Space of Local Fields in Integrable Field Theory and Deformed Abelian Differentials

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Abstract. In this talk I consider the space of local operators in integrable field theory. This space allows two different descriptions. The first of them is due to conformal field theory which provides a universal picture of local properties in quantum field theory. The second arises from counting solutions to form factors equations. Considering the example of the restricted Sine-Gordon model I show that these two very different descriptions give the same result. I explain that the formulae for the form factors are given in terms of deformed hyper-elliptic integrals. The properties of these integrals, in particular the deformed Riemann bilinear relation, are important for describing the space of local operators.

1 Quantum field theory in two dimensions.

Consider a massive relativistic quantum field theory (QFT) in two dimensional Minkowski space M^2. For $x = (x_0, x_1) \in M^2$ we put $x^2 = x_0^2 - x_1^2$. Let us take for simplicity the case when there is only one stable particle of mass m in the spectrum. To this particle we associate the creation-annihilation operators $a^*(\beta), a(\beta)$ where the rapidity β parameterizes the energy-momentum of particle: $p_0(\beta) = m\cosh\beta$, $p_1(\beta) = m\sinh\beta$. The only non-vanishing commutator is

$$[a(\beta_1), a^*(\beta_2)] = \delta(\beta_1 - \beta_2)$$

The space of states of the theory is the Fock space created by the action of an arbitrary finite number of operators $a^*(\beta)$ on the vacuum $|0\rangle$ which is annihilated by $a(\beta)$. We denote this space by $\mathcal{H}_p$. The action of the operators of energy and momentum P_μ in $\mathcal{H}_p$ is defined by $P_\mu|0\rangle = 0$, $[P_\mu, a^*(\beta)] = p_\mu(\beta)a^*(\beta)$.

In local QFT there exist local operators $\mathcal{O}_i(x) = e^{iP_\mu x_\mu}\mathcal{O}_i(0)e^{-iP_\mu x_\mu}$ acting in the space $\mathcal{H}_p$ and satisfying

$$[\mathcal{O}_i(x), \mathcal{O}_j(x')] = 0 \qquad \text{for} \qquad (x - x')^2 < 0$$

Obviously, these local operators create a linear space which will be denoted by $\mathcal{H}_o$. The Lehmann-Symanzik-Zimmerman axiomatic requires the existence of two special local operators. One of them is the symmetric energy-momentum tensor $T_{\mu\nu}$ such that $\partial_\mu T_{\mu\nu} = 0$ and $P_\mu = \int T_{\mu 0}(x)dx_1$. The other one is the interpolating field $\phi(x)$ weakly

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approaching, when $x_0 \to \pm\infty$, the free "out" and "in" fields constructed via the creation-annihilation operators $a^*_{out}(\beta), a_{out}(\beta)$, $a^*_{in}(\beta), a_{in}(\beta)$. It is required that they are unitary equivalent: $a_{out}(\beta) = \mathcal{S} a_{in}(\beta)\mathcal{S}^{-1}$ with the unitary operator $\mathcal{S}$ (S-matrix) which leaves invariant the vacuum and one particle state, and commutes with P_μ. We identify the original operator $a(\beta)$ with $a_{in}(\beta)$.

This axiomatic has an obvious generalization to the case when the particle has internal (isotopic) degrees of freedom, and to the case of fermionic statistic or even generalized statistic, the latter case is also possible in two dimensions.

Let us consider in some details the space of local operators $\mathcal{H}_o$. The general philosophy teaches us that in order to understand the structure of this space one has to consider the ultra-violet (short-distance) limit of the original QFT. At least intuitively this idea is quite natural. The ultra-violet limit of massive QFT is described by a certain conformal field theory (CFT). The spaces of local operators of two theories must coincide, and, since the CFT in two dimensions allows in many cases a complete solution [1], we get a description of "universality classes" of two-dimensional QFT.

