Arnold Diffusion: A Variational Construction

Zhihong Xia[1]

Abstract. We use variational method to study Arnold diffusion and instabilities in high dimensional Hamiltonian systems. Our method is based on a generalization of Mather's theory on twist maps and their connecting orbits to a higher dimensional setting. Under some generic nondegeneracy conditions, we can construct transition chains of arbitrary fixed length, crossing gaps of any size between invariant KAM (lower dimensional) tori. One of notable features of our result is that, instead of using transition tori alone for diffusion as in Arnold's construction, we also use cantori from Aubry-Mather theory in our mechanism for diffusion. Other results, such as shadowing properties, symbolic dynamics and transitivity, etc., can also be obtained by our method. Our nondegeneracy condition is a condition on the splitting of separatrix and in the so-called *a priori* unstable systems, this condition can be verified by the so-called Poincaré-Melnikov integrals.

In Arnold's original example for the instability, the perturbation is carefully chosen so that it does not touch any invariant tori on the normally hyperbolic invariant manifold. As an application of our results, we can choose arbitrary perturbations and are able to conclude the same results (in fact stronger), as long as the Poincaré-Melnikov integrals are in some sense non-degenerate.

1 Introduction

Perhaps one of the most important problems in Hamiltonian dynamics, after the celebrated KAM theory, is the topological stability of near integrable Hamiltonian systems. KAM theory completely answered the problem for two-degree of freedom autonomous Hamiltonian systems, where we have generic stability. However, the higher dimensional situation is much more complicated. A standing conjecture, due to Arnold (cf. [2], [6]), is that genericly we have topological instability. To support his conjecture, Arnold [1] gives an example of a two-degree of freedom, time-periodically forced Hamiltonian system where arbitrary small perturbation produces orbits whose action variables changes arbitrarily in size, resulting in a phenomenon known as *Arnold diffusion.*

[1]Research is supported in part by National Science Foundation.

The mechanism in Arnold's example for diffusion uses transition tori, which are lower dimensional (weakly) hyperbolic KAM tori. In typical situations, some of these lower dimensional invariant tori (near resonances) break, hence breaking the transition chain, which severely limits the size of diffusion. This difficulty is termed as the gap problem and it is one of the main problems in Arnold diffusion. To avoid this difficulty in his example, Arnold chooses a very special perturbation so that all the invariant tori (resonant or non-resonant) preserve under the perturbation, hence achieving diffusion of arbitrary size. Subsequent results on Arnold diffusion all ignored the gap problem, resulting in much limited results and leaving much to be desired. In some cases ([7], [12], [15], [16]) one obtains diffusion in a weaker sense of size depending on perturbations, while in other cases [5], relying on the density of surviving invariant tori in certain restricted region in the space, one obtains stronger diffusion of length order one (independent of small perturbation), but very limited physical size.

We solve this gap problem for the most interesting and common cases where the invariant tori are two-dimensional for the Hamiltonian flow (one-dimensional for symplectic map). We use variational method and it is based on a generalization to a higher dimensional setting of the Aubry-Mather theory on twist maps (cf. [3]) and Mather's theory on connecting orbits of action-minimizing sets [10]. One of the most prominent feature of our method is that, instead of using transition tori alone for diffusion as in Arnold's mechanism, we also use the cantori, which are cantor sets and the remains of the broken invariant tori, for transition and diffusion. We call this mechanism of diffusion Mather's mechanism, which can be thought of as a generalization of Arnold's mechanism.

We think that the variation method and Mather's mechanism is more natural in studying Arnold diffusion and various instability problems in Hamiltonian dynamics. This is because that the hyperbolicity, often required in geometric method, is lacking in diffusion problems. However, variational method requires much weaker hyperbolicity and much weaker smoothness assumptions. Many results that are known to be difficult to obtain with geometric method can be obtained in our settings with relative ease. Recently, Mather was able to construct orbits with infinite energy for Lagrangian systems on a torus using variational method. Our results in this paper can be regarded also as a generalization of this remarkable work.

