Geometry and Analytic Theory of Frobenius Manifolds

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Abstract. Main mathematical applications of Frobenius manifolds are in the theory of Gromov - Witten invariants, in singularity theory, in differential geometry of the orbit spaces of reflection groups and of their extensions, in the hamiltonian theory of integrable hierarchies. The theory of Frobenius manifolds establishes remarkable relationships between these, sometimes rather distant, mathematical theories.

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WDVV equations of associativity is the problem of finding of a quasihomogeneous, up to at most quadratic polynomial, function $F(t)$ of the variables $t = (t^1, \dots, t^n)$ and of a constant nondegenerate symmetric matrix $\left(\eta^{\alpha\beta}\right)$ such that the following combinations of the third derivatives $c_{\alpha\beta}^\gamma(t) := \eta^{\gamma\epsilon}\partial_\epsilon\partial_\alpha\partial_\beta F(t)$ for any t are structure constants of an asociative algebra $A_t = \mathrm{span}\,(e_1, \dots, e_n),\ e_\alpha \cdot e_\beta = c_{\alpha\beta}^\gamma(t) e_\gamma,\ \ \alpha,\, \beta = 1, \dots, n$ with the unity $e = e_1$ (summation w.r.t. repeated indices will be assumed). These equations were discovered by physicists E.Witten, R.Dijkgraaf, E.Verlinde and H.Verlinde in the beginning of '90s. I invented Frobenius manifolds as the coordinate-free form of WDVV.

1. Definition of Frobenius manifold (FM).

1.1. Frobenius algebra (over a field k; we mainly consider the case $k = \mathbf{C}$) is a pair $(A, < \ , \ >)$, where A is a commutative associative k-algebra with a unity e, $< \ , \ >$ is a symmetric nondegenerate *invariant* bilinear form $A \times A \to k$, i.e. $< a \cdot b, c > = < a, b \cdot c >$ for any $a,\, b \in A$. A *gradation of the charge d* on A is a k-derivation $Q : A \to A$ such that $< Q(a), b > + < a, Q(b) > = d < a, b >,\ d \in k$. More generally, graded of the charge $d \in k$ Frobenius algebra $(A, < \ , \ >)$ over a graded commutative associative k-algebra R by definition is endowed with two k-derivations $Q_R : R \to R$ and $Q_A : A \to A$ satisfying the properties $Q_A(\alpha a) = Q_R(\alpha) a + \alpha Q_A(a),\ \alpha \in R,\ a \in A\ < Q_A(a), b > + < a, Q_A(b) > - Q_R < a, b > = d < a, b >,\ a,\, b \in A$.

1.2. Frobenius structure of the charge d on the manifold M is a structure of a Frobenius algebra on the tangent spaces $T_t M = (A_t, < \ , \ >_t)$ depending (smoothly, analytically etc.) on the point $t \in M$. It must satisfy the following axioms.

FM1. The metric $< \ , \ >_t$ on M is flat (but not necessarily positive definite). Denote ∇ the Levi-Civita connection for the metric. The unity vector field e must be covariantly constant, $\nabla e = 0$.

FM2. Let c be the 3-tensor $c(u,v,w) :=< u\cdot v, w >$, u, v, $w \in T_tM$. The 4-tensor $(\nabla_z c)(u,v,w)$ must be symmetric in u, v, w, $z \in T_tM$.

FM3. A linear vector field $E \in Vect(M)$ must be fixed on M, i.e. $\nabla\nabla E = 0$, such that the derivations $Q_{Func(M)} := E$, $\quad Q_{Vect(M)} := \mathrm{id} + \mathrm{ad}_E$ introduce in $Vect(M)$ the structure of graded Frobenius algebra of the given charge d over the graded ring $Func(M)$ of (smooth, analytic etc.) functions on M. We call E *Euler vector field*.

Locally, in the flat coordinates $t^1, \ldots, t^n$ for the metric $< \, , \, >_t$, a FM with diagonalizable (1,1)-tensor ∇E is described by a solution $F(t)$ of WDVV associativity equations, where $\partial_\alpha\partial_\beta\partial_\gamma F(t) =< \partial_\alpha \cdot \partial_\beta, \partial_\gamma >$, and vice versa. We will call $F(t)$ *the potential* of the FM (physicists call it *primary free energy*; in the setting of quantum cohomology it is called Gromov - Witten potential [KM]).

