Invariants in Contact Topology

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Abstract. Contact topology studies contact manifolds and their Legendrian submanifolds up to contact diffeomorphisms. It was born, together with its sister Symplectic topology, less than 20 years ago, essentially in seminal works of D. Bennequin and M. Gromov (see [2, 18]). However, despite several remarkable successes the development of Contact topology is still significantly behind its symplectic counterpart. In this talk we will discuss the state of the art and some recent breakthroughs in this area.

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1 Contact preliminaries

A 1-form α on a $(2n-1)$-dimensional manifold V is called *contact* if the restriction of $d\alpha$ to the $(2n-2)$-dimensional tangent distribution $\xi = \{\alpha = 0\}$ is non-degenerate (and hence symplectic). A codimension 1 tangent distribution ξ on V is called a *contact structure* if it can be locally (and in the co-orientable case globally) defined by the Pfaffian equation $\alpha = 0$ for some choice of a contact form α. The pair (V, ξ) is called a *contact manifold*. According to Frobenius' theorem the contact condition is a condition of maximal non-integrability of the tangent hyperplane field ξ. In particular, all integral submanifolds of ξ have dimension $\leq n-1$. On the other hand, $(n-1)$-dimensional integral submanifolds, called *Legendrian*, always exist in abundance. Any non-coorientable contact structure can be canonically double-covered by a coorientable one. If a contact form α is fixed then one can associate with it the *Reeb vector field* R_α, which is transversal to the contact structure $\xi = \{\alpha = 0\}$. The field R_α is uniquely determined by the equations $R_\alpha \lrcorner d\alpha = 0$; $\alpha(R_\alpha) = 1$.

The $2n$-dimensional manifold $M = (T(V)/\xi)^* \setminus V$, called the *symplectization* of (V, ξ), carries the natural symplectic structure ω induced by the embedding $M \to T^*(V)$ which assigns to each linear form $T(V)/\xi \to \mathbb{R}$ the corresponding form $T(V) \to T(V)/\xi \to \mathbb{R}$. A choice of a contact form α (if ξ is co-orientable) defines a splitting $M = V \times (\mathbb{R} \setminus 0)$. We will usually pick the positive half $V \times \mathbb{R}_+$ of M, and call it symplectization as well. The symplectic structure ω can be written in terms of this splitting as $d(\tau\alpha), \tau > 0$. It will be more convenient for us,

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however, to use additive notations and write ω as $d(e^t\alpha)$, $t \in \mathbb{R}$, on $M = V \times \mathbb{R}$. Notice that the vector field $T = \frac{\partial}{\partial t}$ is conformally symplectic: we have $\mathcal{L}_T\omega = \omega$, as well as $\mathcal{L}_T(e^t\alpha) = e^t\alpha$. All the notions of contact geometry can be formulated as the corresponding symplectic notions, invariant or equivariant with respect to this conformal action. For instance, any contact diffeomorphism of V lifts to an equivariant symplectomorphism of M; contact vector fields on V (i.e. vector fields preserving the contact structure) are projections of $\mathbb{R}$-invariant contact symplectic (and automatically Hamiltonian) vector fields on M; Legendrian submanifolds in M correspond to cylindrical (i.e. invariant with respect to the $\mathbb{R}$-action) Lagrangian submanifolds of M.

The symplectization of a contact manifold is an example of a symplectic manifold with *cylindrical* (or rather conical) ends, which is a possibly non-compact symplectic manifold (W, ω) with ends of the form $E_+ = V_+ \times [0, \infty)$ and $E_- = V_- \times (-\infty, 0]$, such that $V_\pm$ are compact manifolds, and $\omega|_{V_\pm} = d(e^t\alpha_\pm)$, where $\alpha_\pm$ are a contact forms on $V_\pm$. In other words, the ends $E_\pm$ of (W, ω) are symplectomorphic, respectively, to the positive, or negative halves of the symplectizations of the contact manifolds $(V_\pm, \xi_\pm = \{\alpha_\pm = 0\})$. We will consider the splitting of the ends and the the contact forms $\alpha_\pm$ to be parts of the structure of a symplectic manifold with cylindrical ends. We will also call (W, ω) a *directed symplectic cobordism* between the contact manifolds (V_+, ξ_+) and (V_-, ξ_-), and denote it, sometimes, by $\overrightarrow{V_+V_-}$. Let us point out that this is not an equivalence relation, but rather a partial order. Existence of a directed symplectic cobordism $\overrightarrow{M_+M_-}$ does not imply the existence of a directed symplectic cobordism $\overrightarrow{M_-M_+}$, but directed symplectic cobordisms $\overrightarrow{M_0M_1}$ and $\overrightarrow{M_1M_2}$ can be glued, in an obvious way, into a directed symplectic cobordism $\overrightarrow{M_0M_2}$. Suppose now that the symplectic form ω is exact and equal $d\beta$, where $\beta_{|E_\pm} = e^t\alpha_\pm$, and that there exists a Morse function $\varphi : W \to \mathbb{R}$ which coincides with the function t at infinity and such that for any $c \in \mathbb{R}$ the restriction $\beta|_{\{\varphi = c\}}$ is a contact form away from the critical points of the function φ. In this case we say that (W, ω) is a *directed Stein cobordism* between the contact manifolds (V_+, ξ_+) and (V_-, ξ_-). Notice that indices of critical points of the function φ are bounded in this case by $n = \frac{1}{2}\dim W$. If there exists a directed symplectic (resp. Stein) cobordism between a contact manifold (V_+, ξ_+) and $V_- = \emptyset$, then (V_+, ξ_+) is called *symplectically* (resp. *Stein*) *fillable.*[2] The Stein filling W is called *subcritical* if the function φ can be chosen without critical points of the maximal index n.

