Vortices in Ginzburg-Landau Equations

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Abstract. GL models were first introduced by V.Ginzburg and L.Landau around 1950 in order to describe superconductivity. Similar models appeared soon after for various phenomena: Bose condensation, superfluidity, non linear optics. A common property of these models is the major role of topological defects, termed in our context vortices.

In a joint book with H.Brezis and F.Helein, we considered a simple model situation, involving a bounded domain Ω in R^2, and maps v from Ω to R^2. The Ginzburg-Landau functional, then writes

$$E_\epsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{4\epsilon^2}\int_\Omega (1-|v|^2)^2$$

Here ϵ is a parameter describing some characteristic lenght. We are interested in the study of stationary maps for that energy, when ϵ is small (and in the limit ϵ goes to zero). For such map the potential forces $|v|$ to be close to 1 and v will be almost S^1-valued. However at some point $|v|$ may have to vanish, introducing defects of topological nature, the vortices. An important issue is then to determine the nature and location of these vortices.

We will also discuss recent advances in more physical models like superconductivity, superfluidity, as well as for the dynamics: as previously the emphasis is on the behavior of the vortices.

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1 Introduction

Ginzburg-Landau functionals were introduced around 1950 by V.Ginzburg and L.Landau in order to model energy states of superconducting materials and their phase transitions. Related functionals appeared soon therafter in various fields as superfluidity, Bose condensation, nonlinear optics, fluid mechanics and particle physics. A common feature of these models is that they involve non convex potentials, which allow the existence of topological defects for stationary states: here we will mainly focus on two-dimensional situations, where theses defects are often termed vortices. In recent years, very importants efforts have been devoted to their study from a mathematical point of view: we will try here to survey parts of these works.

We begin with a simple model, which was studied extensively, in particular in a joint book with H.Brezis and F.Helein [BBH]. Consider a smooth bounded domain in R^2 (for instance a disk), and complex valued functions v on Ω (i.e maps v from Ω to R^2). The simplest possible Ginzburg-Landau functional then takes the form, for these functions

$$E_\epsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{4\epsilon^2}\int_\Omega (1-|v|^2)^2 \,.$$

Here ϵ is a parameter describing some characteristic lenght and we will mainly be interested in the case ϵ is small and in the limit ϵ tends to zero. The potential $V(v) = \epsilon^{-2}(1-|v|^2)$ forces $|v|$, for critical maps for E_ϵ to be close to 1 and therefore, stationnary (or low energy) maps will be almost S^1-valued. However, at some points $|v|$ may have to vanish, introducing "defects".

To have a well-posed mathematical problem, we prescribe next Dirichlet boundary conditions: let g be a smooth map from $\partial\Omega$ to S^1, and prescribe v to be equal to g on $\partial\Omega$. Therefore we introduce the Sobolev space

$$H^1_g(\Omega; R^2) = \{v \in H^1(\Omega; R^2),\, v = g \text{ on } \partial\Omega\}\,.$$

It is then easy to verify that E_ϵ is a C^∞ functional on H^1_g, and that its critical points verify the Ginzburg-Landau equation

$$\Delta v = \frac{1}{\epsilon^2} v(1-|v|^2) \text{ on } \Omega, \quad v = g \text{ on } \partial\Omega. \tag{1}$$

Standard elliptic estimate show that, any solution to (1) is smooth, that

$$|v| \leq 1 \text{ on } \Omega \quad \text{(maximum principle)}, \tag{2}$$

$$|\nabla v| \leq \frac{C}{\epsilon} \text{ on } \Omega \quad \text{for C, some constant depending on } g, \tag{3}$$

$$\frac{1}{4\epsilon^2}\int_\Omega (1-|v|^2)^2 \leq C\,, \quad \text{provided } \Omega \text{ is starshaped.} \tag{4}$$

Since E_ϵ is strictly positive, one easily verifies that it achieves its infimum k_ϵ on H^1_g and hence (1) possesses minimizing solutions (not necessarily unique). We will denote u_ϵ these solutions.

