Representation Theory of Affine Superalgebras at the Critical Level

Dedicated to the memory of my father Shoji Wakimoto

Minoru Wakimoto

1991 Mathematics Subject Classification: Primary 17B65; Secondary 17B67, 81R10.
Keywords and Phrases: Superalgebra, character, atypical representation, free field construction, mock theta function, modular transformation

0. Introduction.

As is known well, the representation theory of affine Lie algebras has a number of important connection with various area of mathematics and physics, while the representation of superalgebras still remains mysterious, and only a few has been known about it because of serious technical and intrinsic difficulties arising in its analysis.

The representation theory of superalgebras, however, seems quite interesting and fascinating, since the associated Macdonald identities are of different kinds from those of the usual affine Lie algebras, and also the Ramanujan's famous mock theta functions take place closely related to the denominator identities for certain affine superalgebras.

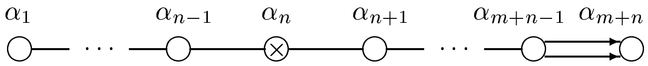
The effective theory for superalgebras are not well established, and some of the results exposed in this note are obtained with the help of a computer and Reduce 3.6.

1. Characters of affine superalgebras.

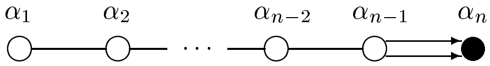
Our interest in this note is a finite-dimensional or affine simple Lie superalgebra, with a non-degenerate even super-invariant super-symmetric bilinear form $(\ | \ )$. The finite-dimensional ones other than Lie algebras are listed in the following table:

$$\begin{array}{c} A(m,n) \\ (m,n \geq 0) \\ (m+n \geq 1) \end{array} \quad : \quad \overset{\alpha_1}{\bigcirc} - \cdots - \overset{\alpha_m}{\bigcirc} - \overset{\alpha_{m+1}}{\bigotimes} - \overset{\alpha_{m+2}}{\bigcirc} - \cdots - \overset{\alpha_{m+n+1}}{\bigcirc}$$

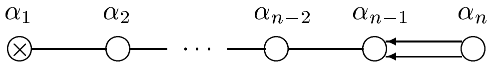
$B(m,n)$ $(m,n \geq 1)$:



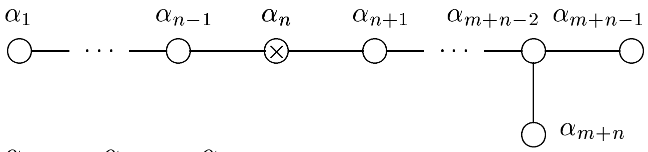
$B(0,n)$ $(n \geq 1)$:



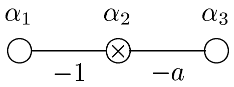
$C(n)$ $(n \geq 3)$:



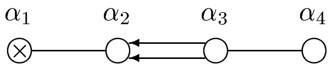
$D(m,n)$ $(m \geq 2)$ $(n \geq 1)$:



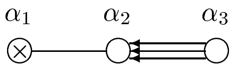
$D(2,1;a)$ $(-1,0 \neq a \in \mathbb{C})$:



$F(4)$:



$G(3)$:



We note that, for a superalgebra with isotropic odd roots, its Dynkin diagram is not determined uniquely, and the diagrams in the above list are *standard* ones. But *non-standard* ones are never less important, and a suitable choice of Dynkin diagrams has sometimes a crucial importance in the representation theory. For example, Dynkin diagrams in Fig 1.1 $\sim$ 1.3 are useful diagrams of $A(1,0) = \mathfrak{sl}(2,1)$, $A(1,1) = \mathfrak{sl}(2,2)/\mathfrak{z}$ and $A(2,2) = \mathfrak{sl}(3,3)/\mathfrak{z}$ respectively (here and henceforth $\mathfrak{z}$ stands for the center of the superalgebra in question) with their affinizations shown in Fig $1.1' \sim 1.3'$:

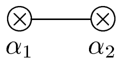


Fig.1.1

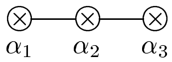


Fig.1.2

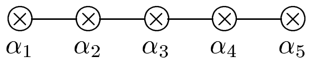


Fig.1.3

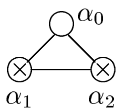


Fig.1.1$'$

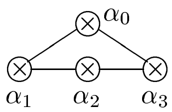


Fig.1.2$'$

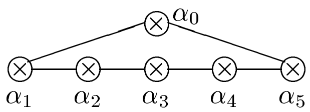


Fig.1.3$'$

Let $\mathfrak{g}$ be a finite-dimensional or affine Lie superalgebra and $\mathfrak{h}$ its Cartan subalgebra. The inner product $(\ |\ )$ is normalized so that the dual Coxeter number $h^\vee$ is a non-negative rational number and the square length of the longest roots is equal to 2 or -2. For usual notations and terminologies of superalgebras, we refer to [5] and [9]. In particular the concept of *integrable* weight is never obvious for superalgebras and given in Chapter 6 of [9].

The character of a *suitable* irreducible integrable highest weight $\mathfrak{g}$-module

$L(\Lambda)$ is given in [9]:

$$\mathrm{ch}L(\Lambda) = \frac{c_\Lambda}{e^\rho R} \sum_{w \in W^\sharp} \varepsilon(w) w \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^k (1+e^{-\beta_i})} \right) \tag{1}$$

with a rational number c_Λ, where $\{\beta_1, \cdots, \beta_k\}$ is a maximal set of mutually orthogonal isotropic positive odd roots satisfying $(\Lambda+\rho|\beta_i) = 0$ for all i, and R is the denominator:

$$R := \frac{\prod_{\alpha \in \Delta_{0+}} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_{1+}} (1+e^{-\alpha})^{\mathrm{mult}(\alpha)}} \tag{2}$$

In the above, $W^\sharp$ is not always the *full* Weyl group W but its subgroup (cf. [9]), and Δ_+ stands, as usual, for the set of all positive roots, and the suffix "0" or "1" implies the set of "even roots" or "odd roots" respectively.

The highest weight Λ is called *typical* when $k=0$, and *k-atypical* or simply *atypical* when $k>0$. The number k is the *atypicality* of a highest weight Λ.

A particular feature and difficulty of the representation theory of superalgebras is that the formula (1) is not necessarily valid for *all* representations, but fails to hold in some cases. The formula (1) is true in case when (i) Λ is typical ([6]) or (ii) $\beta_1, \cdots, \beta_k$ are chosen from simple roots with a suitable choice of the Dynkin diagram ([9]).

The trivial representation $L(0)$ for an affine superalgebra with non-zero dual Coxeter number satisfies the condition (ii) above, and the formula (1) applied to this, called the denominator identity, provides a Lie superalgebraic interpretation to some of number theoretic formulas as is discussed in [9].

For affine superalgebras with $h^\vee = 0$, whose complete list is $\widehat{A}(n,n)$(=the affinization of $\mathfrak{sl}(n+1|n+1)/\mathfrak{z}$) and $\widehat{D}(n+1,n)(=\widehat{\mathfrak{osp}}(2n+2,2n))$ and $\widehat{D}(2,1;a)$, the trivial representation is a representation of critical level. The simplest among them is $\widehat{A}(1,1)$ with the Dynkin diagram Fig $1.2'$. In this case, we claim the following denominator identity, which is obtained from the analysis of the super Boson-Fermion correspondence ([10]) :

$$R = \sum_{w \in \langle r_{\alpha_1+\alpha_2}, t_{\alpha_1+\alpha_2} \rangle} \varepsilon(w) w \left(\frac{1}{(1+e^{-\alpha_0}) \prod_{n=1}^\infty (1+e^{\alpha_2 - n\delta})(1+e^{-\alpha_2-(n-1)\delta})} \right), \tag{3}$$

where r_α is the reflection with respect to α, and t_α is the linear transformation on $\mathfrak{h}$ defined by

$$t_\alpha(\lambda) := \lambda + (\lambda|\delta)\alpha - \left(\frac{|\alpha|^2}{2}(\lambda|\delta) + (\lambda|\alpha) \right) \delta.$$