Contact structures have no local invariants. Moreover, any contact form is locally isomorphic to the form $\alpha_0 = dz - \sum\limits_1^{n-1} y_i dx_i$ (Darboux' normal form). The contact structure ξ_0 on $\mathbb{R}^{2n-1}$ given by the form α_0 is called *standard.* Standard contact structure on S^{2n-1} is formed by complex tangent hyperplanes to the unit sphere in $\mathbb{C}^n$. The standard contact structure on S^{2n-1} is isomorphic in the complement of a point to the standard contact structure on $\mathbb{R}^{2n-1}$. According to

[2] The Stein fillabillity of (V, ξ) is equivalent to the existence of a compact complex manifold with a strictly pseudoconvex boundary V and a Stein interior, such that ξ is the field of complex tangencies to the boundary V. See [9] for the discussion of different notions of symplectic fillabillity.

a theorem of J. Gray (see [17]) contact structures on closed manifolds have the following stability property: *Given a family ξ_t, $t \in [0,1]$, of contact structures on a closed manifold M, there exists an isotopy $f_t : M \to M$, such that $df_t(\xi_0) = \xi_t; t \in [0,1]$.* Notice that for contact *forms* the analogous statement is wrong. For instance, the topology of the 1-dimensional foliation determined by the Reeb vector field R_α is very sensitive to deformations of the contact form α.

The conformal class of the symplectic form $d\alpha|_\xi$ depends only on the cooriented contact structure ξ and not a choice of the contact form α. In particular, one can associate with ξ an almost complex structure $J : \xi \to \xi$, compatible with $d\alpha$ which means that $d\alpha(X, JY); X, Y \in \xi$, is an Hermitian metric on ξ. The space of almost complex structures J with this property is contractible, and hence the choice of J is homotopically canonical. Thus a cooriented contact structure ξ defines on M a *stable almost complex structure* $\widetilde{J} = \widetilde{J}_\xi$, i.e. a splitting of the tangent bundle $T(V)$ into the Whitney sum of a complex bundle of (complex) dimension $(n-1)$ and a trivial 1-dimensional real bundle. The existence of a stable almost complex structure is necessary for the existence of a contact structure on V. If V is open (see [19]) or $\dim V = 3$ (see [25, 24]) this property is also sufficient for the existence of a contact structure in the prescribed homotopy class. It is still unknown whether this condition is sufficient for the existence of a contact structure on a closed manifold of dimension > 3. However, the positive answer to this question is extremely unlikely. Similarly, the homotopy class of $\widetilde{J}_\xi$, which we denote by $[\xi]$ and call the *formal* homotopy class of ξ, serves as an invariant of ξ. For an open V it is a complete invariant (see [19]) up to a deformation of contact structures, but not up to a contact diffeomorphism. For closed manifolds this is known to be false in all dimensions, see the discussion below. The main goal of this talk is the construction of invariants which would allow to distinguish contact manifolds in the same formal homotopy class.

2 Invariants of open manifolds

We concentrate in this section on 3-dimensional contact manifolds, although some part of the discussion can be generalized to higher dimensions. First of all 3-dimensional contact manifolds are orientable, and any contact structure determines an orientation of M. If M is a priori oriented then contact structures can be divided into positive and negative. We will consider here only positive contact structures.

It is proven to be useful to divide all 3-dimensional contact manifolds into two complementary classes: tight and overtwisted. A contact 3-manifold (M, ξ) is called *overtwisted* if there exists an embedded disc $D^2 \subset M$ such that its boundary ∂D^2 is tangent to ξ (i.e. ∂D^2 is a Legendrian curve), while the disc itself is transverse to ξ along its boundary. A non-overtwisted contact structures are called *tight*. D. Bennequin (see [2]) was the first who discovered the phenomenon of overtwisting. He proved that the standard contact structure ξ_0 on S^3 is tight and constructed an overtwisted contact structure ξ_1 in the same formal homotopy class.