In the conformal case the theory essentially splits into two chiral sectors, which means that any operator $\mathcal{O}(x)$ can be rewritten as $\mathcal{O}^-(x_-)\mathcal{O}^+(x_+)$ where $x_\pm = x_0 \pm x_1$ are light-cone coordinates. The space of local operators in CFT is described in terms of two Virasoro algebras with generators $\mathcal{L}^\pm_n$ satisfying

$$[\mathcal{L}^\pm_m, \mathcal{L}^\pm_n] = (m-n)\mathcal{L}^\pm_{m+n} + c\,\delta_{m,-n}\tfrac{n^3-n}{12}$$

where the central charge c is an important characteristic of the theory. These Virasoro algebras act on the space $\mathcal{H}_o$ which happens to be organized as follows. There are primary fields ϕ_m satisfying

$$\mathcal{L}^\pm_n \phi_m = 0,\ n < 0, \qquad \mathcal{L}^\pm_0 \phi_m = \Delta_m \phi_m$$

where Δ_m is the scaling dimension of primary field. Different local operators are obtained by acting with $\mathcal{L}^\pm_{-n}$ on the primary fields. So, the one has

$$\mathcal{H}_o = \bigoplus_m W^-_m \otimes W^+_m$$

where W_m is a Verma module of the Virasoro algebra.

In this talk I shall consider a particular example of CFT with $c < 1$. The coupling constant ξ which we use is related to c as follows

$$c = 1 - \tfrac{6}{\xi(\xi+1)}$$

Considering the coupling constant in generic position we have infinitely many primary fields ϕ_m , $m \geq 0$ with scaling dimensions $\Delta_m = -\frac{m}{2} + \frac{m}{2}(\frac{m}{2}+1)\xi$. We shall concentrate on one chirality considering only one Virasoro algebra with generators $\mathcal{L}_n \equiv \mathcal{L}^-_n$. The Verma module W_m has a singular vector on level $m+1$. The irreducible representation of the Virasoro algebra is obtained by factorizing over the Verma submodule created over this singular vector. The vectors from this submodule are called "null-vectors". It must be emphasized that the process of factorizing over the null-vectors has the dynamical meaning of imposing the equations of motion. The latter statement can be clearly understood in the classical limit $\xi \to 0$ when the chiral CFT gives the classical Korteweg-de-Vris (KdV) hierarchy with second Poisson structure. The space of local operators turns into the space of functions on the phase space of KdV. It is shown in [2] that the null-vectors in that case provide all the equations of motion of the KdV hierarchy.

Let us return to massive QFT. Consider a local operator $\mathcal{O}(x)$ and define

$$f_{\mathcal{O}}(\beta_1, \cdots, \beta_n) = \langle 0| \, \mathcal{O}(0) \, a^*(\beta_1) \cdots a^*(\beta_n)|0\rangle$$

where $\beta_1 < \cdots < \beta_n$. To other ranges of β's the function $f_{\mathcal{O}}$ is continued analytically. The function $f_{\mathcal{O}}$ is called form factor. The matrix elements of a local operator between two arbitrary states of $\mathcal{H}_p$ can be obtained by certain analytical continuation of the form factors due to crossing symmetry. The dependence on x can be taken into account trivially because the matrix element is taken between the eigen-states of the energy-momentum. Thus the form factors define the local operator completely. On the other hand the set of form factors define a pairing between the spaces $\mathcal{H}_o$ and $\mathcal{H}_p$.

2 INTEGRABLE FIELD THEORY.

The problem of finding the form factors of local operators for any massive QFT looks rather hopeless. However, in the special case of integrable field theory (IFT) this problem can be solved. In IFT the scattering is factorizable which means that every scattering process is reduced to two-particle scattering [3]. The two-particle S-matrix $S(\beta_1 - \beta_2)$ depends analytically on the difference of rapidities. As it has been already said the particle can carry internal degrees of freedom lying in finite-dimensional isotopic space. In that case $S(\beta_1 - \beta_2)$ is an operator acting in the tensor product of the isotopic spaces attached to the particles scattered. The S-matrix must satisfy certain requirements, the most important of which being the Yang-Baxter equation [4].

