Rogers-Ramanujan Identities: A Century of Progress from Mathematics to Physics

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Abstract. In this talk we present the discoveries made in the theory of Rogers-Ramanujan identities in the last five years which have been made because of the interchange of ideas between mathematics and physics. We find that not only does every minimal representation $M(p,p')$ of the Virasoro algebra lead to a Rogers-Ramanujan identity but that different coset constructions lead to different identities. These coset constructions are related to the different integrable perturbations of the conformal field theory. We focus here in particular on the Rogers-Ramanujan identities of the $M(p,p')$ models for the perturbations $\phi_{1,3},\ \phi_{2,1},\ \phi_{1,2}$ and $\phi_{1,5}$.

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1 Introduction

In 1894 L.J. Rogers [1] proved the following identities for $a = 0,1$ between infinite series and products valid for $|q| < 1$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1-a})(1-q^{5n-4+a})}$$
$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (q^{n(10n+1+2a)} - q^{(5n+2-a)(2n+1)}) \text{ with } (q)_n = \prod_{j=1}^{n}(1-q^j). \quad (1)$$

For about the first 85 years after their discovery interest in these identities and their generalizations was confined to mathematicians and many ingenious proofs and relations with combinatorics, basic hypergeometric functions and Lie algebras were discovered by MacMahon, Rogers, Schur, Ramanujan, Watson, Bailey, Slater, Gordon, Göllnitz, Andrews, Bressoud, Lepowsky and Wilson and by 1980 there were over 130 isolated identities and several infinite families of identities known.

The entry of these identities into physics occurred in the early '80's when Baxter [2], Andrews, Baxter and Forrester [3, 4], and the Kyoto group [5] encountered (1) and various generalizations in the computation of order parameters of certain lattice models of statistical mechanics. A further glimpse of the relation to physics is seen in the development of conformal field theory by Belavin, Polyakov and Zamolodchikov [6] and the form of computation of characters of representations of Virasoro algebra by Kac [7], Feigin and Fuchs [8] and Rocha-Caridi [9]. The occurrence of (1) in this context led Kac [10] to suggest that "every modular invariant representation of Vir should produce a Rogers-Ramanujan type identity."

The full relation, however, between physics and Rogers-Ramanujan identities is far more extensive than might be supposed from these first indications. Starting in 1993 the authors [11]-[17] have fused the physical insight of solvable lattice models in statistical mechanics with the classical work of the first 85 years and the recent developments in conformal field theory to greatly enlarge the theory of Rogers-Ramanujan identities. In this talk we will summarize the results of this work and present some of the current results. Our point of view will be dictated by our background in statistical mechanics but we will try to indicate where alternative viewpoints exist. Hopefully in this way some of the inevitable language barriers between physicists and mathematicians can be overcome.

2 What is a Rogers-Ramanujan Identity?

The work of the last 5 years originating in physics problems has provided a new framework and point of view in the study of Rogers-Ramanujan identities. The emphasis is not the same as in the earlier mathematical investigations and thus it is worthwhile to discuss generalities before the presentation of detailed results.

2.1 Sums instead of products

The equation (1) is the equality of three objects; an infinite sum involving $(q)_n$, an infinite product, and a second sum with $(q)_\infty$ in the denominator. For the first 85 years since (1) was proved it was the equality of the first infinite series with the infinite product which was called the Rogers-Ramanujan identity. The second sum while present in the intermediate steps of the proofs was always eliminated in favor of the product by use of the triple or pentuple product formula. The first important insight that was recognized when Rogers-Ramanujan identities arose in physics is that, contrary to this long history, it is not the product but rather the second sum on the right which arises in the statistical mechanical and conformal field theory applications. Indeed by now it is true that in most cases where we have generalizations of the identities between the two sums a product form is not known. Consequently by Rogers-Ramanujan identity we will mean the equality of the sums without further reference to possible product forms.

