Characters of Irreducible Representations of Simple Lie Superalgebras

Vera Serganova

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1. Introduction

Simple Lie superalgebras were classified in 1977 by V. Kac [6]. These superalgebras can be divided into three groups

1. *Contragredient* Lie superalgebras, i.e. Lie superalgebras which can be determined by a Cartan matrix. These superalgebras have an invariant symmetric form and Cartan involution. There are two families of such algebras $sl\,(m|n)$ (factored by the center when $m=n$) and $osp\,(m|2n)$. The Lie superalgebra $osp\,(4|2)$ has a one-parameter deformation, called $D\,(\alpha)$. There are also two exceptional Lie superalgebras G_3 and F_4.
2. New "*strange*" superalgebras $Q\,(n)$ and $P\,(n)$, the former consists of operators commuting with an odd nondegenerate operator, the latter consists of operators preserving a non-degenerate odd symmetric form.
3. *Cartan type superalgebras* W_n, S_n, S'_n and SH_n, i.e. superalgebra of vector fields on a supermanifold of pure odd dimension and its simple subalgebras.

In [7] it was shown that all finite-dimensional irreducible representations of a simple Lie superalgebra $\mathfrak{g}$ are enumerated by a highest weight $\lambda \in \mathfrak{h}^*$, satisfying certain conditions of dominance, $\mathfrak{h}$ being a Cartan subalgebra.

The problem of finding the character of an irreducible highest weight module L_λ for a general dominant λ appeared to be unexpectedly difficult. It was solved first for Lie superalgebras W_n and S_n of Cartan type by J. Bernstein and D. Leites in [1] and for SH_n by A. Shapovalov [20]. They considered a module M_λ of tensor fields on a supermanifold of purely odd dimension and described completely its Jordan–Hölder series. Since $\operatorname{ch} M_\lambda$ and the multiplicity $[M_\lambda : L_\mu]$ are known, one can obtain $\operatorname{ch} L_\lambda$ by solving a simple system of linear equations.

For contragredient and strange Lie superalgebras the problem was solved in [7] in the particular case of a generic (*typical*) highest weight λ. The character is given by a nice Weyl type formula:

$$\operatorname{ch} L_\lambda = D \sum_{w\in W} \operatorname{sgn} w \cdot e^{w(\lambda)},\ D = \frac{\prod_{\alpha\in\Delta_1^+}\left(e^{\alpha/2}+e^{-\alpha/2}\right)}{\prod_{\alpha\in\Delta_0^+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)}, \tag{1.1}$$

where W is the Weyl group of $\mathfrak{g}_0$.

This formula can be obtained from the Borel–Weil–Bott theorem for a flag supermanifold and a typical invertible sheaf $\mathcal{O}_\lambda$. Geometry of flag supermanifolds was studied in 1980*s* by Yu. Manin and his pupils I. Penkov, I. Skornyakov and A. Voronov. It was shown by Penkov and Skornyakov in [16] that the invertible sheaf $\mathcal{O}_\lambda$ with a typical dominant weight λ satisfies the Borel–Weil–Bott theorem, namely $H^0\mathcal{O}_\lambda = L_\lambda$ and $H^i\mathcal{O}_\lambda = 0$ for $i > 0$.

On the other hand, any invertible sheaf $\mathcal{O}_\lambda$ on G/B can be considered as a sheaf $\mathcal{L}_\lambda$ on the underlying flag manifold G_0/B_0. $\mathcal{L}_\lambda$ has a filtration by invertible sheaves $\mathcal{O}_{\lambda+\nu}(G_0/B_0)$, where ν runs over the set of sums of odd negative roots $\sum_{\alpha_i\in\Delta_1^-}\alpha_i$. Therefore one can write down the Euler characteristic of $\mathcal{L}_\lambda$

$$E_\lambda = \sum_i (-1)^i \operatorname{ch} H^i\mathcal{O}_\lambda = \sum_\nu \sum_i (-1)^i \operatorname{ch} H^i_{G_0/B_0}\mathcal{O}_{\lambda+\nu}, \tag{1.2}$$

using additivity of Euler characteristic. It happens to be exactly the Kac character formula (1.1).

If a highest weight λ is *atypical* the Borel–Weil–Bott theorem fails. Depending on *the degree of atypicality* of λ the $\mathfrak{g}$-module structure of $H^i\mathcal{O}_\lambda$ becomes more and more complicated. There were several conjectures and partial results about a character formula for a general dominant weight (see [2, 13, 22, 21, 9, 14, 4]). For the case $\mathfrak{g} = gl(m|n)$ two different formulae were conjectured in [17] and in [5]. The first conjecture was proven in [18]. The second one is believed to be equivalent to the first one.[1] For $\mathfrak{g} = Q(n)$ the problem was solved in [15]. For $\mathfrak{g} = osp(m|2n)$ it is solved just recently, we announce the results here.

We use the same method to solve the problem for gl, osp and Q. Namely we calculate the "Euler-multiplicities" $a_{\lambda,\mu} = \sum(-1)^i\left[H^i_{G/B}\mathcal{O}_\lambda : L_\mu\right]$ (in some cases we have to use a suitable parabolic subgroup P instead of Borel subgroup B). Since a lot of numbers $a_{\lambda,\mu}$ vanish, one can find $\operatorname{ch} L_\lambda$ from the system $\sum_\mu a_{\lambda,\mu}\operatorname{ch} L_\mu = E_\lambda$ and (1.2). To calculate the coefficients $a_{\lambda,\mu}$, we represent B as the end of a flag of parabolic subgroups $G = P^{(1)} \supset P^{(2)} \supset \cdots \supset P^{(n)} = B$. Composing push down of sheaves, the coefficients $a_{\lambda,\mu}$ can be expressed in terms of similar coefficients $a^{(j)}_{\lambda,\mu}$ for the supermanifold $P^{(j)}/P^{(j+1)}$.

