Quaternionic Analysis on Riemann Surfaces and Differential Geometry

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Abstract. We present a new approach to the differential geometry of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ that treats this theory as a "quaternionified" version of the complex analysis and algebraic geometry of Riemann surfaces.

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1 Introduction

1.1 Meromorphic functions

Let M be a Riemann surface. Thus M is a two-dimensional differentiable manifold equipped with an almost complex structure J, i.e. on each tangent space T_pM we have an endomorphism J satisfying $J^2 = -1$, making T_pM into a one-dimensional complex vector space. J induces an operation $*$ on 1-forms ω defined as

$$*\omega(X) = \omega(JX)\,. \tag{1}$$

A map $f : M \to \mathbb{C}$ is called holomorphic if

$$*df = i\,df\,.$$

A map $f : M \to \mathbb{C} \cup \{\infty\}$ is called meromorphic if at each point either f or f^{-1} is holomorphic. Geometrically, a meromorphic function on M is just an orientation preserving (possibly branched) conformal immersion into the plane $\mathbb{C} = \mathbb{R}^2$ or rather the 2-sphere $\mathbb{CP}^1 = S^2$.

Now consider $\mathbb{C}$ as embedded in the quaternions $\mathbb{H} = \mathbb{R}^4$. Every immersed surface in $\mathbb{R}^4$ can be descibed by a conformal immersion $f : M \to \mathbb{R}^4$, where M is a suitable Riemann surface. In Section 2 we will show that conformality can again be expressed by an equation like the Cauchy-Riemann equations:

$$*df = N df\,, \tag{2}$$

where now $N : M \to S^2$ in $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ is a map into the purely imaginary quaternions of norm 1. In the important special case where f takes values in

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$\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$, N is just the unit normal vector for the surface f. In the differential geometry of surfaces in $\mathbb{R}^4$, N is called the "left normal vector" of f. "Meromorphic functions" $f : M \to \mathbb{R}^4 \cup \{\infty\} = S^4 = \mathbb{HP}^1$ are defined as in the complex case.

To summarize: from the quaternionic viewpoint i is just one special imaginary quaternion of norm one. The transition from complex analysis to surface theory is done by

i. leaving the Riemann surface as it is.

ii. allowing the whole of $\mathbb{H} \cup \{\infty\}$ as the target space of meromorphic functions.

iii. writing the Cauchy Riemann equations with a "variable i".

1.2 Line bundles

A classical method to construct meromorphic functions on a Riemann surface M is to take the quotient of two holomorphic sections of a holomorphic line bundle over M. For example, if M is realized as an algebraic curve in $\mathbb{CP}^n$, then the affine coordinate functions on $\mathbb{CP}^n$ are quotients of holomorphic sections of the inverse of the tautological bundle over M. Another common way to construct meromorphic functions is to take quotients of theta functions, which also can be viewed as sections of certain holomorphic line bundles over M.

In Section 3 we introduce the notion of a holomorphic quaternionic line bundle L over M. Quotients of holomorphic sections of such bundles are meromorphic conformal maps into $\mathbb{H}$ and every conformal map can be obtained as such a quotient in a unique way.

Every complex holomorphic line bundle E gives rise to a certain holomorphic quaternionic bundle $L = E \oplus E$. The deviation of a general holomorphic quaternionic bundle L from just being a doubled complex bundle can be globally measured by a quantity

$$W = \int |Q|^2$$

called the Willmore functional of L. Here Q is a certain tensor field, the Hopf field.

On compact surfaces, W is (up to a constant) the Willmore functional in the usual sense of surface theory of $f : M \to \mathbb{H} = \mathbb{R}^4$, where f is the quotient of any two holomorphic sections of L.

1.3 Abelian differentials

A second classical method to construct meromorphic functions on a Riemann surface M is to use Abelian differentials, i.e. integrals of meromorphic 1-forms. In the quaternionic theory there is no good analog of the canonical bundle K. On the other hand, also in the complex case 1-forms often arise as products of sections of two line bundles E and KE^{-1}. Notably, this is the case in situations where the Riemann-Roch theorem is applied. This setup carries over perfectly to the quaternionic case, including the Riemann-Roch theorem itself.