1.3. Deformed flat connection $\tilde\nabla$ on M is defined by the formula $\tilde\nabla_u v := \nabla_u v + z\, u\cdot v$. Here u, v are two vector fields on M, z is the parameter of the deformation. (In [Gi1] another normalization is used $\tilde\nabla \mapsto \hbar\tilde\nabla$, $\hbar = z^{-1}$.) We extend this to a meromorphic connection on the direct product $M\times \mathbf{C}$, $z \in \mathbf{C}$, by the formula $\tilde\nabla_{d/dz} v = \partial_z v + E\cdot v - z^{-1}\mu\, v$ with $\mu := 1/2(2-d)\cdot \mathbf{1} - \nabla E$, other covariant derivatives are trivial. Here u, v are tangent vector fields on $M\times\mathbf{C}$ having zero components along $\mathbf{C} \ni z$. The curvature of $\tilde\nabla$ is equal to zero. This can be used as a definition of FM [Du3]. So, there locally exist n independent functions $\tilde t_1(t;z), \ldots, \tilde t_n(t;z)$, $z \neq 0$, such that $\tilde\nabla\, d\tilde t_\alpha(t;z) = 0$, $\alpha = 1, \ldots, n$. We call these functions *deformed flat coordinates*.

2. Examples of FMs appeared first in 2D topological field theories [W1, W2, DVV].

2.0. Trivial FM: $M = A_0$ for a graded Frobenius algebra A_0. The potential is a cubic, $F_0(t) = \frac{1}{6} < 1, (t)^3 >$, $\quad t \in A_0$. Nontrivial examples of FM are

2.1. FM with good analytic properties. They are analytic perturbations of the cubic. That means that, in an appropriate system of flat coordinates $t = (t', t'')$, where all the components of t' have $Lie_E t' = \mathrm{const}$, all the components of t'' have $Lie_E t'' \neq \mathrm{const}$, we have $F(t) = F_0(t) + \sum_{k,l\geq 0} A_{k,l}(t'')^l e^{k\,t'}$ and the series converges in some neiborghood of $t'' = 0$, $t' = -\infty$.

2.2. K.Saito theory of primitive forms and Frobenius structures on universal unfoldings of quasihomogeneous singularities. Let $f_s(x)$, $s = (s_1, \ldots, s_n)$ be the universal unfolding of a quasihomogeneous isolated singularity $f(x)$, $x \in \mathbf{C}^N$, $f(0) = f'(0) = 0$. Here n is the Milnor number of the singularity. The Frobenius structure on the base $M \ni s$ of the universal unfolding can be easily constructed [BV] using the theory of primitive forms [Sai2]. For the example [DVV] of the A_n singularity $f(x) = x^{n+1}$ the universal unfolding reads $f_s(x) = x^{n+1} + s_1 x^{n-1} + \ldots + s_n$, $M = \mathbf{C}^n \ni (s_1, \ldots, s_n)$. On the FM $e = \partial/\partial s_n$, $E = \sum (k+1) s_k \partial/\partial s_k$, the metric has the form

$$< \partial_{s_i}, \partial_{s_j} >= -(n+1) \operatorname*{res}_{x=\infty} \frac{\partial f_s(x)/\partial s_i\, \partial f_s(x)/\partial s_j}{f_s'(x)} \tag{2.1}$$

the multiplication is defined by

$$< \partial_{s_i}\cdot\partial_{s_j}, \partial_{s_k} >= -(n+1) \operatorname*{res}_{x=\infty} \frac{\partial f_s(x)/\partial s_i\, \partial f_s(x)/\partial s_j\, \partial f_s(x)/\partial s_k}{f_s'(x)}. \tag{2.2}$$

This is a polynomial FM. The deformed flat coordinates are given by oscillatory integrals

$$\tilde{t}_c = \frac{1}{\sqrt{z}} \int_c e^{z\, f_s(x)} dx \tag{2.3}$$

Here c is any 1-cycle in $\mathbf{C}$ that goes to infinity along the direction $\mathrm{Re}\, z\, f_s(x) \to -\infty$.

2.3. Quantum cohomology of a $2d$-dimensional smooth projective variety X is a Frobenius structure of the charge d on a domain $M \subset H^*(X, \mathbf{C})/2\pi i H^2(X, \mathbf{Z})$ (we assume that $H^{odd}(X) = 0$ to avoid working with supermanifolds, see [KM]). It is an analytic perturbation in the sense of n.2.1 of the cubic for $A_0 = H^*(X)$ defined by a generating function of the genus zero Gromov - Witten (GW) invariants of X [W1, W2, MS, RT, KM, Beh]. They are defined as intersection numbers of certain cycles on the moduli spaces of stable maps [KM]

$$X_{[\beta],l} := \left\{ \beta : (S^2, p_1, \ldots, p_l) \to X, \;\; \text{given homotopy class } [\beta] \in H_2(X; \mathbf{Z}) \right\}.$$

