Inverse Boundary Value Problems for Partial Differential Equations

Dedicated to the memory of my father
and to the memory of Warren Ambrose

Gunther Uhlmann

Abstract. We survey the role of complex geometrical optics solutions to partial differential equations in the solution of several inverse boundary value problems.

1991 Mathematics Subject Classification: 35R30, 35P05, 35J05, 78A70
Keywords and Phrases: Inverse boundary problems, complex geometrical optics, Dirichlet to Neumann map, inverse conductivity problem

0. Introduction

Inverse boundary problems are a class of problems in which one seeks to determine the internal properties of a medium by performing measurements along the boundary of the medium. These inverse problems arise in many important physical situations, ranging from geophysics to medical imaging to the non-destructive evaluation of materials.

The appropriate mathematical model of the physical situation is usually given by a partial differential equation (or a system of such equations) inside the medium. The boundary measurements are then encoded in a certain boundary map. The inverse boundary problem is to determine the coefficients of the partial differential equation inside the medium from knowledge of the boundary map.

In this paper we will survey part of the significant progress which has been made in the last twenty years in this area. Many of the advances have been a consequence of the construction of *complex geometrical optics* solutions for the class of partial differential equations under consideration. The prototypical example of an inverse boundary problem is the inverse conductivity problem, also called electrical impedance tomography, first proposed by A. P. Calderón [7]. In this case the boundary map is the voltage to current map; that is, the map assigns to a voltage potential on the boundary of a medium the corresponding induced current flux at the boundary of the medium. The inverse problem is to recover the electrical conductivity of the medium from the boundary map.

We will also discuss in this paper other examples of inverse boundary problems, including examples associated to the Schrödinger equation in the presence of a magnetic field, Maxwell's equations and the Lamé system of elasticity. The unifying theme of the paper is the role of complex geometrical optics solutions in inverse boundary value problems and our selection of problems reflects this choice. We list a series of basic open problems in the field. For an account of the close connection between inverse boundary value problems and inverse scattering problems at a fixed energy see [40]. Another important omission is the discussion of inverse boundary value problems for hyperbolic equations, in particular the Boundary Control Method. See the review paper [4] for more details.

1. The inverse conductivity problem for an isotropic conductivity

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary (many of the results are valid for Lipschitz boundaries). We denote by γ the conductivity of Ω, which we assume is in $L^\infty(\Omega)$ and strictly positive. The potential u in Ω with voltage f on $\partial\Omega$ satisfies

$$L_\gamma u = \operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega; \quad u|_{\partial\Omega} = f. \tag{1.1}$$

The voltage to current map, or Dirichlet to Neumann map (DN), is defined by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\bigg|_{\partial\Omega}, \tag{1.2}$$

where u is the solution of (1.1), and ν denotes the unit outer normal to $\partial\Omega$.

The inverse problem is to determine γ knowing Λ_γ. More precisely we want to study properties of the map

$$\gamma \xrightarrow{\Lambda} \Lambda_\gamma. \tag{1.3}$$

Note that $\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is bounded. We can divide this problem into several parts.

a) Injectivity of Λ (identifiability).
b) Continuity of Λ and its inverse if it exists (stability).
c) What is the range of Λ? (characterization problem).
d) Formula to recover γ from Λ_γ (reconstruction).
e) Give a numerical algorithm to find an approximation. of the conductivity given a finite number of voltage and current measurements at the boundary (numerical reconstruction).

In this section we outline the proof of the following identifiability result proven in [36].

1.1 Theorem. *Let $n \geq 3$. Let $\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$ be strictly positive functions in $\bar{\Omega}$ such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then $\gamma_1 = \gamma_2$ in $\bar{\Omega}$.*

Sketch of the proof. Using Green's thorem it is easy to prove that

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f dS, \tag{1.4}$$

where u is the solution of (1.1). In other words $Q_\gamma(f)$ is the quadratic form associated to the selfadjoint linear map $\Lambda_\gamma(f)$, i.e., to know $\Lambda_\gamma(f)$ or $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ is equivalent. $Q_\gamma(f)$ measures the energy needed to maintain the potential f at the boundary.

Formula (1.4) suggests that instead of prescribing voltage measurements at the boundary to determine the conductivity in the interior, we find solutions of the equation (1.1). This is the point of view of Calderón [7] in his analysis of the linearized problem at a constant conductivity.