We note the following. For $\widehat{A}(1,1)$ with the Dynkin diagram Fig $1.2'$, one can choose

$$\{\alpha_0 + n\delta, \;\; \alpha_2 + n\delta\}_{n \geq 0} \cup \{-\alpha_0 + n\delta, \;\; -\alpha_2 + n\delta\}_{n>0} \tag{4}$$

as a maximal system of mutually orthogonal isotropic positive odd roots which are orthogonal to $\Lambda+\rho$ since $\Lambda=\rho=0$, where $\delta:=\sum_{i=0}^{3}\alpha_i$ is the canonical imaginary root as usual. So $\Lambda=0$ is "infinitely atypical" in this case. A very remarkable fact is that not all of β_i's in this maximal system (4) does take a part in the denominator identity (3).

The formula becomes more complicated in higher rank cases. For example, the denominator identity of $\widehat{A}(2,2)$ is given as follows with respect to the Dynkin diagram Fig 1.3$'$:

$$
\begin{aligned}
& R\cdot\prod_{n=1}^{\infty}\frac{(1-e^{\alpha_1+\alpha_3+\alpha_5-2n\delta})(1-e^{-\alpha_1-\alpha_3-\alpha_5-2(n-1)\delta})}{(1-e^{-n\delta})(1-e^{-(2n-1)\delta})} \\
= & \sum_{w\in W^{\sharp}}\varepsilon(w)w\left(\frac{1}{(1+e^{-\alpha_0})\prod\limits_{i=2,4}\prod\limits_{n=1}^{\infty}(1+e^{\alpha_i-n\delta})(1+e^{-\alpha_i-(n-1)\delta})}\right)
\end{aligned}
\tag{5}
$$

where $W^{\sharp}:=\langle r_{\alpha_1+\alpha_2},\,r_{\alpha_3+\alpha_4},\,t_{\alpha_1+\alpha_2},\,t_{\alpha_3+\alpha_4}\rangle$.

We now look at other simplest examples of representations at the critical level, namely *integrable* $\widehat{\mathfrak{sl}}(2,1)$-modules of level -1. For a linear form $\Lambda:=k\Lambda_0-(k+1)\Lambda_1$ $(k\in\mathbb{Z})$ which is atypical with respect to α_2, we claim the following:

$$
\begin{aligned}
\mathrm{ch}L(\Lambda) &= \frac{\prod\limits_{n=1}^{\infty}(1-e^{-n\delta})}{e^{\rho}R}\times \\
&\begin{cases}
\sum\limits_{w\in\langle r_{\alpha_0}\rangle}\sum\limits_{j=0}^{\infty}\varepsilon(w)wt_{j\alpha_0}\left(\dfrac{e^{\Lambda+\rho}}{\prod\limits_{n=1}^{\infty}(1+e^{\alpha_2-n\delta})(1+e^{-\alpha_2-(n-1)\delta})}\right) & (\text{if } k\geq 0)\\
\sum\limits_{w\in\langle r_{\alpha_1+\alpha_2}\rangle}\sum\limits_{j=0}^{\infty}\varepsilon(w)wt_{j(\alpha_1+\alpha_2)}\left(\dfrac{e^{\Lambda+\rho}}{\prod\limits_{n=1}^{\infty}(1+e^{\alpha_2-n\delta})(1+e^{-\alpha_2-(n-1)\delta})}\right) & (\text{if } k<0).
\end{cases}
\end{aligned}
\tag{6}
$$

Note that the sum in the right-hand side is *not* taken over a subgroup of the affine Weyl group. From this formula, $\mathrm{ch}L(-\rho)$ is obtained as follows:

$$
\begin{aligned}
&\mathrm{ch}L(-\rho)\\
&= \frac{\prod\limits_{n=1}^{\infty}(1-e^{-n\delta})}{e^{\rho}R}\left\{\sum_{k=1,2}\frac{\sum\limits_{j=0}^{\infty}e^{j\alpha_k-\frac{j(j+1)}{2}\delta}}{\prod\limits_{n=1}^{\infty}(1+e^{\alpha_k-n\delta})(1+e^{-\alpha_k-(n-1)\delta})}-\prod_{n=1}^{\infty}(1-e^{-n\delta})\right\}\\
&= \frac{\prod\limits_{n=1}^{\infty}(1-e^{-n\delta})}{e^{\rho}R}\times\\
&\left\{\frac{\sum\limits_{j\geq 0}e^{j\alpha_1-\frac{j(j+1)}{2}\delta}}{\prod\limits_{n=1}^{\infty}(1+e^{\alpha_1-n\delta})(1+e^{-\alpha_1-(n-1)\delta})}-\frac{\sum\limits_{j<0}e^{j\alpha_2-\frac{j(j+1)}{2}\delta}}{\prod\limits_{n=1}^{\infty}(1+e^{\alpha_2-n\delta})(1+e^{-\alpha_2-(n-1)\delta})}\right\}.
\end{aligned}
$$

The second equality in the above is due to the Jacobi triple product identity

$$\prod_{n=1}^{\infty}(1-e^{-n\delta}) = \frac{\sum_{j\in\mathbb{Z}} e^{j\alpha}e^{-\frac{j(j+1)}{2}\delta}}{\prod_{n=1}^{\infty}(1+e^{\alpha-n\delta})(1+e^{-\alpha-(n-1)\delta})}. \tag{7}$$

2. Free field construction of atypical $\widehat{\mathfrak{sl}}(2,1)$-modules at the critical level.

It is known in [2], [3] and [7] that, in case of affine algebras, irreducible highest weight representations and their characters display a remarkable behavior at the critical level. This may also be expected for superalgebras. To get an information on the structure of these representations, we give an explicit construction of atypical highest weight modules at the critical level for the simplest affine superalgebra $\widehat{\mathfrak{sl}}(2,1)$.

Let $\{h_i,\, e_i,\, f_i\}_{i=1,2}$ be a system of Chevalley generators of $\mathfrak{sl}(2,1)$ with respect to its Dynkin diagram Fig 1.1. Then these elements together with $e_{12} := -[e_1,\, e_2]$ and $f_{12} := [f_1,\, f_2]$, satisfying $[e_{12},\, f_{12}] = h_1+h_2$, form a basis of $\mathfrak{sl}(2,1)$. Choosing the super-invariant super-symmetric bilinear form $(\ |\ )$ such that $(e_i|f_i) = 1$ $(i = 1,2)$, we consider its affinization

$$\widehat{\mathfrak{sl}}(2,1) := \mathfrak{sl}(2,1)\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\cdot K\oplus\mathbb{C}\cdot t\frac{\partial}{\partial t}$$

with bracket

$$[X\otimes t^m,\, Y\otimes t^n] := [X,\, Y]\otimes t^{m+n} + m(X|Y)K\delta_{m+n,0},$$

which is written as follows

$$X(z)Y(w) = \frac{[X,\, Y](w)}{z-w} + \frac{(X|Y)K}{(z-w)^2},$$

in terms of operator products of fields $X(z) := \sum_{n\in\mathbb{Z}} X\otimes t^n\cdot z^{-n-1}$.

Let $a_j,\, a_j^*,\, b_j,\, \psi_j,\, \psi_j^*$ $(j\in\mathbb{Z})$ be linear operators on a linear space

$$V := \mathbb{C}[x_n\,;\, n\in\mathbb{Z}]\otimes\mathbb{C}[y_n\,;\, n\in\mathbb{Z}_{>0}]\otimes\wedge[\xi_n;\, n\in\mathbb{Z}],$$

defined by

$$a_j \;:=\; \begin{cases} \frac{\partial}{\partial x_j} & \text{if } j\geq 0\\ x_j & \text{if } j<0,\end{cases} \qquad a_j^* := \begin{cases} -x_j & \text{if } j\geq 0\\ \frac{\partial}{\partial x_j} & \text{if } j<0,\end{cases}$$

$$b_j \;:=\; \begin{cases} \frac{\partial}{\partial y_j} & \text{if } j>0\\ -jy_{-j} & \text{if } j<0,\end{cases}$$

$$\psi_j \;:=\; \begin{cases} \frac{\partial}{\partial \xi_j} & \text{if } j \geq 0 \\ \xi_j \wedge & \text{if } j < 0, \end{cases} \qquad \psi_j^* := \begin{cases} \xi_j \wedge & \text{if } j \geq 0 \\ \frac{\partial}{\partial \xi_j} & \text{if } j < 0, \end{cases}$$