The holomorphic maps β of the Riemann sphere S^2 with $l \geq 1$ distinct marked points are considered up to a holomorphic change of parameter. The markings define evaluation maps $p_i : X_{[\beta],l} \to X, \;\; (\beta, p_1, \ldots, p_l) \mapsto \beta(p_i)$.

$$\begin{aligned} F(t) &= F_0(t) + \sum_{[\beta] \neq 0} \sum_l \left\langle e^{t''} \right\rangle_{[\beta],l} \exp \int_{S^2} \beta^*(t') \\ \left\langle e^t \right\rangle_{[\beta],l} &:= \frac{1}{l!} \int_{X_{[\beta],l}} p_1^*(t) \wedge \ldots \wedge p_l^*(t) \end{aligned} \tag{2.4}$$

for $t = (t', t'') \in H^*(X)$, $t' \in H^2(X)/2\pi i H^2(X, \mathbf{Z})$, $t'' \in H^{*\neq 2}(X)$. This potential together with the Poincaré pairing on $TM = H^*(X)$, the unity vector field $e = 1 \in H^0(X)$, the Euler vector field $E(t) = \sum (1 - q_\alpha) t^\alpha e_\alpha + c_1(X), \;\; t = t^\alpha e_\alpha, \; e_\alpha \in H^{2q_\alpha}(X)$ gives the needed Frobenius structure. The deformed flat coordinates are generating functions of certain "gravitational descendents" [Du5], see also [DW, Ho, Gi1] $\tilde{t}_\alpha(t; z) = \sum_{p=0}^\infty \sum_{[\beta],l} \left\langle z^{\mu+p} z^{c_1(X)} \tau_p(e_\alpha) \otimes 1 \otimes e^{t''} \right\rangle_{[\beta],l} e^{\int_{S^2} \beta^*(t')}$, $\alpha = 1, \ldots, n = \dim H^*(X)$, $\mu(e_\alpha) = (q_\alpha - d/2) e_\alpha$, The definition of the descendents $\left\langle \tau_{p_1}(a_1) \otimes \tau_{p_2}(a_2) \otimes \ldots \otimes \tau_{p_l}(a_l) \right\rangle_{[\beta],l}$ see in [W2], [KM]. The definition of GW invariants can be extended on a certain class of compact symplectic varieties X using Gromov's theory [Gr] of pseudoholomorphic curves, see [W2, MS, RT].

3. Classification of semisimple FMs.

3.1. Definition. A point $t \in M$ is called *semisimple* if the algebra on $T_t M$ is semisimple. A connected FM M is called semisimple if it has at least one semisimple point. Classification of semisimple FMs can be reduced, by a nonlinear change of coordinates, to a system of ordinary differential equations. First we will describe these new coordinates.

3.2. Canonical coordinates on a semisimple FM. Denote $u_1(t), \ldots, u_n(t)$ the roots of the characteristic polynomial of the operator of multiplication by the Euler vector field $E(t)$ ($n = \dim M$). Denote $M^0 \subset M$ the open subset where all the roots are pairwise distinct. It turns out [Du2] that the functions $u_1(t)$, $\ldots$, $u_n(t)$ are independent local coordinates on $M^0 \neq \emptyset$. In these coordinates $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$, where $\partial_i := \partial/\partial u_i$, and $E = \sum_i u_i \partial_i$. The local coordinates u_1, $\ldots$, u_n on M^0 are called *canonical*.

3.3. DEFORMED FLAT CONNECTION IN THE CANONICAL COORDINATES AND ISOMONODROMY DEFORMATIONS. Staying in a small ball on M^0, let us order the canonical coordinates and choose the signs of the square roots $\psi_{i1} := \sqrt{< \partial_i, \partial_i >}$, $i = 1, \ldots, n$. The orthonormal frame of the normalized idempotents ∂_i establishes a local trivialization of the tangent bundle TM^0. The deformed flat connection $\tilde{\nabla}$ in TM^0 is recasted into the following flat connection in the trivial bundle $M^0 \times \mathbf{C} \times \mathbf{C}^n$

$$\tilde{\nabla}_i = \partial_i - z\, E_i - V_i, \quad \tilde{\nabla}_{d/dz} = \partial_z - U - z^{-1} V, \tag{3.1}$$

other components are obvious. Here the $n \times n$ matrices E_i, U, $V = (V_{ij})$ read $(E_i)_{kl} = \delta_{ik}\delta_{il}$, $U = \mathrm{diag}\,(u_1, \ldots, u_n)$, $V = \Psi\, \mu\, \Psi^{-1} = -V^T$ where the matrix $\Psi = (\psi_{i\alpha})$ satisfying $\Psi^T \Psi = \eta$ is defined by $\psi_{i\alpha} := \psi_{i1}^{-1} \partial t_\alpha / \partial u_i$, $i, \alpha = 1, \ldots, n$. The skew-symmetric matrices V_i are determined by the equations $[U, V_i] = [E_i, V]$.