As it turned out, overtwisted contact structures on all closed 3-manifolds are classified up to isotopy by their formal homotopy classes (see [7]). On open manifolds one should subdivide furthermore overtwisted contact structures into *overtwisted at infinity* and *tight at infinity*. A contact structure, which is overtwisted at infinity, is determined up to isotopy by its formal homotopy class (see [6]). Overtwisted, but tight at infinity contact structure on $M \setminus F$, where M is a closed 3-manifold and F is its finite subset, can always be canonically extended to M see [6]), and thus an isotopical classification of such structures coincides with the formal homotopical classification on M (and not on $M \setminus F$!).

Let us now restrict ourself to the class of tight contact structures. It is not so easy to provide non-trivial invariants of tight contact structures on open manifolds. The problem is that all standard symplectic invariants (the Gromov width, capacities, etc.) take infinite values for symplectizations of contact manifolds. One knows, for instance, that on $\mathbb{R}^3$ any tight contact structure is isotopic to the standard one (see [6]). However, on the closed half-space $\mathbb{R}^3_+ = \{y \geq 0\} \subset \mathbb{R}^3$ there are non-isomorphic contact structures, which we are describing below. Let us denote by Ξ_0 the space of tight contact structures on $\mathbb{R}^3_+$ which coincide with the standard contact structure ξ_0 near the plane $\Pi = \{y = 0\} = \partial\mathbb{R}^3_+$. We are interested in invariants of contact half-spaces $(\mathbb{R}^3_+, \xi)$, where $\xi \in \Xi_0$, up to diffeomorphisms fixed near Π.

Given a contact structure $\xi \in \Xi_0$ let us consider an embedded plane $\widetilde{\Pi} \subset \mathbb{R}^3_+$ which coincides with Π at infinity, and which is transversal to ξ_0. The 1-dimensional line field $\xi \cap T(\widetilde{\Pi})$ integrates into a 1-dimensional *characteristic foliation* $\widetilde{\Pi}_\xi$ on $\widetilde{\Pi}$. The foliation $\widetilde{\Pi}_\xi$ coincides with the foliation by lines $\{z = \text{const}\}$ at infinity, and thus the holonomy along its leaves defines a compactly supported diffeomorphism $h_{\widetilde{\Pi}} : \mathbb{R} \to \mathbb{R}$, where we identify the source $\mathbb{R}$ with the line $\{x = -N\} \subset \Pi$, and the target $\mathbb{R}$ with the line $\{x = N\} \subset \Pi$ for a sufficiently large $N > 0$. Let us define $c_\xi(z) = \sup\limits_{\widetilde{\Pi}} \left(h_{\widetilde{\Pi}}(z) - z\right)$, and call the function $c_\xi(z)$ the *contact shape* of $(\mathbb{R}^3, \xi)$. Of course, sometimes we have $c(\xi) \equiv +\infty$. For instance this is the case for the standard contact structure $\xi = \xi_0$. On the other hand, the following construction (see [12]) shows that any positive continuous [3] function $f : \mathbb{R} \to \mathbb{R}$, such that $f(z) + z$ is a monotone function, can be realized as the invariant c_ξ for some contact structure $\xi \in \Xi_0$.

For a positive Lipschitz function φ on $\mathbb{R}^2$ we denote by S_φ its graph $\{y = \varphi(x, z)\} \subset \mathbb{R}^3$. If the function φ decays sufficiently fast when $|x| \to \infty$ (say, $\varphi(x, z) < \frac{C}{x^2}$), then the holonomy diffeomorphism $h_\varphi : \mathbb{R} \to \mathbb{R}$ along the leaves of the characteristic foliation of the graph $\Pi_\varphi = \{y = \varphi(x, z)\}$ is well defined. It is easy to find a Lipshitz function φ with the prescribed continuous holonomy $h_\varphi(z) = z + f(z)$. Consider the domain $\Omega_\varphi = \{0 \leq y < \varphi(x, z)\}$. Clearly, the contact manifold (Ω_φ, ξ_0) belongs to the class Ξ_0. We have (see [12])

PROPOSITION 2.1
$$c_{(\Omega_\varphi, \xi_0)}(z) = h_\varphi(z) - z = f(z)\,.$$

A similar invariant can be defined for open manifolds (without boundary) when $H_1(M) \neq 0$. (see [5])

[3] In fact, it need not to be even continuous.