2 Asymptotic analysis of minimisers

The winding number d of g (as map from $\partial\Omega$ to S^1) is crucial in this analysis, forcing, when $d \neq 0$, vortices to appear.

2.1 THE CASE $d = 0$.

In this case, there exists ψ from $\partial\Omega$ to R such that $g = \exp i\psi$. Next let φ_* be the solution of $\Delta\varphi_* = 0$ on Ω, $\varphi_* = \psi$ on $\partial\Omega$ and consider $u_* = \exp i\varphi_*$. Clearly u_* is S^1-valued, so that

$$E_\epsilon(u_*) = \frac{1}{2}\int_\Omega |\nabla u_*|^2 = \frac{1}{2}\int_\Omega |\nabla \varphi_*|^2$$

is bounded independently on ϵ. Hence k_ϵ remains bounded as $\epsilon \to 0$. It is that easy to show that $u_\epsilon \to u_*$ in H^1. Finally in [BBH2] we carried out more refined asymptotics, in particular

$$\|u_* - u\|_{L^\infty} \le C\epsilon^2.$$

2.2 THE CASE $d \neq 0$.

We may assume, for instance $d > 0$. In this case there are no maps in H^1_g which are S^1-valued (the fact that there are no continuous S^1-valued maps reduces to standard degree theory). In particular $k_\epsilon \longrightarrow +\infty$, and we are facing a singular limit. Since u_ϵ is smooth, the topology of the boundary data forces u_ϵ to vanish somewhere in Ω. The points where u_ϵ vanishes play an important role: the Dirichlet energy will concentrate in there neighborhood, accounting for the divergence of k_ϵ. In [BBH], we established

THEOREM 1 *i) There exists a constant $C > 0$ depending only on g such that*

$$|k_\epsilon - \pi d|\log\epsilon|| \le C\,,\ \forall\, 0 < \epsilon < 1\,. \tag{5}$$

ii) The map u_ϵ has exactly d zeroes, provided ϵ is sufficiently small (these result relies on a work by P.Baumann, N.Carlson and D.Philips [BCP]) .
iii) There exists exactly d points $a_1, a_2, \cdots, a_d$ in Ω such that up to a subsequence , $\epsilon_n \to 0$,

$$u_{\epsilon_n} \longrightarrow u_*, \text{ on any compact subset of } \Omega\setminus\bigcup_{i=1}^{d}\{a_i\}\,,$$

where

$$u_* = \prod_{i=1}^{d}\frac{z-a_i}{|z-a_i|}\exp i\varphi \ \ (\varphi \text{ being a harmonic function}).$$

In particular, the winding number around each singularity is $+1$.
iv) The configuration a_i is not arbitrary, but minimizes on $\Omega^d\setminus\Delta$ (where Δ denotes the diagonal) a renormalized energy which has the form

$$W_g(a_1,\cdots,a_d) = \pi\sum_{i\neq j}\log|a_i - a_j| + \text{ boundary conditions.} \tag{6}$$

v) The energy has the expansion, as $\epsilon \to 0$

$$k_\epsilon = \pi d|\log\epsilon| + W_g(a_1,\cdots,a_d) + d\gamma_0 + o(1)$$

where γ_0 is some absolute constant.

REMARKS 1) Theorem 1 was established in [BBH] under the additional assumption that Ω is starshaped. This assumption was removed by M.Struwe [Str] (see also Del Pino-Felmer [DF]).
2) Similar results have been obtained by André and Shafrir, when the potential depends also on x, [AS], in [BR] for the abelian Higgs models, and in [HJS] for a self-dual model.
3) Hardt and Lin have studied in [HL] a different singular limit problem, with the same renormalized energy.
4) A three dimensional analog was studied by Rivière in [R].