In fact cohomology of "dominant sheaves" on $P^{(j)}/P^{(j+1)}$ can be completely described by using a certain analogue of *translation* and *reflection* functors. The space of dominant weights can be stratified by the degree of atypicality. A translation functor allows us to move inside a stratum (compare Theorem 2.5), while a reflection functor increases the degree of atypicality by one. This together with the fact that cohomology of irreducible vector bundles on $P^{(j)}/P^{(j+1)}$ are almost semisimple $P^{(j)}$-modules (see Lemma 4.2) helps to write down the crucial recurrence relations on $a^{(j)}_{\lambda,\mu}$ (compare Theorems 4.1 and 4.3).

In this paper we outline the schematic of the character formulae for gl and osp omitting all the details needed for the proof only. We also omit the case of Q_n containing additional complications, since irreducible representations of $\mathfrak{h}$ are not 1-dimensional.

[1] Note that in [18] references to [5] and [22] were corrupted.

Currently the problem of finding irreducible characters remains open for exceptional Lie superalgebras and $P(n)$. While similar methods should work for exceptional superalgebras, in the case of $P(n)$ it is unclear how to define translation and reflection functors: the center of the universal enveloping algebra of $P(n)$ is too small and central characters do not separate blocks.

2. Dominant weights, central characters and blocks

Throughout this paper $\mathfrak{g}$ stands for one of the Lie superalgebras $gl(m|n)$, $osp(2m|2n)$ or $osp(2m+1|2n)$. The Lie superalgebra $\mathfrak{g}$ has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

The roots are called *even* or *odd* depending on the parity of the root space $\mathfrak{g}_\alpha$. Denote the set of even (correspondingly odd) roots by Δ_0 (correspondingly Δ_1). Clearly, $\Delta = \Delta_0 \cup \Delta_1$. Many odd roots are *isotropic*, put $\Delta^{\text{is}} = \{\alpha \mid (\alpha,\alpha) = 0\}$, where $(,)$ stands for the Killing form.

Describe the set of roots in the standard basis $\{\delta_1,\dots,\delta_n,\varepsilon_1,\dots,\varepsilon_m\}$ of $\mathfrak{h}^*$. Note that $(\delta_i,\delta_j) = \delta_{i,j}$, $(\varepsilon_i,\varepsilon_j) = -\delta_{i,j}$.

Let $\mathfrak{g} = gl(m|n)$. Then

$$\Delta_0 = \{\varepsilon_i - \varepsilon_j \mid i,j = 1,\dots,m\} \cup \{\delta_i - \delta_j \mid i,j = 1,\dots,n\},$$
$$\Delta_1 = \Delta^{\text{is}} = \{\pm(\varepsilon_i - \delta_j) \mid i = 1,\dots,m, j = 1,\dots,n\}.$$

Let $\mathfrak{g} = osp(2m|2n)$. Then $\Delta_1 = \Delta^{\text{is}}$ is the same as for $\mathfrak{g} = gl(m|n)$,

$$\Delta_0 = \{\varepsilon_i \pm \varepsilon_j \mid i,j = 1,\dots,m, i \neq j\} \cup \{\delta_i \pm \delta_j \mid i,j = 1,\dots,n\}.$$

Let $\mathfrak{g} = osp(2m+1|2n)$. Then

$$\Delta_0 = \{\varepsilon_i \pm \varepsilon_j \mid i,j = 1,\dots,m, i \neq j\} \cup \{\pm\varepsilon_i \mid i = 1,\dots,m\} \cup \{\delta_i \pm \delta_j \mid i,j = 1,\dots,n\},$$
$$\Delta^{\text{is}} = \{\pm(\varepsilon_i - \delta_j) \mid i = 1,\dots,m, j = 1,\dots,n\}.$$
$$\Delta_1 = \Delta^{\text{is}} \cup \{\pm\delta_i \mid i = 1,\dots,n\}.$$

Fix a subdivision $\Delta = \Delta^+ \cup \Delta^-$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, defined by $\mathfrak{n}^\pm = \bigoplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha$. A choice of Δ^+ is not unique, here we fix it by enumerating simple roots in each case.

For $\mathfrak{g} = gl(m|n)$ choose simple roots $\sigma_1 = \delta_1 - \delta_2, \dots, \sigma_{n-1} = \delta_{n-1} - \delta_n$, $\sigma_n = \delta_n - \varepsilon_1, \dots, \sigma_{m+n-1} = \varepsilon_{m-1} - \varepsilon_m$.

For $\mathfrak{g} = osp(2m|2n)$ choose simple roots $\sigma_1 = \delta_1 - \delta_2, \dots, \sigma_{n-1} = \delta_{n-1} - \delta_n$, $\sigma_n = \delta_n - \varepsilon_1, \dots, \sigma_{m+n-1} = \varepsilon_{m-1} - \varepsilon_m$, $\sigma_{m+n} = \varepsilon_{m-1} + \varepsilon_m$.

For $\mathfrak{g} = osp(2m+1|2n)$ choose simple roots $\sigma_1 = \delta_1 - \delta_2, \dots, \sigma_{n-1} = \delta_{n-1} - \delta_n$, $\sigma_n = \delta_n - \varepsilon_1, \dots, \sigma_{m+n-1} = \varepsilon_{m-1} - \varepsilon_m$, $\sigma_{m+n} = \varepsilon_m$.