We show that for each holomorphic quaternionic line bundle L there exists a certain holomorphic quaternionic line bundle KL^{-1} such that any holomorphic section ψ of L can be multiplied with any holomorphic section ϕ of KL^{-1}, the product being a closed $\mathbb{H}$–valued 1-form (ψ, ϕ) that locally integrates to a conformal map f into $\mathbb{H}$:

$$df = (\psi, \phi)\,. \tag{3}$$

In the case where KL^{-1} is isomorphic to L itself, we call L a spin bundle. If ψ is a nowhere vanishing holomorphic section of a spin bundle then

$$df = (\psi, \psi)$$

defines a conformal immersion into $\mathbb{R}^3$. This construction is in fact a more intrinsic version of the "Weierstrass-representation for general surfaces in 3-space" that has received much attention in the recent literature [5], just as (3), when expressed in coordinates, gives a representation for surfaces in $\mathbb{R}^4$. The Hopf field Q mentioned above can be identified as the "Dirac-potential" or "mean curvature half-density" of the surface f:

$$Q = \frac{1}{2}H|df|\,.$$

Here H is the mean curvature, and $|df|$ is the square root of the induced metric.

1.4 Applications

The only geometric application discussed in some detail in this paper is a rigidity theorem for spheres: if $f, g : S^2 \to \mathbb{R}^3$ are two conformal immersions which are not congruent up to scale but have the same mean curvature half-density, then

$$\int H^2 \geq 16\pi\,.$$

This inequality is sharp.

Many other applications, to be discussed in a more elaborate future paper [1], will concern the geometry of Willmore surfaces (critical points of the Willmore functional) both in $\mathbb{R}^3$ and $\mathbb{R}^4$. Moreover, rudiments of an "algebraic geometry of holomorphic curves" in quaternionic projective space $\mathbb{HP}^n$ can be developed.

2 Conformal surfaces: the standard example

Let M be a Riemann surface and $f : M \to \mathbb{R}^3$ a smooth map. The map f is a conformal immersion if

i. $df(v)$ is perpendicular to $df(Jv)$ for any tangent vector v, where J is the complex structure on M, and

ii. $|df(v)| = |df(Jv)| \neq 0$ for $v \neq 0$.

If $N : M \to S^2$ is the oriented unit normal to f, then the conformality condition can be rephrased as

$$df(Jv) = N \times df(v)\,.$$

To see the similarity with complex function theory, we rewrite this condition using quaternions $\mathbb{H} = \mathbb{R} \oplus \mathrm{Im}\,\mathbb{H}$ [4]. We will always think of $\mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$ as the imaginary quaternions. If $x, y \in \mathbb{R}^3$ then

$$xy = - < x, y > + x \times y\,.$$

With the notation (1) the conformality condition for f becomes (2). For the rest of the article we will take this to be the defining equation for conformality, also in the case of maps (not necessarily immersions) into $\mathbb{R}^4$:

DEFINITION 2.1. A map $f : M \to \mathbb{R}^4 = \mathbb{H}$ is *conformal* if there exists a map $N : M \to \mathbb{H}$ such that $N^2 = -1$ and

$$*df = N df\,.$$

At immersed points this is equivalent to the usual notion of conformality, and f determines N uniquely. If f is $\mathbb{R}^3$–valued then N is the oriented unit normal, but otherwise N is not normal to f. We will call N the *left normal* to f. Moreover, if f is conformal so is its Moebius inversion f^{-1}, with left normal $f^{-1}Nf$. Thus, the above definition is Moebius invariant and hence defines conformality of maps $f : M \to S^4 = \mathbb{HP}^1 = \mathbb{H} \cup \{\infty\}$.

3 HOLOMORPHIC QUATERNIONIC LINE BUNDLES

A *quaternionic line bundle* L over a base manifold is a smooth rank 4 real vector bundle whose fibers have the structure of 1-dimensional quaternionic right vector spaces varying smoothly over the base. Two quaternionic line bundles L_1 and L_2 are *isomorphic* if there exists a smooth bundle isomorphism $A : L_1 \to L_2$ that is quaternionic linear on each fiber. We adopt the usual notation $\mathrm{Hom}_{\mathbb{H}}(L_1, L_2)$ and $\mathrm{End}_{\mathbb{H}}(L) = \mathrm{Hom}_{\mathbb{H}}(L, L)$, etc., for the spaces of quaternionic linear maps.

