The Mathematics of Fivebranes

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Abstract.

Fivebranes are non-perturbative objects in string theory that generalize two-dimensional conformal field theory and relate such diverse subjects as moduli spaces of vector bundles on surfaces, automorphic forms, elliptic genera, the geometry of Calabi-Yau threefolds, and generalized Kac-Moody algebras.

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1 Introduction

This joint session of the sections Mathematical Physics and Algebraic Geometry celebrates a historic period of more than two decades of remarkably fruitful interactions between physics and mathematics. The 'unreasonable effectiveness', depth and universality of quantum field theory ideas in mathematics continue to amaze, with applications not only to algebraic geometry, but also to topology, global analysis, representation theory, and many more fields. The impact of string theory has been particularly striking, leading to such wonderful developments as mirror symmetry, quantum cohomology, Gromov-Witten theory, invariants of three-manifolds and knots, all of which were discussed at length at previous Congresses.

Many of these developments find their origin in two-dimensional conformal field theory (CFT) or, in physical terms, in the first-quantized, perturbative formulation of string theory. This is essentially the study of sigma models or maps of Riemann surfaces Σ into a space-time manifold X. Through the path-integral over all such maps a CFT determines a partition function Z_g on the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces. String amplitudes are functions $Z(\lambda)$, with λ the string coupling constant, that have asymptotic series of the form

$$Z(\lambda) \sim \sum_{g\geq 0} \lambda^{2g-2} \int_{\mathcal{M}_g} Z_g. \tag{1}$$

But string theory is more than a theory of Riemann surfaces. Recently it has become possible to go beyond perturbation theory through conceptual breakthroughs such as string duality [23] and D-branes [19]. Duality transformations can interchange the string coupling λ with the much better understood geometric moduli of the target space X. D-Branes are higher-dimensional extended objects that give rise to special cycles $Y \subset X$ on which the Riemann surface can end, effectively leading to a *relative* form of string theory.

One of the most important properties of branes is that they can have multiplicities. If k branes coincide a non-abelian $U(k)$ gauge symmetry appears. Their 'world-volumes' carry Yang-Mills-like quantum field theories that are the analogues of the two-dimensional CFT on the string world-sheet. The geometric realization as special cycles (related to the theory of calibrations) has proven to be a powerful tool to analyze the physics of these field theories. The mathematical implications are just starting to be explored and hint at an intricate generalization of the CFT program to higher dimensions.

This lecture is a review of work done on one of these non-perturbative objects, the fivebrane, over the past years in collaboration with Erik Verlinde, Herman Verlinde and Gregory Moore [4, 5, 8, 7]. I thank them for very enjoyable and inspiring discussions.

2 Fivebranes

One of the richest and enigmatic objects in non-perturbative string theory is the so-called fivebrane, that can be considered as a six-dimensional cycle Y in space-time. Dimension six is special since, just as in two dimensions, the Hodge star satisfies $*^2 = -1$ and one can define *chiral* or 'holomorphic' theories. The analogue of a free chiral field theory is a 2-form 'connection' B with a self-dual curvature H that is locally given as $H = dB$ but that can have a 'first Chern class' $[H/2\pi] \in H^3(Y, \mathbb{Z})$. (Technically it is a Deligne cohomology class, and instead of a line bundle with connection it describes a 2-gerbe on Y.) A system of k coinciding fivebranes is described by a 6-dimensional conformal field theory, that is morally a $U(k)$ non-abelian 2-form theory. Such a theory is not known to exist at the classical level of field equations, so probably only makes sense as a quantum field theory.

One theme that we will not further explore here is that (at least for $k = 1$) the fivebrane partition function Z_Y can be obtained by quantizing the intermediate Jacobian of Y, very much in analogy with the construction of conformal blocks by geometric quantization of the Jacobian or moduli space of vector bundles of a Riemann surface [24]. This leads to interesting relations with the geometry of moduli spaces of Calabi-Yau three-folds and topological string theory. In fact there is even a seven-dimensional analogue of Chern-Simons theory at play.