Consider now the form factors. The first examples of exact form factors in IFT are given in [5]. I gave a complete solution of the problem in a series of papers (partly in collaboration with A.N. Kirillov) summarized in the monograph [6]. If the particles have internal degrees of freedom the form factor takes values in the tensor product of isotopic spaces. It is convenient to consider the form factors as row-vectors. Then we act from the right by the operators like $S(\beta_i - \beta_j)$ (which act non-trivially only in the tensor product of i-th and j-th spaces). It has been shown that for the operator $\mathcal{O}$ to be local it is necessary and sufficient that the following requirements are satisfied [6]:

1. ANALYTICITY. The form factor $f_{\mathcal{O}}(\beta_1, \cdots, \beta_n)$ is a meromorphic function of all its arguments in the finite part of the complex plane.

2. SYMMETRY.

$$f_{\mathcal{O}}(\beta_1, \cdots \beta_i, \beta_{i+1} \cdots, \beta_n) S(\beta_i - \beta_{i+1}) = f_{\mathcal{O}}(\beta_1, \cdots \beta_{i+1}, \beta_i \cdots, \beta_n) \tag{1}$$

3. TOTAL EUCLIDEAN ROTATION.

$$f_{\mathcal{O}}(\beta_1, \cdots, \beta_{n-1}, \beta_n + 2\pi i) = f_{\mathcal{O}}(\beta_n, \beta_1, \cdots, \beta_{n-1}) \tag{2}$$

3. ANNIHILATION POLE. In the absence of bound states there are no other singularities in variable β_n in the strip $0 < \mathrm{Im}\beta_n < 2\pi$ but simple poles at the points $\beta_n = \beta_j + \pi i$.

The residue at the one of them ($\beta_n = \beta_{n-1} + \pi i$) is given below, other residues can be obtained from the symmetry property.

$$2\pi i \,\mathrm{res} f_{\mathcal{O}}(\beta_1, \cdots \beta_{n-1}, \beta_n) = \\ = f_{\mathcal{O}}(\beta_1, \cdots \beta_{n-2}) \otimes c_{n-1,n} \left(I - S(\beta_{n-1} - \beta_1) \cdots S(\beta_{n-1} - \beta_{n-2})\right) \qquad (3)$$

where $c_{n-1,n}$ is a certain vector from the tensor product of n-th and $(n-1)$-th isotopic spaces which is canonically related to the S-matrix.

Two comments are in order here. First, clearly the IFT is completely defined by the S-matrix in agreement with the general idea of Heisenberg. Second, the space $\mathcal{H}_o$ is in one-to-one correspondence with the space of solutions of the system of linear equations (1, 2, 3). So, we have to establish the relation of this description to the one given by CFT.

Let me consider my favorite example of IFT which is the restricted Sine-Gordon model (RSG)[7]. I will not give the traditional Lagrangian definition of the model, instead I shall present the S-matrix which, as it has been said, defines the IFT completely. The particles in RSG are two-component (soliton-antisoliton), so, the S-matrix is an operator acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$ as follows

$$S(\beta) = S_0(\beta) \left(e^{-\frac{\beta}{\xi}}\, R(q) - e^{\frac{\beta}{\xi}}\, \widehat{R}(q)^{-1}\right) \qquad (4)$$

where $\beta = \beta_1 - \beta_2$, ξ is a coupling constant, $S_0(\beta)$ is certain c-number multiplier which is not very relevant for our goals. The matrix $R(q)$ is the R-matrix of the quantum group $U_q(sl_2)$ [8] with $q = \exp(\frac{2\pi i}{\xi})$ acting in the tensor product of two-dimensional representations:

$$R(q) = q^{\frac{1}{4}(\sigma^3 \otimes \sigma^3 + 1)} \left(I + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\sigma^+ \otimes \sigma^-\right),$$

where $\sigma^3, \sigma^\pm$ are Pauli matrices. Finally, $\widehat{R}(q) = PR(q)$, where P is the operator of permutation. The S-matrix (4) gives the famous Sine-Gordon (SG) S-matrix found by Zamolodchikov [9].

The RSG model is a sector of SG model. Let us consider the isotopic spaces as spaces of two-dimensional representations of the quantum group. The S-matrix (4) is written in a manifestly $U_q(sl_2)$-invariant form. If one introduces the action of $U_q(sl_2)$ in the space of particles of the SG model, restriction to RSG corresponds to considering $U_q(sl_2)$-invariant subspace. This restriction looks at the first glance as a kinematical one, but it has important dynamical consequences. The space $\mathcal{H}_o$ of RSG corresponds to $U_q(sl_2)$-invariant solutions of the equations (1, 2, 3). From certain physical consideration we know that this space must coincide with the space of operators of CFT with $c = 1 - \frac{6}{\xi(\xi+1)}$ defined above.

