Elements of a Qualitative Theory of Hamiltonian PDEs

Sergei B. Kuksin

We discuss nonlinear Hamiltonian partial differential equations (PDEs) and consider the finite-volume case only. That is, we are concerned with equations for functions (or vector-functions) $u(t,x)$, where the space-variable x belongs to a bounded domain and the equations are supplemented by appropriate boundary conditions. We treat them as ordinary differential equations in infinite-dimensional function spaces formed by functions of x and assume that they can be written in the Hamiltonian form:

$$\dot{u}(t) = J\nabla H(u(t)). \tag{1}$$

Here J is an anti self-adjoint operator in the space of square-integrable functions, H is a hamiltonian of the equation and ∇H is its L_2-gradient (if H is a functional of the calculus of variations, then ∇H equals to its variational derivative). The equation (1) is Hamiltonian with respect to a symplectic structure, defined in the function space by the form α_2,

$$\alpha_2(\xi(x),\eta(x)) = \langle(-J)^{-1}\xi(x),\eta(x)\rangle_{L_2}.$$

Hamiltonian PDEs are of extreme physical importance since they describe processes without dissipation of energy: (usually) the system's energy equals the hamiltonian H and preserves due to the same trivial arguments as in the finite-dimensional case.

Below we discuss three groups of results concerning qualitative behaviour of Hamiltonian PDEs. We have selected them according to our own taste, the references are by no means complete.

1. Nearly integrable PDEs

Some of nonlinear Hamiltonian PDEs with one-dimensional space variable x in a finite segment, supplemented by appropriate boundary conditions, are known to be integrable. For example, the Korteweg - de Vries equation (KdV) under zero-meanvalue periodic boundary conditions:

$$\dot{u} = \frac{\partial}{\partial x}(-u_{xx}+3u^2), \qquad x\in S^1=\mathbb{R}/2\pi\mathbb{Z}, \quad \int_0^{2\pi} u(t,x)\,dx = 0, \tag{KdV}$$

and the Sine-Gordon equation (SG) under Dirichlet boundary conditions:

$$\text{(SG)} \qquad \ddot{u} = u_{xx} - A\sin Bu, \qquad u(t,0) = u(t,\pi) = 0,$$

where $A, B > 0$. We view the equations as dynamical systems in appropriate Sobolev spaces Z^s, formed by functions $u(x)$ which respect the boundary conditions. (For the KdV, Z^s is the Sobolev space $H_0^s(S^1)$, formed by zero-meanvalue functions on the circle S^1. For the SG equation Z^s is the Sobolev space formed by odd 2π-periodic functions – these functions vanish for $x = 0$ and $x = \pi$).

Integrability of the KdV equation manifests itself in the following properties of a dynamical system which the equation defines in the spaces Z^s, discovered twenty years ago by P.Lax and S.P.Novikov (see [DMN]): For $n = 1, 2, \dots$ the space $\cap Z^s$ contains a smooth $2n$-dimensional manifold $\mathcal{T}^{2n}$, invariant for the KdV-flow, such that:

a) restriction of the equation to $\mathcal{T}^{2n}$ defines a Liouville-Arnold integrable Hamiltonian system,

b) $\mathcal{T}^{2n} \subset \mathcal{T}^{2m}$ if $m > n$,

c) union of all manifolds $\mathcal{T}^{2n}$ is dense in each space Z^s.

For the SG equation everything is much the same but the manifolds $\mathcal{T}^{2n}$ have algebraic singularities and their union is only proven to be dense in the vicinity of the origin.