We illustrate our method and ideas in the so-called *a priori* unstable systems. We consider a near integrable Hamiltonian system of the form:

$$H_\epsilon(p,q,I,\theta,t) = F(x) + G(I) + \epsilon H^1(x,I,\theta,t) \tag{1}$$

Where we assume that $x \in M$ for some symplectic manifold M^{2n} with symplectic form ω_M, $I \in \mathbb{R}$ and $\theta \in S^1$ are a pair of action angle variables and H_ϵ is periodic in t with period one. We also assume the non-degeneracy condition

$$G'(I) > 0 \ \text{ for } \ I \in \mathbb{R}$$

For appropriate change of coordinates, we may assume that $G(I) = I^2/2$.

For $\epsilon = 0$, the Hamiltonian $H_0(x, I, \theta, t) = F(x) + G(I)$ is time independent and it defines a decoupled flow on $M \times (\mathbb{R} \times S^1)$. The Hamiltonian flow on M is given by the vector field X_F, where X_F is defined by $\omega_M(X_F, \cdot) = dF(\cdot)$. We assume that the Hamiltonian vector field X_F has a hyperbolic periodic point at $p \in M$ and p is connected to itself by its stable and unstable manifolds, or in other words, $W^s(p) \equiv W^u(p)$.

The flow on $\mathbb{R} \times S^1$ defined by the Hamiltonian $G(I)$ is completely integrable. For each fixed constant c, the circle $I = c$ is left invariant by the flow.

Since the perturbation of the Hamiltonian system is time periodic, it is convenient to reduce the system to a symplectic map. For this purpose, we fix a cross section Σ^{t_0}, for some $t_0 \in [0, 1)$, define by

$$\Sigma^{t_0} = \big\{(x, I, \theta, t) \in M \times \mathbb{R} \times S^1 \times S^1 \mid t = t_0\big\}$$

Let P_ϵ be the Poincaré map, of the Hamiltonian H_ϵ, defined on Σ^{t_0}. P_ϵ preserves the symplectic form $\omega_M + dI \wedge d\theta$.

For simplicity of notations, we identify points on Σ^{t_0} with the points on $M \times \mathbb{R} \times S^1$. In terms of the Poincaré map P_ϵ for $\epsilon = 0$, the invariant set $A_0 = \{p\} \times \mathbb{R} \times S^1$ is normally hyperbolic. This normally hyperbolic invariant set is foliated by invariant circles of the form $T_c = \{p\} \times \{I = c\} \times S^1 \subset \Sigma^{t_0}$. Each invariant circle is connected to itself by a P_0 invariant $n+1$ dimensional manifold which serves as both the stable manifold and the unstable manifold of T_c.

We are interested in what happens when we perturb the map P_0 to P_ϵ for $\epsilon \neq 0$. Restricting to a bounded domain, say $I \in [a, b] \subset \mathbb{R}$, the normally hyperbolic invariant manifold A_0 persists under small perturbations. Let A_ϵ be the new P_ϵ invariant normally hyperbolic manifold. Then P_ϵ restricted to A_ϵ is area preserving and with the nondegeneracy assumption, KAM theory states that all the invariant curves with diophantine rotation numbers of fixed diophantine constants survive under small perturbations. Let $T_{c,\epsilon}$ be one surviving invariant torus with a diophantine rotation number c. Before the perturbation, $W^s(T_{c,\epsilon}) \equiv W^u(T_{c,\epsilon})$ for $\epsilon = 0$. For $\epsilon \neq 0$ small, $W^s(T_{c,\epsilon})$ typically intersects $W^u(T_{c,\epsilon})$ transversally at some point. This type of transversal intersections results in some very complicated dynamics.

A P_ϵ-invariant torus $T_{c,\epsilon}$ (invariant circle in this particular case) on the normally hyperbolic invariant set A_ϵ is said to be a transition torus if the stable manifold of $T_{c,\epsilon}$ intersects transversally the unstable manifold of $T_{c,\epsilon}$ at some point. A sequence of transition tori $T_{c_1,\epsilon}$, $T_{c_2,\epsilon}$, $\ldots$, $T_{c_k,\epsilon}$ is said to form a transition chain if the stable manifold of $T_{c_i,\epsilon}$ intersect transversally the unstable manifold of $T_{c_j,\epsilon}$ for all $|i - j| = 1$. We define the length of the chain to be $\max_{1 \le i,j \le k}\{|c_i - c_j|\}$.

The concept of transition chain was introduced by Arnold to construct unstable near-integrable Hamiltonian systems. By introducing the concept of obstructing sets, Arnold was able to show that for any two small neighborhoods of two transition tori in a transition chain, one can find an orbit that connects these two neighborhoods.