Flatness of the connection (3.1) reads as the system of commuting time-dependent Hamiltonian flows on the Lie algebra $so(n) \ni V$ equipped with the standard linear Poisson bracket

$$\partial_i V = \{V, H_i(V; u)\}, \quad i = 1, \ldots, n \tag{3.2}$$

with the quadratic Hamiltonians $H_i(V; u) = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij}^2}{u_i - u_j}$, $i = 1, \ldots, n$. For the first nontrivial case $n = 3$ (3.2) can be reduced to a particular case of the classical Painlevé-VI equation. The monodromy of the operator $\tilde{\nabla}_{d/dz}$ (i.e., the monodromy at the origin, the Stokes matrix, and the central connection matrix, see definitions in [Du3, Du5]) does not change with small variations of a point $u = (u_1, \ldots, u_n) \in M$.

3.4. PARAMETRIZATION OF SEMISIMPLE FMS BY MONODROMY DATA OF THE DEFORMED FLAT CONNECTION. We now reduce the above system of nonlinear differential equations to a linear boundary value problem of the theory of analytic functions. First we will describe the set of parameters of the boundary value problem.

3.4.1. MONODROMY AT THE ORIGIN (defined also for nonsemisimple FMs) consists of:

- a linear n-dimensional space $\mathcal{V}$ with a symmetric nondegenerate bilinear form $< \, , \, >$, a skew-symmetric linear operator $\mu : \mathcal{V} \to \mathcal{V}$, $< \mu(a), b > + < a, \mu(b) >= 0$, and a marked eigenvector e_1 of μ, $\mu(e_1) = -d/2\, e_1$. In main examples the operator μ will be diagonalizable.

- A linear operator $R : \mathcal{V} \to \mathcal{V}$ satisfying the following properties: (1) $R = R_1 + R_2 + \ldots$ where $R_k(\mathcal{V}_\lambda) \subset \mathcal{V}_{\lambda+k}$ for the root decomposition of $\mathcal{V} = \oplus_\lambda \mathcal{V}_\lambda$, $\mu(v_\lambda) = \lambda v_\lambda$ for $v_\lambda \in \mathcal{V}_\lambda$. (2) $\{Rx, y\} + \{x, Ry\} = 0$ for any x, $y \in \mathcal{V}$ where $\{x, y\} := \left\langle e^{\pi i \mu} x, y \right\rangle$.

3.4.2. Stokes matrix is an arbitrary $n \times n$ upper triangular matrix $S = (s_{ij})$ with $s_{ii} = 1$, $i = 1, \dots, n$. We treat it as a bilinear form $< a, b >_S := a^T S b$, $a, b \in \mathbf{C}^n$.

3.4.3. Central connection matrix is an isomorphism $C : \mathbf{C}^n \to \mathcal{V}$ satisfying $< a, b >_S = < Ca, e^{\pi i \mu} e^{\pi i R} C b >$ for any $a, b \in \mathbf{C}^n$. The matrices S and C are defined up to a transformation $S \mapsto DSD$, $C \mapsto CD$, $D = \operatorname{diag}(\pm 1, \dots, \pm 1)$.

3.4.4. Riemann - Hilbert boundary value problem (RH b.v.p.). Let us fix a radius $R > 0$ and an argument $0 \le \varphi < 2\pi$. Denote $\ell = \ell_+ \cup \ell_-$ the oriented line $\ell_+ = \{z \,|\, \arg z = \varphi\}$, $\ell_- = \{z \,|\, \arg z = \varphi + \pi\}$. It divides the complex z-plane into two halfplanes Π_{right} and Π_{left}. For a given $u = (u_1, \dots, u_n)$ with $u_i \ne u_j$ for $i \ne j$ and for given monodromy data we are looking for: (1) $n \times n$ matrix-valued functions $\Phi_{\text{right}}(z)$, $\Phi_{\text{left}}(z)$ analytic for $|z| > R$ and $z \in \Pi_{\text{right}}$ and $z \in \Pi_{\text{left}}$ resp., continuous up to the boundaries $|z| = R$ or $z \in \ell$ and satisfying $\Phi_{\text{right/left}}(z) = 1 + O(1/z)$ for $|z| \to \infty$ within the correspondent half-plane $\Pi_{\text{right/left}}$; (2) $n \times n$ matrix-valued function $\Phi_0(z)$ (with values in $Hom(\mathcal{V}, \mathbf{C}^n)$) analytic for $|z| < R$ and continuous up to the boundary $|z| = R$, such that $\det \Phi_0(0) \ne 0$. The boundary values of the functions must satisfy