The fivebrane theory is best understood on manifolds of the product form $Y = X \times T^2$, with X a 4-manifold. In the limit where the volume of the two-torus goes to zero, it gives a $U(k)$ Yang-Mills theory as studied in [21]. In that case the partition function computes the Euler number of the moduli space of $U(k)$ instantons on X. In the $k = 1$ case this relation follows from the decomposition of the 3-form

$$H = F_+ \wedge dz + F_- \wedge d\overline{z} \tag{2}$$

with $F_\pm$ (anti)-self-dual 2-forms on X. In this way holomorphic fields on T^2 are coupled to self-dual instantons on X. The obvious action of $SL(2, \mathbb{Z})$ on T^2

translates in a deep quantum symmetry (S-duality) of the 4-dimensional Yang-Mills theory.

Actually, the full fivebrane theory is much richer than a 6-dimensional CFT. It is believed to be a six-dimensional *string* theory that does not contain gravity and that reduces to the CFT in the infinite-volume limit. We understand very little about this new class of string theories, other than that they can be described in certain limits as sigma models on instanton moduli space [20, 7, 1, 25]. As we will see, this partial description is good enough to compute certain topological indices, where only so-called BPS states contribute.

3 Conformal field theory and modular forms

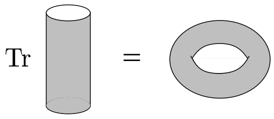
One of the striking properties of conformal field theory is the natural explanation it offers for the modular properties of the characters of certain infinite-dimensional Lie algebras such as affine Kac-Moody algebras. At the hart of this explanation—and in fact of much of the applications of quantum field theory to mathematics—lies the equivalence between the Hamiltonian and Lagrangian formulation of quantum mechanics [22]. For the moment we consider a *holomorphic* or chiral CFT.

In the Hamiltonian formulation the partition function on an elliptic curve T^2 with modulus τ is given by a trace over the Hilbert space $\mathcal{H}$ obtained by quantization on $S^1 \times \mathbb{R}$. For a sigma model with target space X, this Hilbert space will typically consist of L^2-functions on the loop space $\mathcal{L}X$. It forms a representation of the algebra of quantum observables and is $\mathbb{Z}$-graded by the momentum operator P that generates the rotations of S^1. For a chiral theory P equals the holomorphic Hamiltonian $L_0 = z\partial_z$. The character of the representation is then defined as

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}\, q^{P-\frac{c}{24}} \tag{3}$$

with $q = e^{2\pi i\tau}$ and c the central charge of the Virasoro algebra. The claim is that this character is always a suitable modular form for $SL(2,\mathbb{Z})$, *i.e.*, it transforms covariantly under linear fractional transformations of the modulus τ.

In the Lagrangian formulation $Z(\tau)$ is computed from the path-integral over maps from T^2 into X. The torus T^2 is obtained by gluing the two ends of the cylinder $S^1 \times \mathbb{R}$, which is the geometric equivalent of taking the trace.



Modularity is therefore built in from the start, since $SL(2,\mathbb{Z})$ is the 'classical' automorphism group of the torus T^2.

The simplest example of a CFT consists of c free chiral scalar fields $x : \Sigma \to V \cong \mathbb{R}^c$. Ignoring the zero-modes, the chiral operator algebra is then given by an infinite-dimensional Heisenberg algebra that is represented on the graded Fock space

$$\mathcal{H}_q = \bigotimes_{n>0} S_{q^n} V. \tag{4}$$

Here we use a standard notation for formal sums of (graded) symmetric products

$$S_q V = \bigoplus_{N \geq 0} q^N S^N V, \qquad S^N V = V^{\otimes N}/S_N. \tag{5}$$

The partition function is then evaluated as

$$Z(\tau) = q^{-\frac{c}{24}} \prod_{n>0} (1-q^n)^{-c} = \eta(q)^{-c} \tag{6}$$

and is indeed a modular form of $SL(2,\mathbb{Z})$ of weight $-c/2$ (with multipliers if $c \not\equiv 0$ mod 24.) The 'automorphic correction' $q^{-c/24}$ is interpreted as a regularized sum of zero-point energies that naturally appear in canonical quantization.