To find these solutions we first reduce the problem to studying the Schrödinger equation at zero energy. Let $\gamma \in C^2(\overline{\Omega})$ be a positive function. We have

$$\gamma^{-\frac{1}{2}} L_\gamma \gamma^{-\frac{1}{2}} u = (\Delta - q)u, \qquad q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{1.5}$$

For any $q \in L^\infty(\Omega)$ we can define the set of Cauchy data

$$C_q = \{(f,g); f = u|_\Omega, g = \frac{\partial u}{\partial \nu}|_\Omega, u \in H^1(\Omega) \text{ solution of } (1.1)\}$$

If 0 is not a Dirichlet eigenvalue of $\Delta - q$ then C_q is the graph of a map which is, by definition, the DN map. Theorem 1.1 follows from Theorem 1.2 and the fact that Λ_γ determines both γ at the boundary and the normal derivative of γ at the boundary (see [15], [37]).

1.2 THEOREM. *Assume $q_i \in L^\infty(\Omega), i = 1,2$ and $C_{q_1} = C_{q_2}$. Then $q_1 = q_2$.*

Sketch of the proof of Theorem 1.2. The key result is the construction of complex geometrical optics solutions to the Schrödinger equation. This was motivated by Calderón's analysis of the linearized problem at a constant conductivity [7].

1.1 LEMMA. *Let $q \in L^\infty(\mathbb{R}^n)$ with compact support. Let $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$. Let $-1 < \delta < 0$. Then if $|\rho| \geq C(\delta) sup_{x\in\mathbb{R}^n} |(1+|x|^2)q(x)|$ for some $C(\delta) > 0$, there exists a unique solution of $(\Delta - q)u = 0$ in $\mathbb{R}^n$ of the form*

$$u = e^{x\cdot\rho}(1 + \psi_q(x,\rho)) \tag{1.6}$$

with $\psi_q(\cdot,\rho) \in L^2_\delta(\mathbb{R}^n)$. Moreover $\|\psi_q(\cdot,\rho)\|_{L^2_\delta(\mathbb{R}^n)}$ goes to 0 as $|\rho|$ goes to infinity. A more precise estimate is proven in [36]. *(Here $L^2_\delta(\mathbb{R}^n)$ denotes the weighted L^2 space with norm $\|f\|^2_{L^2_\delta(\mathbb{R}^n)} = \int (1+|x|^2)^\delta |f(x)|^2 dx$.)*

Let $q_i \in L^\infty(\Omega)$ as in the statement of Theorem (1.2). We define $q_i = 0$ in $\mathbb{R}^n - \Omega$. Let $\rho_i, i = 1,2$ as in Lemma (1.1) with $\rho_1 = \eta + i(k+l), \quad \rho_2 = -\eta + i(k-l)$ with $\eta, k, l \in \mathbb{R}^n$ satisfying $\langle \eta, k\rangle = \langle k, l\rangle = \langle \eta, l\rangle = 0, \quad |\eta|^2 = |k|^2 + |l|^2$ and $|l| \geq R_i$, with R_i sufficiently large so that Lemma 1.1 is valid for $q_i, i = 1,2$ (here we use $n \geq 3$). We take

$$u_i = e^{x\cdot\rho_i}(1 + \psi_{q_i}(x,\rho_i)), \quad i = 1,2. \tag{1.7}$$

The next important ingredient is the following identity which follows easily from Green's theorem.

1.2 LEMMA. *Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$ and $C_{q_1} = C_{q_2}$. Then*

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0 \tag{1.8}$$

for every solution $u_i \in H^1(\Omega)$ of $(\Delta - q_i) u_i = 0$ in $\mathbb{R}^n$.

Now we plug (1.7) into (1.8). Taking the limit as $|l| \to \infty$, we easily conclude that the Fourier transform of q_1 and q_2 coincide.

In order to construct ψ_q as in (1.6) we solve the equation

$$\Delta_\rho \psi_q = q(1 + \psi_q) \text{ with } \Delta_\rho f = e^{-x\cdot\rho} \Delta(e^{x\cdot\rho} f). \tag{1.9}$$

We note that the characteristic variety of Δ_ρ is a codimension two real submanifold. We can construct an inverse Δ_ρ that satisfies the following estimate proven for $n \geq 3$ in [36] and for $n = 2$ in [35].

$$\|\Delta_\rho^{-1}\|_{\delta+1,\delta} \leq \frac{C}{|\rho|} \tag{1.10}$$

with $-1 < \delta < 0$, C is a positive constant, and $\| \; \|_{\delta+1,\delta}$ denotes the operator norm.