The zero section of a quaternionic line bundle over an oriented surface has codimension 4, so that transverse sections have no zeros. Thus any quaternionic line bundle over a Riemann surface M is smoothly isomorphic to $M \times \mathbb{H}$.

3.1 COMPLEX QUATERNIONIC LINE BUNDLES

EXAMPLE. Given a conformal map $f : M \to \mathbb{H}$ with left normal N, the quaternionic line bundle $L = M \times \mathbb{H}$ also has a complex structure $J : L \to L$ given by $J(\psi) = N\psi$ for $\psi \in L$.

We make this additional complex structure part of our theory:

DEFINITION 3.1. A *complex quaternionic line bundle* over a base manifold is a pair (L, J) where L is a quaternionic line bundle and $J \in \mathrm{End}_{\mathbb{H}}(L)$ is a quaternionic linear endomorphism such that $J^2 = -1$.

Put differently, a complex quaternionic line bundle is a rank two left complex vector bundle whose complex structure is compatible with the right quaternionic structure. Two complex quaternionic line bundles are isomorphic if the quaternionic linear isomorphism is also left complex linear.

The dual of a quaternionic line bundle L,

$$L^{-1} = \{\omega : L \to \mathbb{H};\ \omega \text{ quaternionic linear}\},$$

has a natural structure of a left quaternionic line bundle via $(\lambda\omega)(\psi) = \lambda\omega(\psi)$ for $\lambda \in \mathbb{H}$, $\omega \in L^{-1}$ and $\psi \in L$. Using conjugation, we can regard L^{-1} as a right quaternionic line bundle, $\omega \cdot \lambda = \bar{\lambda}\omega$. If L has a complex structure then the complex structure on L^{-1} is given by

$$J\omega := \omega \circ J\,,$$

so that L^{-1} is also complex quaternionic.

Any complex quaternionic line bundle L can be tensored on the left by a complex line bundle E, yielding the complex quaternionic line bundle EL. On a Riemann surface M we have the canonical and anti-canonical bundles K and $\bar{K}$. It is easy to see that

$$KL = \{\omega : TM \to L\,;\ *\omega = J \circ \omega\}\,,$$

and

$$\bar{K}L = \{\omega : TM \to L\,;\ *\omega = -J \circ \omega\}\,.$$

In this way, we have split the quaternionic rank 2 bundle $\mathrm{Hom}_{\mathbb{R}}(TM, L)$, which has a left complex structure given by $*$, as a direct sum $KL \oplus \bar{K}L$ of two complex quaternionic line bundles.

If E is a complex line bundle, then $L_E := E \oplus E$ becomes a complex quaternionic line bundle with $J(\psi_1, \psi_2) = (i\psi_1, i\psi_2)$ and right quaternionic structure given by

$$(\psi_1, \psi_2)i = (i\psi_1, -i\psi_2)\,, \qquad (\psi_1, \psi_2)j = (-\psi_2, \psi_1)\,.$$

Conversely, for a given complex quaternionic line bundle (L, J) we let $E_L := \{\psi \in L\,;\ J\psi = \psi i\}$ be the $+i$ eigenspace of J. Then $E \subset L$ is a complex line subbundle and $E_L \oplus E_L$ is isomorphic to L. This leads to the following

Theorem 3.1. *The above correspondences*

$$E \longmapsto L_E\,, \qquad L \longmapsto E_L$$

give a bijection between isomorphism classes of complex line bundles and isomorphism classes of complex quaternionic line bundles. This bijection is equivariant with respect to left tensoring by complex line bundles and respects dualization.

Definition 3.2. The degree of a complex quaternioninc line bundle L over a compact Riemann surface is the degree of the underlying complex line bundle E_L, i.e. $\deg L := \deg E_L$.