b_0 being the scalar operator with a complex number b_0. As usual, we introduce the following fields :

$$a(z) := \sum_{j \in \mathbb{Z}} a_j z^{-j-1}, \qquad a^*(z) := \sum_{j \in \mathbb{Z}} a_j^* z^j, \qquad b(z) := \sum_{j \in \mathbb{Z}} b_j z^{-j-1},$$
$$\psi(z) := \sum_{j \in \mathbb{Z}} \psi_j z^{-j-1}, \qquad \psi^*(z) := \sum_{j \in \mathbb{Z}} \psi_j^* z^j.$$

PROPOSITION 2.1. *The space V is an $\widehat{\mathfrak{sl}}(2,1)$-module by the following action:*

$$\begin{aligned}
h_1(z) &:= -:a(z)a^*(z): + :\psi(z)\psi^*(z): \\
h_2(z) &:= -:a(z)a^*(z): -b(z) \\
e_1(z) &:= a(z)\psi^*(z) \\
e_2(z) &:= -\psi(z) \\
e_{12}(z) &:= a(z) \\
f_1(z) &:= -a^*(z)\psi(z) \\
f_2(z) &:= \partial\psi^*(z) + :a(z)a^*(z)\psi^*(z): + b(z)\psi^*(z) \\
f_{12}(z) &:= -\partial a^*(z) - :a(z)a^*(z)a^*(z): + :a^*(z)\psi(z)\psi^*(z): - a^*(z)b(z).
\end{aligned}$$

And the constant function 1 is a highest weight vector in V of weight $(b_0-1)\Lambda_0 - b_0\Lambda_2$.

In this case, the weights of variables are given by

$$\begin{aligned}
\mathrm{wt}(x_j) &= \begin{cases} -j\delta - (\alpha_1 + \alpha_2) & \text{if } j \geq 0 \\ j\delta + (\alpha_1 + \alpha_2) & \text{if } j < 0, \end{cases} \\
\mathrm{wt}(y_j) &= -j\delta, \\
\mathrm{wt}(\xi_j) &= \begin{cases} -j\delta - \alpha_2 & \text{if } j \geq 0 \\ j\delta + \alpha_2 & \text{if } j < 0, \end{cases}
\end{aligned}$$

and so, by counting the character of V, one obtains the following:

COROLLARY 2.1. *Let $\mathfrak{g} = \widehat{\mathfrak{sl}}(2,1)$ with Dynkin diagram Fig 1.1$'$, and $\Lambda \in \mathfrak{h}^*$ be a linear form of level -1, atypical with respect to α_i $(i = 1$ or $2)$. Then*

$$\begin{aligned}
\mathrm{ch}L(\Lambda) &\leq \frac{e^\Lambda}{R} \prod_{n=1}^{\infty} \frac{(1-e^{-n\delta})}{(1+e^{-n\delta+\alpha_i})(1+e^{-(n-1)\delta-\alpha_i})} \\
&\leq \frac{e^\Lambda}{R} \prod_{n=1}^{\infty} (1-e^{-n\delta})^2.
\end{aligned}$$

The second inequality in the above is due to (7).

3. Mock theta identities and the associated modular functions.

The formula (1) applied to the trivial representation gives rise to some kinds of identities of Lambert series. The simplest and most remarkable ones among them are mock theta identities obtained from the denominator identities of affine superalgebras $\widehat{\mathfrak{sl}}(2,1)$ and $\widehat{B}(1,1)$ (cf. [9]). The mock theta function associated to $\widehat{\mathfrak{sl}}(2,1)$, already appearing in the classical book [12], has a crucial importance in conformal field theory since it gives rise to the formula of the modular transformation of the characters of the minimal series representations of N=2 superconformal algebras (cf. [9]). In this section, we give an exposition of the modular transformation of another simplest mock theta functions associated to $\widehat{B}(1,1)$.