Since any given KAM torus $T_{c,\epsilon}$ is usually non-isolated, at least for small perturbations, it is easy to construct transition chains of small lengths as long as one can find just one transition torus. Hence one can easily find complicated dynamics associated to transition chain. However, to apply the above mechanism to show instabilities in near integrable Hamiltonian systems, one needs to construct transition chain of arbitrary fixed length. A detailed estimate shows that the resonant gaps between invariant KAM tori can be as large as of order $O(\sqrt{\epsilon})$, while the transversality in the stable manifold and the unstable manifold, as best as one can hope, is only of order $O(\epsilon)$, smaller than the gap size for small ϵ. Hence one can only obtain transition chain of limited size, due to breaking down of invariant tori in resonance gap.

The main result of this paper is that we can overcome the above apparent difficulties by introducing a new approach: the variational method. The variational approach has been successfully applied to the study of the twist maps in the so-called Aubry-Mather theory. In the twist maps, one obtains a collection of action-minimizing orbits, known as Aubry-Mather sets, enjoying many interesting properties. Mather was able to further obtain the connecting orbits among these action minimizing orbits whenever there is no obvious topological obstruction.

We first construct a local variational principle near the homoclinic loop of p, using the fact that the stable manifold and unstable manifold of the hyperbolic fixed point $p \in M$ are Lagrangian submanifolds in the symplectic manifold M. All of orbits we construct are action minimizing in the local sense. To construct the connecting orbits between the action minimizing orbits, we generalize Peierls' Barrier functions to high dimensions. It turns out that the barrier function measures the splitting of the stable and unstable manifold of these normally hyperbolic invariant tori and cantori. The gradient of this Barrier function, in first order approximation, is precisely the so-called Poincaré-Melnikov function (vector). This enable us to verify our barrier conditions in specific systems using the Poincaré-Melnikov functions.

The variational method also enables us to obtain some fine structure and to unfold the underlying complicated dynamics. These structure are known to be hard to obtain with the traditional geometric method. As an example, we can prove that transition chains are transitive and stable manifold of any transition torus intersects transversally the unstable manifold of any other transition torus at some point. In fact, we can construct orbit that "shadows" any sequence of transition tori in a transition chain. Thus diffusion can be observed and understood without the help of the obstructing sets, as introduced by Arnold. Existence of a large number of different types of periodic points, symbolic dynamics, etc., can all be obtained with relative ease.

In the following sections, we will introduce our main ideas and outlines of the proofs of our main results. Complete results and proofs will appear elsewhere.

The author is grateful to Professor John Mather for his interest and patience in listening and discussing with the author on this and other works, and for making useful suggestions.

2 A variational principle

In this section, we construct a local variational principle near homoclinic loops of the hyperbolic periodic point p. The construction relies on the fact that stable and unstable manifolds of any periodic points are lagrangian submanifolds of the ambient symplectic manifold.

We begin with the integrable case, $\epsilon = 0$. In this case, the system is decoupled, the second component of the system in (I, θ) is a integerable twist map and it has a natural variational principle $h_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $h_2(x_1, x_2) = (x_1 - x_2)^2/2$.

The first component of the system in M is more complicated and needs a little more work. It may not have a global variation principle for the Poincar'e map. We can construct a local one in a small neighborhood of the homoclinic loop of p. Topologically, $W^(p)$ $(\equiv W^u(p))$ is homeomorphic to S^n with two points, both corresponding to p, identified. Lifting p to two distinct points and labeling these two points p^- and p^+, we obtain S^n topologically, with poincaré map moves the points near p^- to p^+. By a global Darboux theorem [14], a small neighborhood of S^n is symplecticly diffeomorphic to a small neighborhood of the zero section of the cotangent bundle T^*S^n. It turns that the Poincaré map in this small neighborhood can be obtained by a generating function $h_2 : S^n \times S^n \to \mathbb{R}$. This is based on the following two facts: (1). p is a hyperbolic periodic point, near $p^\pm$, there is a local coordinate system such that the map is given by, for example, a generating function of the type: $h_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with

$$h_1(\mathbf{x_1}, \mathbf{x_2}) = \sum_{i=1}^{n} \frac{1}{2}(x_1^i - x_2^i)^2 + \frac{1}{2}(x_1^i)^2$$

where we assume that all eigenvalues of p are simple and real. The cases with complex eigenvalues and multiple eigenvalues require more careful analysis; (2). Away from the fixed points $p^\pm$, S^n is invariant lagrangian submanifold and all orbits are non-recurrent.