On a compact Riemann surface (isomorphism classes of) complex line bundles are characterized by their degrees. Thus, complex quaternionic line bundles also are characterized by their degrees. Given a trivializing section ψ of L we have $J\psi = \psi N$ for some $N : M \to S^2 \subset \operatorname{Im}\mathbb{H}$, $N^2 = -1$, and one easily checks that $\deg L = \deg N$.

3.2 Holomorphic quaternionic line bundles

Definition 3.3. Let (L, J) be a complex quaternionic line bundle over a Riemann surface M and let $\Gamma(L)$ denote the smooth sections of L. A *holomorphic structure* on L is given by a quaternionic linear map

$$D : \Gamma(L) \to \Gamma(\bar{K}L)$$

satisfying

$$D(\psi\lambda) = (D\psi)\lambda + \frac{1}{2}(\psi d\lambda + J\psi * d\lambda) \tag{4}$$

for $\lambda : M \to \mathbb{H}$.

The quaternionic linear subspace $\ker D \subset \Gamma(L)$ is called the space of *holomorphic sections* and is denoted by $H^0(L)$.

One can check that the $\bar{K}L$–part of a quaternionic connection on L gives a holomorphic structure D, which may be used as motivation for the above formula.

Any complex holomorphic structure $\bar{\partial}$ on the underlying complex line bundle E_L is an example of a holomorphic structure $D = \bar{\partial} \oplus \bar{\partial}$. These holomorphic structures on L are characterized by the condition that D and J commute. The failure to commute is measured by

$$Q = \frac{1}{2}(D + JDJ)$$

which is a section of $T^*M \otimes \operatorname{End}_{\mathbb{H}}(L)$. Now

$$\operatorname{End}_{\mathbb{H}}(L) = \operatorname{End}_+(L) \oplus \operatorname{End}_-(L)$$

splits into linear maps commuting and anti-commuting with J. The former is a trivial complex bundle with global sections Id and J. The latter is a non-trivial complex line bundle isomorphic to $\overline{E_L^{-1}} \otimes E_L$. Since Q anti-commutes with J and satisfies $*Q = -J \circ Q$ we see that Q is a section of the complex line bundle $\bar{K}\operatorname{End}_-(L)$. We call Q the *Hopf field* of the holomorphic quaternionic line bundle L. Thus any holomorphic structure D is uniquely decomposed into

$$D = \bar{\partial} + Q$$

with $\bar{\partial}$ commuting with J. Vanishing of the Hopf field Q characterizes the usual complex holomorphic structures. Two quaternionic holomorphic line bundles are isomorphic if there is an isomorphism of complex quaternionic line bundles which intertwines the respective holomorphic structures. On a compact Riemann surface this implies that the underlying complex holomorphic structures are isomorphic and that the Hopf fields are related up to a constant phase $u \in S^1$. Thus the moduli space of quaternionic holomorphic structures fibers over the Picard group of M. The fiber $\Gamma(\bar{K}\mathrm{End}_-(L_E))/S^1$ over $(E, \bar\partial)$ is given by the Hopf fields.

A global invariant of the quaternionic holomorphic line bundle L is obtained by integrating the length of the Hopf field Q: we define the density $|Q|^2$ by

$$Q_v \circ Q_v = -|Q|^2(v) Id\,, \qquad v \in TM\,,$$

where we identify J^M–invariant quadratic forms with 2-forms on M. The *Willmore functional* of D is the L^2–norm of the Hopf field

$$||Q||^2 = \int |Q|^2\,.$$

The vanishing of $||Q||$ characterizes the complex holomorphic theory.

EXAMPLE. We have already seen that a conformal map $f : M \to \mathbb{H}$ with left normal $N : M \to S^2$ induces the complex quaternionic bundle $L = M \times \mathbb{H}$ with $J\psi = N\psi$. We define the *canonical* holomorphic structure D on L to be the one for which the constant sections are holomorphic, i.e. D is characterized by $D(1) = 0$. Any other section of L is of the form $\psi = 1\lambda$ for some $\lambda : M \to \mathbb{H}$, and (4) implies that ψ is holomorphic iff

$$*d\lambda = N d\lambda\,.$$

Thus the holomorphic sections of L are precisely the conformal maps with the same left normal as f. In particular, $\dim H^0(L) \geq 2$.