First we look at the formula (3). Putting $u := e^{-\alpha_1}, w := e^{-\alpha_2}, v := e^{-\alpha_3}$ and $q := e^{-\delta}$, this is written as follows:

$$\begin{aligned}
&\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-uwq^{n-1})(1-(uw)^{-1}q^n)(1-vwq^{n-1})(1-(vw)^{-1}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+vq^{n-1})(1+v^{-1}q^n)}\\
\times\ &\prod_{n=1}^{\infty}\frac{1}{(1+wq^{n-1})(1+w^{-1}q^n)(1+uvwq^{n-1})(1+(uvw)^{-1}q^n)}\\
=\ &\frac{1}{\prod_{n=1}^{\infty}(1+wq^{n-1})(1+w^{-1}q^n)}\cdot\sum_{k\in\mathbb{Z}}\frac{w^{-k}q^{k(k+1)/2}}{1+(uvw)^{-1}q^{k+1}}\\
-\ &\frac{1}{\prod_{n=1}^{\infty}(1+uq^{n-1})(1+u^{-1}q^n)}\cdot\sum_{k\in\mathbb{Z}}\frac{u^{k+1}q^{k(k+1)/2}}{1+v^{-1}q^{k+1}}. \qquad (8)
\end{aligned}$$

Letting w=u, we obtain the following formula, which coincides with the mock theta identity associated to $\widehat{B}(1,1)$ (cf. [1] and [4]):

$$\begin{aligned}
&\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-u^2q^{n-1})(1-u^{-2}q^n)(1-uvq^{n-1})(1-u^{-1}v^{-1}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+vq^{n-1})(1+v^{-1}q^n)(1+u^2vq^{n-1})(1+u^{-2}v^{-1}q^n)}\\
=\ &\sum_{k\in\mathbb{Z}}\frac{u^{-k}q^{k(k+1)/2}}{1+u^{-2}v^{-1}q^{k+1}} - \sum_{k\in\mathbb{Z}}\frac{u^{k+1}q^{k(k+1)/2}}{1+v^{-1}q^{k+1}} \qquad (9)\\
=\ &\Big\{\sum_{j,k\geq 0}-\sum_{j,k<0}\Big\}(-1)^j\left(u^{-2j-k}v^{-j}-u^{1+k}v^{-j}\right)q^{(k+1)(k+2j)/2}\\
=\ &\Big\{\sum_{\substack{m,n\geq 0\\ \text{s.t. } m\equiv n \mod 2}}-\sum_{\substack{m,n<0\\ \text{s.t. } m\equiv n \mod 2}}\Big\}(-1)^{\frac{m-n}{2}}v^{\frac{m}{2}}(u^{-2}v^{-1})^{\frac{n}{2}}q^{\frac{(m+1)n}{2}}. \qquad (10)
\end{aligned}$$

We consider the theta function

$$f(\tau,z) := e^{\frac{\pi i\tau}{4}}e^{-\pi iz}\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi iz}q^{n-1})(1-e^{-2\pi iz}q^n),$$

defined for $\tau \in \mathbb{C}_+ := \{\tau \in \mathbb{C};\ \mathrm{Im}\tau > 0\}$ and $z \in \mathbb{C}$, where $q = e^{2\pi i\tau}$ as usual. This function satisfies the modular transformation:

$$f\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}} e^{\frac{\pi i z^2}{\tau}} f(\tau, z). \tag{11}$$

We put

$$F(\tau, z_1, z_2) := \frac{\eta(\tau)^3 f(\tau, z_1 + z_2) f(\tau, \frac{z_1 - z_2}{2})}{f(\tau, z_1) f(\tau, z_2) f(\tau, \frac{z_1 + z_2}{2})}. \tag{12}$$

Then by (11), we have

$$F\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}\right) = \tau e^{\frac{\pi i z_1 z_2}{\tau}} F(\tau, z_1, z_2). \tag{13}$$

We now fix a positive even integer M, and consider the set Ω of all equivalence classes in $\mathbb{Z} \times \mathbb{Z}$ with respect to the equivalence relation

$$(j, k) \sim (j', k') \Longleftrightarrow \begin{cases} j - k,\ \ j' - k' & \in M\mathbb{Z}, \\ (j + k) - (j' + k') & \in 2M\mathbb{Z}. \end{cases}$$

For $(j, k) \in \Omega$, we put

$$\begin{aligned} G_{j,k}(\tau, z_1, z_2) &:= e^{\frac{\pi i}{M\tau}\{(z_1 + j\tau)(z_2 + k\tau) - z_1 z_2\}} F(M\tau, z_1 + 1 + j\tau, z_2 + k\tau), \\ H_{j,k}(\tau, z_1, z_2) &:= G_{j,k}(\tau, z_1 - 1, z_2), \end{aligned}$$

whose Lambert series expressions and power series expansions are easily calculated from (9) and (10).