2.2 Polynomials instead of infinite series

The second insight which is also present in the very first papers on the connection of Rogers-Ramanujan identities with physics [2, 3, 4] is the fact that the physics will often lead to polynomial identities (with an order depending on an integer L) which yield infinite series identities as $L \to \infty$. The polynomial generalization of

(1) is the identity first proven in 1970 [18]

$$F_a(L,q) = B_a(L,q) \tag{2}$$

where

$$F_a(L,q) = \sum_{n=0}^{\infty} q^{n(n+a)} \begin{bmatrix} L-n-a \\ n \end{bmatrix} \tag{3}$$

and

$$B_a(L,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1+2a)/2} \begin{bmatrix} L \\ \lfloor \frac{1}{2}(L-5n-a) \rfloor \end{bmatrix} \tag{4}$$

where $\lfloor x \rfloor$ denotes the integer part of x and the Gaussian polynomials (q-binomial coefficients) are defined for integer m,n by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q)_n}{(q)_m (q)_{n-m}} & 0 \le m \le n \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The identity (1) is obtained by using $\lim_{n\to\infty} \begin{bmatrix} n \\ m \end{bmatrix} = 1/(q)_m$ It is generalizations of the polynomial identity (2) which we will call a Rogers-Ramanujan identity.

2.3 The generalizations of $F_a(L,q)$

All known generalizations of $F_a(L,q)$ can be written in terms of the following function [12]

$$f = \sum_{\text{restrictions}} q^{\frac{1}{2}\mathbf{mBm} - \frac{1}{2}\mathbf{Am}} \prod_{\alpha=1}^{n} \begin{bmatrix} ((1-\mathbf{B})\mathbf{m} + \frac{\mathbf{u}}{2})_\alpha \\ m_\alpha \end{bmatrix} \tag{6}$$

where $\mathbf{m}, \mathbf{u}$ and $\mathbf{A}$ are n dimensional vectors and $\mathbf{B}$ is an $n \times n$ dimensional matrix and the sum is over all values of the variables m_α possibly subject to some restrictions (such as being even or odd). In many cases the q-binomials are defined by (5) but there do occur cases in which an extended definition

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \begin{cases} \frac{(q^{n+1})_m}{(q)_m} & \text{for } m \ge 0,\ n \text{ integers} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

which allows n to be negative needs to be used.

The function (6) has the interpretation as the partition function for a collection of n differerent species of free massless (right moving) fermions with a linear energy momentum relation $e(P_j^\alpha)_\alpha = vP_j^\alpha$ where the momenta are quantized in units of $2\pi/M$ and are chosen from the sets

$$P_j^\alpha \in \{P_{\min}^\alpha(\mathbf{m}), P_{\min}^\alpha(\mathbf{m}) + \frac{2\pi}{M}, P_{\min}^\alpha(\mathbf{m}) + \frac{4\pi}{M}, \cdots, P_{\max}^\alpha(\mathbf{m})\} \tag{8}$$

with the Fermi exclusion rule $P_j^\alpha \neq P_k^\alpha$ for $j \neq k$ and all $\alpha = 1, 2, \cdots, n$,

$$P_{\min}^\alpha(\mathbf{m}) = \frac{\pi}{M}[((\mathbf{B}-1)\mathbf{m})_\alpha - \mathbf{A}_\alpha + 1] \text{ and } P_{\max}^\alpha = -P_{\min}^\alpha + \frac{2\pi}{M}(\frac{\mathbf{u}}{2} - \mathbf{A})_\alpha \tag{9}$$

where if some $u_\alpha = \infty$ the corresponding $P^\alpha_{\max} = \infty$. The $F_a(L,q)$ of (3) is regained in the very special case of $n = 1$, $\mathbf{B} = 2, \mathbf{u} = 2L - 2a$ and $\frac{1}{2}\mathbf{A} = -a$. Because of the Fermi exclusion rule we call these sums which generalize $F_a(L,q)$ Fermi forms. The generalization which the selection rule (8) makes over the usual exclusion rule of fermions is of great physical importance in the physics of the fractional quantum Hall effect [19].

2.4 THE GENERALIZATIONS OF $B_a(L,q)$

The first polynomial found which generalizes $B_a(L,q)$ is $B^{(p,p')}_{r,s}(L,a,b;q)$ given by[3, 4]

$$\sum_{j=-\infty}^{\infty}\left(q^{j(jpp'+rp'-sp)}\begin{bmatrix} L \\ \frac{L+a-b}{2} - jp' \end{bmatrix} - q^{(jp+r)(jp'+s)}\begin{bmatrix} L \\ \frac{L-a-b}{2} - jp' \end{bmatrix}\right). \tag{10}$$

with $L + a - b$ even. When $L \to \infty$ this polynomial reduces to

$$\lim_{L\to\infty} B^{(p,p')}_{r,s}(L,a,b;q) = \frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left(q^{j(jpp'+rp'-sp)} - q^{(jp+r)(jp'+s)}\right) \tag{11}$$

which is (multiplied by $q^{\Delta^{(p,p')}_{r,s} - c/24}$) the well known character [8, 9] of the minimal model $M(p,p')$ of the Virasoro algebra with central charge $c = 1 - 6(p-p')^2/pp'$ and conformal dimension $\Delta^{(p,p')}_{r,s} = [(rp' - sp)^2 - (p-p')^2]/4pp'$ $(1 \leq r \leq p-1,\ 1 \leq s \leq p'-1)$. In the method of Feigin and Fuchs [8] this formula is obtained by modding out null vectors from the Fock space of one free boson. For this reason we call generalizations of $B_a(L,q)$ bosonic forms.