The Moebius invariance of the holomorhic structure follows since f^{-1} induces an isomorphic holomorphic structure on $M \times \mathbb{H}$. Thus we have assigned to each conformal map into $\mathbb{HP}^1 = S^4$ a quaternionic holomorphic line bundle with at least two holomorphic sections.

The Hopf field for this holomorphic structure is $Q = \frac{1}{4}N(dN + *dN)$, and

$$|Q|^2 = \frac{1}{4}(|H|^2 - K - K^\perp)|df|^2\,,$$

where H is the mean curvature vector of f and $K^\perp$ is the curvature of the normal bundle. We see that $|Q|^2$ is a Moebius invariant density, which is consistent with the Moebius invariance of our setup. Thus, the Willmore energy of our holomorphic structure, $||Q||^2 = \int |Q|^2 = W(f)$, is (up to topological constants) just the Willmore energy of f.

So far we have seen how a conformal map induces a holomorphic line bundle with at least two holomorphic sections. As in the classical complex theory we have the converse construction, i.e. all conformal maps into S^4 arise as quotients of holomorphic sections.

EXAMPLE. Let $L \to M$ be a quaternionic holomorphic line bundle and assume that $\dim H^0(L) \geq 2$ with ψ, ϕ holomorphic sections such that ψ has no zeros. Then $J\psi = \psi N$ for some $N : M \to S^2$. We define $f : M \to \mathbb{H}$ by

$$\phi = \psi f\,,$$

then (4) implies that $*df = Ndf$, i.e. f is conformal with left normal N.

An interesting special case comes from conformal maps with $Q = 0$. It can be shown that they are superconformal in the sense that their curvature ellipse is a circle. Since $Q = 0$, superconformal maps are critical for the Willmore energy and thus Willmore surfaces in S^4. In case f is $\mathbb{R}^3$–valued $Q = 0$ simply means that f is a conformal map into the 2-sphere. The superconformal maps all arise as projections from holomorphic maps into the twistor space $\mathbb{CP}^3$ over S^4 and have been studied by various authors [2, 3]. In our theory these maps arise as quaternionic quotients of holomorphic sections of (doubled) complex holomorphic line bundles.

In the above construction the structure of zeros of quaternionic holomorphic sections becomes important. Applying a result of Aronszajn we can show

THEOREM 3.2. *Let ψ be a non-trivial holomorphic section of a quaternionic holomorphic line bundle L over a Riemann surface M. Then the zeros of ψ are isolated and, if z is a centered local coordinate near a zero $p \in M$,*

$$\psi = z^k \phi + O(|z|^{k+1})$$

where ϕ is a local nowhere vanishing section of L. The integer k and the value $\phi(p) \in L_p$ are well-defined independent of choices. We define the order *of the zero p of ψ by $ord_p\psi = k$.*

We conclude this section with a degree formula:

THEOREM 3.3. *Let ψ be a non-trivial section of a quaternionic holomorphic line bundle L over a compact Riemann surface M. Then*

$$\pi \deg L + ||Q||^2 \geq \pi \sum_{p \in M} ord_p\psi \,. \tag{5}$$

In contrast to the complex holomorphic case, where negative degree bundles do not have holomorphic sections, we see that in the quaternionic theory the Willmore energy of the bundle compensates for this failure and we still can have holomorphic sections.

Equality in (5) is attained by holomorphic bundles L^{-1} where $L = E \oplus E$ is a doubled complex holomorphic bundle E and L has a nowhere vanishing meromorphic section ψ. The holomorphic structure on L^{-1} then is obtained by defining ψ^{-1} to be holomorphic.

We conjecture the following lower bound for the Willmore energy on holomorphic line bundles over the 2-sphere: let $n = \dim H^0(L)$ and $d = \deg L$ then

$$\frac{1}{\pi}||Q||^2 \geq n^2 - n(d+1)\,. \tag{6}$$

Examples are known where equality holds. Using the degree formula we can prove (6) under certain non-degeneracy assumptions [1]. For $d = -1$, the case of spin bundles (see the next section), this estimate has been conjectured by Taimanov [6].