$$\Phi_{\text{right}}(z) e^{zU} = \Phi_{\text{left}}(z) e^{zU} S \quad \text{for } z \in \ell_+,\ |z| > R;$$

$$\Phi_{\text{right}}(z) e^{zU} = \Phi_{\text{left}}(z) e^{zU} S^T \quad \text{for } z \in \ell_-,\ |z| > R;$$

$$\Phi_{\text{right}}(z) e^{zU} = \Phi_0(z) z^\mu z^R C \quad \text{for } |z| = R,\ z \in \Pi_{\text{right}};$$

$$\Phi_{\text{left}}(z) e^{zU} S = \Phi_0(z) z^\mu z^R C \quad \text{for } |z| = R,\ z \in \Pi_{\text{left}}.$$

The branchcut in the definition of the multivalued functions z^μ and z^R is chosen along ℓ_-. For solvability of the above RH b.v.p. we have also to require the complex numbers $u_1, \dots, u_n$ to be ordered in such a way, depending on φ, that

$$\mathcal{R}_{jk} := \{z = -ir(\bar u_j - \bar u_k) | r \ge 0\} \subset \Pi_{\text{left}} \quad \text{for } j < k. \tag{3.3}$$

Denote $\mathcal{U}(\varphi) \subset \mathbf{C}^n$ the set of all points $u = (u_1, \dots, u_n)$ with $u_i \ne u_j$ for $i \ne j$ satisfying (3.3). Let $\mathcal{U}_0(\varphi)$ be the subset of points $u \in \mathcal{U}(\varphi)$ such that: (1) the RH b.v.p. is solvable and (2) all the coordinates of the vector $\Phi_0(0) e_1$ are distinct from zero. It can be shown (cf. [Mi], [Mal]) that the solution $\Phi_{\text{right/left}} = \Phi_{\text{right/left}}(z; u)$, $\Phi_0 = \Phi_0(z; u)$ of the RH b.v.p depends analytically on $u \in \mathcal{U}_0(\varphi)$. Let $\Phi_0(z; u) = \sum_{p=0}^\infty \phi_p(u) z^p$. Denote (only here) $(\ ,\ )$ the standard sum of squares quadratic form on $\mathbf{C}^n$. Choose a basis $e_1, e_2, \dots, e_n$ of eigenvectors of μ, $\mu(e_\alpha) = \mu_\alpha e_\alpha$, $\mu_1 = -d/2$, and put $\eta_{\alpha\beta} := < e_\alpha, e_\beta >$, $(\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}$.

Theorem 1 [Du2, Du3, Du5]. *The formulae*

$$t_\alpha(u) = (\phi_0(u) e_\alpha, \phi_1(u) e_1), \quad t^\alpha = \eta^{\alpha\beta} t_\beta,\ \alpha = 1, \dots, n,$$

$$F = 1/2\,[(\phi_0 t, \phi_1 t) - 2(\phi_0 t, \phi_1 e_1) + (\phi_1 e_1, \phi_2 e_1) - (\phi_3 e_1, \phi_0 e_1)]$$

$$E(t) = \sum_{\alpha=1}^n (1 + \mu_1 - \mu_\alpha) t^\alpha \partial_\alpha + \sum_\alpha (R_1)_1^\alpha \partial_\alpha$$

define on $\mathcal{U}_0(\varphi)$ a structure of a semisimple FM $Fr(\mathcal{V},<\ ,\ >,\mu,e_1,R,S,C)$. Any semisimple FM locally has such a form.

3.5. REMARK. The columns of the matrices $\Phi_0(z;u)z^\mu z^R$ and $\Phi_{\rm right}(z;u)e^{zU}$ correspond to two different bases in the space of deformed flat coordinates. The first basis is a deformation, $z \to 0$, of the original flat coordinates, $\tilde t = [t + O(z)]z^\mu z^R$. The second one, defined only in the semisimple case, corresponds to a system of deformed flat coordinates given by oscillatory integrals (see (2.3) and Section 6 below).