The invariant manifolds $\mathcal{T}^{2n}$ are filled with time-quasiperiodic solutions $u_n(t,x)$ (so-called *n-gap solutions*). An n-gap solution u_n depends on an n-dimensional action $p \in \mathbb{R}_+^n$ and on n-dimensional angle $q \in \mathbb{T}^n$: $u_n(t,x;p,q) = \Phi_n(W_p t + q, x, p)$. The function $\Phi_n(q,x,p)$ is analytic and can be explicitly written in terms of theta-functions (the Its-Matveev formula, see [DMN]); this is another manifestation of integrability of KdV and SG equations. The n-vector W_p is called the *frequency vector*. The union in q and t of the curves $u_n(t,\cdot;p,q)$ is a smooth invariant n-torus in the space $\cap Z^s$, called the *n-gap torus*.

1.1. The problem of persistence. Since both KdV and SG equations do not arise in mathematical physics in their exact form (as, for example, Navier - Stokes equations do), but only present simplified forms of some real physical equations, then it is important to understand if the finite-gap solutions u_n have something to do with "real" equations. Assuming that a "real" equation is Hamiltonian, that (say) the KdV equation comprises its highest derivatives and that the equation is local[1] (i.e. it does not contain integral terms), we write it as

$$\dot{u} = \frac{\partial}{\partial x}\left(-u_{xx} + 3u^2 + \varepsilon \frac{\partial}{\partial u} h(u,x)\right), \tag{2}$$

where the function h is assumed to be analytic in u.

1.2. KAM for PDEs. The question we posed in the previous section can be understood in the following way: Does a finite-gap solution $u_n(t,x)$ of the KdV equation persist as a time-quasiperiodic solution for equation (2) (i.e., does (2)

[1] this assumption is imposed only for simplicity

have a time-quasiperiodic solution u_n^ε, close to u_n)? The affirmative answer is given by the following KAM for PDEs theorem:

For most (in the sense of measure) values of the action p, the n-gap solution $u_n(t,x;p,q)$ for the KdV-equation persists as a time-quasiperiodic solution $u_n^\varepsilon(t,x;p,q)$ for equation (2). The solution u_n^ε is linearly stable. Its closure in any space Z^s forms an invariant smooth n-torus.

The persisted solutions u_n^ε have the form $u_n^\varepsilon(t,x;p,q)=\Phi_n^\varepsilon(W_p^\varepsilon t+q,\, x,\, p)$. The new frequency vector W_p^ε is $O(\varepsilon)$-close to W_p and the function Φ_n^ε is $O(\varepsilon^\rho)$-close to Φ_n for any $\rho<1$. In particular, for most p the theta-formula for an n-gap solution with the corrected frequency vector gives the function $\Phi_n(W_p^\varepsilon t+q,\, x,p)$, which is *forever* $O(\varepsilon^\rho)$-close to an exact solution of (2). The corrected frequency vector is $W_p^\varepsilon=W_p+\varepsilon W_p^1+o(\varepsilon)$, where W_p^1 equals to averaging along the corresponding n-gap torus of a vector-function, constructed in terms of a hamiltonian of the perturbation.

A union (in n, p and q) of all persisted solutions, treated as curves in a space Z^s, becomes dense in Z^s as $\varepsilon\to 0$.

Similar results hold for the perturbed SG equation:

$$u_{tt}=u_{xx}-A\sin Bu+\varepsilon g(u,x)=0. \tag{3}$$

The differences are that, first, large-amplitude solutions both for SG equation and for (3) are not linearly stable and, second, we do not know if the persisted solutions jointly are asymptotically dense as $\varepsilon\to 0$.

The KAM-theorem for PDEs is an infinite-dimensional version of the classical finite-dimensional theorem due to Kolmogorov-Arnold-Moser. Essential difference is that in the finite-dimensional case persisted time-quasiperiodic solutions fill the phase-space up to a set of small measure, while in the PDE-case the solutions which persist *due to the theorem* jointly have zero measure (for any reasonable measure in the corresponding function space).

The theorem we discussed in this section applies to quasilinear perturbations of all "classical" integrable PDEs with one-dimensional space variable, including all equations from the KdV hierarchy, etc. It is based on an abstract infinite-dimensional KAM-theorem. For exact statements and proofs see [K1, K3, K4, P1].