3 Invariants of closed manifolds

Until very recently there was known only one result allowing to distinguish contact structures on manifolds of dimension > 3 within a given formal homotopy class. Namely, we have

Theorem 3.1 *For any $n > 2$ the sphere S^{2n-1} has a contact structure ξ_1 in the standard formal homotopy class, which is not isomorphic to the standard contact structure ξ_0.*

For odd values of n this was shown by the author in [4], and later extended to even values of n by H. Geiges (see [15]). [4]

We will discuss in this section some new powerful algebraic invariants of closed contact manifolds, related to Gromov-Witten invariants of symplectic manifolds, which were recently developed jointly by H. Hofer, A. Givental and the author. See also Hofer's talk at the current proceedings for the discussion of other related aspects of this theory.

Contact homology algebra. To define the invariants of a contact manifold (V, ξ) let us fix a contact form α and an almost complex structure $J : \xi \to \xi$ compatible with the symplectic form $d\alpha$. The symplectization M of (V, ξ) can be identified, as was explained in Section 1, with $(V \times \mathbb{R}, d(e^t\alpha))$. The complex structure J extends from ξ to $T(M)$ by setting $J\frac{\partial}{\partial t} = R_\alpha$, where R_α is the Reeb vector field of the contact form α. For a generic choice of α there are only countably many periodic trajectories (including multiple ones) of the vector field R_α. Moreover, these trajectories can be assumed *non-degenerate* which means that the linearized Poincaré return map along any of these trajectories has no eigenvalues equal to 1.

Let $\mathcal{P} = \mathcal{P}_\alpha$ be the set of all periodic trajectories of R_α. We do not fix initial points on periodic trajectories, and include all multiples as separate points of $\mathcal{P}$. Let us first assume that $H_1(V) = 0$. For each $\gamma \in \mathcal{P}$ let us choose and fix a surface F_γ spanning the trajectory γ in V. This enable us to define the *Conley-Zehnder index* $\mu(g)$ of γ as follows. Choose a homotopically unique trivialization of the symplectic vector bundle $(\xi, d\alpha)$ over each trajectory $\gamma \in \mathcal{P}$ which extends to $\xi|_{F_\gamma}$. The linearized flow of R_α along γ defines then a path in the group $Sp(2n-2, \mathbb{R})$ of symplectic matrices, which begins at the unit matrix and ends at a matrix with all eingenvalues different from 1. The Maslov index of this path (see [1, 26]) is, by the definition, the Conley-Zehnder index $\mu(\gamma)$ of the trajectory γ. For our purposes it will be convenient to use the *reduced* Conley-Zehnder index $\overline{\gamma} = \mu(\gamma) + n - 3$, also called the degree of γ. Notice that by changing the spanning surfaces for the trajectories from $\mathcal{P}$ one can change Conley-Zehnder indices by the values of the cohomology class $2c_1(\xi)$, where $c_1(\xi)$ is the first Chern class of the contact bundle

[4]Although this result sounds similar to Bennequin's theorem asserting that the standard contact structure on S^3 is not overtwisted, non-standard contact structures on high-dimensional spheres provided by Theorem 3.1 are quite different: they are *symplectically*, and even *Stein fillable*, while an overtwisted contact structure on S^3 is not.

ξ. In particular, mod 2 indices can be defined independently of any spanning surfaces, and even in the case when $H_1(V) \neq 0$.

Next, we consider certain moduli spaces of holomorphic curves in the manifold $M = V \times \mathbb{R}$, which are essential for all our algebraic constructions. Let us observe that for each periodic orbit $\gamma \in \mathcal{P}$ the cylinder $\gamma \times \mathbb{R}$ is a J-holomorphic curve. Let D_r be the disc of radius r in $\mathbb{C}$ centered at the origin. Given any orbit $\gamma \in \mathcal{P}$ we say that a J-holomorphic map $f : D_r \setminus 0 \to M = V \times \mathbb{R}$ converges near 0 to the periodic trajectory γ at $\pm\infty$ if $f(z) = (g(z), h(z))$, $h(z) \underset{|z|\to 0}{\to} = \pm\infty$, and there exits the limit $\bar{g}(\varphi) = \lim\limits_{\rho\to 0} g(\rho e^{i\varphi})$ which parametrizes the periodic trajectory γ. Notice that the orientation which is defined this way on γ coincides with the orientation given by the Reeb vector field $\mathbb{R}_\alpha$ at $+\infty$, and opposite to this orientation at $-\infty$.