One remark should be made. I have said that particles in two dimensional QFT can have generalized statistics which means that their interpolating fields are quasi-local (some phases appear in the commutation relations on the space-like interval). In that case the equations (1, 2, 3) are satisfied for the operators which are not only local, but also mutually local with the interpolating fields, otherwise some minor modification is needed. This is the situation which takes place in RSG model: solitons are particles with generalized statistics. Only the primary fields ϕ_{2m} and their Virasoro descendents are mutually local with the interpolating fields of solitons. For simplicity we shall take for $\mathcal{H}_o$ of RSG the space span by these "truely local" operators.

3 Deformed hyper-elliptic differentials.

The formulae for the form factors of the RSG model are given in terms of deformed hyper-elliptic integrals. Let me explain what these integrals are. Consider a hyper-elliptic Riemann surface of genus $n-1$ defined by the equation $c^2 = p(a)$ with $p(a) = \prod_{j=1}^{2n}(a-b_j)$. Take the abelian differentials regular everywhere except at the two points lying over the point $a = \infty$, and having no simple poles. Up to exact forms there are $2n-2$ such differentials, everyone of them is written in the form

$$\omega = \frac{l(a)}{\sqrt{p(a)}}da \tag{5}$$

with some polynomial $l(a)$. Introduce the intersection form

$$\omega_1 \circ \omega_2 = \sum_{a=\infty} \text{res } (\omega_1 \Omega_2)$$

where $d\Omega = \omega$. The basis of dual differentials can be constructed as follows. Consider the anti-symmetric polynomial of two variables:

$$c(a_1, a_2) = \sqrt{p(a_1)}\frac{\partial}{\partial a_1}\left(\frac{\sqrt{p(a_1)}}{a_1 - a_2}\right) - \sqrt{p(a_2)}\frac{\partial}{\partial a_2}\left(\frac{\sqrt{p(a_2)}}{a_2 - a_1}\right)$$

For any decomposition of this polynomial of the form

$$c(a_1, a_2) = \sum_{i=1}^{n-1}(r_i(a_1)s_i(a_2) - r_i(a_2)s_i(a_1))$$

the differentials η_i and ζ_i defined by using r_i and s_i respectively in equation (5) are dual:

$$\eta_i \circ \eta_j = 0,\ \zeta_i \circ \zeta_j = 0,\ \eta_i \circ \zeta_j = \delta_{ij}$$

Consider the canonical homology basis with a-cycles α_i and b-cycles β_i. The Riemann bilinear relation (as A. Nakayashiki pointed out to me, the hyper-elliptic case was found by Weierstrass) says that the matrix of periods

$$\mathcal{P} = \begin{pmatrix} \int_\alpha \eta\ , & \int_\beta \eta \\ \int_\alpha \zeta\ , & \int_\beta \zeta \end{pmatrix}$$

belongs to the symplectic group $Sp(2n-2)$.

Now I am going to describe a deformation of these abelian differentials which is needed for the description of RSG form factors. Obviously only tensor product of even number of two-dimensional representations can have a $U_q(sl_2)$-invariant subspace. So, we have only form factors with even number of particles with rapidities $\beta_1, \cdots, \beta_{2n}$. Let us introduce the notations $b_j = \exp(\frac{2\beta_j}{\xi})$, $B_j = \exp(\beta_j)$. Consider two polynomials $l(a)$ and $L(A)$ which can depend respectively on b_j and B_j as parameters. We define the following pairing for these polynomials [10]:

$$\langle l, L\rangle = \int_{-\infty}^{\infty} \Phi(\alpha)l(a)L(A)d\alpha \tag{6}$$

where $a = \exp(\frac{2\alpha}{\xi})$, $A = \exp(\alpha)$. The function $\Phi(\alpha)$ satisfies the equations