It is unknown what happens to infinite-gap solutions (and the corresponding infinite-gap tori) under Hamiltonian perturbations of the integrable equations.

1.3. SMALL OSCILLATIONS. The persistence problem posed in section 1.1 admits another understanding: does a small-amplitude finite-gap solution for the KdV or for SG equation persist after we have perturbed the equation by a higher-order at zero term? The affirmative answer follows from the same abstract KAM-theorem which implies the results of the previous section, see [BoK]. In particular, since $\sin Bu=Bu-B^3u^3/6+O(u^5)$, then most of small amplitude finite-gap solutions of the SG equation (with appropriate A and B) persist in the φ^4-equation:

$$u_{tt}=u_{xx}-mu+\gamma u^3,\quad m,\gamma>0. \tag{φ^4}$$

Small solutions for this (and similar) equations can be also constructed treating (φ^4) as a perturbation of another integrable infinite-dimensional system, namely its Birkhoff normal form at zero, see [KP, P2] (also see [W] and the Introduction to [K2] for related results).

Nothing is known about small time-quasiperiodic solutions for the φ^4-equation with $m = 0$.

1.4. Closely related persistence problem arises when we examine an *equation (1) with a small nonlinearity* and with a linear part with pure imaginary spectrum in order to prove that time-quasiperiodic solutions of the linear equation persist in the nonlinear equation (1). If the linear equation depends on an additional finite-dimensional parameter in a non-degenerate way, then any its time-quasiperiodic solution persists in the nonlinear equation for most values of the parameter, provided that: 1) the space-variable x belongs to a finite segment and 2) the perturbed equation is quasi-linear (i.e., the nonlinear term of (1) contains less derivatives than its linear part). – This follows from the same abstract KAM-theorem as above, see [K2, P1].

Some years ago J.Bourgain [B1] developed another KAM-approach, originally proposed by Craig - Wayne in [CW], and successfully used it to study the persistence problem which we discuss in this subsection. The main advantage of this approach is that it applies to two-dimensional (in space) Schrödinger equation. A disadvantage is that it applies only to semilinear equations (i.e. to equations where the nonlinear term contains no derivatives).

We do not know what happens to invariant tori of an n-dimensional (in space) linear Schrödinger equation with $n \geq 3$ and of a linear wave equation with $n \geq 2$ under Hamiltonian perturbations.

1.5. Averaging theorems. Due to the KAM-results presented in section 1.2, the perturbed KdV equation (2) contains invariant finite-dimensional tori, filled with linearly stable time-quasiperiodic solutions, and union of these tori is asymptotically dense in any space Z^s as $\varepsilon \to 0$. Hence, any solution of equation (2) with sufficiently small ε for long time stays close to some n-gap torus. This result does not specify the persistence time. For a *finite-dimensional* nearly-integrable system this time is known to grow at least as $\exp \varepsilon^{-a}$, $a > 0$ (Nekhoroshev's theorem). To obtain an analogy of this result for equation (2) is an intriguing open problem. What is known is a local theorem which applies to a class of nearly integrable PDEs and states that for solutions of these equations with small analytical initial data the persistence time is bigger than $C_M \varepsilon^{-M}$ for each M, see [Bam] (also see there references for related results concerning some parameter-depending equations with small nonlinearities).

2. Symplectic invariants and Gromov's non-squeezing property.

2.1. Gibbs measure. Flow-maps $\{S_t\}$ of any Hamiltonian PDE (1) preserve the symplectic form α_2 (see the introduction), provided that they are C^1-smooth. For a finite-dimensional Hamiltonian system in the space $(\mathbb{R}^{2N}_{p,q}, dp \wedge dq)$ symplecticity of the flow-maps of a Hamiltonian vector-field yields that they preserve the