3.6. GLOBAL STRUCTURE OF SEMISIMPLE FMS AND ACTION OF THE BRAID GROUP ON THE MONODROMY DATA. Let B_n be the group of braids with n strands. We will glue globally the FM from the charts described in n.3.4 with different S and C. So, for brevity, we redenote here the charts $Fr(\mathcal{V},<\ ,\ >,\mu,e_1,R,S,C) =: Fr(S,C)$. The charts will be labelled by braids $\sigma \in B_n$. By definition in the chart $Fr(S^\sigma,C^\sigma)$ the functions $t^\alpha(u)$, $F(u)$ are obtained as the result of analytic continuation from $Fr(S,C)$ along the braid σ. The action $S \mapsto S^\sigma$, $C \mapsto C^\sigma$ of the standard generators $\sigma_1,\dots,\sigma_{n-1}$ of B_n is given by $S^{\sigma_i} = KSK$, $C^{\sigma_i} = CK$ where the only nonzero entries of the matrix $K = K^{(i)}(S)$ are $K_{kk} = 1$, $k = 1,\dots,n$, $k \neq i, i+1$, $K_{i,i+1} = K_{i+1,i} = 1$, $K_{i,i} = -s_{i,i+1}$. Let $B_n(S,C) \subset B_n$ be the subgroup of all braids σ such that $S^\sigma = DSD$, $C^\sigma = CD$, $D = \mathrm{diag}(\pm1,\dots,\pm1)$.

THEOREM 2 [Du3, Du5]. *Any semisimple FM has the form*

$$M = \cup_{\sigma\in B_n/B_n(S,C)} Fr(\mathcal{V},<\ ,\ >,\mu,e_1,R,S^\sigma,C^\sigma)$$

where the glueing of the charts is given by the above action of B_n.

3.7. TAU-FUNCTION OF THE ISOMONODROMY DEFORMATION AND ELLIPTIC GW INVARIANTS. Like in n.2.2, the genus g GW invariants can be defined in terms of the intersection theory on the moduli space $X_{[\beta],l}(g)$ of stable maps $\beta : C_g \to X$ of curves of genus g with markings [KM, Beh]. It turns out that, assuming semisimplicity of quantum cohomology of X, the elliptic (i.e., of $g = 1$) GW invariants can still be expressed via isomonodromy deformations. To this end we define, following [JM], the τ-function $\tau(u_1,\dots,u_n)$ of a solution $V(u)$ of the system (3.2) by the quadrature of a closed 1-form $d\log\tau = \sum_{i=1}^n H_i(V(u);u)du_i$. We define G-function of the FM by $G = \log(\tau/J^{1/24})$ where $J = \det\left(\partial t^\alpha/\partial u_i\right) = \pm\prod_{i=1}^n \psi_{i1}(u)$.

THEOREM 3 [DZ2]. *For an arbitrary semisimple FM the G-function is the unique, up to an additive constant, solution to the system of [Ge] for the generating function of elliptic GW invariants satisfying $Lie_e G = 0$, $Lie_E G = \mathrm{const}$.*

3.8. PROBLEM OF SELECTION OF SEMISIMPLE FMS WITH GOOD ANALYTIC PROPERTIES of n.2.1 is still open. Experiments for small n [Du3] show that such solutions are rare exceptions among all semisimple FMs. Analyticity of the G-function near the point $t' = -\infty$, $t'' = 0$ imposes further restrictions on M [DZ2]. To solve the problem one is to study the behaviour of solutions of the RH b.v.p. in the limits when two or more among the canonical coordinates merge. At the point $t' = -\infty$, $t'' = 0$ all $u_1 = \ldots = u_n = 0$.

4. EXAMPLES OF MONODROMY DATA.

4.1. UNIVERSAL UNFOLDINGS OF ISOLATED SINGULARITIES. The subspace $M_0 \subset M$ consists of the parameters s for which the versal deformation $f_s(x)$ has $n = \dim M$ distinct critical values $u_1(s), \dots, u_n(s)$. These will be our canonical coordinates. The monodromy at the origin is the classical monodromy operator [AGV] of the singularity, the Stokes matrix coincides with the matrix of the variation operator in the Gabrielov's distinguished basis of vanishing cycles (see [AGV]; we may assume that $\dim x \equiv 1 \,(\mathrm{mod} 4)$).

4.2. Quantum cohomology of Fano varieties. The following two questions are to be answered in order to apply the above technique to the quantum cohomology of a variety X.

Problem 1. When does the generating series (2.4) converge?

Problem 2. For which X the quantum cohomology of X is semisimple?

Hopefully, in the semisimple case the convergence can be proved on the basis of the differential equations of n.3. To our opinion the problem 2 is more deep. A necessary condition to have a semisimple quantum cohomology is that X must be a Fano variety. It was conjectured to be also a sufficient condition [TX], [Man1]. We analyze below one example and suggest some more modest conjecture describing also a part of the monodromy data.