4 String theories and automorphic forms

The partition function of a *string* theory on a manifold Y will have automorphic properties under a larger symmetry group that reflects the 'stringy' geometry of Y. For example, if we choose $Y = X \times S^1 \times \mathbb{R}$, with X compact and simply-connected, quantization will lead to a Hilbert space $\mathcal{H}$ with a natural $\mathbb{Z} \oplus \mathbb{Z}$ gradation. Apart from the momentum P we now also have a winding number W that labels the components of the loop space $\mathcal{L}Y$. Thus we can define a two-parameter character

$$Z(\sigma,\tau) = \mathrm{Tr}_{\mathcal{H}}\left(p^W q^P\right), \tag{7}$$

with $p = e^{2\pi i\sigma}, q = e^{2\pi i\tau}$, with both σ,τ in the upper half-plane $\mathbb{H}$. We claim that $Z(\sigma,\tau)$ is typically the character of a generalized Kac-Moody algebra [2] and an automorphic form for the arithmetic group $SO(2,2;\mathbb{Z})$.

The automorphic properties of such characters become evident by changing again to a Lagrangian point of view and computing the partition function on the compact manifold $X \times T^2$. The T-duality or 'stringy' symmetry group of T^2 is

$$SO(2,2;\mathbb{Z}) \cong PSL(2,\mathbb{Z}) \times PSL(2,\mathbb{Z}) \ltimes \mathbb{Z}_2, \tag{8}$$

where the two $PSL(2,\mathbb{Z})$ factors act on (σ,τ) by separate fractional linear transformations and the mirror map $\mathbb{Z}_2$ interchanges the complex structure τ with the complexified Kähler class $\sigma \in H^2(T^2,\mathbb{C})$. This group appears because a string moving on T^2 has both a winding number $w \in \Lambda = H_1(T^2;\mathbb{Z})$ and a momentum vector $p \in \Lambda^*$. The 4-vector $k = (w,p)$ takes value in the even, self-dual Narain lattice $\Gamma^{2,2} = \Lambda \oplus \Lambda^*$ of signature $(2,2)$ with quadratic form $k^2 = 2w \cdot p$ and automorphism group $SO(2,2;\mathbb{Z})$.

In the particular example we will discuss in detail in the next sections, where the manifold X is a Calabi-Yau space, there will be an extra $\mathbb{Z}$-valued quantum number and the Narain lattice will be enlarged to a signature $(3,2)$ lattice. Correspondingly, the automorphic group will be given by $SO(3,2,\mathbb{Z}) \cong Sp(4,\mathbb{Z})$.

5 QUANTUM MECHANICS ON THE HILBERT SCHEME

As we sketched in the introduction, in an appropriate gauge the quantization of fivebranes is equivalent to the sigma model (or quantum cohomology) of the moduli space of instantons. More precisely, quantization on the six-manifold $X \times S^1 \times \mathbb{R}$, gives a graded Hilbert space

$$\mathcal{H}_p = \bigoplus_{N \geq 0} p^N \mathcal{H}_N, \tag{9}$$

where $\mathcal{H}_N$ is the Hilbert space of the two-dimensional supersymmetric sigma model on the moduli space of $U(k)$ instantons of instanton number N on X. If X is an algebraic complex surface, one can instead consider the moduli space of stable vector bundles of rank k and $ch_2 = N$. This moduli space can be compactified by considering all torsion-free coherent sheaves up to equivalence. In the rank one case it coincides with the Hilbert scheme of points on X. This is a smooth resolution of the symmetric product $S^N X$. (We note that for the important Calabi-Yau cases of a $K3$ or abelian surface the moduli spaces are all expected to be hyper-Kähler deformations of $S^{Nk}X$.)