Using the complex geometrical optics solutions of Lemma 1.1 Alessandrini proved stability estimates for the map (1.3). A reconstruction method using these solutions was proposed in [19], [25]. We remark that the construction of the solutions (1.6) is in the whole of $\mathbb{R}^n$. Complex geometrical solutions in compact sets have been constructed in [8], [10].

Theorem 1.1 extends to non-linear conductivities [29]. Theorem 1.2 extends to the non-linear Schrödinger equation under some additional assumptions on the potential [14]. These results use a linearization procedure due to Isakov [11].

MAXWELL'S EQUATIONS.

One obtains the conductivity equation (1.1) if one neglects the time variation of the electromagnetic field in Maxwell's equations. We now describe the boundary map in this case.

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary. The electromagnetic field (E, H) satisfies the frequency domain Maxwell's equation which are given by

$$\operatorname{rot} E = i\omega\mu H, \quad \operatorname{rot} H = (-i\omega\varepsilon + \sigma)E \text{ in } \Omega \tag{1.11}$$

where $\omega > 0$ is the time-harmonic frequency of the field, $\varepsilon > 0$ denotes the electrical permittivity, $\mu > 0$ the magnetic permeability, and $\sigma \geq 0$ the conductivity. We assume that all the functions are smooth. The boundary map is given by

$$\Lambda_{\varepsilon,\mu,\sigma}(\omega) : \nu \wedge E|_{\partial\Omega} \to \nu \wedge H|_{\partial\Omega}$$

where E, H satisfies (1.11). A global identifiability result was proven in this case in [26]. The proof was simplified in [27], where the problem is reduced to constructing geometrical optics solutions for a Schrödinger equation with q an 8×8 matrix. Lemma 1.1 applies also in this case.

Open problem 1. How much smoothness should one assume on the conductivity for Theorem 1.1 to be valid? R. Brown extended Theorem 1.1 to conductivities in $C^{\frac{3}{2}+\epsilon}(\overline{\Omega})$, with ϵ any positive number. The natural conjecture is that the theorem holds for Lipschitz conductivities since unique continuation is valid in this case. There are no known counterexamples for rough conductivites. Kohn and Vogelius proved identifiability for piecewise real-analytic conductivities [16]. In [12] the case of a conductivity having a jump discontinuity across the boundary of a subdomain is considered.

Open problem 2. Is Theorem 1.1 valid if we measure the DN map only on part of the boundary?

Open problem 3. Is it possible to characterize the boundary values of the complex geometrical optics solutions (1.6)? This might have implications in the characterization and reconstruction problem.

Open problem 4. Is it possible to develop the reconstruction method based on the complex geometrical optics solutions into a convergent numerical algorithm?

Open problem 5. (The anisotropic case.) Conductivities may depend also on direction. Muscle tissue in the human body is an example. In this case the conductivity is represented by a positive definite matrix. It seems like a difficult problem to find complex geometric optics solutions in the anisotropic case. Moreover, it is not true that the DN map in this case determines uniquely the conductivity. See [38] for a discussion of the obstruction to identifiability in this case. The case of real analytic conductivities was considered in [17]. The case of quasilinear real-analytic anisotropic conductivities is discussed in [31]. For further results see [38].

2. The two dimensional case

Nachman proved in [20] that, in the two dimensional case, one can uniquely determine conductivities in $W^{2,p}(\Omega)$ for some $p > 1$ from Λ_γ. An essential part of Nachman's argument is the construction of the complex geometrical optics solutions (1.6) for all complex frequencies $\rho \in \mathbb{C}^2$, $\rho \cdot \rho = 0$, for potentials of the form (1.5). Then he applies the $\overline{\partial}$-method in inverse scattering, pioneered in one dimension by Beals and Coifman [2] and extended to higher dimensions by several authors (see [25] for further discussions and applications of the $\overline{\partial}$ method). The analog of Theorem 1.2 is open, in two dimensions, for a general potential $q \in L^\infty(\Omega)$. We outline a different approach to [20] that allows less regular conductivities.