When $p = 2, p' = 5, r = 1$ and $s = 2 - a$ the character (11) is identical with the righthand side of (1). This is the original inspiration for the belief that there is a connection between conformal field theory and Rogers-Ramanujan identities.

Moreover we note that the relation between the exclusion rules (8) with the character formula (11) provided by Rogers-Ramanujan identities explains why conformal field theory and related Kac-Moody algebra [20] methods have been successfully applied to the fractional quantum Hall effect. In particular the Rogers-Ramanujan identities of [21] guarantee that starting from the $U(1)$ Kac-Moody algebra description of edge states in the fractional quantum hall effect [20] there must be corresponding description in terms of fermionic quasiparticles.

But unlike the generalizations of $F_a(L,q)$ there are other quite distinct generalizations of $B_a(L,q)$ which have been found to occur. One of the more widely studied uses, instead of q-binomials (5), the q-trinomials of Andrews and Baxter [22]

$$\binom{L}{A}^{p}_{2} = \sum_{j=0}^{\infty} q^{j(j+A-p)}\frac{(q)_L}{(q)_j(q)_{j+A}(q)_{L-2j-A}} \tag{12}$$

and replaces (10) by either $B^{(1)(p,p')}_{r,s}(L,a,b;q)$ given by

$$\sum_{j=-\infty}^{\infty}\left[q^{j(pp'j+rp'-sp)}\binom{L}{2pj+a-b}^{0}_{2} - q^{(jp+r)(jp'+s)}\binom{L}{2pj+a+b}^{0}_{2}\right], \tag{13}$$

which appear in the computation of the order parameters of the dilute A models [23], or $B^{(2)(p,p')}_{r,s}(L,a,b;q)$ given by

$$\sum_{j=-\infty}^{\infty}\left[q^{j(pp'j+rp'-sp)}\binom{L}{p'j+a-b}_2^0 - q^{(jp+r)(jp'+s)}\binom{L}{p'j+a+b}_2^0\right]. \tag{14}$$

These q-trinomials have the property that $\lim_{L\to\infty}\binom{L}{A}_2^0=\frac{1}{(q)_\infty}$ and thus we see that although the polynomials $B^{(1)(p,p')}_{r,s}(L,a,b;q)$ and $B^{(2)(p,p')}_{r,s}(L,a,b;q)$ are not the same as $B^{(p,p')}_{r,s}(L,a,b;q)$ all three polynomials have the the same $L\to\infty$ limit (11). Further generalizations to q-multinomials have also been investigated [24, 25, 26, 27].

2.5 Proof by L-difference equations

The polynomial Roger-Ramanujan identities which generalize (2) are proven by demonstrating that the generalizations of $F_a(L,q)$ and $B_a(L,q)$ each satisfy the same difference equation in the variable L and are explicitly identical for suitably small values of L. Thus (2) is proven by demonstrating [18] that both $F_a(L,q)$ and $B_a(L,q)$ satisfy

$$h(L,q)=h(L-1,q)+q^{L-1}h(L-2,q)\quad \text{for} L\geq a+2 \tag{15}$$

and that they are identical for $L=a,a+1$. We refer to such equations as L-difference equations.

For the Fermi forms (6) the L-difference equations are derived by the general technique of telescopic expansions [13] which uses the two recursion relations for q-binomial coefficients (5)

$$\begin{bmatrix}n\\m\end{bmatrix}=\begin{bmatrix}n-1\\m-1\end{bmatrix}+q^m\begin{bmatrix}n-1\\m\end{bmatrix}=q^{n-m}\begin{bmatrix}n-1\\m-1\end{bmatrix}+\begin{bmatrix}n-1\\m\end{bmatrix} \tag{16}$$

which hold for all positive integers m,n or the identical recursion relations for generalized q-binomial coefficients (7) which hold for all integer m,n without restriction.