The simplest type of partition function will correspond to the Witten index. For this computation it turns out we can replace the Hilbert scheme by the more tractable orbifold $S^N X$. For a smooth manifold M the Witten index computes the superdimension of the graded space of ground states or harmonic forms, which is isomorphic to $H^*(M)$, and therefore equals the Euler number $\chi(M)$.

For an orbifold M/G the appropriate generalization is the *orbifold* Euler number. If we denote the fixed point locus of $g \in G$ as M^g and centralizer subgroups as C_g, this is defined as a sum over the conjugacy classes $[g]$

$$\chi_{orb}(M/G) = \sum_{[g]} \chi_{top}(M^g/C_g). \tag{10}$$

For the case of the symmetric product $S^N X$ this expression can be straightforwardly computed, as we will see in the next section, and one finds

THEOREM 1 [13] — *The orbifold Euler numbers of the symmetric products $S^N X$ are given by the generating function*

$$\chi_{orb}(S_p X) = \prod_{n>0} (1-p^n)^{-\chi(X)}.$$

Quite remarkable, if we write $p = e^{2\pi i \tau}$, the formal sum of Euler numbers is (almost) a modular form for $SL(2,\mathbb{Z})$ of weight $\chi(X)/2$. This is in accordance with the interpretation as a partition function on $X \times T^2$ and the S-duality of the corresponding Yang-Mills theory on X [21].

A much deeper result of Göttsche tells us that the same result holds for the Hilbert scheme [9]. In fact, in both cases one can also compute the full cohomology and express it as the Fock space, generated by an infinite series of copies of $H^*(X)$

shifted in degree [10]

$$H^*(S_pX) \cong \bigotimes_{n>0} S_{p^n} H^{*-2n+2}(X). \tag{11}$$

Comparing with (4) we conclude that the Hilbert space of ground states of the fivebrane is the Fock space of a chiral CFT. This does not come as a surprise given the remarks in the introduction. One can also derive the action of the corresponding Heisenberg algebra using correspondences on the Hilbert scheme [16].

6 The elliptic genus

We now turn from particles to strings. To compute the fivebrane string partition function on $X \times T^2$, we will have to study the two-dimensional supersymmetric sigma model on the moduli space of instantons on X. Instead of the full partition function we will compute again a topological index — the elliptic genus. Let us briefly recall its definition.

For the moment let X be a general complex manifold of dimension d. Physically, the elliptic genus is defined as the partition function of the corresponding $N=2$ supersymmetric sigma model on a torus with modulus τ [15]

$$\chi(X;q,y) = \mathrm{Tr}_{\mathcal{H}}\Big((-1)^{F_L+F_R} y^{F_L} q^{L_0-\frac{d}{8}}\Big), \tag{12}$$

with $q=e^{2\pi i\tau}$, $y=e^{2\pi i z}$, z a point on T^2. Here $\mathcal{H}$ is the Hilbert space obtained by quantizing the loop space $\mathcal{L}X$ (formally the space of half-infinite dimensional differential forms). The Fermi numbers $F_{L,R}$ represent (up to an infinite shift that is naturally regularized) the bidegrees of the Dolbeault differential forms representing the states. The elliptic genus counts the number of string states with $\overline{L}_0 = 0$. In terms of topological sigma models, these states are the cohomology classes of the right-moving BRST operator Q_R. In fact, if we work modulo Q_R, the CFT gives a *cohomological* vertex operator algebra.

