Wave Equations with Low Regularity Coefficients

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Abstract. We illustrate how harmonic analysis techniques that were developed to understand the L^p mapping properties of oscillatory integral and Fourier integral operators lead to an understanding of solutions to the wave equation on Riemannian manifolds with metrics of limited differentiability.

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1. L^p mapping properties of Fourier integral operators

For the purposes of this section, a standard Fourier integral operator of order m is a finite sum of operators of the form

$$Tf(x) = \int e^{i\varphi(x,\xi)}\, a(x,\xi)\, \widehat{f}(\xi)\, d\xi\,. \tag{1.1}$$

The phase function $\varphi(x,\xi)$ is real, homogeneous of degree 1 in ξ, smooth for $\xi \neq 0$, and satisfies the nondegeneracy condition

$$\det\left[\frac{\partial^2\varphi}{\partial x_i \partial \xi_j}\right] \neq 0\,.$$

The amplitude $a(x,\xi)$ is a standard amplitude of order m, which for convenience is also taken compactly supported in x:

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \le C_{\alpha,\beta}\left(1+|\xi|\right)^{m-|\alpha|}\,.$$

The most important examples are the two terms of the wave group:

$$\begin{aligned} \mathbf{C}_t f(x) &= \int e^{i\langle x,\xi\rangle}\cos\bigl(t\,|\xi|\bigr)\,\widehat{f}(\xi)\,d\xi\,, \\ \mathbf{S}_t f(x) &= \int e^{i\langle x,\xi\rangle}\frac{\sin\bigl(t\,|\xi|\bigr)}{|\xi|}\,\widehat{f}(\xi)\,d\xi\,. \end{aligned}$$

For each fixed t, these are standard Fourier integral operators, respectively of order 0 and -1, with two phases $\varphi^\pm(x,\xi) = \langle x,\xi\rangle \pm t|\xi|$. The importance of these operators is that the solution to the Cauchy problem for the wave equation

$$\begin{aligned} \partial_t^2 u(t,x) &= \sum_{j=1}^n \partial_{x_j}^2 u(t,x)\,, \\ u(0,x) &= f(x)\,, \\ \partial_t u(0,x) &= g(x)\,, \end{aligned}$$

is given by $u(t,x) = \mathbf{C}_t f(x) + \mathbf{S}_t g(x)$.

By a theorem of Hörmander [H] and Eskin [E], Fourier integral operators of order 0 are bounded on the space $L^2(\mathbb{R}^n)$. For the spaces $L^p(\mathbb{R}^n)$, $p \neq 2$, this is not the case, and examples of Littman [Li] show that the following result of Seeger-Sogge-Stein [SSS] is of the best possible nature.

THEOREM. *Let T be a Fourier integral operator of order $m = -(n-1)\,|1/p - 1/2|$, and $1 < p < \infty$. Then T is a bounded operator on $L^p(\mathbb{R}^n)$. If $m = -(n-1)/2$, then T is a bounded operator on the local Hardy space $h^1(\mathbb{R}^n)$.*

Lipschitz and L^p estimates for the wave equation on compact manifolds were obtained by Colin de Vèrdiere and Frisch [CFr]. For operators related to the wave equation the above was demonstrated by Peral [Per], M. Beals [Be], and Miyachi [M].

The key to establishing the above theorem is to break up the operator (1.1) into simple pieces using a partition of unity in the ξ variable. The first step is to make a Littlewood-Paley decomposition by splitting the ξ space into dyadic annuli $2^{k-1} \leq |\xi| \leq 2^{k+1}$. A finer decomposition involving the angular variable is then made in a parabolic manner: the shell $|\xi| \approx \lambda$ is divided into conic sets of opening angle $\lambda^{-1/2}$.

The motivation behind this decomposition is that on each resulting region in ξ, the homogeneous phase function $\varphi(x,\xi)$ is well approximated by a phase which is *linear* in ξ , in the sense that the error is uniformly controlled. Each piece of the operator is then essentially a localisation of $\widehat{f}(\xi)$ to a cube, followed by a change of coordinates, and has uniformly bounded norm on $L^1(\mathbb{R}^n)$. The loss of $(n-1)/2$ derivatives on the local Hardy space results from the fact that, at frequencies comparable to λ, the operator is a sum of $\lambda^{(n-1)/2}$ pieces, each of which acts independently on $h^1(\mathbb{R}^n)$. For details, see [SSS] or chapter IX of [St].