For an even root α put $\alpha^\vee = \alpha/(\alpha,\alpha)$. We say that $\lambda \in \mathfrak{h}^*$ is *integral* if $(\lambda,\alpha^\vee) \in \mathbb{Z}$ for any $\alpha \in \Delta_0$. Denote the set of integral weights by Λ.

Denote by L_λ an irreducible module generated by highest vector v of weight $\lambda-\rho$, i.e., $\mathfrak{n}^+v=0$, $hv=\langle\lambda-\rho,h\rangle v$ for $h\in\mathfrak{h}$, and

$$\rho=1/2\sum_{\alpha\in\Delta_0^+}\alpha-1/2\sum_{\alpha\in\Delta_1^+}\alpha.$$

Call a weight $\lambda\in\Lambda$ *dominant* if $\dim L_\lambda<\infty$. Denote the set of dominant weights by Λ^+. The conditions on λ to be dominant were first calculated in [6]. We reproduce them here in our notations.

PROPOSITION 2.1. *Let $\mathfrak{g}=gl\,(m|n)$, $osp\,(2m|2n)$ or $osp\,(2m+1|2n)$. Let $\lambda=a_1\delta_1+\cdots+a_n\delta_n+b_1\varepsilon_1+\cdots+b_m\varepsilon_m\in\Lambda$. Then $\lambda\in\Lambda^+$ iff the following conditions on a_i and b_j hold:*

1. *for $gl\,(m|n): a_i-a_{i+1},\ b_j-b_{j+1}\in\mathbb{Z}_{>0}$;*
2. *for $osp\,(2m|2n): a_i,b_j\in\mathbb{Z}$, $a_1>a_2>\cdots>a_n>-m$, $b_1>b_2>\cdots>b_{m-1}>|b_m|$ and for each $l\leq 0$, $l\geq a_n$, $b_{m+l}=-l$;*
3. *for $osp\,(2m+1,2n)$, $a_i\in 1/2+\mathbb{Z}$, $b_j\in 1/2+\mathbb{Z}$ or $\mathbb{Z}$, $a_1>a_2>\cdots>a_n\geq 1/2-m$, $b_1>b_2>\cdots>b_m>0$ and for each $l\leq 0$, $a_n\leq l-1/2$, $b_{m+l}=1/2-l$.*

Remark 2.2. For *osp* type superalgebras one can not choose Δ^+ in such a way that the set of simple roots for Δ_0^+ is the subset of simple roots for Δ^+. That is why in this case the conditions on dominance with respect to $\mathfrak{g}_0$ differ from the conditions on dominance with respect to $\mathfrak{g}$.

Let $\mathfrak{g}=osp\,(2m|2n)$ or $osp\,(2m+1|2n)$. By Proposition 2.1 if $\lambda\in\Lambda^+$ and $a_r>0\geq a_{r+1}$, one can find odd isotropic roots $\delta_{r+1}-\varepsilon_{i_1},\dots,\delta_n-\varepsilon_{i_s}$ orthogonal to λ. Call the set of such roots the *tail* of λ and denote it by T_λ. Call the number $t_\lambda=s=n-r$ *the tail length* of λ. For $\mathfrak{g}=gl\,(m|n)$ put $t_\lambda=0$.

Note also that in the case $\mathfrak{g}=osp\,(2m|2n)$ there is a symmetry of Dynkin diagram which induces an outer automorphism τ such that $\tau\,(\sigma_{m+n-1})=\sigma_{m+n}$. The automorphism τ acts on the set of dominant weights. In what follows we always assume that $b_m\geq 0$. If $b_m<0$ we can obtain all coefficients using the rule $a_{\tau(\lambda),\tau(\mu)}=a_{\lambda,\mu}$, $a_{\tau(\lambda),\mu}=0$ if $\tau\,(\mu)<\mu$, $\tau\,(\lambda)<\lambda$.

Let $\mathcal{F}=\mathcal{F}\,(\mathfrak{g})$ be the category formed by finite-dimensional $\mathfrak{g}$-modules. To describe the structure of $\mathcal{F}$, consider the center $Z\,(\mathfrak{g})$ of the universal enveloping algebra $U\,(\mathfrak{g})$. Recall that a *central character* is a homomorphism $\chi\colon Z\,(\mathfrak{g})\to\mathbb{C}$. We say that a $\mathfrak{g}$-module M has a central character χ if for any $z\in Z\,(\mathfrak{g})\,,x\in M$ there is $N\in\mathbb{Z}_{\geq 0}$ such that $(z-\chi\,(z)\,\mathrm{id})^N\cdot x=0$. Clearly any finite-dimensional indecomposable $\mathfrak{g}$-module has some central character, and any finite-dimensional $\mathfrak{g}$-module decomposes into a direct sum of submodules with central characters.

We use a Harish–Chandra homomorphism $HC\colon Z\,(\mathfrak{g})\hookrightarrow \mathrm{Pol}\,(\mathfrak{h}^*)$. The construction of this homomorphism is the same as for semi-simple Lie algebras (see for example [3]). Thus, any $\lambda\in\mathfrak{h}^*$ defines a central character χ_λ by the rule $\chi_\lambda\,(z)=HC\,(z)\,(\lambda)$. Definition of HC immediately implies that an irreducible module L_λ has a central character χ_λ. A central character χ is *dominant* if $\chi=\chi_\lambda$ at least for one $\lambda\in\Lambda^+$.

The following statement was first formulated in [8] and proved in [19] and [12].