Let us denote by S_{sr}, $s, r = 0, 1, \ldots$, the 2-sphere S^2 with $s+r$ fixed punctures $y_1, \ldots, y_s, x_1, \ldots, x_r$. Given $s + r$ periodic orbits $\gamma_1, \ldots, \gamma_s, \delta_1, \ldots, \delta_r \in \mathcal{P}$ we consider the space $\widetilde{\mathcal{M}}^A(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots, \delta_r)$, which consists of pairs (f, j), where j is a conformal structure on S_{sr}, and $f : S_{sr} \to M$ is a (j, J)-holomorphic curve, such that near each puncture y_k, $k = 1, \ldots, s$, the map f converges to γ_k at $+\infty$, and near each puncture x_l, $l = 1, \ldots, r$, it converges to δ_l at $-\infty$. As usual we pass to the corresponding moduli space $\mathcal{M}(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots, \delta_r)$ by identifying pairs (f, j) and $(\tilde{f}, \tilde{j})$ which differ by a diffeomorphism of the sphere S^2 which fixes the punctures $y_1, \ldots, y_s, x_1, \ldots, x_r$. The space $\mathcal{M}$ can be written as a disjoint union $\mathcal{M} = \bigcup\limits_{A \in H_2(V)} \mathcal{M}^A$, where $\mathcal{M}^A$ consists of holomorphic curves which together with the surfaces spanning in V the trajectories $\gamma_1, \ldots, \gamma_s, \delta_1, \ldots, \delta_r \in \mathcal{P}$ represent the homology class $A \in H_2(M) = H_2(V)$. Then we have

PROPOSITION 3.2 *For a generic choice of J, and any periodic orbits $\gamma_1, \ldots, \gamma_s, \delta_1, \ldots \delta_r \in \mathcal{P}$ the moduli space $\mathcal{M}^A(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots \delta_r)$ is an orbifold of dimension* [5]

$$\sum_{k=1}^{s} \overline{\gamma_k} - \sum_{l=1}^{r} \overline{\delta}_l + (2 - 2s)(n - 3) + 2c_1(\xi)[A].$$

REMARK 3.3 The additive group $\mathbb{R}$ acts on $M = V \times \mathbb{R}$ by J-biholomorphic translations $(x, t) \mapsto (x, t+c)$. The moduli spaces $\mathcal{M}(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots, \delta_r)$ are invariant under this action, and hence, with the exception of trivial spaces $\mathcal{M}(\gamma; \gamma)$ (which consist of cylinders $\gamma \times \mathbb{R}$), a non-empty moduli space $\mathcal{M}(\gamma_1, \ldots, \gamma_s; \delta_1, \ldots, \delta_r)$ always has a positive dimension.

Let us consider now a free (super-)commutative graded algebra $\Theta = \Theta_\alpha$ over $\mathbb{C}$ with the unit element generated by elements of $\mathcal{P}_\alpha$. In other words, Θ is a polynomial algebra with complex coefficients of generators of even degree and an exterior algebra of odd degree generators. Let us recall that we count all

[5] It is a standard difficulty in the Floer homology theory and the theory of holomorphic curve invariants in general, that in the presence of multiply-covered curve it is, sometimes, impossible to achieve transversality needed for this dimension formula just by perturbing the almost complex structure J. The appropriate virtual cycles technique which works in this case and involves multivalued perturbations was recently developed by several authors, see [14], [23] et al.

multiples of a given trajectories as independent generators of Θ. Each monomial element $\theta \in \Theta$ is graded by its total degree $\bar{\theta}$. Let Θ^{H_2} be the group algebra of $H_2(V) = H_2(M)$ with coefficients in Θ. Thus elements of Θ^{H_2} can be written as polynomials $\sum\limits_{A \in H_2(V)} \theta_A t^A$, $\theta_A \in \Theta$.

We define now a sequence of operations $\underbrace{\Theta^{H^2} \otimes \dots \Theta^{H_2}}_{s} \to \Theta^{H_2}$, which make Θ into a L_∞-algebra (see, for instance, [22]), or rather a P_∞-algebra, where P stands for the Poisson structure. This is done by an appropriate counting of components of the moduli spaces $\mathcal{M}^A(\gamma_1, \dots, \gamma_s; \delta_1, \dots \delta_r)$.

Take any $s \geq 1$ periodic orbits $\gamma_1, \dots, \gamma_s \in \mathcal{P}$ and set

$$[\gamma_1, \dots, \gamma_s]_s = \sum_{A \in H_2(V)} \sum_{\Delta} a^A_\Delta t^A \Delta,$$

where $a^A_\Delta \in \mathbb{C}$, and the second sum is taken over all monomials $\Delta = \delta_1^{j_1} \cdots \delta_r^{j_r}$ of (distinct) generators $\delta_1, \dots, \delta_r, \dots \in \Theta$ (notice that we allow the case $r = 0$). We set $a^A_\Delta = 0$ if the dimension $\sum\limits_{k=1}^{s} \overline{\gamma_k} - \sum\limits_{l=1}^{r} j_l \overline{\delta_l} + (2-2s)(n-3) + 2c_1(\xi)[A]$ of the moduli space $\mathcal{M}^A = \mathcal{M}^A(\gamma_1, \dots, \gamma_s; \underbrace{\delta_1 \dots, d_1}_{j_1}, \dots, \underbrace{d_r, \dots, \delta_r}_{j_r})$ is different from 1. Otherwise, we define the coefficient a^A_Δ as the sum $\sum\limits_C w(C)$ of weights $w(C)$ assigned to 1-dimensional components of the moduli space $\mathcal{M}^A$ Given a component C of $\mathcal{M}$ we set