For the Bose form (10) which involves q-binomials the recursion relation (16) is sufficient to derive an L-difference equation but for the Bose forms (13) and (14) which involve q-trinomials we need not only the trinomial recursion relations such as

$$\binom{L}{A}_2^1=q^{L-1}\binom{L-1}{A}_2^1+q^A\binom{L-1}{A+1}_2^0+\binom{L-1}{A-1}_2^0 \tag{17}$$

but also so-called "tautological" equations such as

$$\binom{L}{A-1}_2^1-q^{A-1}\binom{L}{A+1}_2^1=\binom{L}{A-1}_2^0-q^{2A}\binom{L}{A+1}_2^0 \tag{18}$$

which reduce to trivialities when $q=1$. These "tautological" identities are what make the results involving q-trinomials more intricate to prove.

3 Results for minimal models $M(p,p')$

The irreducible representations $M(p,p')$ with central charge less than one are parameterized by two relatively prime integers p and p' and the characters are given by (11). Thus the suggestion of Kac [10] can by taken to mean that each bosonic form of the character has a fermionic form. We have recently proven [14, 15] that such identities do indeed exist, even generalized to polynomial identities, for all p and p'.

But there is much more to the theory than this. The minimal models $M(p,p')$ can be realized in terms of the coset construction of fractional level [28, 29]

$$\frac{(A_1^{(1)})_1 \times (A_1^{(1)})_m}{(A_1^{(1)})_{m+1}} \quad \text{with } m = \frac{p}{p'-p} - 2 \text{ or } -\frac{p'}{p'-p} - 2. \tag{19}$$

However, these constructions are not unique and as an example we note that the model $M(3,4)$ in addition to the coset (19) with $m=1$ has the representation $(E_8^{(1)})_1 \times (E_8^{(1)})_1/(E_8^{(1)})_2$. It may thus be asked whether or not the Rogers Ramanujan identity is a unique property of the model $M(p,p')$ or is it a property of the several different coset constructions. For the $M(3,4)$ it is known that just as there are two coset constructions so there are two very different fermionic representations of the characters. For example

$$\chi_{1,1}^{(3,4)} = \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{q^{\frac{m^2}{2}}}{(q)_m} = \sum_{n_1,\cdots,n_8=0}^{\infty} q^{\mathbf{n}\mathbf{C}_{E_8}^{-1}\mathbf{n}} \prod_{j=1}^{8} \frac{1}{(q)_j}. \tag{20}$$

Thus it is natural to extend the suggestion of Kac to the conjecture that *to every coset construction of conformal field theory there exists a Rogers-Ramanujan polynomial identity.*

Physically there are even more reasons to make such a conjecture. Conformal field theories represent integrable massless systems. But it is not needed for a system to be massless for it to be integrable and it is known [30] that the operators $\phi_{1,3},\ \phi_{2,1}, \phi_{1,5}$ and $\phi_{1,2}$ provide integrable massive perturbations of $M(p,p')$ whenever they are relevant. Each of these massive models has a fermionic quasi-particle spectrum which is a basis of states in the Hilbert space. As a basis this is independent of mass and thus still is a basis in the massless limit. We identity these quasi-particles with the fermionic representations (6). But the different massive perturbations will in general have a differerent number of quasi-particles and thus each integrable perturbation is expected to give a different fermionic form and hence a different Rogers-Ramanujan identity. However, even though at the level of the field theory these characters are the same at the level of finite statistical mechanical models the polynomials will be different. Thus we expect that each coset will lead to a different polynomial identity.

In the remainder of this section we will summarize how much of this conjecture has been proven.

3.1 The perturbation $\phi_{1,3}$

The integrable perturbation $\phi_{1,3}$ corresponds to the coset (19) and the bosonic polynomial is the original $B^{(p,p')}_{r,s}(L,a,b;q)$ (10) of [3, 4].

For the unitary case $M(p,p+1)$ the Rogers-Ramanujan identities were first proven in [13]. Here the matrix $\mathbf{B}$ is $\frac{1}{2}$ the Cartan matrix of A_{p-2}

$$B_{j,k} = \frac{1}{2}C_{A_{p-2}}|_{j,k} = \delta_{j,k} - \frac{1}{2}\delta_{j,k+1} - \frac{1}{2}\delta_{j,k-1} \quad 1 \leq j,k \leq p-2 \tag{21}$$

and $u_j = L\delta_{j,1}$ for $r = s = 1$ The general case of arbitrary p and p' is treated in [14, 15] and here $\mathbf{B}$ is a "fractional" generalization of a Cartan matrix which is obtained from the analysis of Bethe's Ansatz equations of the XXZ spin chain of Takahashi and Suzuki [31]. There are families of r,s for which the vector $\mathbf{A}$ is known but results for all cases have not been explicitly written down although an algorithm exists which allows the identity for any r,s to be found. For $p' = p+1$ only the conventional binomial coefficients (5) are needed and the Fermi form consists of a single term of the form (6). However, for general values of p' the modified binomials (7) arise and in addition there are many values of r,s where the Fermi form consists of a linear combination of terms of the form (6). It is essentially the existence of these linear combinations which makes the complete set of results difficult to explicitly write down.