Proposition 2.3. *Let $\lambda, \mu \in \Lambda$, W be the Weyl group of $\mathfrak{g}_0$. Then $\chi_\lambda = \chi_\mu$ iff there is a sequence of isotropic roots $\alpha_1, \ldots, \alpha_s \in \Delta^{is}$ and $w \in W$ such that $\mu = w(\lambda + \alpha_1 + \cdots + \alpha_s)$ and $(\lambda + \alpha_1 + \cdots + \alpha_{i-1}, \alpha_i) = 0$ for $i = 1, \ldots, s$.*

Let $\mathcal{F}_\chi = \mathcal{F}_\chi(\mathfrak{g})$ be the full subcategory of $\mathcal{F}$ consisting of modules with central character χ. Obviously, $\mathcal{F} = \bigoplus \mathcal{F}_\chi$, where the summation is taken over all dominant central characters χ. Different categories $\mathcal{F}_\chi$ may be equivalent. They fall into one of 4 series as we will see in Theorem 2.6.

State more constructive condition for $\chi_\lambda = \chi_\mu$. For $\lambda \in \Lambda^+$ let $A_\lambda = \{\alpha_1, \ldots, \alpha_k\}$ be a maximal set of mutually orthogonal positive isotropic roots such that $(\lambda, \alpha_i) = 0$ for $i = 1, \ldots, k$. If $\mathfrak{g} = gl(m|n)$, then the set A_λ is uniquely defined. For *osp*-type $\mathfrak{g}$ we choose A_λ in such way that $T_\lambda \subseteq A_\lambda$. The number $k = |A_\lambda|$ is called *the degree of atypically* of λ and is denoted by $\#\lambda$. A weight $\lambda \in \Lambda^+$ is *typical* if $\#\lambda = 0$.

Let $\bar{\mathfrak{h}}^*_\lambda$ be the subspace of $\mathfrak{h}^*$ generated by all basis vectors ε_i, δ_j orthogonal to the roots from A_λ. Denote by $\bar{\lambda}$ the image of λ under the orthogonal projection $\mathfrak{h}^* \to \bar{\mathfrak{h}}^*_\lambda$. One can see that if $\#\lambda = \#\mu$, then $w\left(\bar{\mathfrak{h}}^*_\lambda\right) = \bar{\mathfrak{h}}^*_\mu$ for some $w \in W$.

Proposition 2.4. *Let $\lambda, \mu \in \Lambda^+$. Then $\chi_\lambda = \chi_\mu$ iff $\#\lambda = \#\mu$ and $w\left(\bar{\lambda}\right) = \bar{\mu}$ for some $w \in W$.*

By Proposition 2.4 one can correctly define $\#\chi$ for a dominant central character χ by putting $\#\chi \stackrel{\text{def}}{=} \#\lambda$ for any dominant λ with $\chi_\lambda = \chi$. Moreover, one can define $\bar{\chi}$ as a typical central character for Lie superalgebra $\bar{\mathfrak{g}}$, here $\bar{\mathfrak{g}}$ is an appropriate subalgebra of $\mathfrak{g}$ isomorphic to $gl(m-k|n-k)$, $osp(2(m-k)|2(n-k))$ or $osp(2(m-k)+1|2(n-k))$ depending on the type of $\mathfrak{g}$, and $k = \#\chi$.

Theorem 2.5. *A category $\mathcal{F}_\chi$ is indecomposable for any dominant central character χ. If $\mathfrak{g} = gl(m|n)$ or $osp(2m+1|2n)$, then two categories $\mathcal{F}_{\chi_1}$ and $\mathcal{F}_{\chi_2}$ are equivalent iff $\#\chi_1 = \#\chi_2$.*

Let $\mathfrak{g} = osp(2m|2n)$ and $\bar{\tau}$ be the outer automorphism of $\bar{\mathfrak{g}}$ induced by the symmetry of Dynkin diagram. Then two categories $\mathcal{F}_{\chi_1}$ and $\mathcal{F}_{\chi_2}$ are equivalent iff $\#\chi_1 = \#\chi_2$, and for both $i = 1, 2$ either $\bar{\tau}(\bar{\chi}_i) = \bar{\chi}_i$ or $\bar{\tau}(\bar{\chi}_i) \neq \bar{\chi}_i$.

An indecomposable category $\mathcal{F}_\chi$ is called a *block* of $\mathcal{F}$.

Theorem 2.6. *Let $\mathfrak{g} = gl(m|n)$, $osp(2m+1|2n)$ or $osp(2m|2n)$. A block $\mathcal{F}_\chi$ with $\#\chi = k$ is equivalent to $\mathcal{F}'_{\chi_{\rho'}} \stackrel{def}{=} \mathcal{F}_{\chi_{\rho'}}(\mathfrak{g}')$, where $\mathfrak{g}' = gl(k|k)$ if $\mathfrak{g} = gl(m|n)$, $\mathfrak{g}' = osp(2k+1|2k)$ if $\mathfrak{g} = osp(2m+1|2n)$, $\mathfrak{g}' = osp(2k+2|2k)$ if $\mathfrak{g} = osp(2m|2n)$ and $\bar{\tau}(\bar{\chi}) = \bar{\chi}$, $\mathfrak{g}' = osp(2k|2k)$ if $\mathfrak{g} = osp(2m|2n)$ and $\bar{\tau}(\bar{\chi}) \neq \bar{\chi}$, here ρ' is the analogue of ρ for $\mathfrak{g}'$.*

Let $\Phi\colon \mathcal{F}_\chi \to \mathcal{F}'_{\chi_{\rho'}}$ be a functor establishing equivalence of categories of theorem 2.6. Then Φ sends irreducible objects to irreducible objects, describe the corresponding mapping of highest weights:

Proposition 2.7. *Let $L_\lambda \in \mathcal{O}b\,\mathcal{F}_\chi$ and $L_{\lambda'} = \Phi L_\lambda$. Let $\lambda = \sum_{i=1}^n a_i \delta_i + \sum_{j=1}^m b_j \varepsilon_j$.*