Lebesgue measure $dpdq$ as well as the Gibbs measure $\exp(-\mathcal{H}(p,q))dpdq$, where $\mathcal{H}$ is the hamiltonian. In an infinite-dimensional function space $\{u(x)\}$ a Lebesgue measure $du(\cdot)$ does not exist, but the Gibbs measure $\mu_H = \exp(-H(u(\cdot)))\,du(\cdot)$, where H is a hamiltonian of the PDE, often is well defined if $\dim x = 1$ or 2. Its construction is well known from the quantum field theory (see [GJ]). A difficulty is that the measure μ_H is supported by a space of functions of low smoothness. To prove invariance of μ_H, a flow of the equation (1) has to be proven to exist in the corresponding low-smoothness space and to possess some regular properties. This can be done for many one-dimensional and for some two-dimensional equations, see [B2, B5, MV] and references therein.

It is an open problem whether a non-integrable Hamiltonian PDE has an invariant measure, supported by smooth functions (this measure should not be supported by a trivial invariant set like a periodic trajectory of the equation). This problem is closely related to the following question: is it true that high Sobolev norms of typical solutions for a non-integrable Hamiltonian PDE grow with time unboundedly, see [B3, B4].

2.2. Symplectic capacity. The Gibbs measure μ_H corresponds to a subset of the function phase-space of a Hamiltonian PDE a flow-invariant quantity, namely its measure. This is not a unique invariant characteristic of subsets. Existence of another symplectic invariant for finite-dimensional Hamiltonian systems, called *symplectic capacity*, follows from Gromov's non-squeezing theorem (or can be constructed independently to prove the theorem), see in [HZ]. To discuss a version of this invariant applicable to (1), we need a notion of a *Darboux phase-space* Z_D for this equation[2] : Z_D is a Hilbert space which admits an orthonormal Hilbert basis $\{\varphi_j \mid j \in \mathbb{Z}_0\}$ ($\mathbb{Z}_0$ is the set of non-zero integers), which is a Darboux basis for the equation's symplectic structure, i.e., $\alpha_2[\varphi_j, \varphi_{-k}] = \delta_{j,k}$ for any $j \in \mathbb{N}$ and for all k.

Examples. 1) For the KdV equation (and its perturbation (2)), Z_D is the Sobolev space $H_0^{-1/2}(S^1)$. 2) A nonlinear wave equation

$$\ddot{u} - \delta\triangle u + mu + f(u,x) = 0, \quad u = u(t,x),\ x \in \mathbb{T}^n,$$

where $m > 0$ and f is a smooth function, can be written in the following Hamiltonian form:

$$\dot{u} = -Lw, \ \ \dot{w} = Lu + L^{-1}f(u,x), \tag{4}$$

where $L = (-\delta\triangle + m)^{1/2}$. For this equation $Z_D = Z^{1/2} = H^{1/2}(\mathbb{T}^n) \times H^{1/2}(\mathbb{T}^n)$ (see [K5, K6]). 3) If $f = 0$ (so the equation (4) is linear), then any space Z^s is a Darboux space. (On the contrary, for a typical *nonlinear* equation (4) a space Z^s with $s \geq 5$ is not Darboux. It is plausible that $Z^{1/2}$ is the unique Darboux space).

Let Z_D be a Darboux space for a symplectic form α_2 and $\{\varphi_j \mid j \in \mathbb{Z}_0\}$ be its basis as above. A map c which corresponds to an open subset $O \subset Z_D$ a number $c(O) \in [0, \infty]$ is called a (symplectic) capacity if

[2] in fact, for its symplectic structure.

α) c is translationary invariant, i.e., $c(O) = c(O+\xi)$ for $\xi \in Z_D$; β) c is monotonic, i.e. a bigger set has a bigger capacity; γ) c is 2-homogeneous, i.e. $c(\tau O) = \tau^2 c(O)$; δ) $c(B_r) = c(\Pi_r^{(k)}) = \pi r^2$, where B_r is the r-ball in Z_D, centered at the origin, and $\Pi_r^{(k)}$ is the cylinder formed by all vectors $\sum z_l \varphi_l$ such that $z_k^2 + z_{-k}^2 \le r^2$.