$$w(C) = \pm \frac{1}{r!d} m(\delta_1)^{j_1} \cdots m(\delta_r)^{j_r},$$

where $m(\delta_l)$ is the multiplicity of the periodic orbit δ_l, $l = 1, \dots, r$; $d = 1$ if the curves from C are not multiply-covered and d is the order of the group of deck transformations of the corresponding branched covering in the multiply-covered case. Finally the sign $\pm$ is determined by an algorithm, similar to the one used in the traditional Floer theory (see [13]). This algorithm shows, in particular, that the operations $[\cdot, \dots, \cdot]_s$ are skew-symmetric: a transposition of any two elements γ_1, γ_2' in the bracket changes the sign by $(-1)^{\overline{\gamma_1 \gamma_2}}$. It is important to point out that compactness theorems for holomorphic curves (see [18, 20, 11]) garantee that the operations $[\,\cdot, \dots, \cdot\,]_s$ take values in *polynomial* functions (and not in formal power series).

The operation $[\,\cdot, \dots, \cdot\,]_s$ which was just defined on the generators of Θ admits a unique extension to a skewsymmetric multilinear operation $\underbrace{\Theta^{H_2} \otimes \dots, \otimes \Theta^{H_2}}_{s} \to \Theta^{H_2}$ which satisfies the Leibnitz rule:

$$(-1)^t [\theta_1, \dots, \theta_l \theta_l', \dots, \theta_s]_s = \theta_l [\theta_1, \dots, \theta_l', \dots, \theta_s]_s + (-1)^{\overline{\theta_l}} \theta_l' [\theta_1, \dots, \theta_l, \dots, \theta_s]_s,$$

where $t = \sum\limits_{1}^{l-1} \overline{\theta_i}$. Let us now take a closer look to the operation $[\theta]_1$ which will also be denoted by $d\theta$, and called the *differential* of θ. Notice that it decreases the grading by 1, i.e. $\overline{d\theta} = \overline{\theta} - 1$ for any monomial $\theta \in \Theta$.

THEOREM 3.4 *1.* $d^2 = 0$.

2. Any homotopy $\alpha_t, t \in [0,1]$, of contact forms, together with a compatible homotopy J_t of almost complex structures, induces a quasi-isomorphism $\Phi_{\{a_t,J_t\}}$ of the corresponding algebras. In particular, the graded contact homology algebra $H\Theta^{H_2} = \operatorname{Ker} d/\operatorname{Coker} d$ is an invariant of the contact manifold (M,ξ).

Sometimes it is more convenient to consider the *reduced* contact homology algebra $\widetilde{H\Theta}^{H_2}(M,\xi)$ of a closed contact manifold (M,ξ), which is defined similarly to $H\Theta^{H_2}$, except that instead of contact forms for ξ on the whole M, we use contact forms on the punctured manifold $M \setminus x$, $x \in M$, which are isomorphic at infinity to the standard contact form $dz - \sum\limits_{i=1}^{n-1} y_i dx_i$ on $\mathbb{R}^{2n-1}$.

The contact homology algebras $H\Theta^{H_2}$ and/or $\widetilde{H\Theta}^{H_2}$ can be explicitely computed in several interesting examples. Let us formulate here some of these results.

THEOREM 3.5 *1. $\widetilde{H\Theta}(S^{2n-1},\xi_0) = \mathbb{C}$; $H\Theta(S^{2n-1},\xi_0)$ is a graded polynomial algebra of generators $\gamma_1,\gamma_2,\dots,$ of degrees $\overline{\gamma_i} = 2(n+i-1)$, $i = 0,\dots.$*

2. If ξ is an overtwisted contact structure on a 3-manifold V then $H\Theta(V,\xi) = 0$.

3. For any Stein fillable contact manifold (V,ξ) we have $H\Theta(V,\xi) \neq 0$.

4. Suppose that a contact manifold (V,ξ) of dimension $2n-1$ with $H_1(V) = 0$ has a subcritical Stein filling W. Then $\widetilde{H\Theta}^{H_2}(V,\xi)$ is a group algebra of $H^2(V)$ over a free graded commutative algebra with generators $\gamma_{ikl}, k = 1,\dots,n-1$, $l = 1,\dots,\dim H_k(W;\mathbb{R}), i = 0,\dots,$ of degree $\overline{\gamma_{ikl}} = 2(n+i-1)-k$.