3 ASYMPTOTICS FOR NON MINIMIZING SOLUTION

A similar analysis can be carried out for solution which are not necessarily minimizing. Assume Ω is starshaped. Then, we have, [BBH], for v_ϵ solution to (1):

THEOREM 2 *i) There exists some constant $C > 0$, such that, for $0 < \epsilon < 1$*

$$E(v_\epsilon) \leq C(|\log \epsilon| + 1)\,.$$

ii) there exists a subsequence ϵ_n, l points $a_1, \cdots, a_l$ and l integer $d_1, \cdots, d_l$ such that

$$v_{\epsilon_n} \longrightarrow v_* = \prod_{i=1}^{l} \left(\frac{z - a_i}{|z - a_i|} \right)^{d_i} \exp i\varphi, \text{ where } \varphi \text{ is harmonic.}$$

iii) The configuration $(a_i,\, d_i)$ is critical for the renormalized energy.

Note that an important difference between minimizing and non-minimizing solutions is that, for the later one, the multiplicity of vortices has not to be $+1$, and the vortices of opposite degree might coexist.

4 THE EXISTENCE PROBLEM

In view of Theorem 2, a natural question is to determine whether non-minimizing solutions do really exist, and if one is able to prescribe the multiplicity of the vortices. We begin with an elementary example.

4.1 AN EXAMPLE:

Take $\Omega = D^2$ and $g(\theta) = \exp id\theta$ (here $(r,\,\theta)$ denote polar coordinate). In view of the symmetries, one can find a solution $v(r,\,\theta)$ of the form $v_d(r,\,\theta) = f_d(r)\exp id\theta$, where f_d verifies the ODE

$$r^2 f'' + r f' - d^2 f + \frac{1}{\epsilon^2} r^2 f(1 - f^2) = 0\,, \quad f(0) = 0\,, \quad f(1) = 1\,.$$

Computing the energy of these solutions, one sees that they are of order $\pi d^2 |\log \epsilon|$: hence, if $|d| \geq 2$, and ϵ is sufficiently small they are non minimizing. [In the case $d = 1$, v is minimizing thanks to results by P. Mironescu [Mi] and Pacard and Rivière [PaR]].

Actually, for large d, there are much more solutions. Indeed, the Morse Index of the solution v_d is of order $|d|^2$, for large d (see [AB1], [BeH]). Therefore, using symmetries and the index theory of Faddell and Rabinowitz [FR] (a Lyusternik-Schnirelmann theory in the presence of compact group actions), one obtains the existence of at least $\mu_0|d|^2$ orbits of solutions, for large d, where μ_0 is some positive constant (the orbit of a solution v is the set $\{\exp(-id\alpha)v(\exp i\alpha z),\ \alpha \in [0, 2\pi[\ \})$.

4.2 VARIATIONAL METHODS

A complete Morse theory for (1) has yet to be constructed. In view of (6), one might expect that the level sets for E_ϵ are related to the level sets of W_g on $\Sigma = \Omega_d \setminus \Delta$, and hence that the topologie of Σ might yield solution for (1). This idea was introduced in [AB1], and then extended by Zhou and Zhou [ZZ]: they proved that (1) has at least $|d|+1$ distinct solutions, for sufficiently small ϵ. They are using crucially the fact that the cuplenght of Σ is (at least), $|d|-1$, a result due to V. Arnold [Ar].

We conjecture actually that the number of solutions is much higher. In order to find solutions with vortices of higher multiplicity, one has also to take into account vortices of opposite charges and also the fact that they might annihilate. For that reason, $\Omega^d \setminus \Delta$ is no longer the good model, and one has to turn to spaces as studied by D.Mc. Duff [McD].

REMARK: Another construction of (stable) solutions has been introduced in [Li1].

5 SUPERCONDUCTIVITY

We turn now to the original model for superconductivity, as introduced by Ginzburg and Landau. Here Ω represents a superconducting sample, h_{ex} denotes the external applied magnetic field. The functional to minimize is now

$$F_\epsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |dA - h_{ex}|^2 + \frac{1}{4\epsilon^2}\int_\Omega (1-|u|^2)^2 \,.$$

Here $A = A_1 dx_1 + A_2 dx_2$ is a connection accounting for electromagnetic effects, and u represents a condensated wave function for Cooper pairs of electrons, the carrier of superconductivity. In the above renormalized units, $|u|^2$ represents the density of Cooper pairs, so that if $|u| \simeq 0$ the sample is in the normal state, whereas if $|u| \simeq 1$ the material is in the superconducting state. We will see that for certain applied fields h_{ex}, the two states may coexist in the same sample (phase transition of second order). This model leads therefore to many interesting mathematical questions, often related to physical experiments.

