They are also the degrees of the irreducible representations of the binary icosahedral group. A similar thing happens for the baby monster: this time there are 5 classes of elements that are the product of two involutions of type $2A$ and their orders are 2, 4, 3, 2, 1. (This is connected with the fact that the baby monster is a "3,4-transposition group".) These are the numbers on an affine F_4 Dynkin diagram, and if we take the "double cover" of an F_4 Dynkin diagram we get an E_7 Dynkin diagram. The number on an E_7 Dynkin diagram are 1, 1, 2, 2, 3, 3, 4, 2 which are the dimensions of the irreducible representations of the binary octahedral group. The double cover of the baby monster is the centralizer of an element of order 2 in the monster. Finally a similar thing happens for $Fi_{24}.2$: this time there are 3 classes of elements that are the product of two involutions of type $2A$ and their orders are 2, 3, 1. (This is connected with the fact that $F_{24}.2$ is a "3-transposition group".) These are the numbers on an affine G_2 Dynkin diagram, and if we take the "triple cover" of an G_2 Dynkin diagram we get an E_6 Dynkin diagram. The number on an E_6 Dynkin diagram are 1, 1, 1, 2, 2, 2, 3, which are the dimensions of the irreducible representations of the binary tetrahedral group. The triple cover of $Fi_{24}.2$ is the centralizer of an element of order 3 in the monster.

The connection between Dynkin diagrams and 3-dimensional rotation groups is well understood (and is called the McKay correspondence), but there is no known explanation for the connection with the monster.

REFERENCES

[B94] R. E. Borcherds, *Simple groups and string theory*, First European congress of mathematics, Paris July 1992, Ed. A. Joseph and others,vol. 1 p. 411-421, Birkhauser 1994.

[B98] R. E. Borcherds, *Automorphic forms and Lie algebras*. Current developments in mathematics 1996, to be published by international press, 1998. Also available from www.dpmms.cam.ac.uk/~reb.

[C-G] C. J. Cummins, T. Gannon, *Modular equations and the genus zero property of moonshine functions*. Invent. Math. 129 (1997), no. 3, 413–443.

[C-N] J. H. Conway, S. Norton, *Monstrous moonshine*, Bull. London. Math. Soc. 11 (1979) 308-339.

[D] F. J. Dyson, *Missed opportunities*. Bull. Amer. Math. Soc. 78 (1972), 635–652.

[D-L-M] C. Dong, H. Li, G. Mason, *Modular invariance of trace functions in orbifold theory*, preprint q-alg/9703016.

[F-L-M] I. B. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the monster*, Academic press 1988. (Also see the announcement A natural representation of the Fischer-Griess monster with the modular function J as character, Proc. Natl. Acad. Sci. USA 81 (1984), 3256-3260.)

[Ge] R. W. Gebert, *Introduction to vertex algebras, Borcherds algebras and the monster Lie algebra*, Internat. J. Modern Physics A8 (1993) no 31, 5441-5503.

[Go] The Work of R. E. Borcherds, this volume, p. 99.

[G-T] P. Goddard and C. B. Thorn, *Compatibility of the dual Pomeron with unitarity and the absence of ghosts in the dual resonance model*, Phys. Lett., B 40, No. 2 (1972), 235-238.

[G] R. L. Griess, *The friendly giant*. Invent. Math. 69 (1982), no. 1, 1–102.

[Hi] F. Hirzebruch, T. Berger, R. Jung, *Manifolds and modular forms*. Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, 1992. ISBN: 3-528-06414-5

[H] G. Höhn, *Selbstduale Vertexoperatorsuperalgebren und das Babymonster*. [Self-dual vertex-operator superalgebras and the Baby Monster] Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1995. Bonner Mathematische Schriften 286. Universität Bonn, Mathematisches Institut, Bonn, 1996.

[J] E. Jurisich, *Generalized Kac-Moody Lie algebras, free Lie algebras and the structure of the Monster Lie algebra.* J. Pure Appl. Algebra 126 (1998), no. 1-3, 233–266.

[K90] V. G. Kac, *Infinite dimensional Lie algebras*, third edition, Cambridge University Press, 1990. (The first and second editions (Birkhauser, Basel, 1983, and C.U.P., 1985) do not contain the material on generalized Kac-Moody algebras.)

[Ko] M. Koike, *On Replication Formula and Hecke Operators*, Nagoya University preprint.

[LZ] B. Lian, G. Zuckerman, *New perspectives on the BRST-algebraic structure in string theory*, hepth/921107, Communications in Mathematical Physics 154, (1993) 613-64, and *Moonshine cohomology*, q-alg/ 950101 Finite groups and Vertex Operator Algebras, RIMS publication (1995) 87-11.

[M] Y. Martin, *On modular invariance of completely replicable functions. Moonshine, the Monster, and related topics* (South Hadley, MA, 1994), 263–286, Contemp. Math., 193, Amer. Math. Soc., Providence, RI, 1996.

[N] S. P. Norton, Appendix to G. Mason's paper in *The Arcata Conference on Representations of Finite Groups* (Arcata, Calif., 1986), 181–210, Proc. Sympos. Pure Math., 47, Part 1, Amer. Math. Soc., Providence, RI, 1987.

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