The inverse conductivity problem.

We describe here an extension of Nachman's result to $W^{1,p}(\Omega), p > 2$, conductivities by Brown and the author [6]. We follow an earlier approach of Beals and Coifman [3], who studied scattering for a first order system whose principal part is $\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}$.

2.1 THEOREM. *Let $n = 2$. Let $\gamma \in W^{1,p}(\Omega), p > 2$, γ strictly positive. Assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

We first reduce the conductivity equation to a first order system. We define the scalar potential q and matrix potential Q by

$$q = -\frac{1}{2}\partial \log \gamma, \quad Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}. \tag{2.1}$$

We let D be the operator

$$D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}. \tag{2.2}$$

An easy calculation shows that if u satisfies the conductivity equation $\operatorname{div}(\gamma \nabla u) = 0$, then

$$D \begin{pmatrix} v \\ w \end{pmatrix} - Q \begin{pmatrix} v \\ w \end{pmatrix} = 0 \quad \text{with} \quad \begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{\frac{1}{2}} \begin{pmatrix} \partial u \\ \overline{\partial} u \end{pmatrix}. \tag{2.3}$$

In [6] matrix solutions of (2.3) are constructed which have the form

$$u_k = m(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\overline{z}k} \end{pmatrix}, \tag{2.4}$$

where $z = x_1 + ix_2$, $k \in \mathbb{C}$, with $m \to 1$ as $|z| \to \infty$ in a sense to be described below. To construct m we solve the integral equation

$$m - D_k^{-1} Q m = 1 \tag{2.5}$$

where, for a matrix-valued function A,

$$D_k A = E_k^{-1} D E_k A; \quad E_k A = A^d + \Lambda_k^{-1} A^{\text{off}}; \quad \Lambda_k(z) = \begin{pmatrix} e^{i(z\overline{k} + \overline{z}k)} & 0 \\ 0 & e^{-i(zk + \overline{z}\overline{k})} \end{pmatrix}.$$

Here A^d denotes the diagonal part of A and A^{off} the antidiagonal part.

The next result gives the solvability of (2.5) in an appropriate space.

2.1 LEMMA. *Let $Q \in L^p(\mathbb{R}^2), p > 2$, and compactly supported. Assume that Q is a hermitian matrix. Choose r so that $\frac{1}{p} + \frac{1}{r} > \frac{1}{2}$ and then β so that $\beta r > 2$. Then the operator $(I - D_k^{-1} Q)$ is invertible in $L^r_{-\beta}$. Moreover the inverse is differentiable in k in the strong operator topology. Here L^r_β denotes a weighted L^r space.*

Lemma 2.1 implies the existence of solutions of the form (2.4) with $m - 1 \in L^r_{-\beta}(\mathbb{R}^2)$ with β, r as in Lemma 2.1. The next step, following the $\overline{\partial}$ method, consists in relating $\frac{\partial}{\partial \overline{k}} m(z,k)$ and scattering data that in turn is determined from the DN map. For more details see [6].

Problem 5 has been solved in the anisotropic case in two dimensions for sufficiently smooth conductivities. By using isothermal coordinates, one can reduce the anisotropic case to the isotropic case, and therefore construct complex geometrical optics solutions in this case (see [34].) The case of quasilinear anisotropic conductivities is considered in [31].

Open problem 6: The potential case. Problems 1-4 are also open for the inverse conductivity problem in two dimensions. As we mentioned at the beginning of this section, the analog of Theorem 1.2 is unknown at present for a general potential $q \in L^\infty(\Omega)$. By Nachman's result it is true for potentials of the form $q = \frac{\Delta u}{u}$ with $u \in W^{2,p}(\Omega), u > 0$ for some p, $p > 1$. Sun and Uhlmann proved generic uniqueness for pairs of potentials in [32]. The semilinear case, under additional assumptions on the potential, was considered in [13]. In [33] it is shown that one can determine the singularities of an L^∞ potential from its Cauchy data.

3. First order perturbations of the Laplacian

There are several inverse boundary value problems associated to first order perturbations of the Laplacian. We consider briefly here an inverse boundary value problem associated to the Lamé system in elasticity theory.

We first discuss how to construct complex geometrical optics solutions for any scalar first order perturbation of the Laplacian.