1. *If* $\mathfrak{g} = gl\,(m|n)$, $A_\lambda = \{\alpha_1 = \delta_{i_1} - \varepsilon_{j_1}, \dots, \alpha_k = \delta_{i_k} - \varepsilon_{j_k}\}$, *then*

$$\lambda' = \sum_{p=1}^{k} a'_p\,(\delta_p - \varepsilon_{k-p+1})\,,\ a'_p = a_{i_p} + (\rho, \alpha_p)\,;$$

2. *If* $\mathfrak{g} = osp\,(2m|2n)$ *or* $osp\,(2m+1|2n)$, *and*

$$A_\lambda = \left\{\delta_{i_1} + \varepsilon_{j_1}, \dots, \delta_{i_r} + \varepsilon_{j_r}, \delta_{i_{r+1}} - \varepsilon_{j_{r+1}}, \dots, \delta_{i_k} - \varepsilon_{j_k}\right\},$$

then $\lambda' = \sum_{p=1}^{k}\left(a'_p\delta_p + |a'_{s(p)}|\varepsilon_p\right)$, $a'_p = a_{i_p} - x_p \cdot \operatorname{sgn} a_{i_p}$, *where*

$$x_p = \#\left\{q, l \mid \left(\bar\lambda, \delta_q\right) < |a_{i_p}|, 0 \neq -\left(\bar\lambda, \varepsilon_l\right) < |a_{i_p}|\right\},$$

and s is a permutation such that $|a_{s(1)}| > \cdots > |a_{s(k)}|$. *In particular* $t_\lambda = t_{\lambda'}$.

EXAMPLE 2.8. Let $\mathfrak{g} = gl\,(3|3)$, $\lambda = 5\delta_1 + 3\delta_2 - \delta_3 + 4\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3$. Then $A_\lambda = \{\delta_3 - \varepsilon_3\}$, $\#\lambda = 1$, $\bar{\mathfrak{g}} \simeq gl\,(2|2)$, $\mathfrak{g}' \simeq gl\,(1|1)$, $\bar\lambda = 5\delta_1 + 3\delta_2 + 3\varepsilon_2 + \varepsilon_3$, $\lambda' = -3\delta_1 + 3\varepsilon_1$.

EXAMPLE 2.9. Let $\mathfrak{g} = osp\,(8|6)$, $\lambda = 4\delta_1 + \delta_2 - 3\delta_3 + 3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + 0\varepsilon_4$. Then $A_\lambda = \{\delta_2 + \varepsilon_3, \delta_3 - \varepsilon_1\}, t_\lambda = 1$, $\#\lambda = 2$, $\bar{\mathfrak{g}} \simeq osp\,(4|2)$, $\mathfrak{g}' \simeq osp\,(6|4)$, $\bar\lambda = 4\delta_1 + 2\varepsilon_2 + 0\varepsilon_4$, $\lambda' = \delta_1 - 2\delta_2 + 2\varepsilon_1 + \varepsilon_2 + 0\varepsilon_3$.

3. BOREL–WEIL–BOTT THEOREM AND GEOMETRIC INDUCTION

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a Borel subalgebra. For a *parabolic* subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$ denote by $\Delta_\mathfrak{p}$ the set of roots α such that $\mathfrak{g}_{\pm\alpha} \subseteq \mathfrak{p}$. Denote by $L_\lambda\,(\mathfrak{p})$ an irreducible $\mathfrak{p}$-module with highest weight λ, which of course remains irreducible after restriction to a reductive subalgebra $\mathfrak{g}_\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_\mathfrak{p}} \mathfrak{g}_\alpha$. Due to the geometric origin of the following argument we will need representations of the *supergroup P* corresponding to $\mathfrak{p}$. A *flag supermanifold G/P* is a compact homogeneous supermanifold with the underlying manifold G_0/P_0 (for definitions see [11] or [10]). Any finite-dimensional P-module M induces a vector bundle $\mathcal{O}\,(M)$ on G/P with the fiber M over P. Let[2]

$$\Gamma_i^P\,(M) = \left[H^i_{G/P}\mathcal{O}\,(M^*)\right]^*.$$

Since G/P is compact, $\Gamma_i^P\,(M)$ is a finite-dimensional G-module. One can consider Γ_i^P as a derived functor from the category $\mathcal{F}\,(P)$ of finite dimensional P-modules to the category $\mathcal{F}$.

LEMMA 3.1. *Let $\mathfrak{n}_\mathfrak{p}$ be the nilpotent ideal in $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{g}_\mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p}$. If $X \in \operatorname{Ob}\mathcal{F}$, then coinvariants $X_{\mathfrak{n}_\mathfrak{p}}$ form a P-module, there is a natural inclusion $X \hookrightarrow H^0_{G/P}\mathcal{O}\left(X_{\mathfrak{n}_\mathfrak{p}}\right)$.*

If $M \in \operatorname{Ob}\mathcal{F}_\chi$, then $\Gamma_i^P\,(M^{\mathfrak{n}_\mathfrak{p}}) \in \operatorname{Ob}\mathcal{F}_\chi$, and there is a natural projection $\Gamma_0^P\,(M^{\mathfrak{n}_\mathfrak{p}}) \to M$. In particular when $M = L_\lambda$ there is a natural projection $\Gamma_0^P\,(L_\lambda\,(P)) \to L_\lambda$.

[2]We use doubled duality to avoid a problematic notion of antidominant weight.