This dyadic-parabolic decomposition is implicit in the work of C. Fefferman [F], where it was exploited to understand spherical summation multipliers. For the wave operators, it is related to approximate plane-wave decompositions of solutions: if the function $f(x)$ has Fourier transform localised to a dyadic shell $|\xi| \approx \lambda$, and within angle $\lambda^{-1/2}$ about some direction ω, then for $|t| \lesssim 1$,

$$\mathbf{C}_t f(x) \approx \frac{1}{2}\big(\, f(x + t\omega) - f(x - t\omega)\,\big)\,,$$

with errors that can be uniformly controlled as $\lambda \to \infty$.

The above theorem can be sharpened by the following result [Sm1], which is natural in view of the fact that order 0 Fourier integral operators form an algebra.

THEOREM. *There exists a function space $\mathcal{H}^1_{\mathrm{FIO}}(\mathbb{R}^n)$, with continuous mappings*

$$D^{-(n-1)/2} : h^1(\mathbb{R}^n) \longrightarrow \mathcal{H}^1_{\mathrm{FIO}}(\mathbb{R}^n) \longrightarrow h^1(\mathbb{R}^n)\,,$$

on which the order 0 Fourier integral operators are bounded mappings.

The norm of a function f in $\mathcal{H}^1_{\mathrm{FIO}}$ is defined by the integral of a quadratic expression in f, analogous to the Lusin area characterisation of functions in the real Hardy space of Fefferman-Stein [FS]. The appropriate area function for $\mathcal{H}^1_{\mathrm{FIO}}$ is evaluated at a spatial point and a direction; in essence, each dyadic-parabolic piece of f is treated independently.

2. Strichartz Estimates

Of greater interest for nonlinear wave equations than the preceeding fixed time estimates for solutions of the wave equation are the family of *Strichartz estimates*, which control mixed L^p norms of a solution over space and time, in terms of Sobolev norms of the initial data. For simplicity, we restrict attention here to space dimension $n = 3$.

Theorem. *Let $u(t,x) = \mathbf{C}_t f(x) + \mathbf{S}_t g(x)$ be the solution to the Cauchy problem for the wave equation. Then for $2 \le q < \infty$, if $1/p + 1/q = 1/2$, the following hold:*

$$\|u\|_{L^p_t L^q_x(\mathbb{R}^{1+3})} \le C_q \left(\, \|f\|_{\dot H^{1-2/q}(\mathbb{R}^3)} + \|g\|_{\dot H^{-2/q}(\mathbb{R}^3)} \, \right). \tag{2.1}$$

In the form stated here, (2.1) is due to Pecher [Pec]. The original Strichartz estimate [Str1,2] is the case $p = q = 4$. More general estimates of this type, in general dimensions, have been developed by several authors, including Brenner [Br], Ginibre and Velo [GV1,2], Kapitanski [K], Keel and Tao [KT], and Lindblad and Sogge [LS].

In case $q = 2\,, p = \infty$, estimate (2.1) is an energy inequality. The other endpoint estimate at $q = \infty$ does not hold; this would state that

$$\|u\|_{L^2_t L^\infty_x(\mathbb{R}^{1+3})} \le C \left(\, \|f\|_{\dot H^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} \, \right).$$

(The failure of this estimate motivates the study of the important *null form* estimates of Klainerman-Machedon [KM].) However, a substitute estimate does hold, which is sufficient to obtain (2.1) for $q < \infty$ by interpolation. To state this estimate, let

$$\mathbf{C}^\lambda_t f(x) = \int e^{i\langle x,\xi\rangle} \cos\bigl(t\,|\xi|\,\bigr)\, \phi\bigl(\lambda^{-1}\,|\xi|\,\bigr)\, \widehat f(\xi)\, d\xi\,,$$

where $\phi(s)$ is supported in $1/2 \le s \le 2$. Then

$$\|\mathbf{C}^\lambda_t f\|_{L^\infty(\mathbb{R}^3)} \le C\,\lambda^2\,t^{-1}\,\|f\|_{L^1(\mathbb{R}^3)}\,. \tag{2.2}$$

This says that the convolution kernel associated to $\mathbf{C}^\lambda_t$ is pointwise bounded by $\lambda^2\,t^{-1}$, which can be demonstrated by stationary phase arguments.