We remark that the case $n = 1$ is easier and since there are two components in the stable (unstable) manifold, one need only to consider one branch.

Let

$$h = h_1 + h_2 : (S^n \times \mathbb{R}) \times (S^n \times \mathbb{R}) \to \mathbb{R}$$

then h gives a local variational principle for the map near the homoclinic loop for $\epsilon = 0$. When $\epsilon \neq 0$, h is slightly perturbed and we no longer have a decoupled system. However, the normally hyperbolic surfaces $\{p^\pm\} \times (\mathbb{R} \times S^1)$, when restricted to bounded domain, persists for small ϵ. Without losing generality, we may assume that these normally hyperbolic surface takes the same form: $\{p^\pm\} \times (\mathbb{R} \times S^1)$.

For simplicity of statements, from now on we assume that our map is extended to whole space $T^*(S^n \times S^1)$, keeping in mind that only these orbits that stay in our original domain give arises to true orbits of our system. Also we will drop the dependences in ϵ in our notations.

3 Action-Minimizing Orbits

We consider the space

$$(S^n \times \mathbb{R})^{\mathbb{Z}} = \{x \mid x : \mathbb{Z} \to (S^n \times \mathbb{R})\}$$

of bi-infinite sequences of points on $(S^n \times \mathbb{R})$ with the usual product topology. An element $x \in (S^n \times \mathbb{R})^{\mathbb{Z}}$ will also be denoted by $(x_i)_{i\in\mathbb{Z}}$ and will be called an orbit, or a configuration, in the configuration space. The orbits in the configuration space may or may not correspond to any true orbits in the phase space. Only the critical configurations have corresponding true orbits.

Given an action function $h : (S^n \times \mathbb{R}) \times (S^n \times \mathbb{R}) \to \mathbb{R}$, we extend h to finite segments $(x_j, \dots, x_k), j < k$ of an orbit x by

$$h(x_j, \dots, x_k) = \sum_{i=j}^{k-1} h(x_i, x_{i+1})$$

We say that the segment $(x_j, \dots, x_k)$ is minimal or action-minimizing with respect to h if

$$h(x_j, \dots, x_k) \leq h(x_j^*, \dots, x_k^*)$$

for all $(x_j^*, \dots, x_k^*)$ with $x_j = x_j^*$ and $x_k = x_k^*$.

An orbit x in the configuration space is said to be minimal or action-minimizing, with respect to h if every finite segment of x is minimal. We denote the set of action-minimizing orbits with respect to h by M_h. It's easy to see that M_h is closed.

Now, we restrict ourselves to the normally hyperbolic surface and consider the minimal orbits in that surface. This is the situation where we have a monotone twist map. This problem has been well-studied and the results are collectively known as the Aubry-Mather theory. We recall some of the basic results.

Let $M_h^{\pm} \subset M_h$ be the set of all minimal orbits that are supported, in the phase space, on the two normally hyperbolic surfaces, say $N^{\pm}$, corresponding to $p^{\pm}$. In other words, if $x = (x_i)_{i\in\mathbb{Z}} \in M_h^{\pm}$, then $\pi_1(x_i) = \{p^{\pm}\} \in S^n$ for all $i \in \mathbb{Z}$. Where $\pi_1 : (S^n \times \mathbb{R}) \to S^n$ is the natural projection into the first component. For simplicity in the notations, we identify x_i for all $i \in \mathbb{Z}$ with its projection into the second component $\mathbb{R}$. M^+ and M^- are identical copies. The following results are well-known.