$$\begin{aligned} &p(aq)\Phi(\alpha + 2\pi i) = q^2 p(a)\Phi(\alpha), && p(a) = \prod(a - b_j) \\ &P(-AQ)\Phi(\alpha + i\xi) = QP(A)\Phi(\alpha), && P(A) = \prod(A + B_j) \end{aligned} \tag{7}$$

where $Q = e^{i\xi}$. We require that $\Phi(\alpha)$ is regular for $0 < \mathrm{Im}\alpha < \pi$, that it behaves as aA when $\alpha \to -\infty$, and that it has the following asymptotics when $\alpha \to +\infty$:

$$\begin{aligned} &\Phi(\alpha) \sim f(a)F(A), \\ &f(a) = a^{-(n-1)}\big(1 + \sum_{k>0} c_k a^{-k}\big), \qquad F(A) = A^{-(2n-1)}\big(1 + \sum_{k>0} C_k A^{-k}\big) \end{aligned}$$

where the coefficients c_k, C_k can be found using (7). These requirements fix the function $\Phi(\alpha)$ completely, the explicit formula is available, but we shall not need it. The following functionals are defined for arbitrary polynomials $l(a)$ and $L(A)$:

$$\mathbf{r}l \equiv \mathrm{res}_{a=\infty}(a^{-1}l(a)f(a)), \qquad \mathbf{R}L \equiv \mathrm{res}_{A=\infty}(A^{-1}L(A)F(A)) \tag{8}$$

What is the relation between the pairing $\langle l, L\rangle$ and the hyper-elliptic integrals? Take the limit when $\xi \to \infty$ keeping b_j finite. In RSG model this is the strong coupling limit. In this limit the integral (6) goes asymptotically to the period of the differential defined by $l(a)$ (5) over a cycle which is fixed by the polynomial $L(A)$. Thus we have a deformation of hyper-elliptic integrals in which the differentials and the cycles enter in much more symmetric way than they do classically. We shall call l and L respectively q-form and q-cycle. The striking feature of this deformation is that it preserves all the important properties of classical hyper-elliptic integrals. Let me explain this point.

After appropriate regularization [6], the pairing $\langle l, L\rangle$ can be defined for every pair of polynomials l and L satisfying $\mathbf{r}l = 0$, $\mathbf{R}L = 0$. However, only a finite number of them give really different results because it can be shown that the value of the integral does not change if we add to l or L polynomials of the form

$$\begin{aligned} &\mathbf{d}[h](a) \equiv a^{-1}\big(p(a)h(a) - p(aq^{-1})h(aq^{-2})\big) \\ &\mathbf{D}[H](A) \equiv A^{-1}\left(P(A)H(A) - P(AQ)H(-A)\right) \end{aligned} \tag{9}$$

where the polynomials h and H are arbitrary. The first polynomial from (9) can be considered as an exact q-form and the second one as a q-boundary. It is easy to see that modulo (9) we have $2n-2$ q-forms and $2n-2$ q-cycles, so, the dimensions of cohomologies and homologies do not change after the deformation.

Consider now two anti-symmetric polynomials:

$$\begin{aligned} c(a_1, a_2) = &\frac{p(a_1)}{a_1(a_1q - a_2)} - \frac{p(a_1q^{-1})}{a_1(a_1q^{-1} - a_2)} - \\ &- \frac{p(a_2)}{a_2(a_2q - a_1)} + \frac{p(a_2q^{-1})}{a_2(a_2q^{-1} - a_1)} \\ C(A_1, A_2) = &\frac{1}{A_1A_2}\Big(\frac{A_1 - A_2}{A_1 + A_2}(P(A_1)P(A_2) - P(-A_1)P(-A_2)) + \\ &+ (P(-A_1)P(A_2) - P(A_1)P(-A_2))\Big) \end{aligned} \tag{10}$$

Suppose that modulo exact q-forms and q-boundaries we have the decompositions:

$$c(a_1,a_2)=\sum_{i=1}^{n-1}\left(r_i(a_1)s_i(a_2)-r_i(a_2)s_i(a_1)\right)$$
$$C(A_1,A_2)=\sum_{i=1}^{n-1}\left(R_i(A_1)S_i(A_2)-R_i(A_2)S_i(A_1)\right)$$

then the following deformed Riemann bilinear relation holds [10].
PROPOSITION. *The matrix*

$$\mathcal{P}=\begin{pmatrix}\langle r,R\rangle, & \langle r,S\rangle\\ \langle s,R\rangle, & \langle s,S\rangle\end{pmatrix}$$

belongs to the symplectic group $Sp(2n-2)$.