We note the following important lemma which is deduced from (13) and the power series expansion (10):

LEMMA 3.1. *The function F satisfies the following transformation property:*

$$\begin{aligned} F\left(-\frac{M}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}\right) &= \frac{\tau}{M} e^{\frac{\pi i z_1 z_2}{\tau M}} \\ \times \sum_{\substack{a, b \in \mathbb{Z}/M\mathbb{Z} \\ \text{s.t. } a \equiv b + 1 \mod 2}} & (-1)^a \Big\{ e^{\frac{\pi i \tau a b}{M}} e^{\frac{\pi i (a z_1 + b z_2)}{M}} F(M\tau, z_1 + 1 + b\tau, z_2 + a\tau) \\ & + e^{\frac{\pi i \tau a (b + M)}{M}} e^{\frac{\pi i (a z_1 + (b + M) z_2)}{M}} F(M\tau, z_1 + 1 + (b + M)\tau, z_2 + a\tau) \Big\}. \end{aligned}$$

By this lemma, we obtain the modular transformation of $G_{j,k}$ and $H_{j,k}$ $(j, k \in \Omega)$ as follows:

THEOREM 3.1.

$$G_{j,k}\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}\right) = (-1)^k \frac{\tau}{M} e^{\frac{\pi i z_1 z_2}{M\tau}} \times$$

$$\begin{cases} \sum\limits_{\substack{(a,b)\in\Omega \\ \text{s.t. } a\equiv b \mod 2}} e^{-\frac{\pi i(jb+ka)}{M}} G_{a,b}(\tau,z_1,z_2) & \text{if } j\equiv k \mod 2 \\ \sum\limits_{\substack{(a,b)\in\Omega \\ \text{s.t. } a\equiv b \mod 2}} e^{-\frac{\pi i((j-1)b+ka)}{M}} H_{a,b}(\tau,z_1,z_2) & \text{if } j\equiv k+1 \mod 2, \end{cases}$$

$$H_{j,k}\left(-\frac{1}{\tau},\frac{z_1}{\tau},\frac{z_2}{\tau}\right) = (-1)^k\frac{\tau}{M}e^{\frac{\pi i z_1 z_2}{M\tau}}\times$$

$$\begin{cases} \sum\limits_{\substack{(a,b)\in\Omega \\ \text{s.t. } a\equiv b+1 \mod 2}} e^{-\frac{\pi i(jb+k(a+1))}{M}} G_{a,b}(\tau,z_1,z_2) & \text{if } j\equiv k \mod 2 \\ \sum\limits_{\substack{(a,b)\in\Omega \\ \text{s.t. } a\equiv b+1 \mod 2}} e^{-\frac{\pi i((j-1)b+k(a+1))}{M}} H_{a,b}(\tau,z_1,z_2) & \text{if } j\equiv k+1 \mod 2. \end{cases}$$

The transformation under $\tau \longrightarrow \tau+1$ is easily obtained:

Theorem 3.2.

$$G_{j,k}(\tau+1,z_1,z_2) = (-1)^k e^{-\frac{\pi i M}{4}}\times\begin{cases} e^{\frac{\pi i jk}{M}}G_{j,k}(\tau,z_1,z_2) \\ \qquad \text{if } j\equiv k \mod 2 \\ e^{\frac{\pi i(j+1)k}{M}}H_{j,k}(\tau,z_1,z_2) \\ \qquad \text{if } j\equiv k+1 \mod 2, \end{cases}$$

$$H_{j,k}(\tau+1,z_1,z_2) = (-1)^k e^{-\frac{\pi i M}{4}}\times\begin{cases} e^{\frac{\pi i jk}{M}}H_{j,k}(\tau,z_1,z_2) \\ \qquad \text{if } j\equiv k \mod 2 \\ e^{\frac{\pi i(j-1)k}{M}}G_{j,k}(\tau,z_1,z_2) \\ \qquad \text{if } j\equiv k+1 \mod 2. \end{cases}$$