A finite-dimensional symplectic space $(\mathbb{R}^{2n}_{p,q}, dp \wedge dq)$ admits a symplectic capacity, invariant for symplectomorphisms [HZ]. A Darboux space Z_D also admits one. This capacity is invariant for flow-maps $\{S_t\}$ of a Hamiltonian equation (1), provided that

$$S_t = \text{linear operator} + \text{compact smooth operator}, \tag{5}$$

where the linear operator is a direct sum of rotations in the planes, spanned by the vectors φ_j and φ_{-j}, $j = 1, 2, \dots$ (see [K5]).

The assumption (5) is met by the nonlinear wave equation (4) if $n = 1$ and $f(u, x)$ has a polynomial growth in u, or $n = 2, 3$ and f as a function of u is a polynomial of a sufficiently low degree, see [K5,K6] and [B4].

The symplectic capacity is an invariant of the flow of a Hamiltonian PDE in a function space of low smoothness, as well as the Gibbs measure. An essential difference between these two invariants is that the former is constructed in terms of the equation's symplectic structure, while the latter – in terms of its hamiltonian (the same is true for the corresponding function spaces, so usually they are different).

An immediate consequence of existence of a symplectic capacity is that the flow-maps $\{S_t\}$, satisfying (5), can not squeeze a ball in a Darboux space Z_D to a cylinder of a smaller radius[3]; cf. the properties α), β) and δ). This is Gromov's non-squeezing property.

On the contrary, the squeezing (and a closely related pulling-through phenomenon, see below) both are possible (and are typical under some circumstances) if we consider the equation in a function space of high smoothness, i.e. study its classical solutions rather than generalised ones. In particular, the flow $\{S_t\}$ of equation (4) in a Sobolev space Z^s, $s \ge 5$, squeezes a typical ball of a radius of order one to a cylinder $\Pi_\rho^{(k)}$ with $\rho \sim (\ln \delta^{-1})^{-1}$, provided that the nonlinearity f is also typical,[4] see [K6].

3. Small-dispersion/dissipation equations

Let us consider the following class of PDEs:

$$\begin{aligned}&\langle\text{non-linear homogeneous Hamiltonian equation}\rangle + \langle\delta_1\text{-small linear damping}\rangle\\ &\qquad + \langle\delta_2\text{-small linear dispertion}\rangle = \zeta(t, x),\end{aligned} \tag{6}$$

where $\delta_1 \ge 0$, $\delta_2 \ge 0$ and $\delta := \sqrt{\delta_1^2 + \delta_2^2} > 0$. If $\delta_1 = 0$, then this equation is Hamiltonian. Still, the most important are equations with $\delta_1 > 0$ since they describe turbulence in different physical media.

[3] It is unknown if the assumption (5) is superfluous and can be dropped.

[4] Clearly, $\rho \ll 1$ if $\delta \ll 1$. So Gromov's property fails in this space.

The Navier-Stokes (NS) equations have the form (6) with the Euler equations for the homogeneous Hamiltonian equation and with $\delta_1 > 0$, $\delta_2 = 0$. Another good example of equation (6) is given by the damped/driven nonlinear Schrödinger equation:

$$\dot{u} - \delta_1 \triangle u + i\delta_2 \triangle u - i|u|^{2p}u = \zeta(t,x), \quad p \in \mathbb{N},\ \delta_1, \delta_2 \geq 0, \tag{7}$$

which we shall consider for $x \in \mathbb{R}^n$, $n \leq 3$, under the odd periodic boundary conditions:

$$u(t,x) = u(t,x_1,\dots,x_j+2,\dots) = -u(t,x_1,\dots,-x_j,\dots) \quad \forall j$$

(they imply that $u(t,x)$ vanishes at the boundary of the cube of half-periods $\{0 \leq x_j \leq 1\}$). It is known that (7) has a unique smooth in x solution for any smooth odd periodic initial data $u(0,x) = u_0(x)$ (and for any continuous in t, smooth odd periodic in x function ζ). We shall discuss qualitative behaviour of solutions for equation (7) in the turbulent limit, i.e. when $\delta \ll 1$. We shall state results for equation (7), using some terminology which comes from the hydrodynamical turbulence, i.e. from the NS equations.