Let $L_N = \Delta + N(x,D)$ with $N(x,D)$ a first order differential operator in $\mathbb{R}^n$ with smooth coefficients with compact support. We attempt to construct solutions u_ρ of $L_N u_\rho = 0$ of the form $u_\rho = e^{x\cdot\rho} m_\rho$. The equation for $m(x,\rho)$ is $M_\rho m_\rho := (\Delta_\rho + N_\rho) m_\rho = 0$ where $N_\rho f = e^{-x\cdot\rho} N(e^{x\cdot\rho} f)$ and Δ_ρ as in (1.9).

The difficulty in finding m_ρ is that the operator $\Delta_\rho^{-1} N_\rho$ contain terms that don't decay in $|\rho|$ in any reasonable norm. We get around this difficulty by conjugating the operator $\Delta_\rho + N_\rho$ to an operator that behaves like a zeroeth order perturbation of Δ_ρ. This idea is motivated by formula (1.5). To do this we consider pseudodifferential operators depending on a complex parameter [28]. For these operators the variable ρ behaves like the variable ξ. More precisely, we define $Z = \{\rho \in \mathbb{C}^n - 0; \rho\cdot\rho = 0\}$ and $A \in L^m(\mathbb{R}^n, Z)$ if we can write

$$Af(x) = \int e^{i\langle x,\xi\rangle} a_\rho(x,\xi) \widehat{f}(\xi) d\xi, \quad f \in C_0^\infty(\mathbb{R}^n), \text{ where } a_\rho \in S^m(\mathbb{R}^n, Z), i.e.$$

.

$$\sup_{x\in K} |\partial_x^\alpha \partial_\xi^\beta a_\rho(x,\xi)| \le C_{\alpha,\beta,K}(1 + |\xi| + |\rho|)^{m-|\beta|}, \forall K \subset\subset \mathbb{R}^n.$$

.

We have that $\Delta_\rho \in L^2(\mathbb{R}^n, Z), N_\rho \in L^1(\mathbb{R}^n, Z)$. The key result proved in [21] is that one can conjugate $\Delta_\rho + N_\rho$ to $\Delta_\rho + C_\rho$, with $C_\rho \in L^0(\mathbb{R}^n, Z)$.

3.1 Lemma. *Let $K \subset\subset \mathbb{R}^n$ be a compact subset. Let $M_\rho(x,D)$ be as defined above. Then there exist A_ρ, $B_\rho \in L^0(\mathbb{R}^n, Z)$ such that*

$$M_\rho A_\rho = B_\rho (\Delta_\rho + C_\rho), \tag{3.1}$$

where $C_\rho \in L^0(\mathbb{R}^n, Z)$. Moreover $\phi A_\rho \phi$ and $\phi B_\rho \phi$ are invertible on $L^2(K)$ for large $|\rho|$ for all $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi = 1$ on K.

Now it is easy to construct many solutions l_ρ of $(\Delta_\rho + C_\rho)l_\rho = 0$ in any compact set since the operator $\phi C_\rho \phi$ is bounded on $L^2(\mathbb{R}^n)$, with operator norm independent of $|\rho|$ being a pseudodifferential operator of order zero depending on the parameter ρ (see [28] for more details on thiese operators.) Therefore, by the intertwining property (3.1), $m_\rho = A_\rho l_\rho$ is a solution of $M_\rho m_\rho = 0$. The construction of A_ρ, B_ρ is quite explicit.

In the paper [23], building on early work of Sun [30], these complex geometrical optics solutions were used to prove a global identifiability result for an inverse boundary value problem associated to the Schrödinger equation in the presence of smooth magnetic potential and electric potential. C. Tolmasky reduced the regularity needed in [39] to just one derivative for the magnetic potential, and a bounded electric potential, by using non-smooth symbols depending on the complex parameter ρ. The paper [18] also uses these solutions to prove a global identifiability result for Maxwell's equations in chiral media by reducing this case to a first order system perturbation of the Laplacian.

An inverse boundary value problem for the elasticity system.