LEMMA 3.2. *Let M be a P-module and $E^P(M) = \sum (-1)^i \operatorname{ch} \Gamma_i^P(M)$. Consider D from* (1.1). *Then*

$$E^P(M) = D \sum_{w \in W} \operatorname{sgn} w \cdot w \left(\frac{e^\rho \operatorname{ch} M}{\prod_{\alpha \in \Delta_{\mathfrak{p}} \cap \Delta_1^+} (1 + e^{-\alpha})} \right). \tag{3.1}$$

The following theorem is a generalization of a result by Penkov–Skornyakov. We say that $\lambda \in \Lambda^+$ is *P-typical* if $A_\lambda \subseteq \Delta_{\mathfrak{p}}$.

THEOREM 3.3. *Let λ be a P-typical dominant weight. Then $\Gamma_0^P(L_\lambda(\mathfrak{p})) = L_\lambda$ and $\Gamma_i^P(L_\lambda(\mathfrak{p})) = 0$ for $i > 0$.*

Generalizing this theorem, denote by $[M : L_\mu]$ the multiplicity of an irreducible module L_μ in a $\mathfrak{g}$-module M. Define the *Kazhdan–Lusztig polynomials* and coefficients by:

$$K_{\lambda,\mu}^P(q) = \sum_{i=0}^{\dim G/P} \left[\Gamma_i^P(L_\lambda(\mathfrak{p})) : L_\mu\right] q^i, \qquad a_{\lambda,\mu}^P = K_{\lambda,\mu}^P(-1).$$

Let $E_\lambda^P \stackrel{\text{def}}{=} E^P(L_\lambda(\mathfrak{p}))$. Clearly

$$\sum_\mu a_{\lambda,\mu}^P \operatorname{ch} L_\mu = E_\lambda^P. \tag{3.2}$$

PROPOSITION 3.4. *Let $\mathfrak{p}$ be a parabolic subalgebra, $\lambda \in \Lambda^+$ and $T_\lambda \subseteq \Delta_{\mathfrak{p}}$. Then $K_{\lambda,\lambda}^P = 1$, and $K_{\lambda,\mu}^P \neq 0$ implies $\mu \leq \lambda$, $\chi_\lambda = \chi_\mu$.*

Let $\mathfrak{g} = gl(m|n)$, then $t_\lambda = 0$. Put $P = B$. By Proposition 3.4 the matrix $\left(a_{\lambda,\mu}^B\right)$ is unipotent, thus easy to invert. Let $(b_{\lambda,\mu}) = \left(a_{\lambda,\mu}^B\right)^{-1}$. Then the equations (3.2) imply $\operatorname{ch} L_\lambda = \sum_{\mu \leq \lambda} b_{\lambda,\mu} E_\mu^B$.

If $\mathfrak{g}$ is an algebra of *osp* type then the matrix $\left(a_{\lambda,\mu}^B\right)$ is not invertible. Let $\Lambda_s^+ = \{\lambda \in \Lambda^+ \mid t_\lambda = s\}$ and $\mathfrak{p}^{(r)}$ be the parabolic subalgebra such that $\Delta_{\mathfrak{p}^{(r)}}$ is generated by the simple roots $\sigma_r, \dots, \sigma_{n+m}$. One can see that $T_\lambda \subseteq \Delta_{\mathfrak{p}^{(r+1)}}$ for any $\lambda \in \Lambda_{n-r}^+$. By Proposition 3.4 the matrix $\left(a_{\lambda,\mu}^{P^{(r+1)}}\right)_{\lambda,\mu \in \Lambda_{n-r}^+}$ is again unipotent, thus easy to invert. Since $\mu \leq \lambda$ implies $t_\mu \geq t_\lambda$, the equation

$$\sum_{\mu \in \Lambda_{n-r}^+} a_{\lambda,\mu}^{P^{(r+1)}} \operatorname{ch} L_\mu = E_\lambda^{P^{(r+1)}} - \sum_{t_\nu > t_\lambda} a_{\lambda,\nu}^{P^{(r+1)}} \operatorname{ch} L_\nu.$$

This taken together with (3.1) expresses $\operatorname{ch} L_\lambda$ in terms of $\operatorname{ch} L_\mu\left(\mathfrak{p}^{(r+1)}\right) = \operatorname{ch} L_\mu\left(\mathfrak{g}_{\mathfrak{p}^{(r+1)}}\right)$, $a_{\mu,\nu}^{P^{(r+1)}}$ for $r = n - t_\lambda$, $\mu, \nu \in \Lambda^+$ with $t_\mu = t_\lambda$, $t_\nu \geq t_\lambda$, and $\operatorname{ch} L_{\nu'}$ for $t_{\nu'} > t_\lambda$. If $t_\lambda < n$, $\operatorname{rk} \mathfrak{g}_{\mathfrak{p}^{(r+1)}} < \operatorname{rk} \mathfrak{g}$, which gives a recurrence relation for $\operatorname{ch} L_\lambda$.

What remains is the case $t_\lambda = n$. Then $T_\lambda = A_\lambda$, and λ is Q-typical for the parabolic subalgebra $\mathfrak{q}$ with $\Delta_{\mathfrak{q}}$ generated by $\sigma_1, \dots, \sigma_{m+n-1}$. By theorem 3.3

$$\operatorname{ch} L_\lambda = E_\lambda^Q = D \sum_{w \in W} \operatorname{sgn} w \cdot w \left(\frac{e^\rho \operatorname{ch} L_\lambda(\mathfrak{q})}{\Pi_{\alpha \in \Delta_{\mathfrak{q}} \cap \Delta_1^+} (1 + e^{-\alpha})} \right) \ldots$$

On the other hand, $\operatorname{ch} L_\lambda(\mathfrak{q}) = \operatorname{ch} L_\lambda(\mathfrak{g}_\mathfrak{q})$, and $\mathfrak{g}_\mathfrak{q}$ is isomorphic to $gl(m|n)$. Since the case $\mathfrak{g} = gl(m|n)$ is already covered, we can calculate $\operatorname{ch} L_\lambda$.