- For any $x = (x_i)_{i\in\mathbb{Z}} \in M_h^{\pm}$, let $\alpha(x) = \lim_{|i|\to\infty} x_i/i$. The limit always exists and it is called the rotation number for x. Moreover, $\alpha : M_h^{\pm} \to \mathbb{R}$ is a continuous function.
- For any $\alpha \in \mathbb{R}$ the set $M_\alpha^{\pm} = \{x \in M^{\pm} \mid \alpha(x) = \alpha\}$ is not empty.
- If C is an invariant curve for the twist map on $N^{\pm}$, with rotation number α, then the lift of all the orbits in C belongs to $M_\alpha^{\pm}$, i.e., all orbits in the invariant curve are minimal.
- If $\alpha \in \mathbb{Q}$, then $M_\alpha^{\pm}$ can be decomposed into three subsets: (1). the set of all minimal periodic points of period $\alpha = p/q$, still labeled as $M_\alpha^{\pm}$; (2). the set of all minimal orbits whose α-limit set is smaller that its ω-limit set. We label this set $M_{(p/q)^+}^{\pm}$ and (3). the set of all minimal orbits whose α-limit set is larger that its ω-limit set. We label this set $M_{(p/q)^-}^{\pm}$.

In various stability problems in Hamiltonian systems, it is very important and desirable to construct orbits that connect one region or invariant set to another set or region. It is often also important to construct orbits that visit prescribed sequences of regions in the phase space. In his remarkable works on the monotone twist maps, Mather was able to obtain various connecting orbits between minimal orbits whenever there is no obvious topological obstruction. Since Mather's work is very instrumental in our construction of Arnold diffusion, we need to recall his results first.

Consider the monotone twist map f on the cylinder N. N is either N^+ or N^-. Let Γ_1 and Γ_2 be two f-invariant homotopically non-trivial Jordan curve on N, $\alpha(\Gamma_1) < \alpha(\Gamma_2)$. Where $\alpha(\Gamma_1)$ and $\alpha(\Gamma_2)$ are rotation numbers of Γ_1 and Γ_2 respectively. Assume that there is no such invariant curve for any rotation number α, $\alpha(\Gamma_1) < \alpha < \alpha(\Gamma_2)$. The region bounded by Γ_1 and Γ_2 are called the *Birkhoff region of instability.*

Theorem 3.1 (Mather) *Suppose $\alpha(\Gamma_1) < \alpha_1, \alpha_2 < \alpha(\Gamma_2)$. Then there is an orbit of f whose α-limit set lies in M_{α_1} and whose ω-limit set lies in M_{α_2}. Furthermore, if $\alpha(\Gamma_1)$ (resp. $\alpha(\Gamma_2)$) is irrational, then this conclusion still holds with the weaker hypothesis $\alpha(\Gamma_1) \leq \alpha_1, \alpha_2$ (resp. $\alpha_1, \alpha_2 \leq \alpha(\Gamma_2)$).*

Moreover, for each $i \in \mathbb{Z}$ a real number $\alpha(\Gamma_1) \leq \alpha_i \leq \alpha(\Gamma_2)$ and a positive number ϵ_i, there exists an orbit in the phase space $(\dots, P_j, \dots)$ and an increasing bi-infinite sequence of integers $j(i)$ such that distance between $P_{j(i)}$ and M_{α_i} is smaller than ϵ_i.

The connecting orbits Mather constructed are constraint minima. The main technical difficulty is to construct the constraints so that the constraint minima do not bump up against the constraints. i.e., the constraint minima have to take place in the interior of the constraints rather than on the boundary. Therefore, certain *a priori* estimates on the boundary of the constraints are required. One of the important idea here is the introduction of the so-called *Peierl's energy barrier*. We shall discuss the energy barrier and it's generalizations in the next section.

4 Barrier functions

In this section, we define Peierl's energy barrier function. Our definition is different from that of Mather's. We choose this definition so that it works in high dimensions. When applying our definition to twist maps, ours is consistent with that of Mather's, even though it appears a little bit different. Mather also has given a generalization of the barrier function to high dimensions in terms of action-minimizing measure (cf. [8], [9], [10],[11]). Ours is different from that generalization.

Now we come back to the full system. Our construction of diffusion orbits are based on two types of action-minimizing orbits. The first type is the one we already discussed: for any given $\alpha \in \mathbb{R}$, we have the action-minimizing set $M_\alpha^\pm$. The second type of action-minimizing set is the set of connecting orbits between M_α^- and M_α^+. This is a set of action-minimizing orbit whose α-limit set is contained in M_α^- and whose ω-limit set is contained in M_α^+. We denote this set by $M_{(0^+,\alpha)}$, where 0^+ indicates the rotation number of the action-minimizing orbit in its first component. In this notation, the rotation numbers for the sets $M_\alpha^\pm$ are both $(0,\alpha)$. We can also denote the union of the two sets $M_\alpha^\pm$ by $M_{(0,\alpha)}$. It is easy to show that the set $M_{(0^+,\alpha)}$ is non-empty.