An inverse boundary value problem arising in the mechanics of materials is to determine the elastic parameters of a medium by making displacement and traction measurements at the boundary of the medium. We describe briefly below the boundary map in this case.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary. We consider Ω as an elastic, isotropic, inhomogeneous medium with Lamé parameters λ, μ. The generalized Hooke's law states that under the assumption of no body forces acting on Ω, the displacement u satisfies

$$(3.2) \qquad (Lu)_i = (L_{\lambda,\mu}u)_i = \sum_{j,k,l=1}^{n} \frac{\partial}{\partial x_j} C_{ijkl} \frac{\partial}{\partial x_l} u_k = 0 \text{ in } \Omega, \quad i = 1, \dots, n,$$

$$u|_{\partial\Omega} = f$$

where

$$(3.3) \qquad C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{ij} + \delta_{il}\delta_{jk}) \quad (1 \le i, j, k, l \le n),$$

with δ_{ij} the Kronecker delta and $(Lu)_i$ denotes the i-th component of Lu.

$C = (C_{ijkl})$ is the elastic tensor. The boundary value problem (3.2) has a unique solution under the strong convexity condition $\mu > 0, n\lambda + 2\mu > 0$ in $\bar{\Omega}$.

The Dirichlet to Neumann map is defined in this case by

$$(3.4) \qquad (\Lambda_{\lambda,u}(f))_i = \sum_{l,k,l=1}^{n} \nu_j C_{ijkl} \left.\frac{\partial u_k}{\partial x_l}\right|_{\partial\Omega}, \quad i = 1, ..., n$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the unit outer normal to $\partial\Omega$ and u is the solution of (3.2). Physically the DN map sends the displacement at the boundary to the traction at the boundary. The following global identifiability result was proven in [21].

3.1 Theorem. *Let* $n \geq 3$. *Let* $(\lambda_i, \mu_i) \in C^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega}), i = 1, 2$ *satisfy the strong convexity condition (3.3). Assume* $\Lambda_{(\lambda_1,\mu_1)} = \Lambda_{(\lambda_2,\mu_2)}$. *Then* $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$ *in* $\bar{\Omega}$.

The proof of Theorem 3.1 follows the general outline of the proof of Theorem 1.2. Namely, one proves an identity similar to (1.8) by using Green's theorem. Second, one reduces the elasticity system to a first order system (a more direct way to do this was given in [9]). Now one constructs geometrical optics solutions for the elasticity system using Lemma 3.1, which also applies to the first order system under consideration. The details of this outline can be found in [21].

Open problem 7. The analog of problems 1-5 are also open for the elasticity system. The analog of Theorem 2.1 is not known for the elasticity system in two dimensions. It is known that one can uniquely identify from the DN map Lamé parameters close to constant (see [22].) The methods of section 2 might be useful to prove a global identifiability result in this case.

Acknowledgements

The author would like to acknowledge support from the National Science Foundation and the Office of Naval Research, for the research presented in this paper.

I would also like to thank Eduardo Chappa, Rafe Mazzeo and Hart Smith for their helpful comments on an earlier version of the manuscript.

References

1. Alessandrini, G.,, *Stable determination of conductivity by boundary measurements*, Appl. Anal. (1988), 153–172.
2. Beals R. and Coifman R., *Transformation spectrales et equation d' evolution non-lineares*, Seminaire Goulaouic-Meyer-Schwarz, exp. 21 (1981-1982).
3. ———, *The spectral problem for the Davey-Stewarson and Ishimori hierarchies*, Nonlinear evolution equations: Integrability and spectral methods, Manchester University Press (1988), 15–23.
4. Belishev M., *Boundary control in reconstruction of manifolds and metrics (the BC method)*, Inverse Problems **13** (1997), R1-R45.
5. Brown R., *Global uniqueness in the impedance imaging problem for less regular conductivities*, SIAM J. Math. Anal. **27** (1996), 1049–1056.
6. Brown R. and Uhlmann G., *Uniqueness in the inverse conductivity problem with less regular conductivities in two dimensions*, Comm. P.D.E. **22** (1997), 1009-1027.
7. A. P. Calderón, *On an inverse boundary value problem*, Soc. Brasileira de Matemática (1980), 65–73.
8. Hähner P, *A periodic Faddeev-type solution*, J. Diff. Eq. **128** (1996), 300–308.
9. Ikehata M., *A remark on an inverse boundary value problem arising in elasticity*, preprint.
10. Isakov V., *Completeness of products of solutions and some inverse problems for PDE*, J. Diff. Eq. **119** (1992), 59–70.
11. ———, *On uniqueness in inverse problems for semilinear parabolic equations*, Arch. Rat. mech. Anal. **92** (1993), 1–12.
12. ———, *On uniqueness of recovery of discontinuity conductivity*, Comm. Pure Appl. Math. **41** (1988), 865–877.
13. Isakov V. and Nachman A., *Global uniqueness for a two dimensional semilinear elliptic inverse problem*, Trans. AMS **347** (1995), 3375–3390.
14. Isakov V. and Sylvester J., *Global uniqueness for a semilinear elliptic inverse problem*, Comm. Pure Appl. Math. **47** (1994), 1403–1410.