4 EXACT FORM FACTORS AND SPACE OF OPERATORS.

The quantum group invariance means that the $2n$-particle form factors in the RSG model belong to $U_q(sl_2)$-invariant subspace of the tensor product $(\mathbb{C}^2)^{\otimes 2n}$. The dimension of this invariant subspace equals $\binom{2n}{n}-\binom{2n}{n-1}$. There is a nice coincidence of dimensions

$$\binom{2n}{n}-\binom{2n}{n-1}=\binom{2n-2}{n-1}-\binom{2n-2}{n-3}$$

where the RHS gives the dimension of $(n-1)$-th fundamental irreducible representation of $Sp(2n-2)$ (explicitly described later). This representation is naturally related to the construction of the previous section.

Consider the space $\mathfrak{h}_k$ of anti-symmetric polynomials of k variables $a_1,\cdots,a_k$. We can define the following operators acting between different $\mathfrak{h}_k$:
1. The operator $\mathbf{r}$ acting from $\mathfrak{h}_k$ to $\mathfrak{h}_{k-1}$ by applying the "residue" (8) to one argument, obviously $\mathbf{r}^2=0$.
2. For every $h\in\mathfrak{h}_1$ (a polynomial of one variable) define the operator $\mathbf{d}[h]$ acting from $\mathfrak{h}_{k-1}$ to $\mathfrak{h}_k$ by

$$(\mathbf{d}[h]l_{k-1})(a_1,\cdots,a_k)=\sum_{i=1}^{k}(-1)^i\mathbf{d}[h](a_i)\,l_{k-1}(a_1,\cdots,\widehat{a}_i,\cdots,a_k)$$

3. The operator $\mathbf{c}$ acts from $\mathfrak{h}_{k-2}$ to $\mathfrak{h}_k$ by

$$(\mathbf{c}l_{k-2})(a_1,\cdots,a_k)=\sum_{i<j}^{k}(-1)^{i+j}c(a_i,a_j)\,l_{k-2}(a_1,\cdots,\widehat{a}_i,\cdots,\widehat{a}_j\cdots,a_k)$$

Denote by $\widehat{\mathfrak{h}}_k$ the following subspace of $\mathfrak{h}_k$:

$$\widehat{\mathfrak{h}}_k=\mathrm{Ker}\big(\mathbf{r}\mid_{\mathfrak{h}_k\to\mathfrak{h}_{k+1}}\big)/\bigoplus_{h\in\mathfrak{h}_1}\mathrm{Im}\big(\mathbf{d}[h]\mid_{\mathfrak{h}_{k-1}\to\mathfrak{h}_k}\big)$$

The space $\widehat{\mathfrak{h}}_k$ is finite-dimensional of dimension $\binom{2n-2}{k}$. The action of the operator $\mathbf{c}$ can be restricted to the spaces $\mathfrak{h}_k$. We denote by $\mathfrak{h}_k^0$ the subspace:

$$\mathfrak{h}_k^0 = \widehat{\mathfrak{h}}_k/\mathrm{Im}\big(\mathbf{c}\,|_{\widehat{\mathfrak{h}}_{k-2}\to\widehat{\mathfrak{h}}_k}\big)$$

which is isomorphic to $Sp(2n-2)$-irreducible submodule of maximal dimension in the space of anti-symmetric tensors of rank k. We are interested in the biggest possible: $\mathfrak{h}_{n-1}^0$.

The construction of form factors starts by describing a certain linear isomorphism:

$$(\mathbb{C}^2)^{\otimes 2n}_{\mathrm{inv}} \simeq \mathfrak{h}_{n-1}^0 \tag{11}$$

of which we shall not need the explicit form. Using this isomorphism we identify every $e \in (\mathbb{C}^2)^{\otimes 2n}_{\mathrm{inv}}$ with a polynomial $l[e]_{n-1} \in \mathfrak{h}_{n-1}^0$.