3.1. ESSENTIAL PART OF A PHASE-SPACE. Let us first consider equation (7) with $\zeta = 0$, supplemented by an order-one initial condition

$$u|_{t=0} = u_0(x) \in C^\infty, \quad |u_0|_{L_\infty} = U, \tag{8}$$

$U \sim 1$. Due to a trivial a priori estimate, L_2-norm in x of a solution u decays with t at least as $\exp -\delta_1 t$. Hence, the solution practically vanishes by a time $\gg \delta_1^{-1}$. We are interested in its behaviour for $0 \leq t \leq \delta^{-a}$ with $0 < a \leq 1$.

Denoting by $|u|_m$ the C^m-norm of a function $u(x)$, we define the essential part of the smooth phase-space of equation $(7)|_{\zeta=0}$ (with respect to the C^m-norm, $m \geq 2$) as

$$\mathfrak{A}_m = \{u(x) \in C^\infty \mid u \text{ is odd periodic and } |u|_0^{2m\varkappa+1} < K_m \delta^{m\varkappa} |u|_m\}.$$

Here $\varkappa$ is any fixed number $< 1/3$ and $K_m = K_m(\varkappa)$ is some specific constant. This set is formed by fast oscillating functions since $|u|_m \gg |u|_0$ for any $u \in \mathfrak{A}_m$ if $\|u\|_0 \gtrsim 1$ (when $\delta \ll 1$). The set looks like a narrow tube with respect to the C^m-norm since its intersection with a ball $\{|u|_m \leq R\}$ is contained in the narrow cylinder $\Pi_\rho^{(k)}$, formed by complex functions $u = \sum u_s e^{\pi i s \cdot x}$ such that $|u_k| < \rho$, where $\rho = C_m \delta^{1/2+O(m^{-1})} R^{O(m^{-1})}$.

The set $\mathfrak{A}_m$ is important to understand dynamics of the equation (7) since: *by the time $C_m\delta^{-1}$ the flow of equation $(7)|_{\zeta=0}$ will pull the whole space of smooth odd periodic functions through $\mathfrak{A}_m$.* This pull-through phenomenon can be specified: *a solution u for $(7)_{\zeta=0}$, (8) will visit the set $\mathfrak{A}_m$ by the time $\delta^{-1/3}U^{-4/3}$. By the moment of a first entry to $\mathfrak{A}_m$ the solution will change its supremum-norm no more than twice.*

Hence, by the time $\delta^{-1/3}$ any solution $u(t,x)$ for $(7)_{\zeta=0}, (8)_{U=1}$ will make its C^m-norm as big as $C_m\delta^{-m\varkappa}$.

Equation (7) with $\zeta = 0, \delta_1 = 0$ takes the Hamiltonian form

$$\dot{u} + i\delta\triangle u - i|u|^{2p}u = 0. \tag{7'}$$

It has two integrals of motion: the hamiltonian and the L_2-norm $|u(t,\cdot)|_{L_2}$. Since $|u(t,\cdot)|_{L_\infty} \geq |u(t,\cdot)|_{L_2} =$const, then any non-zero solution for $(7')$ will visit $\mathfrak{A}_m$ during any time-interval longer than $\delta^{-1/3}|u|_{L_2}^{-4/3}$. I.e., $\mathfrak{A}_m$ is a recursion subset for this Hamiltonian PDE.