15. Kohn R. and Vogelius M., *Determining conductivity by boundary measurements*, Comm. Pure Appl. Math. **37** (1984), 289–298.
16. ———, *Determining conductivity by boundary measurements II. Interior results*, Comm. Pure Appl. Math **38** (1985), 643-667.
17. Lee J. and Uhlmann G., *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math. **42** (1989), 1097–1112.
18. McDowall S., *An Electrodynamic inverse problem in chiral media*, Trans. AMS, to appear.
19. Nachman A., *Reconstruction from boundary measurements*, Annals of Math. **128** (1988), 71–96.
20. ———, *Global uniqueness for a two-dimensional inverse boundary value problem*, Annals of Math. **143**.
21. Nakamura G. and Uhlmann G., *Global uniqueness for an inverse boundary value problem arising in elasticity*, Invent. Math. **118** (1994), 457–474.
22. ———, *Identification of Lamé parameters by boundary measurements*, Amer. J. of Math. **115** (1993), 1161–1187.
23. Nakamura G., Sun Z. and Uhlmann G., *Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field*, Math. Annalen **303** (1995), 377–388.
24. Novikov R., *Multidimensional inverse spectral problems for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Functional Analysis and its Applications **22** (1988), 263–272.
25. Novikov R. and Henkin G., $\overline{\partial}$*- equation in the multidimensional inverse scattering problem*, Russian Math. Surveys **42** (1987), 109–180.
26. Ola P., Päivärinta L. and Somersalo E., *An inverse boundary value problem in electrodynamics*, Duke Math. J. **70** (1993), 617–653.
27. Ola P. and Somersalo E., *Electromagnetic inverse problems and generalized Sommerfeld potentials*, SIAM J. Appl. Math. **56** (1996), 1129–1145.
28. Shubin, M. A., *Pseudodifferential operators and spectral theory*, Springer Series in Soviet Mathematics, Springer-Verlag, 1987.
29. Sun Z., *On a quasilinear boundary value problem*, Math. Z. **221** (1996), 293–305.
30. ———, *An inverse boundary value problem for Schrödinger operator with vector potentials*, Trans. AMS **338** (1993), 953–969.
31. Sun Z. and Uhlmann G., *Inverse problems in quasilinear anisotropic media*, Amer. J. Math. **119** (1997), 771–797.
32. ———, *Generic uniqueness for an inverse boundary value problem*, Duke Math. J. **62** (1991), 131–155.
33. ———, *Recovery of singularities for formally determined inverse problems*, Comm. Math. Physics **153** (1993), 431–445.
34. Sylvester, J., *An anisotropic inverse boundary value problem*, Comm Pure Appl. Math. **43** (1990), 202–232.
35. Sylvester J. and Uhlmann G., *A uniqueness theorem for an inverse boundary value problem in electrical prospection Comm. Pure Appl. Math.* **39** (1986), 91–112.
36. ———, *A global uniqueness theorem for an inverse boundary value problem*, Annals of Math. **125** (1987), 153–169.
37. ———, *Inverse boundary value problems at the boundary–continuos dependence*, Comm. Pure Appl. Math. **41** (1988), 197–221.
38. ———, *Inverse problems in anisotropic media*, Contemp. Math. **122** (1991), 105–117.
39. Tolmasky C., *Exponentially growing solutions for non smooth first-order perturbations of the Laplacian*, SIAM J. Math. Anal. **29** (1998), 116–133.
40. Uhlmann G., *Inverse boundary value problems and applications*, Astérisque **207** (1992), 153–211.

Gunther Uhlmann
Department of Mathematics
Box 354350
University of Washington
Seattle, WA 98195-4350, USA