4 Abelian differentials

DEFINITION 4.1. A *pairing* between two complex quaternionic line bundles L and $\tilde{L}$ over M is a nowhere vanishing real bilinear bundle map $(\,,\,): L \times \tilde{L} \to T^*M \otimes \mathbb{H}$ satisfying

$$(\psi\lambda, \phi\mu) = \bar{\lambda}(\psi, \phi)\mu$$
$$*(\psi, \phi) = (J\psi, \phi) = (\psi, J\phi)$$

for all $\lambda, \mu \in \mathbb{H}$, $\psi \in L$, $\phi \in \tilde{L}$.

A pairing between L and $\tilde{L}$ is actually the same as an isomorphism of complex quaternionic line bundles $\tilde{L} \to KL^{-1}$, given as $\phi \mapsto \alpha$, where

$$\alpha_X(\psi) = (\psi, \phi)(X)\,.$$

If ω is a 1-form on M with values in L and ϕ is a section of L, then we define an L-valued 2-form $(\omega \wedge \phi)$ as

$$(\omega \wedge \phi)(X, Y) = (\omega(X), \phi)(Y) - (\omega(Y), \phi)(X)\,.$$

Similarly, for $\psi \in \Gamma(L)$ and η a 1-form with values in $\tilde{L}$, we set

$$(\psi \wedge \eta)(X, Y) = (\psi, \eta(X))(Y) - (\psi, \eta(Y))(X)\,.$$

LEMMA 4.1. *For each $\omega \in \bar{K}End_-(L)$ there is a unique $\bar{\omega} \in \bar{K}End_-(\tilde{L})$ such that*

$$(\omega\psi \wedge \phi) + (\psi \wedge \bar{\omega}\phi) = 0$$

for all $\psi \in \Gamma(L)$, $\phi \in \Gamma(\tilde{L})$. The map $\omega \mapsto \bar{\omega}$ is complex antilinear:

$$\overline{J\omega} = -J\bar{\omega}\,.$$

THEOREM 4.2. *If two complex quaternionic line bundles L and $\tilde{L}$ are paired, then for any holomorphic structure D on L there is a unique holomorphic structure $\tilde{D}$ on $\tilde{L}$ such that for each $\psi \in \Gamma(L)$, $\phi \in \Gamma(\tilde{L})$ we have*

$$d(\psi, \phi) = (D\psi \wedge \phi) + (\psi \wedge \tilde{D}\phi)\,.$$

The Hopf fields Q and $\tilde{Q}$ of D and $\tilde{D}$ are conjugate:

$$\tilde{Q} = \bar{Q}\,.$$

Thus, a holomorphic structure on L determines a unique holomorphic structure on KL^{-1} such that L and KL^{-1} become paired holomorphic bundles. In this situation, the Riemann-Roch theorem is true in the familiar form of the theory of complex line bundles: on compact Riemann surfaces of genus g we have

$$\dim H^0(L) - \dim H^0(KL^{-1}) = \deg(L) - g + 1\,.$$

Theorem 4.2 suggests a way to construct conformal immersions $f: M \to \mathbb{R}^4 = \mathbb{H}$. If L and $\tilde{L}$ are paired holomorphic bundles and $\psi, \phi \in H^0(L)$ are both nowhere vanishing sections, then (ψ, ϕ) is a closed 1-form that integrates to a conformal immersion into $\mathbb{R}^4$, possibly with translational periods. In fact, this construction is completely general:

THEOREM 4.3. *Let $f : M \to \mathbb{H}$ be a conformal immersion. Then there exist paired holomorphic quaternionic line bundles L, $\tilde{L}$ and nowhere vanishing sections $\psi \in H^0(L)$, $\phi \in H^0(\tilde{L})$ such that*

$$df = (\psi, \phi)\,. \tag{7}$$

L, $\tilde{L}$, ψ and ϕ are uniquely determined by f up to isomorphism.