Consider now the spaces $\mathfrak{H}_k$ of anti-symmetric polynomials $L_k(A_1, \cdots, A_k)$. The action of operators $\mathbf{R}$, $\mathbf{D}[H]$, $\mathbf{C}$ is defined in exactly the same way as the action of $\mathbf{r}$, $\mathbf{d}[h]$, $\mathbf{c}$ using the formulae (8) and (9). For $L_n \in \mathfrak{H}_n$ and for $l_{n-1} \in \mathfrak{h}_{n-1}^0$ define the pairing:

$$\langle l_{n-1}, L_n\rangle = \int_{-\infty}^{\infty} d\alpha_1 \cdots \int_{-\infty}^{\infty} d\alpha_{n-1} \prod_{i=1}^{n-1}\Phi(\alpha_i) \times l_{n-1}(a_1,\cdots,a_{n-1})(\mathbf{R}L_n)(A_1,\cdots,A_n) \tag{12}$$

The requirement $l_{n-1} \in \mathfrak{h}_{n-1}^0$ together with the existence of q-boundaries and of deformed Riemann bilinear relation leads to the following remarkable consequence. For arbitrary $H \in \mathfrak{H}_1$, $L_{n-1} \in \mathfrak{H}_{n-1}$ and $L_{n-2} \in \mathfrak{H}_{n-2}$

$$\mathbf{D}[H]L_{n-1} \simeq 0, \qquad \mathbf{C}L_{n-2} \simeq 0 \tag{13}$$

where $L_n \simeq 0$ means that for such L_n the integrals (12) vanish for any $l_{n-1} \in \mathfrak{h}_{n-1}^0$.

The form factors must satisfy three equations (1), (2), (3). Consider first the equations (1) and (2) only. Obviously, one can multiply any solution of these two equations by a quasi-constant, i.e. $2\pi i$-periodic symmetric function of β_j which is the same as symmetric Laurent polynomial of B_j. We have

Proposition. *To every $L_n \in \mathfrak{H}_n$ corresponds a solution to (1), (2) belonging to $(\mathbb{C}^2)^{\otimes 2n}_{\mathrm{inv}}$:*

$$f^{L_n}(\beta_1,\cdots,\beta_{2n}) = \sum_e \langle l_{n-1}[e], L_n\rangle e$$

where the sum is taken over a basis of $(\mathbb{C}^2)^{\otimes 2n}_{\mathrm{inv}}$. These solutions span a vector space over the ring of quasiconstants, and the only possible linear dependence of solutions arises from relations (13).

Let me appeal to the strong coupling limit for explaining the meaning of this construction. In this limit the equations (1), (2) turn into certain linear differential equations. These linear differential equations are solved in terms of hyper-elliptic integrals (b_j are the branch points) and, naturally, different solutions are counted by different cycles. So, it is not a wonder that after the deformation the solutions are counted by L_n which have the meaning of deformed cycles as explained above.

With every local operator $\mathcal{O}$ we identify an infinite tower of polynomials $L[\mathcal{O}]_n$ such that

$$f_{\mathcal{O}}(\beta_1,\cdots,\beta_n)=f^{L_n[\mathcal{O}]}(\beta_1,\cdots,\beta_{2n})$$

The polynomials $L_n[\mathcal{O}]$ must be related for different n in order to satisfy the remaining equation (3). We have

PROPOSITION. *The form factors $f_{\mathcal{O}}(\beta_1,\cdots,\beta_n)$ satisfy (3) if and only if the anti-symmetric polynomials $L_n[\mathcal{O}](A_1,\cdots,A_n)$, which are at the same time symmetric Laurent polynomials of the parameters $B_1,\cdots,B_{2n}$, satisfy the recurrence relations:*

$$L_n[\mathcal{O}](A_1,\cdots,A_n|B_1,\cdots,B_{2n})|_{B_{2n}=-B_{2n-1}}\equiv\sum_{i=1}^{n-1}(-1)^i\prod_{j\neq i}(A_j^2-B_{2n}^2)$$
$$\times L_{n-1}[\mathcal{O}](A_1,\cdots,\widehat{A_i},\cdots,A_n|B_1,\cdots,B_{2n-2})\qquad\big(\mathrm{mod}\prod_{j=1}^{n}(A_j^2-B_{2n}^2)\big)\quad(14)$$

i.e. the difference between LHS and RHS is divisible by $\prod(A_j^2-B_{2n}^2)$ as polynomial of A_j.