These arguments reduce the calculation of $\operatorname{ch} L_\lambda$ to the calculation of the matrix $\left(a^P_{\mu,\nu}\right)$. The next statement reduces the latter problem to the case of the most atypical central character.

PROPOSITION 3.5. *Let $\lambda \in \Lambda^+$, λ' and $\mathfrak{g}'$ be as in theorem 2.6 and proposition 2.7. Let $\mathfrak{p} = \mathfrak{b}$ if $\mathfrak{g} = gl(m|n)$, $\mathfrak{p} = \mathfrak{p}^{(r+1)}$ if $\mathfrak{g} = osp(2m|2n)$ or $osp(2m+1|2n)$ and $t_\lambda = n - r$. Let $\mathfrak{p}'$ be the analogous parabolic subalgebra in $\mathfrak{g}'$ determined by λ'. Then $K^P_{\lambda,\mu} = K^{P'}_{\lambda',\mu'}$.*

4. CALCULATION OF COEFFICIENTS $a_{\lambda,\mu}$ IN $\mathcal{F}_{\chi_\rho}$

In this section we concentrate on calculation of coefficients $a^P_{\lambda,\mu}$. By Proposition 3.5 it is sufficient to find these coefficients only for the most atypical block $\mathcal{F}_{\chi_\rho}$ and $\mathfrak{g} = gl(k|k)$, $osp(2k|2k)$, $osp(2k+2|2k)$ or $osp(2k+1|2k)$.

First, consider the case $\mathfrak{g} = gl(k|k)$. Here $t_\lambda = 0$ and $P = B$. Introduce a formal operator A in the Grothendieck ring of $\mathcal{F}_{\chi_\rho}$ by the formula

$$A[L_\lambda] = \sum_{\mu\in\Lambda} a^P_{\lambda,\mu}[L_\mu].$$

Let $\mathfrak{p}^{(i)}$ be the parabolic subalgebra such that $\Delta_{\mathfrak{p}^{(i)}}$ is generated by the simple roots $\sigma_i, \dots, \sigma_{2k-i}$. Consider the flag of parabolic subalgebras $\mathfrak{g} = \mathfrak{p}^{(1)} \supset \mathfrak{p}^{(2)} \supset \dots \supset \mathfrak{p}^{(k)} \supset \mathfrak{p}^{(k+1)} = \mathfrak{b}$. Note that $P^{(i)}/P^{(i+1)}$ is isomorphic to the supermanifold of $(1|1)$-dimensional subspaces in $\mathbb{C}^{i|i}$. As before define derived functors

$$\Gamma^{(i)}_j : \mathcal{F}\left(\mathfrak{p}^{(i+1)}\right) \to \mathcal{F}\left(\mathfrak{p}^{(i)}\right), \qquad \Gamma^{(i)}_j(M) = \left[H^j_{P^{(i)}/P^{(i+1)}}\mathcal{O}(M^*)\right]^*,$$

generating functions $K^{(i)}$ and coefficients $a^{(i)}$ by

$$K^{(i)}_{\lambda,\mu}(q) = \sum_j \left[\Gamma^{(i)}_j\left(L_\lambda\left(\mathfrak{p}^{(i+1)}\right)\right) : L_\mu\left(\mathfrak{p}^{(i)}\right)\right] q^j, \qquad a^{(i)}_{\lambda,\mu} = K^{(i)}_{\lambda,\mu}(-1).$$

Define the operators $A^{(i)}[L_\lambda] = \sum_\mu a^{(i)}_{\lambda,\mu}[L_\mu]$. Obviously,

$$A = A^{(1)} \circ \dots \circ A^{(k)}. \tag{4.1}$$

Theorem 4.1 below gives recurrence relations for calculating polynomials $K^{(i)}_{\lambda,\mu}$. It is the most difficult result in the paper. Before stating it let us recall that any $\lambda \in \Lambda^+$ with $\chi_\lambda = \chi_\rho$ can be written as $a_1\alpha_1 + \dots + a_k\alpha_k$ where $\alpha_i = \delta_i - \varepsilon_{k+1-i} \in A_\lambda$ and $a_1 > a_2 > \dots > a_k$. For $S(q) \in \mathbb{Z}\left[q, q^{-1}\right]$ denote the polynomial part of S by S_+.

THEOREM 4.1. *Let $\mathfrak{g} = gl(k|k)$. Then the following recurrence relations hold:*

1. $K^{(i)}_{\lambda,\lambda} = 1$;
2. *if $a_i > a_{i+1} + 1$, then $K^{(i)}_{\lambda,\lambda-\alpha_i} = 1$ and $K^{(i)}_{\lambda,\mu} = \left(q^{-1}K^{(i)}_{\lambda-\alpha_i,\mu}\right)_+$ for any $\mu \neq \lambda, \lambda - \alpha$;*
3. *if $a_i = a_{i+1} + 1$, then $K^{(i)}_{\lambda,\mu} = qK^{(i+1)}_{\lambda-\alpha_i,\mu}$ for any $\mu \neq \lambda, \lambda - \alpha_i$;*

4. $K^{(k)}_{\lambda,\lambda-\alpha_k}=1$ *and* $K^{(k)}_{\lambda,\mu}=0$ *for any* $\mu\neq\lambda,\lambda-\alpha_k$.

These relations uniquely determine the polynomials $K^{(i)}_{\lambda,\mu}$.