For any rotation vector $\alpha=(\alpha_1,\alpha_2)$, where $\alpha_1\in\{0,0^+\}$ and $\alpha_2\in\mathbb{R}$, we fix a minimal orbit $x^\alpha=(x_i^\alpha)_{i\in\mathbb{Z}}\in M_\alpha$. For any $a\in(S^n\times\mathbb{R})$, define

$$P_\alpha(a)=\inf_{x^*}\sum_{i\in\mathbb{Z}}\big(h(x_i^*,x_{i+1}^*)-h(x_i^\alpha,x_{i+1}^\alpha)\big)$$

where the infimum is taken among all $x^*\in(S^n\times\mathbb{R})^{\mathbb{Z}}$ such that (1). $x_0^*=a$ and (2). the α-limit set and ω-limit set of x^* are both contained in the closure of M_α.

The infinite series in the above definition may not necessarily be convergent in the usual sense. The above summation is taken in the sense of $(C,1)$ summation. Recall that an infinite series $\sum_{i=1}^\infty a_i$ is said to be $(C,1)$ summable to a real value s if

$$s=\lim_{n\to\infty}(s_1+s_2+\ldots+s_n)/n,$$

where s_1, s_2, …, are partial summations $s_k=(a_1+a_2+\cdots+a_k)$. A convergent series is always $(C,1)$ summable with the same limit.

We remark that the infimum in the definition of $P_\alpha(a)$ can always be realized by an orbit $x^*=(x_i^*)_{i\in\mathbb{Z}}$ with $a=x_0^*$.

$P_\alpha(a)$ is called the energy barrier function. $P_\alpha(a)$ depends on the rotation number α, but it does not on the specific minimal orbit $x^\alpha\in M_\alpha$ used in the definition. $P_\alpha(a)\geq 0$ for all α and a. $P_\alpha(a)=0$ if and only if $a=x_0$ for some $x=(x_i)_{i\in\mathbb{Z}}\in M_\alpha$.

It is easy to see that the barrier function $P_\alpha(a)$ is continuous with respect to $a\in(S^n\times\mathbb{R})$. Its regularity with respect to the rotation number is more complicated. For $\alpha=(0,\alpha_2)$ and $a=(p^\pm,a_2)$, it can be shown that the value of $P_\alpha(a)$ is the same as those defined by Mather and the barrier function $P_\alpha(a)=P_{(0,\alpha_2)}(p^\pm,a_2)$ is continuous at every point $\alpha_2\notin\mathbb{Q}$ and continuous from one side for all $\alpha_2=(\frac{p}{q})^\pm$ for integers p,q. The barrier function is typically discontinuous at the rational points $\alpha_2=\frac{p}{q}$. For $\alpha=(0^+,\alpha_2)$, one can show that the function $P_\alpha(a)$ is continuous at every point $\alpha_2\in\mathbb{R}$.

Fix a rotation vector α. The barrier function $P_\alpha(a)$ is said to have a nondegenerate local minimum at $a^*\in S^n\times\mathbb{R}$ if there exists a neighborhood U of a^*, contractible to a point, such that $P_\alpha(a^*)\leq P_\alpha(a)$ for all $a\in U$ and $P_\alpha(a^*)<P_\alpha(a)$ for all $a\in\partial U$, where ∂U is the non-empty boundary of U. The orbit that realizes $P_\alpha(a^*)$ is a local minimal and it gives arises to a true orbit in the phase space.

In order to construct long connecting orbits, we first construct orbits that connects nearby minimal orbits. For this purpose, we define the joint barrier function.

For any two rotation vectors, α and α', define

$$\begin{aligned}P_{(\alpha,\alpha')}(a) &= \inf_{x^*}\sum_{i=-\infty}^{0}\big(h(x_i^*,x_{i+1}^*)-h(x_i^\alpha,x_{i+1}^\alpha)\big)\\ &+ \inf_{x^*}\sum_{i=0}^{\infty}\Big(h(x_i^*,x_{i+1}^*)-h(x^{\alpha'}{}_i,x_{i+1}^{\alpha'})\Big)\end{aligned}$$

where the infimum is taken among all $x^* \in (S^n \times \mathbb{R})^{\mathbb{Z}}$ such that (1). $x_0^* = a$; (2). the α-limit set is contained in the closure of M_α and (3). the ω-limit set of x^* is contained in the closure of $M_{\alpha'}$.