Recall that the equation (3) concerns the residue at the pole $\beta_{2n}=\beta_{2n-1}+\pi i$ which corresponds to $B_{2n}=-B_{2n-1}$ and $b_{2n}=qb_{2n-1}$. In the strong coupling limit the branch points b_{2n} and b_{2n-1} approach each other, so, we arrive at a singularity of the moduli space. Thus the geometrical analogy of our construction is as follows. With every $2n$-particle space we associate the moduli space of hyper-elliptic curves, the lower moduli space is embedded into the upper one as its singularity. Equation (14) gives a rule for embedding of deformed homologies.

The solution to the relation (14) which describes the primary field ϕ_{2m} is

$$L_n[\phi_{2m}](A_1,\cdots,A_n|B_1,\cdots,B_{2n})=\prod_{i<j}(A_i^2-A_j^2)\prod A_i^{2m}\prod B_j^{-m}$$

One can multiply $L_n(\phi_{2m})$ by the polynomials

$$I_{2k-1}(B)=\sum B_j^{2k-1},\qquad J_{2k}(A|B)=\sum A_i^{2k}-\frac{1}{2}\sum B_j^{2k}$$

which does not spoil the relation (14). It corresponds to the action of operators $\mathcal{I}_{2k-1}$ and $\mathcal{J}_{2k}$ in the space $\mathcal{H}_0$, for example, $L_n[\mathcal{J}_{2k}\mathcal{O}](A|B)=J_{2k}(A|B)L_n[\mathcal{O}](A|B)$. I put forward the following

CONJECTURE. *The space of operators span by $\mathcal{I}_{2k_1-1}\cdots\mathcal{I}_{2k_p-1}\mathcal{J}_{2k_1}\cdots\mathcal{J}_{2k_q}\phi_{2m}$ coincides with the Verma module of Virasoro algebra generated over the primary field.*

Let me say a few words about the meaning of this construction. In RSG model there is an infinite number of local integrals of motion I_{2k-1} which can be written in the form $I_{2k-1}=\int h_{2k}(x)dx_1$ with some local densities $h_{2k}(x)$. For any local operator $\mathcal{O}$ we define an operator $\mathcal{I}_{2k_1-1}\mathcal{O}=[I_{2k-1},\mathcal{O}]$ which is also local. The operator $\mathcal{I}_{2k-1}$ acting on $\mathcal{H}_p$ is the same as before because the eigen-value of I_{2k-1} on $2n$-particle state equals $I_{2k-1}(B)$. The operators $\mathcal{J}_{2k}$ describe certain transverse to the integrals of motion coordinates.

The Verma module of Virasoro algebra is span by the vectors $\mathcal{L}_{k_1}\cdots\mathcal{L}_{k_l}\phi_{2m}$. The operator $\mathcal{L}_0$ defines the grading in this space such that the degree of $\mathcal{L}_k$ equals k. It can be shown that the degrees of $\mathcal{I}_{2k-1}$ and $\mathcal{J}_{2k}$ with respect to the same grading equal respectively $2k-1$ and $2k$. So, the characters of the two graded spaces coincide which makes the above conjecture very plausible. There are other arguments in favour of this conjecture which I cannot explain here.

There is a crucial check for the above conjecture. It has been said that the Verma module of the Virasoro algebra is reducible: there is a submodule of null-vectors which corresponds to the equations of motion of the model. The question is whether it is possible to find these null-vectors describing the space $\mathcal{H}_o$ in terms of $\mathcal{I}_{2k-1}$ and $\mathcal{J}_{2k}$? This can be done because certain local operators vanishes due to the relations (13), moreover, the number of these operators is exactly the same as the number of null-vectors in the Verma module [2]. I think that this statement which links together two very different descriptions of the space $\mathcal{H}_o$ is a good point to finish this talk.

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