For $n = 3$, a proof of (2.2) (for $|t| \lesssim 1$ and $\lambda \ge 1$) can be obtained using the dyadic-parabolic decomposition mentioned in the first section of this paper; this is important since it allows for a broader class of amplitudes in the Fourier integral operator, which is crucial for low regularity wave equations.

To begin, write

$$\phi\bigl(\lambda^{-1}\,|\xi|\,\bigr) = \sum_\omega \widehat\psi^\omega_\lambda(\xi)\,, \tag{2.3}$$

where $\widehat\psi^\omega_\lambda(\xi)$ is supported in a cone of angle $\lambda^{-1/2}$ about the direction ω, and ω varies over λ indices evenly distributed over the unit sphere. The function $\psi^\omega_\lambda(x)$

is of L^∞ norm comparable to λ^2, and is concentrated in a box with two sides of length $\lambda^{-1/2}$, and one side of length λ^{-1}, the last along the direction ω. The function

$$\int e^{i\langle x,\xi\rangle - it|\xi|}\,\widehat{\psi}^\omega_\lambda(\xi)\,d\xi \tag{2.4}$$

is a "coherent wave packet of frequency λ", in the sense that for $|t|\lesssim 1$ it travels along a ray without significantly changing its shape. We remark that this function is also critical for the Strichartz estimates, in that the two sides of (2.1) are comparable as $\lambda\to\infty$.

The convolution kernel of $\mathbf{C}^\lambda_t$ splits into a sum

$$\sum_\omega \int e^{i\langle x,\xi\rangle}\,\cos\bigl(t\,|\xi|\,\bigr)\,\widehat{\psi}^\omega_\lambda(\xi)\,d\xi \approx \frac{1}{2}\sum_\omega \psi^\omega_\lambda(x+t\omega)+\psi^\omega_\lambda(x-t\omega)\,.$$

Then (2.2) follows by showing that the overlap of "supports" of the $\psi^\omega_\lambda(x+t\omega)$ is bounded by t^{-1}, which is a simple exercise in geometry. (We remark that this simple proof fails in space dimension $n\ge 4$, where the overlap count is too high.)

3. The wave equation on Riemannian manifolds

Let

$$\Delta_{\mathbf{g}} f(x) = \frac{1}{\sqrt{\mathbf{g}(x)}}\sum_{i,j=1}^n \partial_{x_i}\Bigl(\sqrt{\mathbf{g}(x)}\;\mathbf{g}^{ij}(x)\,\partial_{x_j} f(x)\Bigr)$$

be the Laplace-Beltrami operator for a smooth Riemannian metric $\mathbf{g}$ in a coordinate patch. The Cauchy problem for the wave equation

$$\begin{aligned} \partial_t^2 u(t,x) &= \Delta_{\mathbf{g}} u(t,x)\,,\\ u(0,x) &= f(x)\,,\\ \partial_t u(0,x) &= g(x)\,, \end{aligned} \tag{3.1}$$

has finite propagation speed, so for small time intervals it suffices to work in a coordinate neighborhood.

To solve the Cauchy problem, one seeks the analogue of the plane wave solutions $\exp(i\langle x,\xi\rangle \pm it|\xi|)$. Lax [Lax] provided an asymptotic construction of solutions for small t of the form

$$e^{i\varphi^\pm(t,x,\xi)}\,a_\pm(t,x,\xi)\,,$$

where $a_\pm(t,x,\xi)$ is a standard amplitude of order 0 (which equals 1 at $t=0$), and the real phase $\varphi^\pm(t,x,\xi)$ satisfies the eikonal equation

$$\begin{aligned} \partial_t\varphi^\pm(t,x,\xi) &= \pm\bigl\|d_x\varphi^\pm(t,x,\xi)\bigr\|_{\mathbf{g}}\,,\\ \varphi^\pm(0,x,\xi) &= \langle x,\xi\rangle\,. \end{aligned}$$

The solution to the Cauchy problem (for initial condition $g = 0$) can be written (up to an error which is a smooth integral kernel aplied to f) in the form

$$u(t,x) = \frac{1}{2}\sum_{\pm}\int e^{i\varphi^{\pm}(t,x,\xi)}\, a_{\pm}(t,x,\xi)\, \widehat{f}(\xi)\, d\xi\,.$$

Using stationary phase techniques, the estimate (2.2) can be shown to hold for small time intervals, and together with L^2 bounds on Fourier integral operators this implies the Strichartz estimates (2.1) locally, as shown by Kapitanski [K], and Mockenhaupt-Seeger-Sogge [MSS].