3.2. BOUNDS FOR AVERAGED HIGH NORMS. It turns out that since C^m-norms of a solution for $(7)_{\zeta=0}, (8)_{U=1}$ become big at least once, then they are big at the average; hence, its Sobolev norms are big at the average as well:

$$\delta^a \int_0^{\delta^{-a}} \|u(t,\cdot)\|_m^2 dt \geq C_m \delta^{-2m\varkappa_m}. \tag{9}$$

Here $a \geq 1/3$, $\varkappa_m = \varkappa_m(a) \nearrow 1/3$ and $\|\cdot\|_m$ stands for the norm in the Sobolev space of odd periodic functions. This estimate is essentially nonlinear since it obviously fails if $p = 0$.

The norms of the solution u satisfy usual upper estimates: if $\delta_2 = 0$ and $\zeta = 0$, then

$$\delta_a \int_0^{\delta^{-a}} \|u(t,\cdot)\|_m^2 dt \leq C'_m \delta^{-m}, \tag{10}$$

where the constants C'_m depend on C^m-norms of the initial condition u_0. We stress that the exponents for δ in the r.h.s.'s of (9) and (10) are universal: they do not depend on the nonlinearity $|u|^{2p}u$, the dimension n and the initial condition u_0.

Estimates similar to (9), (10) remain true for solutions of equation (7) with non-zero forcing ζ if we assume that $\zeta = \zeta^\omega(t,x)$ is a random field, smooth odd periodic in x and stationary in t (such equations are believed to present right mathematical description of physical turbulence, see in [EKMS, K8]): If $u^\omega(t,x)$ is a solution for (7) with, say, zero initial condition at $t = 0$ and $\langle\|u\|_m^2\rangle$ is its averaged squared Sobolev norm, $\langle\|u\|_m^2\rangle = \delta^a \int_0^{\delta^{-a}} \mathbf{E}\|u(t,\cdot)\|_m^2 dt$, then

$$C_m^{-1}\delta^{-2m\nu_m} \leq \langle\|u\|_m^2\rangle \leq C_m \delta^{-2m\mu_m} \quad \text{if} \quad a \geq 1, \tag{11}$$

where $\mu_m \nearrow B < \infty$ and $\nu_m \nearrow A > 0$. Moreover, we know that $\frac{3}{17} < A$, $B \leq \frac{3}{2}$ and that (11) remains true if in the definition of $\langle\|u\|_m^2\rangle$ we replace the time-segment $[0, \delta^{-a}]$ by any segment in $[0,\infty)$, longer than δ^{-a}.

A popular mathematical idealisation of the physically correct forcings ζ as above is given by a random field ζ which is white noise in time [EKMS]. For forcings like that the estimates (11) hold with $A = \frac{1}{2}$, $B = 1$.

An important feature of turbulent behaviour of a solution $u_\delta^\omega(t,x)$ is a short size of its space-scale l_x (see e.g. [LL], § 33 and [CDT]). Defining the space-scale as $l_x = \delta^\gamma$, where

$$\gamma = \gamma(u^\omega) = \liminf_{m\to\infty} \liminf_{\delta\to 0} \frac{\ln\langle\|u_\delta\|_m^2\rangle^{1/2m}}{\ln \delta^{-1}}$$

(see [K8]), we get from (11) that $A \leq \gamma \leq B$.

3.3. ASYMPTOTICAL SPECTRAL PROPERTIES OF SOLUTIONS AND THE KOLMOGOROV - OBUKHOV LAW. The estimates for the space-scale l_x of a solution $u^\omega(t,x)$, discussed above, characterise its infinitesimal in x behaviour. Arguments of Tauberian kind transform these estimates to information on asymptotical as $s \to \infty$ behaviour of Fourier coefficients $\hat{u}_s^\omega(t)$ of the solution.[5] To present it we denote by E_s the averaged squared Fourier coefficient $\hat{u}_s$, $E_s = \delta^a \int_0^{\delta^{-a}} \mathbf{E}|\hat{u}_s^\omega(t)|^2\,dt$. (We remark that if u^ω was a space-periodic solution for the NS equations, then E_s would be the energy of the fluid, corresponding to the wave vector s).