Furthermore, an interesting family of modular functions is obtained by putting

$$g_{j,k}(\tau,z) := G_{j,k}(\tau,z,-z) \qquad \text{and} \qquad h_{j,k}(\tau,z) := H_{j,k}(\tau,z,-z).$$

Since $g_{j,k} = -g_{-k,-j}$ and $h_{j,k} = -e^{-\frac{\pi i}{M}(j+k)}h_{-k,-j}$, the explicit matrices of modular transformation of these functions are written in terms of the trigonometric function "sin". They may look similar to at a glance but turn to be quite different from those of the characters of the N=2 superconformal algebra.

It is known in [9] that a specialization of the denominator identity of $\widehat{\mathfrak{sl}}(2,1)$ gives the formula

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1+zq^{n-1})(1+z^{-1}q^n)} = \sum_{n\in\mathbb{Z}}(-1)^n\frac{q^{n(n+1)/2}}{1+zq^n}, \tag{14}$$

which appeared in [11] and was rediscovered by [8] in connection with Hecke indefinite modular forms. A similar identity

$$\prod_{n=1}^{\infty}\frac{(1-q^{\frac{n}{2}})(1-q^{2n})(1-zq^{2n-1})(1-z^{-1}q^{2n-1})}{(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}})} = \sum_{n\in\mathbb{Z}}(-1)^n\frac{q^{n(n+1)/4}}{1+zq^{n+\frac{1}{2}}} \tag{15}$$

is deduced from (10) by letting $u = q^{\frac{1}{4}}$ and $v = z^{-1}q^{\frac{1}{4}}$.

REFERENCES

[1] B. C. Berndt : Ramanujan's Notebooks Part III, Springer-Verlag, 1991.

[2] B. L. Feigin and E. Frenkel : Representations of affine Kac-Moody algebras and bosonization, in "Physics and Mathematics of Strings ∼ memorial volume of Vadim Knizhnik" ed. by L. Brink, D. Friedan and A. M. Polyakov, World Scientific, 1990, 271-316.

[3] B. L. Feigin and E. Frenkel : Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras, in "Proceedings of the RIMS Research Project 1991 on Infinite Analysis" ed. by A. Tsuchiya, T. Eguchi and M. Jimbo, Advanced Series in Math. Phys. Vol. 16, World Scientific, 1992, 197-215.

[4] D. Hickerson : A proof of the mock theta conjectures, Invent. math. 94 (1988), 639-660.

[5] V. G. Kac : Lie superalgebras, Advances in Math. 26 (1977), 8-96.

[6] V. G. Kac : Contravariant form for infinite-dimensional Lie algebras and superalgebras, in Lecture Notes in Phys. 94, Springer-Verlag, 1979, 441-445.

[7] V. G. Kac and D. A. Kazhdan : Structure of representations with highest weight of infinite-dimensional Lie algebras, Advances in Math. 34 (1979), 97-108.

[8] V. G. Kac and D. Peterson : Infinite-dimensional Lie algebras, theta functions and modular forms, Advances in Math. 53 (1984), 125-264.

[9] V. G. Kac and M. Wakimoto : Integrable highest weight modules over affine superalgebras and number theory, in "Lie Theory and Geometry ∼ in honor of Bertram Kostant" ed. by J. L. Brylinski, R. Brylinski, V. G. Guillemin and V. Kac, Progress in Math. Phys. Vol. 123, Birkhäuser, 1994, 415-456.

[10] V. G. Kac and M. Wakimoto : in preparation.

[11] J. Tannery and J. Molk : Élements de la théorie des fonctions elliptiques, Paris, 1898.

[12] E. T. Whittaker and G. N. Watson : A Course of Modern Analysis, fourth edition, Cambridge Univ. Press, 1927.

Minoru Wakimoto
Graduate School of Mathematics
Kyushu University
Fukuoka, 812-8581, Japan