4. LOW REGULARITY METRICS

Consider the following question: what is the minimal regularity condition on the metric coefficients $\mathbf{g}^{ij}(x)$ which insures that the Strichartz estimates hold for solutions $u(t,x)$ to the Cauchy problem (3.1)?

A natural condition for geometric optics is that the metric coefficients possess two bounded derivatives; that is, $\mathbf{g}^{ij}(x) \in C^{1,1}(\mathbb{R}^n)$. This is the minimal regularity condition in the Hölder classes which yields a unique, bilipschitz geodesic flow. That this condition is also optimal among the Hölder classes for Strichartz estimates is shown by the following counterexamples of the author and Sogge [SS].

THEOREM. *For $n \geq 3$, and any $\alpha < 1$, there exists $h_\alpha(x) \in C^{1,\alpha}(\mathbb{R}^n)$, and a solution $u(t,x)$ to the Cauchy problem for*

$$\partial_t^2 u(t,x) = h_\alpha(x)\, \Delta u(t,x)\,,$$

for which the Strichartz estimates do not hold.

The function $h_\alpha(x)$ is constructed so that the geodesic flow is singularly focused along some ray. This permits the construction of coherent wave packets travelling along the ray which, due to the singular focusing, are contained in smaller sets than the coherent wave packets (2.4) that are critical for the Strichartz estimates.

On the other hand, the arguments at the end of section 2 show that a positive proof of the Strichartz estimates (in space dimensions 2 and 3) for a metric $\mathbf{g}$ can be reduced to studying wave packets. Roughly, one needs to show that the solution to the Cauchy problem with initial condition $\psi_\lambda^\omega(x)$ is a coherent wave packet that travels along the geodesic x in direction ω. Together with a bilipschitz geodesic flow, this implies the analogue of estimate (2.2).

In [Sm2], this idea was coupled with a decomposition of functions into wave packets to construct the wave group for metrics $\mathbf{g}(t,x) \in C^{1,1}(\mathbb{R}^{1+n})$. Modifying techniques of Frazier and Jawerth [FJW] permits the construction of a spanning set of functions for $L^2(\mathbb{R}^n)$ consisting of translates of the $\psi_\lambda^\omega(x)$. The ansatz that the function $\psi_\lambda^\omega(x)$ is rigidly transported along the geodesic flow leads to an inverse for the wave equation, modulo an error that can be eliminated by iteration.

To obtain a manageable class of operators, however, a modification is needed: the function ψ_λ^ω is transported not along the geodesic flow of the metric $\mathbf{g}$, but rather along the flow of a smooth approximation $\mathbf{g}_\lambda$ to $\mathbf{g}$, where the approximation is chosen depending on the frequency λ, analogous to the paraproduct/multilinear Fourier analysis techniques of Bony [Bo], Coifman and Meyer [CM]. We outline this approximation in the next section in the context of a modified parametrix construction for metrics of bounded curvature.

5. Metrics of bounded sectional curvature

In this section, we assume $\mathbf{g}(x)$ to be a Riemannian metric such that all sectional curvatures are pointwise bounded by some constant C; this is the notion of L^∞ pinched curvature. Some a priori regularity is necessary to make sense of the Riemann curvature tensor, which is a nonlinear expression in the derivatives of $\mathbf{g}$; the condition $\nabla_x \mathbf{g}^{ij}(x) \in L^q(\mathbb{R}^n)$ for some $q > n$ is sufficient. This is also sufficient to construct local harmonic coordinates for $\mathbf{g}$. Lanczos [Lan] observed that in these coordinates the Ricci curvature is an elliptic expression in terms of $\mathbf{g}$ (see DeTurk and Kazdan [DK]); consequently in such coordinates the metric has all second partial derivatives belonging to $\mathrm{BMO}(\mathbb{R}^n)$, which we henceforth assume.