The numbers E_s obey the following asymptotic, which hold for any $\varepsilon > 0$ with A, B and γ as in the previous section:

1. $E_s = o(|s|^{-M})$ for $|s| \geq \delta^{-B-\varepsilon}$ with every M, if δ is sufficiently small. If $|s| \geq \delta^{-\gamma-\varepsilon}$, then the same holds true for all δ from an appropriate sequence $\{\delta_j \searrow 0\}$.

2. There exist $c(\varepsilon)$ and $C(\varepsilon)$ such that

$$\delta^c \leq |\mathcal{A}_\varepsilon|^{-1} \sum_{s \in \mathcal{A}_\varepsilon} E_s \leq \delta^C,$$

where $\mathcal{A}_\varepsilon = \{\delta^{-A+\varepsilon} \leq |s| \leq \delta^{-B-\varepsilon}\}$, for all small δ. The same holds true for the smaller set $\mathcal{A}_\varepsilon = \{\delta^{-\gamma+\varepsilon} \leq |s| \leq \delta^{-\gamma-\varepsilon}\}$ with appropriate exponents $c(\varepsilon)$ and $C(\varepsilon)$, for all δ from a sequence $\{\delta_j \searrow 0\}$.

The heuristic Kolmogorov - Obukhov (K-O) law (see [LL], § 33) states that the energy E_s of a wave-vector s is $o(|s|^{-M})$ for every M if $|s| > \delta^{-\gamma^K}$, and

$$\frac{1}{C} \sum_{r \leq |s| \leq r+C} E_s \sim \text{const} \cdot r^\theta \quad \text{for} \quad \delta^{-\gamma^0} < r < \delta^{-\gamma^K}.$$

The inverse threshold δ^{γ^K} is called *Kolmogorov's inner scale* of the turbulent flow. For 3-dimensional NS equations the exponent $\theta = 5/3$ and $\gamma^K = 3/4$, see [LL].

The properties 1 and 2 of a solution $u^\omega(t,x)$ for (7) present a weak form of the K-O law. In particular, if any solution u^ω for (7) satisfies the K-O law, then γ^K must equal the exponent $\gamma(u^\omega)$. Consequently, γ^K must meet the estimate $A \leq \gamma^K \leq B$. It is curious to note that for the forcing $\zeta^\omega(t,x)$ which is white noise in time, the results of section 3.2 imply the bounds $\frac{1}{2} \leq \gamma^K \leq 1$ which remarkably agree with the value $\gamma^K = 3/4$, prescribed by K-O for the 3-dimensional hydrodynamic turbulence.

The property 1 shows that the Fourier modes $\hat{u}_s^\omega e^{\pi i s \cdot x}$ with $|s| > \delta^{-\gamma}$ can be ignored when a solution u is calculated numerically, while the modes with $|s| < \delta^{-\gamma}$ are essential. Hence, a numerical scheme to calculate u has to have dimension of order $\delta^{-2\gamma n}$. This is a very big number since δ corresponding to a turbulent regime is very small (for turbulence in water and in air it is as small as $10^{-7} - 10^{-4}$). – This is why it is so difficult to study turbulence numerically.

Proofs of the results presented in sections 3.1-3.3 see in [K7, K8]. See [EKMS] for the turbulence-limit $\delta \to 0$ in a randomly forced Burgers equation.

[5] We write $u^\omega(t,x)$ as $\sum_{s \in \mathbb{Z}^n} \hat{u}_s^\omega(t) e^{\pi i s \cdot x}$.

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Sergei B. Kuksin
Department of Mathematics
Heriot-Watt University
Riccarton
EH14 4AS Edinburgh. UK
and
Steklov Institute
Moscow
S.B.Kuksin@ma.hw.ac.uk