3.2 The perturbations $\phi_{2,1}$ and $\phi_{1,5}$

Rogers-Ramanujan identities for the character with the minimal conformal dimension for the integrable perturbations $\phi_{2,1}$ and $\phi_{1,5}$ have recently been obtained [16] for models $M(p,p')$ by means of the recently discovered [17] trinomial analogue of Bailey's lemma and some computer tested conjectures. For the unitary case $M(p,p+1)$ we have just completed the proof of the identities for all values of r and s. When $2p > p'$ the perturbation $\phi_{2,1}$ is relevant and the bosonic form $B^{(1)}$ of (13) appears in the identities. We also have identities for $\frac{p'}{3} < p < \frac{p'}{2}$ where the perturbation $\phi_{1,5}$ is relevant and the bosonic form $B^{(2)}$ of (14) is used.

For the unitary case $M(p,p+1)$ the matrix $\mathbf{B}$ is of dimension $p-1$ where

$$\begin{aligned} B_{j,k} &= \frac{1}{2}C_{A_{p-2}}|_{j,k} \quad 2 \leq j,k \leq p-2 \\ B_{0,0} &= B_{1,1} = 1, \quad B_{0,2} = -B_{2,0} = 1/2 \quad B_{1,2} = B_{2,1} = -1/2 \end{aligned} \tag{22}$$

and zero otherwise and $u_j = 2L\delta_{j,0}$ for $r = s = 1$. This matrix differs significantly from the $p-2$ dimensional matrix (21) in that it is not symmetric.

The matrices $\mathbf{B}$ are also known [16] for the nonunitary cases $p' \neq p+1$. However, in many of these nonunitary cases a new phenomena arises not seen in the $\phi_{1,3}$ perturbations, namely that there can be several different fermionic representations (with different dimensions of the $\mathbf{B}$ matrix) of the same bosonic polynomial.

3.3 THE PERTURBATION $\phi_{1,2}$

The final case of integrable perturbations is $\phi_{1,2}$ but this case is not nearly so well understood. For the three very special unitary cases of cases $M(3,4)$, $M(4,5)$ and $M(6,7)$ Rogers-Ramanujan identities are known [11, 32] where the $\mathbf{B}$ matrices are twice the inverse of the Cartan matrix of E_8, E_7 and E_6 respectively and the bosonic form is obtained from (13) with the replacement $p \to p+1$ in the q-trinomials. Beyond these nothing further seems to be known.

4 HOW MANY IDENTITIES?

We demonstrated in [14, 15] that every $M(p,p')$ yields a set of Rogers-Ramanujan identities. But we also found that there are more than one identity for each $M(p,p')$. The question then arises of how many fermionic representations there are for the characters of each model $M(p,p')$ The answer to this is not known and the scope of the problem is perhaps most vividly shown by considering the three state Potts model $M(5,6)$ where in addition to the identities for the $\phi_{2,1}$ perturbation discussed above there is another set of identities which are a special case of the "parafermionic" identities first found by Lepowsky and Primc [33] in 1985 where the matrix $\mathbf{B}$ is twice the inverse Cartan matrix of A_2 and in the limit $L \to \infty$, $\mathbf{u} \to \infty$. This perturbation is also for the $\phi_{2,1}$ perturbation but has two quasi-particles instead of the four quasi-particles of (22). One may speculate that this has something to do with the difference between A and D modular invariants, but the actual explanation and interpretation of this fact is not known nor is it known if such extra representations exist for other models. If this is part of the explanation then we must enlarge the conjecture of sec. 3 to account for the possible modular invariants. But even this suggestion will not give an explanation for all of the various identities found for the nonunitary $\phi_{2,1}$ perturbations in [16]. The full range of Rogers-Ramanujan identities is by no means yet understood and it is anticipated that both in the mathematics and in the physics there is much still left to be discovered.

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