Take a sequence of smooth approximating metrics $\mathbf{g}_k(x)$ to $\mathbf{g}(x)$ by the rule

$$\mathbf{g}_k^{ij}(x) = \big(\, \phi_k * \mathbf{g}^{ij} \,\big)(x)\,,$$

where $\phi_k(x) = 2^{nk/2}\phi(2^{k/2}x)$, with $\phi(x)$ a smooth bump function of integral 1.

It follows from the condition $\nabla_x^2 \mathbf{g}^{ij}(x) \in \mathrm{BMO}(\mathbb{R}^n)$ that

$$\big\| \, \mathbf{g}_k^{ij} - \mathbf{g}^{ij} \, \big\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-k}\,. \tag{5.1}$$

Let $\varphi_k^\pm(t,x,\xi)$ be the solutions to the eikonal equations for $\mathbf{g}_k$:

$$\begin{aligned} \partial_t \varphi_k^\pm(t,x,\xi) \;&=\; \pm \big\| d_x \varphi_k^\pm(t,x,\xi) \big\|_{\mathbf{g}_k}\,, \\ \varphi_k^\pm(0,x,\xi) \;&=\; \langle x, \xi \rangle\,. \end{aligned} \tag{5.2}$$

It follows from the bounded sectional curvature condition that the geodesic flow of $\mathbf{g}_k$ is bilipschitz, uniformly in k, hence that the $\varphi_k^\pm(t,x,\xi)$ form a bounded sequence in $C^2(\mathbb{R}^7)$. Let

$$\mathbf{S}_t g(x) = \frac{1}{2i}\sum_{k=0}^{\infty} \int \Big(e^{i\varphi_k^+(t,x,\xi)} - e^{i\varphi_k^-(t,x,\xi)} \Big)\, \|\xi\|_{\mathbf{g}_k(x)}^{-1}\, \widehat{g}_k(\xi)\, d\xi\,, \tag{5.3}$$

where $g = \sum_k g_k$ is a Littlewood-Paley decomposition of g. It then follows from (5.1) and (5.2) that

$$\big(\partial_t^2 - \Delta_{\mathbf{g}}\big)\mathbf{S}_t = \mathbf{R}_t$$

is a bounded operator on the Sobolev spaces $H^\gamma(\mathbb{R}^n)$, for $-1 \le \gamma \le 2$, with norms uniformly bounded in t.

One then seeks a solution to the inhomogeneous Cauchy problem

$$\begin{aligned} \partial_t^2 u(t,x) \;&=\; \Delta_{\mathbf{g}} u(t,x) + F(t,x)\,, \\ u(0,x) \;&=\; 0\,, \\ \partial_t u(0,x) \;&=\; 0\,, \end{aligned}$$

in the form

$$u(t,x)=\int_0^t \mathbf{S}_{t-s}\, G(s,x)\, ds\,.$$

This leads to a Volterra equation,

$$F(t,x)=G(t,x)+\int_0^t \mathbf{R}_{t-s} G(s,x)\, ds$$

which may be solved by iteration.

Estimates of the form (2.1) are thus reduced to $L^2 \to L^p$ mapping properties of operators of the form (5.3). The symbols and phases of these operators satisfy exactly the estimates needed to use the decomposition of Seeger-Sogge-Stein. In particular, functions of the form ψ_λ^ω (see (2.3)) are mapped to coherent wave packets. Combined with the ideas at the end of section 2, this yields the following (for details see [Sm3]).

Theorem. *Let $\mathbf{g}(x)$ be a Riemannian metric on an open ball in $\mathbb{R}^3$ such that, for $1 \le i,j \le 3$, $\nabla_x^2 \mathbf{g}^{ij}(x) \in \mathrm{BMO}(\mathbb{R}^3)$. Suppose also that the components of the Riemannian curvature tensor satisfy, for all indices, $R^{ijkl}(x) \in L^\infty(\mathbb{R}^3)$. Then for t in some interval about 0, solutions to the Cauchy problem (3.1) satisfy the Strichartz estimates (2.1).*

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