Same as in the definition of the barrier function, the above summation is in the sense of $(C, 1)$. Unlike the barrier function, this joint barrier may take negative values.

Fix $\alpha = (0^+, \alpha_2)$. Let a^* be a nondegenerate local minimum for $P_\alpha(a)$ and let U be the open set such that $P_\alpha(a^*) < P_\alpha(a)$. By the continuity of $P_\alpha(a)$, for α_2' sufficiently close to α_2, $P_{\alpha'}(a)$ also has a local minimum in U, where $\alpha' = (0^+, \alpha_2')$. We can further show that the joint barrier function $P_{(\alpha,\alpha')}(a)$ has a nondegenerate local minimum in U too. This provides us with the existence of local minimum orbits that connect nearby action-minimizing sets, provided that the barrier function has a nondegenerate local minimum. We can summarize this in the following lemma:

Lemma 4.1 *Let α be a real number. Assume that the barrier function $P_{(0^+,\alpha)}(a)$ has a nondegenerate local minimum in some contractible open set U, then there exists a positive number $\delta > 0$ such that if $|\alpha' - \alpha| \leq \delta$ then there exists a local minimum orbit, through the interior of U, that connects $M_{\alpha_1}^-$ to $M_{\alpha_2}^+$.*

To obtain diffusion of arbitrary length, we need to join two or more connecting orbits of the above type. Here the idea is to put barriers very close to $M_\alpha^\pm$. However, a better setting for this construction perhaps would be to lift the phase space to infinite to one covering so that the preimage of p consists of $\cdots$, p_{-1}, p_0, p_1, $\cdots$, one then construct connecting orbits through the normal hyperbolic surfaces for each p_i. In the current setting, p has only two to one covering (p^+ and p^-). We state our main results as follows.

Theorem 4.2 *Suppose that $P_{(0^+,\alpha)}(a)$ has a nondegenerate local minimum in some open set U_α for every $\alpha \in [A, B] \subset \mathbb{R}$, then for any $\alpha_1, \alpha_2 \in [A, B]$, there is an orbit in the phase space whose α-limit set lies in $M_{\alpha_1}^-$ and whose ω-limit set lies in $M_{\alpha_2}^+$.*

Moreover, for each $i \in \mathbb{Z}$ a real number $A \leq \alpha_i \leq B$ and a positive number ϵ_i, there exists an orbit in the phase space $(\ldots, P_j, \ldots)$ and an increasing bi-infinite sequence of integers $j(i)$ such that distance between $P_{j(i)}$ and $M_{(0,\alpha_i)}$ is smaller than ϵ_i.

We finish this section by making the following remarks:

(1). The condition that $P_{(0^+,\alpha)}(a)$ has a nondegenerate local minimum in some open set U_α for every $\alpha \in [A, B] \subset \mathbb{R}$ is an open and dense condition in any smooth or analytic topology.

(2). If $M_{(0,\alpha)}$ is an invariant torus, then for near integrable systems where the perturbation is small, $P_{(0^+,\alpha)}(a)$ measures the splitting of the stable manifold and the unstable manifold of $M_{(0,\alpha)}$. In fact, over a fixed compact neighborhood of a in S^n not containing $p^\pm$, $W^s(M_\alpha)$ and $W^u(M_\alpha)$ are horizontal lagrangian submanifolds, and thus are gradients of some potential functions. $P_{(0^+,\alpha)}(a)$ is precisely the difference of these two potential functions.

(3). The barrier function $P_{(0^+,\alpha)}(a)$ can be estimated, to its first order, by the so-called Poincaré-Melnikov integrals in this *a priori* unstable setting (cf. [13]). In *a priori* stable cases, estimating $P_{(0^+,\alpha)}(a)$ is a much more difficult problem, often requiring very delicate analysis.