In the setup of the theorem choose locally non-vanishing sections $\hat{\psi} \in \Gamma(L)$, $\hat{\phi} \in \Gamma(\tilde{L})$ satisfying $\bar{\partial}\hat{\psi} = 0$, $\bar{\partial}\hat{\phi} = 0$, $J\hat{\psi} = -\hat{\psi}i$, $J\hat{\phi} = \hat{\phi}i$. Then there is a $\mathbb{R} \oplus \mathbb{R}i$–valued coordinate chart z on M satisfying

$$dz = (\hat{\psi}, \hat{\phi})\,.$$

We can write

$$\psi = \hat{\psi}(\psi_1 + \psi_2 j) \qquad \phi = \hat{\phi}(\phi_1 + \phi_2 j)$$

with $\mathbb{R} \oplus \mathbb{R}i$–valued functions ψ_α, ϕ_α. Expanding (7) we obtain a generalization of the Weierstrass representation of surfaces in $\mathbb{R}^3$ [5] to surfaces in $\mathbb{R}^4$. The equations $(\bar{\partial} + Q)\psi = 0$ and $(\bar{\partial} + \tilde{Q})\phi = 0$ unravel to Dirac equations for ψ_α and ϕ_α.

DEFINITION 4.2. A holomorphic line bundle Σ over M is called a *spin bundle* if there exists a pairing of Σ with itself such that the second holomorphic structure on Σ provided by Theorem 4.2 coincides with the original one.

As a direct consequence of the definition of a pairing we obtain in the case of spin bundles the relation

$$(\phi, \psi) = -\overline{(\psi, \phi)}\,.$$

Therefore, for any holomorphic section ψ of a spin bundle Σ the equation

$$df = (\psi, \psi)$$

defines a conformal map into $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$, possibly with translational periods. This is in fact a coordinate-free version of the Weierstrass representation for surfaces in $\mathbb{R}^3$ [5], which could be obtained by a calculation similar to the one given above for $\mathbb{R}^4$. We now show that the "Dirac potential" $H|df|$ featured in this representation can be identified with the Hopf field Q of Σ.

For a spin bundle Σ the map $\bar{K}\mathrm{End}_-(\Sigma) \ni \omega \mapsto \bar{\omega}$ puts a real structure on $\bar{K}\mathrm{End}_-(\Sigma)$ and therefore allows us to define a real line bundle

$$R = \operatorname{Re}(\bar{K}\mathrm{End}_-(\Sigma)) = \{\omega \in \bar{K}\mathrm{End}_-(\Sigma)\,;\, \omega = \bar{\omega}\}\,.$$

We now show that R can be identified with the real line bundle $\mathcal{D}^{-1/2}$ of half densities over M. A half-density U is a function on the tangent bundle TM which is of the form

$$U(X_p) = \rho(p)\sqrt{g(X_p, X_p)}$$

where $\rho \in C^\infty(M)$ and g is a Riemannian metric compatible with the given conformal structure. For each $\psi \in \Gamma(\Sigma)$ the function $X \mapsto |(\psi, \psi)(X)|$ is a half-density.

On the other hand, it can be checked that for each $\psi \in \Gamma(\Sigma)$ we can define a section ω_ψ of R as

$$\omega_\psi(\phi) = \psi(J\psi, \phi)\,.$$

There is a canonical isomorphism $R \to \mathcal{D}^{1/2}$ which takes ω_ψ to $|(\psi,\psi)|$ for all $\psi \in \Sigma$.

Theorem 4.4. *Let ψ be a holomorphic section of a spin bundle Σ over M. Then there is a conformal immersion $f : \tilde{M} \to \mathbb{R}^3$ on the universal cover of M with only translational periods such that*

$$df = (\psi, \psi)\,.$$

Identifying the half-density $|df|$ as explained above with a section of $\bar{K}End_-(\Sigma)$, the mean curvature of f is given in terms of the Hopf field Q of Σ as

$$Q = \frac{1}{2}H|df|\,.$$

We conclude by indicating a proof of the rigidity theorem for spheres stated in Section 1.4. The hypotheses imply that in the situation of the theorem above Σ has a 2–dimensional space of holomorphic sections. Since in the case at hand the conjecture (6) has been proven, we take $n = 2$ and $d = \deg \Sigma = -1$ and obtain

$$\int H^2|df|^2 = 4\int |Q|^2 \geq 16\pi\,.$$

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