4.2.1. Quantum cohomology of projective spaces. For $X = \mathbf{P}^d$: (1) the monodromy at the origin is given by the bilinear form $< e_\alpha, e_\beta >= \delta_{\alpha+\beta,d+2}$ in $H^*(X) = \mathcal{V} = \mathrm{span}\,(e_1, \dots, e_{d+1})$, the matrix $\mu = 1/2\mathrm{diag}\,(-d, 1-d, \dots, d-1, d)$ and R is the matrix of multiplication by the first Chern class $R = R_1 = c_1(X)$, $Re_\alpha = (d+1)e_{\alpha+1}$ for $\alpha \leq d$, $Re_{d+1} = 0$. With obvious modifications these formulae work also for any variety X with $H^{\mathrm{odd}}(X) = 0$ (see [Du3]). (2) The Stokes matrix $S = (s_{ij})$ has the form

$$s_{ij} = \binom{d+1}{j-i} \text{ for } i \leq j, \quad s_{ij} = 0 \text{ for } i > j. \tag{4.1}$$

This form of Stokes matrix was conjectured in [CV], [Zas] but, to our knowledge, it was proved only in [Du5] for $d = 2$ and in [Guz] for any d. (3) The central connection matrix C has the form $C = C'C''$, $C' = \left(C'^{\alpha}_{\beta}\right)$, $C'' = \left(C''^{\beta}_{j}\right)$ where $C''^{\beta}_{j} = [2\pi i(j-1)]^{\beta-1}/(\alpha-1)!$, $j, \beta = 1, \dots, d+1$, $C'^{\alpha}_{\beta} = \frac{(-1)^{d+1}}{(2\pi)^{\frac{d+1}{2}} i^{\bar d}} \begin{cases} A_{\alpha-\beta}(d), & \alpha \geq \beta \\ 0, & \alpha < \beta \end{cases}$ with $\bar d = 1$ for d =even and $\bar d = 0$ for d =odd where the numbers $A_0(d) = 1$, $A_1(d)$, $\dots$, $A_d(d)$ are defined from the Laurent expansion for $x \to 0$: $1/x^{d+1} + A_1(d)/x^d + \dots + A_d(d)/x + O(1) = (-1)^{d+1}\Gamma^{d+1}(-x)e^{-\pi i \bar d x}$. Observe that (4.1) is the Gram matrix of the bilinear form $\chi(E, F) := \sum_k (-1)^k \dim Ext^k(E, F)$ in the basis given by a particular full system $E_j = \mathcal{O}(j-1)$, $j = 1, \dots, d+1$ of exceptional objects in the derived category $Der^b(Coh(\mathbf{P}^d))$ of coherent sheaves on $\mathbf{P}^d$ [Rud]. The columns of the matrix C'' are the components of the Chern character $\mathrm{ch}\,(E_j) = e^{2\pi i c_1(E_j)}$, $j = 1, \dots, d+1$. The geometrical meaning of the matrix C' remains unclear. In other charts of the FM S^σ and $C^\sigma = C'C''^\sigma$, $\sigma \in B_n$, have the same structure for another full system $E_1^\sigma, \dots, E_{d+1}^\sigma \in Der^b(Coh(\mathbf{P}^d))$ of exceptional objects, where the action of the braid group $(E_1, \dots, E_{d+1}) \mapsto (E_1^\sigma, \dots, E_{d+1}^\sigma)$ is described in [Rud]. Warning: the points of the FM corresponding to the restricted quantum cohomology [MM], where $t \in H^2(\mathbf{P}^d)$, do not belong to the chart $Fr(S, C)$ with the matrices S and C as above!

4.2.2. CONJECTURE. We say that a Fano variety X is *good* if $Der^b(Coh(X))$ admits, in the sense of [BP], a full system of exceptional objects $E_1, \ldots, E_n$, $n = \dim H^*(X)$. Our conjecture is that (1) the quantum cohomology of X is semisimple *iff* X is a good Fano variety; (2) the Stokes matrix $S = (s_{ij})$ is equal to $s_{ij} = \chi(E_i, E_j)$, $i, j = 1, \ldots, n$; (3) the central connection matrix has the form $C = C'C''$ when the columns of C'' are the components of $\mathrm{ch}\,(E_j) \in H^*(X)$ and $C' : H^*(X) \to H^*(X)$ is some operator satisfying $C'(c_1(X)a) = c_1(X)C'(a)$ for any $a \in H^*(X)$.

For $X = \mathbf{P}^d$ the validity of the conjecture follows from n.4.2.1 above. The conjecture probably can be derived from more general conjecture [Kon] about equivalence of $Der^b(Coh(X))$ to the Fukaya category of the mirror pair X^* of X. According to it (see also [EHX, Gi1, Gi2]) the basis of horizontal sections of $\tilde{\nabla}$ corresponding to the columns of $\Phi_{\mathrm{right}}(z;u)e^{zU}$ coincides with the oscillatory integrals of the Fukaya category of X^*. However, we do not know who is the first factor C' of the connection matrix in this general setting.