Mathematically, the elliptic genus can be understood as the S^1-equivariant Hirzebruch χ_y-genus of the loop space of X. If X is Calabi-Yau the elliptic genus has nice modular properties under $SL(2,\mathbb{Z})$. It is a weak Jacobi form of weight zero and index $d/2$ (possibly with multipliers). The coefficients in its Fourier expansion

$$\chi(X;q,y) = \sum_{m\geq 0,\ \ell} c(m,\ell) q^m y^\ell \tag{13}$$

are integers and can be interpreted as indices of twisted Dirac operators on X. For a $K3$ surface one finds the unique (up to scalars) Jacobi form of weight zero and index one, that can expressed in elementary theta-functions as

$$\chi(K3;q,y) = 2^3 \cdot \sum_{\text{even }\alpha} \vartheta_\alpha^2(z;\tau)/\vartheta_a^2(0;\tau). \tag{14}$$

7 Elliptic genera of symmetric products

We now want to compute the elliptic genus of the moduli spaces of vector bundles, in particular of the Hilbert scheme. Again, we first turn to the much simpler symmetric product orbifold $S^N X$.

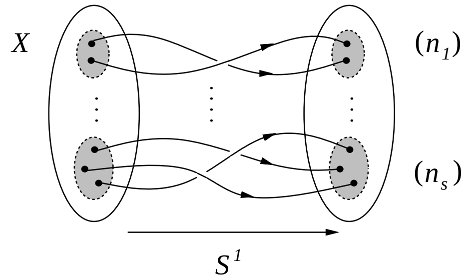
The Hilbert space of a two-dimensional sigma model on any orbifold M/G decomposes in sectors labeled by the conjugacy classes $[g]$ of G, since $\mathcal{L}(M/G)$ has disconnected components of twisted loops satisfying

$$x(\sigma + 2\pi) = g \cdot x(\sigma), \qquad g \in G. \tag{15}$$

In the case of the symmetric product orbifold X^N/S_N these twisted sectors have an elegant interpretation [8]. The conjugacy classes of the symmetric group S_N are labeled by partitions of N,

$$[g] = (n_1) \cdots (n_s), \qquad \sum_i n_i = N, \tag{16}$$

where (n_i) denotes an elementary cycle of length n_i. A loop on $S^N X$ satisfying this twisted boundary condition can therefore be visualized as



As is clear from this picture, one loop on $S^N X$ is *not* necessarily describing N loops on X, but instead can describe $s \leq N$ loops of length $n_1, \ldots, n_s$. By length n we understand that the loop only closes after n periods. Equivalently, the action of the canonical circle action is rescaled by a factor $1/n$.

In this way we obtain a 'gas' of strings labeled by the additional quantum number n. The Hilbert space of the formal sum $S_p X$ can therefore be written as

$$\mathcal{H}(S_p X) = \bigotimes_{n>0} S_{p^n} \mathcal{H}_n(X). \tag{17}$$

Here $\mathcal{H}_n(X)$ is the Hilbert space obtained by quantizing a single string of length n. It is isomorphic to the subspace $P \equiv 0 \pmod n$ of the single string Hilbert space $\mathcal{H}(X)$. From this result one derives

Theorem 2 [8] — *Let X be a Calabi-Yau manifold, then the orbifold elliptic genera of the symmetric products $S^N X$ are given by the generating function*

$$\chi_{orb}(S_p X; q, y) = \prod_{n>0,\, m \geq 0,\, \ell} (1 - p^n q^m y^\ell)^{-c(nm,\ell)}.$$

In the limit $q \to 0$ the elliptic genus reduces to the Euler number and we obtain the results from §5. Only the constant loops survive and, since twisted loops then localize to fixed point sets, we recover the orbifold Euler character prescription and Theorem 1.

8 Automorphic forms and generalized Kac-Moody algebras

The fivebrane string partition function is obtained from the above elliptic genus by including certain 'automorphic corrections' and is closely related to an expression of the type studied extensively by Borcherds [3] with the infinite product representation[1]

$$\Phi(\sigma,\tau,z) = p^a q^b y^c \prod_{(n,m,\ell)>0} (1-p^n q^m y^\ell)^{c(nm,\ell)} \tag{18}$$

For general Calabi-Yau space X it can be shown, using the path-integral representation, that the product Φ is an automorphic form of weight $c(0,0)/2$ for the group $SO(3,2,\mathbb{Z})$ for a suitable quadratic form of signature $(3,2)$ [12, 14, 18].