5.1 NON SIMPLY CONNECTED DOMAINS

In this case, permanent currents have been observed, even when $h_{ex} = 0$. Jimbo, Morita and Zhai [JMZ], Rubinstein and Sternberg [RS] and Almeida [Al1] have related this fact to the existence of configurations minimizing the energy in a topological sector. The threshold energy between different sectors is established in [Al2] and corresponds precisely to the energy of a vortex.

When the external field is non zero interesting phenomena occur (the Little-Parks effect), which have been studied in particular by Berger and Rubinstein ([BgR]).

5.2 CRITICAL FIELDS

Suppose ϵ is small, and let Ω be an arbitrary domain. For $h_{ex} = 0$, the minimizing solution is clearly (up to a gauge transformation) $u = 1$, $A = 0$. It is observed that, until h_{ex} reaches a critical field H_{c_1}, the minimizing solution has no vortex (called a Meissner solution). For $h_{ex} > H_{c_1}$, vortices appear, and their number increases with h_{ex}. Finally, for $h_{ex} > H_{c_2}$, another critical field, superconductivity dissapears, and the minimizing solution is $u = 0$.

Stable solutions near H_{c_1} have been thoroughly investigated by S. Serfaty [S1, S2]. In particular the location of the vortices is determined, and it is proved that many branches of solutions corresponding to various numbers of vortices, coexist at the same time. For larger fields, homogenized equations for the vortex distribution have been proposed and studied (see for instance Chapman, Rubinstein and Schatzman [CRS]).

Finally very precise estimates have been obtained in the one dimensional case by C. Bolley and B. Helffer (see[BoH]), for different critical fields and values of ϵ.

6 EVOLUTION EQUATIONS

Various evolution equations corresponding to the Ginzburg-Landau system have been studied. For the heat-flow equation related to (1), Lin [Li2] has shown that the vortices evolve according to the gradient flow of the renormalised energy (see also [JS]), in a suitable renormalized unit of time. The Schrödinger equation (termed also Gross-Pitaevskii equation)

$$i\frac{\partial u}{\partial t} = \Delta u + u(1 - |u|^2) \tag{7}$$

is of special importance, since it appears as a model for superfluids, Bose condensation, nonlinear optics. It is also related to fluid mechanics, because if $u = \rho \exp i\varphi$, then $\nabla\varphi$ can be interpretated as the velocity in a compressible Euler equation, ρ^2 being the density (with a suitable choice for the pressure).

The dynamics of vortices (on bounded domains) was derived by Colliander and Jerrard [CJ], as the simplectic gradient for the renormalized energy (see also [LX]).

When the domain is R^2, Ovchinnikov and Sigal [OS1] have shown that when the initial data has two vortices of the same sign (and hence infinite GL energy), radiation takes place and the vortices repulse. The existence and behavior of travelling waves solutions to (7) has been widely considered in the physical litterature (see for instance Jones, Putterman and Roberts [JPR], Pismen and

Nepomnyashchy [PN], Josserand and Pomeau [P]). This solutions have the forme $u(x,t) = U(x_1 - ct\,,\, x_2)$ where U is a function on R^2. For $0 < c^2 < 2$, non constant finite energy solutions exists (rigourous proofs are provided in [BS1], [BS2]). When c is small, these solution possess two vortices with degrees $+1$ and -1, the distance separating the vortices is proportional to the inverse of the speed c. The limiting speed $\sqrt{2}$ represents the speed of sound (see [OS2], also for the role of Cherenkov radiation). Stability of these travelling waves has been studied in the physical literature: mathematical proofs are still to be provided as well as for the three dimensional case (vortex rings).

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