Note that the proof of Theorem 4.1 unravelled the following beautiful geometric

Lemma 4.2. *Let* $\lambda\in\Lambda^+$. *The cohomology* $\Gamma^{(i)}_j\left(L_\lambda\left(\mathfrak{p}^{(i+1)}\right)\right)$ *for* $j>0$ *and the kernel of the natural projection* $\Gamma^{(i)}_0\left(L_\lambda\left(\mathfrak{p}^{(i+1)}\right)\right)\to L_\lambda\left(\mathfrak{p}^{(i)}\right)$ *are semisimple* $\mathfrak{p}^{(i)}$*-modules, and any irreducible component of* $\bigoplus_j\Gamma^{(i)}_j\left(L_\lambda\left(\mathfrak{p}^{(i+1)}\right)\right)$ *occurs with multiplicity 1.*

Consider the case $\mathfrak{g}=osp\,(2k+l|2k)$, where $l=0,1$ or 2, and $\chi_\lambda=\chi_\mu$. Consider the flag of parabolic subalgebras $\mathfrak{g}=\mathfrak{p}^{(1)}\supset\mathfrak{p}^{(2)}\supset\cdots\supset\mathfrak{p}^{(k+1)}$, where $\Delta_{\mathfrak{p}^{(i)}}$ is generated by the simple roots $\sigma_i,\ldots,\sigma_{k+[l/2]}$. Note that $P^{(i)}/P^{(i+1)}$ is isomorphic to the supermanifold of $(1|0)$-dimensional subspaces in $\mathbb{C}^{2k-2i|2k+l}$. Let $r=k-t_\lambda$. As it was explained in section 3, we are interested in calculating $a^P_{\lambda,\mu}=a^{P^{(r+1)}}_{\lambda,\mu}$. Using the same notations as for the case $\mathfrak{g}=gl\,(k|k)$ one can write $A=A^{(1)}\circ\cdots\circ A^{(r)}$.

Next, we write recurrence relations for polynomials $K^{(i)}_{\lambda,\mu}$. Write $A_\lambda=\{\alpha_1,\ldots,\alpha_k\}$, where $\alpha_i=\delta_i+\varepsilon_{j_i}$ for $i\leq r$ and $\alpha_i=\delta_i-\varepsilon_{j_i}$ if $i>r$. We assume that $b_k\geq 0$, see remark 2.2. By Proposition 2.7 λ can be written as $\lambda=a_1\alpha_1+\cdots+a_k\alpha_k$, where $a_i\in\mathbb{Z}+l/2$ and $a_1>a_2>\cdots>a_r>0\geq a_{r+1}>\cdots>a_k$.

Theorem 4.3. *Let* $\mathfrak{g}=osp\,(2k+l|2k)$. *Then the following recurrence relations hold:*

1. $K^{(i)}_{\lambda,\lambda}=1$;
2. *if* $a_i>a_{i+1}+1$ *and* $a_i\neq 1-a_j$ *for any* $j>r$, *then* $K^{(i)}_{\lambda,\lambda-\alpha_i}=1$ *and* $K^{(i)}_{\lambda,\mu}=\left(q^{-1}K^{(i)}_{\lambda-\alpha_i,\mu}\right)_+$ *for any* $\mu\neq\lambda,\lambda-\alpha$;
3. *if* $a_i=1-a_j$ *for some* $j>r$, *then* $K^{(i)}_{\lambda,\lambda-\delta_i-\delta_j}=1$ *and* $K^{(i)}_{\lambda,\mu}=\left(q^{-1}K^{(i)}_{\lambda-\delta_i-\delta_j,\mu}\right)_+$ *for any* $\mu\neq\lambda,\lambda-\delta_i-\delta_j$;
4. *if* $a_i=a_{i+1}+1$, *then* $K^{(i)}_{\lambda,\mu}=qK^{(i+1)}_{\lambda-\delta_i,\mu+\varepsilon_{j_i}}$ *for any* $\mu\neq\lambda$;
5. *if* $l=1$ *and* $a_r=1/2$, *then* $K^{(r)}_{\lambda,\lambda-\delta_r}=q^{2s+1}$ *and* $K^{(r)}_{\lambda,\mu}=0$ *for any* $\mu\neq\lambda,\lambda-\delta_r$;
6. *if* $l=2$ *and* $a_r=1$, *then* $K^{(r)}_{\lambda,\lambda-2\delta_r}=q^{2s+1}$ *and* $K^{(r)}_{\lambda,\mu}=0$ *for any* $\mu\neq\lambda,\lambda-2\delta_r$;
7. *if* $l=0$, $a_r=1$ *and* $r\neq k$, *then* $K^{(r)}_{\lambda,\lambda-\delta_r-\delta_{r+1}}=q^{2s}$ *and* $K^{(r)}_{\lambda,\mu}=0$ *for any* $\mu\neq\lambda,\lambda-\delta_r-\delta_{r+1}$;
8. *if* $l=0$, $a_k=1$, *then* $K^{(k)}_{\lambda,\lambda-\delta_k-\varepsilon_k}=1$ *and* $K^{(k)}_{\lambda,\mu}=0$ *for any* $\mu\neq\lambda,\lambda-\delta_k-\varepsilon_k$.

These relations uniquely determine the polynomials $K^{(i)}_{\lambda,\mu}$.

Note that in the case of $gl\,(m|n)$ and $P = B$ one can calculate $K^P_{\lambda,\mu}$ basing on (4.1) and Theorem 4.1, since in this case it happens that $K^P_{\lambda,\mu} = a^P_{\lambda,\mu}$. In the cases $P \neq B$ or $\mathfrak{g} = osp$ formulae for $K^P_{\lambda,\mu}$ are not known.

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Vera Serganova
Dept. of Mathematics
University of California at Berkeley
Berkeley, CA 94720, USA
serganov@@math.berkeley.edu