In the important case of a $K3$ surface Φ is the square of a famous cusp form of $Sp(4,\mathbb{Z}) \cong SO(3,2,\mathbb{Z})$ of weight 10,

$$\Phi(\sigma,\tau,z) = 2^{-12} \prod_{even\ \alpha} \vartheta[\alpha](\Omega)^2 \tag{19}$$

the product of all even theta-functions on a genus-two surface Σ with period matrix

$$\Omega = \begin{pmatrix} \sigma & z \\ z & \tau \end{pmatrix}, \qquad \det \operatorname{Im} \Omega > 0. \tag{20}$$

Note that Φ is the 12-th power of the holomorphic determinant of the scalar Laplacian on Σ, just as η^{24} is on an elliptic curve. The quantum mechanics limit $\sigma \to i\infty$ can be seen as the degeneration of Σ into a elliptic curve.

In the work of Gritsenko and Nikulin [11] it is shown that Φ has an interpretation as the denominator of a generalized Kac-Moody algebra. This GKM algebra is constructed out of the cohomological vertex algebra of X similar as in the work of Borcherds. This algebra of BPS states is induced by the string interaction, and should also have an algebraic reformulation in terms of correspondences as in [12].

9 String interactions

Usually in quantum field theory one first quantizes a single particle on a space X and obtains a Hilbert space $\mathcal{H} = L^2(X)$. Second quantization then corresponds to taking the free symmetric algebra $\bigoplus_N S^N\mathcal{H}$. Here we effective reversed the order of the two operations: we considered quantum mechanics on the 'second-quantized' manifold $S^N X$. (Note that the two operations do not commute.) In this

[1] Here the positivity condition means: $n,m \geq 0$, and $\ell > 0$ if $n=m=0$. The 'Weyl vector' (a,b,c) is defined by $a=b=\chi(X)/24$, and $c = -\frac{1}{4}\sum_\ell |\ell| c(0,\ell)$.

framework it is possible to introduce interactions by deforming the manifold $S^N X$, for example by considering the Hilbert scheme or the instanton moduli space. It is interesting to note that there is another deformation possible.

To be concrete, let X be again a $K3$ surface. Then $S^N X$ or $\mathrm{Hilb}^N(X)$ is an Calabi-Yau of complex dimension $2N$. Its moduli space is unobstructed and 21 dimensional — the usual 20 moduli of the $K3$ surface plus one extra modulus. This follows essentially from

$$h^{1,1}(S^N X) = h^{1,1}(X) + h^{0,0}(X). \tag{21}$$

The extra cohomology class is dual to the small diagonal, where two points coincide, and the corresponding modulus controls the blow-up of this $\mathbb{Z}_2$ singularity. Physically it is represented by a $\mathbb{Z}_2$ twist field that has a beautiful interpretation, that mirrors a construction for the 10-dimensional superstring [6] — it describes the joining and splitting of strings. Therefore the extra modulus can be interpreted as the string coupling constant λ [7, 25].

The geometric picture is the following. Consider the sigma model with target space $S^N X$ on the world-sheet $\mathbb{P}^1$. A map $\mathbb{P}^1 \to S^N X$ can be interpreted as a map of the N-fold *unramified* cover of $\mathbb{P}^1$ into X. If we include the deformation λ the partition function has an expansion

$$Z(\lambda) \sim \sum_{n \geq 0} \lambda^n Z_n, \tag{22}$$

where Z_n is obtained by integrating over maps with n simple branch points. In this way higher genus surfaces appear as non-trivial N-fold branched covers of $\mathbb{P}^1$. The string coupling has been given a geometric interpretation as a modulus of the Calabi-Yau $S^N X$.

It is interesting to note that this deformation has an alternative interpretation in terms of the moduli space of instantons, at least on $\mathbb{R}^4$. The deformed manifold with $\lambda \neq 0$ can be considered as the moduli space of instantons on a *non-commutative* version of $\mathbb{R}^4$ [17].

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