5 Arnold's example

Arnold considered the following periodically forced two degree of freedom Hamiltonian system of the form $H = H_0 + \epsilon H_1$, where

$$\begin{aligned} H_0 &= \frac{1}{2}(I_1^2 + I_2^2) \\ H_1 &= (\cos\phi_1 - 1) + \mu P \\ P &= (\cos\phi_1 - 1)(\sin\phi_2 + \cos t) \end{aligned}$$

For $\mu = 0$ and $\epsilon > 0$, the system decouples into a pendulum and a rotor. One obtains a hyperbolic (weakly) periodic point from the pendulum. The normally hyperbolic invariant tori are in the surface $I_1 = \phi_1 = 0$ For $\mu \neq 0$, Arnold proved the following theorem:

THEOREM 5.1 (Arnold) *Assume $0 < A < B$. For every $\epsilon > 0$ we can find a $\mu_0 > 0$ such that for $0 < \mu < \mu_0$ the system is unstable: there exists a trajectory which connects the region $I_2 < A$ with the region $I_2 > B$.*

A notable feature of Arnold's example, which makes the system much easier to analyze, is that the perturbation term is specificly chosen so that, for $\mu > 0$, it has a factor $(1-\cos\phi_1)$ which vanishes on the normally hyperbolic surface $I_1 = \phi_1 = 0$. This implies that all invariant tori on the surface survive the perturbation. Hence one does not ncounter the difficulties associated with breaking of invariant tori and gaps in the resonant zones.

To apply our results to this setting, we may choose arbitrary perturbation function P, as long as the Melnikov potential has a non-degenerate local minima for every I_2, for $A \leq I_2 \leq B$. One easier example would be just taking $P = \sin\phi_2 + \cos t$. Indeed in this case, the Poincaré-Melnikov integrals have non-degenerate local minima (cf. [5]). Thus all the results in Arnold's theorem hold in this case too.

References

1. Arnold, V.I., *Instabilities of dynamical systems with several degrees of freedom.* Sov. Math. Dokl., 5, 581-585(1964).

2. Arnold, V.I. (ed), *Dynamical Systems III*, Encyclopaedia of Mathematical Sciences, Vol. 3, Springer-Verlag, New York/Berlin, 1998.

3. Bangert, V., *Mather sets for twist maps and geodesics on tori*, Dyn. Rep., 1, 1-56(1988).

4. Bessi, U., *An approach to Arnold diffusion through calculus of variations*, Nonlinear Analysis, 26(6), 1115-1135(1996).

5. Chierchia, L. & Gallavotti, G., *Drift and diffusion in the phase space.* Ann. de la Inst. H. Poincaré Phys. Th., bf 60(1), 1-144 (1994).

6. Douady, R., *Stabilité ou instabilité des points fixes elliptiques*, Ann. scient. Éc. norm. sup., 21(4), 1-46(1988).

7. Holmes, P., & Marsden, J., *Melnikov method and Arnold diffusion for perturbation of integrable Hamiltonian systems*, J. Math. Phys., 234, 669-675(1982).

8. Mañé, R., *Generic properties and problems of minimizing measures of Lagrangian systems*, Nonlinearity, 9, 273-310(1996).

9. Mather, J., *Action minimizing invariant measures for positive definite Lagrangian systems*, Math. Z., 207, 169-207(1991).

10. Mather, J., *Variational construction of orbits of twist diffeomorphisms*, J. Amer. Math. Soc., 4, 207-263(1991).

11. Mather, J., *Variational construction of connecting orbits*, Ann. Inst. Fourier, Grenoble, 435, 1349-1386(1993).

12. Moeckel, R., *Transition tori in five-body problem*, J. Diff. Eqns., 129, 290-314(1996).

13. Robinson, C., *Horseshoes for autonomous Hamiltonian systems using Melnikov integrals,* Ergodic Theory Dynamical Systems, 8, 395-409(1988).

14. Weinstein, A., *Lagrangian submanifolds and hamiltonian systems*, Ann. Math. 98, 377-410 (1973).

15. Xia, Z., *Arnold diffusion in elliptic restricted three-body problem*, J. Dyn. Diff. Eqns, bf 5(2), 219-240 (1993).

16. Xia, Z., *Arnold diffusion and oscillatory solutions in planar three-body problem*, J. Diff. Eqns., 110, 289-321 (1994).

Zhihong Xia
Department of Mathematic
Northwestern University
Evanston, Illinois 60208
xiamath.nwu.edu