5. Let ξ_1 be a non-standard contact structure on S^{2n-1}, $n > 2$, which is provided by Theorem 3.1 above, and the contact manifold (S^{2n-1},ξ_k) be the connected sum of k copies of (S^{2n-1},ξ_1). Then the contact homology algebras $H\Theta(S^{2n-1},\xi_k)$ are pairwise non-isomorphic for all k, and in particular S^{2n-1} has infinitely many distinct contact structures in the standard homotopy class.

We thank Yu. Chekanov who pointed out to us the property 3.5.3. The computations in 3.5.4 were done by M.-L. Yau, and the result in 3.5.5 is due to I. Ustilovsky.

THE CASE $H_1(V) \neq 0$. For a general contact manifold V with $H_1(V) \neq 0$ one may first construct a similar contact homology algebra $H\Theta^{H_2}_{\mathrm{contr}}$ generated by the subset $\mathcal{P}_a^{\mathrm{contr}} \subset \mathcal{P}_a$ of *contractible* periodic orbits, and then for each free loop homotopy class Γ consider a module Θ_Γ over the algebra $\Theta^{H_2}_{\mathrm{contr}}$, generated by elements of $\mathcal{P}$ from the homotopy class Γ. The differential $d : \Theta_\Gamma \to \Theta_\Gamma$ on this module is defined, as above, by counting components of 1-dimensional moduli spaces $\mathcal{M}(\gamma;\delta_1,\dots,\delta_r)$ with an extra condition that γ and δ_1 belong to the class Γ, while all the other trajectories $\delta_2,\dots,\delta_s$ are from $\mathcal{P}^{\mathrm{contr}}$. Then we also have $d^2 = 0$, and thus the homology $H\Theta_\Gamma^{H_2}$, which is a module over the contact homology algebra $H\Theta_{\mathrm{contr}}^{H_2}$ is another invariant of the contact manifold V. For a generic α $H\Theta_\Gamma^{H_2}$ is a finite dimensional module over $H\Theta_{\mathrm{contr}}^{H_2}$, and thus can be effectively computed, especially when the contact homology algebra $H\Theta_{\mathrm{contr}}^{H_2}$ is isomorphic to $\mathbb{C}$ (or to the group algebra of $\mathbb{C}[H_2(V)]$). For instance, let ξ_1 be the standard contact structure on the 3-torus $V = T^3 = T^2 \times S^1$ viewed as the unit cotangent bundle of T^2. For $k = 2, \ldots$, we denote by ξ_k the pull-back of ξ_1 under the k-sheeted covering $T^3 \to T^3$ which unwinds the fiber S^1. Then

THEOREM 3.6 *$H\Theta_{\mathrm{contr}}^{H_2}(\xi_k) = \mathbb{C}[H_2(T^3)]$; $\dim_{\mathbb{C}[H_2(T^3)]}(H\Theta_\Gamma^{H_2}(\xi_k)) = 2k$ for any horizontal 1-dimensional homology class, i.e. a class from $H_1(T^2 \times \mathrm{point}) \subset T^3$, and $H\Theta_\Gamma^{H_2}(\xi_k)) = 0$ for all other classes $\Gamma \in H_1(T^3)$.*

As a corollory of 3.6 we get a theorem of E. Giroux and Y. Kanda (see [16, 21]) which states that the contact structures ξ_k, $k = 1, \ldots$, are pairwise non-isomorphic. It seems likely that the algebra $H\Theta_{\mathrm{contr}}^{H_2}(V, \xi)$ is trivial (i.e. isomorphic to the group algebra of $H_2(V)$ over $\mathbb{C}$) for any *strongly tight* contact manifold (V, ξ), i.e. a contact 3-manifold which is covered by $\mathbb{R}^3$ with a tight, and hence standard contact structure.

HAMILTONIAN FORMALISM. It turns out that the the operations $[\,\cdot, \ldots, \cdot\,]_s$ for $s > 1$ can be viewed as certain cohomological operations on the contact homology algebra. For instance, we have

THEOREM 3.7 *The operation $[\,\cdot, \cdot\,]_2$ is a Poisson bracket on $\Pi(H\Theta)$, where Π is the operator of changing the parity. The quasi-isomorphism $\Phi_{\{\alpha_t, J_t\}}$ from 3.4.2, induced by a deformation of contact forms and almost complex structures, preserves the Poisson bracket.*

Other operations $[\,\cdot, \ldots, \cdot\,]_s$, $s > 2$, define secondary cohomological operations on the contact homology algebra $H\Theta^{H_2}$, which all fit into a structure of a L_∞, or rather a P_∞-algebra on Θ^{H_2}. However, the following Hamiltonian formalism provides a better algebraic framework for all these operations.

Let us associate with each periodic trajectory $\gamma \in \mathcal{P} = \mathcal{P}_\alpha$ two variables, p_γ and q_γ of the same degree $\overline{p_\gamma} = \overline{q_\gamma}$. Let $T\Theta$ be the free graded (super-)commutative algebra over $\mathbb{C}$ with the unit generated by variables p_γ, q_γ associated to each periodic orbit $\gamma \in \mathcal{P}$, and completed with respect to variables p_γ, $\gamma \in \mathcal{P}$. This means that the elements of $T\Theta$ are formal power series in p-variables with coefficients which are polynomial of q-variables. We will also consider the group algebra $T\Theta^{H_2}$ of the group $H_2(V)$ with coefficients in $T\Theta$. Informally, if one thinks about the algebra Θ as the algebra of polynomial functions on an infinite-dimensional (super-)space L with coordinates q_γ, $\gamma \in \mathcal{P}$, then $T\Theta$ is the algebra of functions on the cotangent bundle T^*L of L. This infinite-dimensional cotangent bundle is endowed with an even symplectic form $\sum\limits_{\gamma \in \mathcal{P}} dp_\gamma \wedge dq_\gamma$, which defines, in its turn, Poisson brackets on algebras $T\Theta$ and $T\Theta^{H_2}$.

Next we construct a Hamiltonian function $H \in T\Theta^{H_2}$ which will encode all the information about the brackets $[\cdot, \cdots, \cdot]_k$ introduced above. We set

$$H(p,q) = \sum [\gamma_1, \ldots, \gamma_s]_s p_{\gamma_1} \cdots p_{\gamma_s},$$

where the sum is taken over the set of all monomials in variables p_γ, $\gamma \in \mathcal{P}$, and where we assume that the brackets $[\gamma_1, \ldots, \gamma_s]_s \in \Theta^{H_2}$ are expressed in terms of the variables q_γ, $\gamma \in \mathcal{P}$. Theorems 3.8 and 3.9 below generalize Theorem 3.4 and formalize properties of all considered above operations.

THEOREM 3.8 $\{H, H\} = 0$.[6]

Consider a Poisson subalgebra $Z\Theta^{H_2} = \{f \in T\Theta^{H_2};\ \{f, H\} = 0\} \subset T\Theta^{H_2}$ and its ideal $B\Theta^{H_2}$, generated by functions of the form $\{g, H\}$, $g \in T\Theta^{H_2}$. Then $P\Theta^{H_2} = Z\Theta^{H_2}/B\Theta^{H_2}$ also carries a Poisson structure.

THEOREM 3.9 *Any homotopy α_t, $t \in [0,1]$, of contact forms, together with a compatible homotopy J_t of almost complex structures, induces an isomorphism $\Psi_{\{\alpha_t, J_t\}} : P\Theta^{H_2}(V, \alpha_0, J_0) \to P\Theta^{H_2}(V, \alpha_1, J_1)$ of Poisson algebras.*

These results are only the first steps of a bigger story. For instance, a directed symplectic cobordism between two contact manifolds generate a Lagrangian correspondence between the corresponding Poisson algebras, and the composition of directed cobordisms generates the composition of Lagrangian correspondences. We hope that this would provide tools for effective computations of rational Gromov-Witten invariants of symplectic manifolds by splitting them into compositions of elementary directed symplectic cobordisms. The larger picture also incorporates moduli spaces of holomorphic curves of higher genus, as well as higher-dimensional spaces of holomorphic curves.

INVARIANTS OF LEGENDRIAN SUBMANIFOLDS. Let us briefly mention here a relative analog of the contact homology theory, which provides invariants of pairs (V, L) where $V = (V, \xi)$ is a contact manifold, and L its Legendrian submanifold. For the case when (V, ξ) is the standard contact $(\mathbb{R}^{2n-1}, \xi_0)$ this theory produces invariants of immersed Lagrangian submanifolds in $\mathbb{R}^{2n-2}$ up to contact isotopy (see [10]), i.e up to regular Lagrangian homotopy in $\mathbb{R}^{2n-2}$, which lifts to a Legendrian isotopy in $\mathbb{R}^{2n-1}$. When $n = 2$ all the involved holomorphic curves can be explicitely seen from the combinatorics of the corresponding (Lagrangian) immersion of the curve L into $\mathbb{R}^2$, and thus the theory may be developed via pure combinatorial means. The first part of this combinatorial theory, parallel to the theory of the differential d in the absolute case, was independently done by Yu. Chekanov (see [3]). However, even for $n = 2$ a (non-commutative) analog of the described above Hamiltonian formalism allows us to define many other invariants of Legendrian curves, which can also be computed and studied by pure combinatorial means.

[6]One should remember that in the super-commutative setting the bracket of a function with itself does not vanish automatically.

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