5. INTERSECTION FORM of a FM is a bilinear symmetric pairing on T^*M defined by $(\omega_1, \omega_2)|_t := i_{E(t)}(\omega_1 \cdot \omega_2)$, $\omega_1, \omega_2 \in T_t^*M$. *Discriminant* is the locus $\Sigma = \{t \in M \mid \det(\ ,\ )_t = 0\}$. On $M \setminus \Sigma$ the inverse to $(\ ,\ )$ determines a flat metric and, thus, a local isometry $\pi : \left(M \setminus \Sigma, (\ ,\ )^{-1}\right) \to \mathbf{C}^n$ where $\mathbf{C}^n$ is equipped with a constant complex Euclidean metric $(\ ,\ )_0$. This local isometry is called *period mapping* (our terminology copies that of the singularity theory where the geometrical structures with the same names live on the bases of universal unfoldings, see [AGV]). The image $\pi(\Sigma)$ is a collection of nonisotropic hyperplanes in $\mathbf{C}^n$. Multivaluedness of π is described by the *monodromy representation* $\pi_1(M \setminus \Sigma) \to Iso\,(\mathbf{C}^n, (\ ,\ )_0)$ (for $d \neq 1$ to the orthogonal group $O\,(\mathbf{C}^n, (\ ,\ )_0)$). The image $W(M)$ of the representation is called *monodromy group* of the FM M. In the semisimple case it is always an extension of a reflection group (see details in [Du5]). Our hope is that, for a semisimple FM M with good analytic properties, the monodromy group acts discretely in some domain $\Omega \subset \mathbf{C}^n$, and M is identified with a branched covering of the quotient $\Omega/W(M)$.

5.1. EXAMPLES of a FM with $W(M)$= finite irreducible Coxeter group W acting in $\mathbf{R}^n$ [Du3]. These are polynomial FMs, $M = \mathbf{C}^n/W$, constructed in terms of the theory of invariant polynomials of W. Conjecturally, all polynomial semisimple FMs are equivalent to the above and to their direct sums.

This construction was generalized in [DZ1] to certain extensions of affine Weyl groups and in [Ber] to Jacobi groups of the types A_n, B_n, G_2. For the quantum cohomology of $\mathbf{P}^2$ the monodromy group is isomorphic to $PSL_2(\mathbf{Z}) \times \{\pm 1\}$ [Du5].

6. MIRROR CONSTRUCTION represents certain system of deformed flat coordinates on a semisimple FM by oscillatory integrals $I_j(u;z) = \frac{1}{\sqrt{z}} \int_{Z_j} e^{z\lambda(p;u)}dp$, $\tilde{\nabla} I_j(u;z) = 0$, $j = 1, \ldots, n$ having the phase function $\lambda(p;u)$ depending on the parameters $u = (u_1, \ldots, u_n)$ defined on a certain family of open Riemann surfaces $\mathcal{R}_u \ni p$ realized as a finite-sheeted branched covering $\lambda : \mathcal{R}_u \to D \subset \mathbf{C}$ over a domain in the complex plain. The ramification points of $\mathcal{R}_u$, i.e., the critical values of the phase function, are $u_1, \ldots, u_n$. The 1-cycles $Z_1, \ldots, Z_n$ on $\mathcal{R}_u$ go to infinity in a way that guarantees the

convergence of the integrals. The function $\lambda(p;u)$ satisfies an important property: for any two critical points $p_i^{1,2} \in \mathcal{R}_u$ with the same critical value u_i the equality $d^2\lambda(p_i^1;u)/dp^2 = d^2\lambda(p_i^2;u)/dp^2$ must hold true. The metric $< \, , \, >$ and the trilinear form $c(a_1,a_2,a_3) :=< a_1 \cdot a_2, a_3 >$ are given by the residue formulae similar to (2.1), (2.2). The solutions $p = p(u;\lambda)$ of the equation $\lambda(p;u) = \lambda$ are the flat coordinates of the *flat pencil of the metrics* $(\, , \,) - \lambda < \, , \, >$ on T^*M [Du3-Du5].

For the case when generically there is a unique critical point p_i over u_i for each i and $\mathcal{R}_u$ can be compactfied to a Riemann surface of a finite genus g, we arrive at the Hurwitz spaces of branched coverings [Du1, Du3].

The construction of the Riemann surfaces $\mathcal{R}_u$, of the phase function $\lambda(p;u)$ and of the cycles $Z_1, \ldots, Z_n$ is given in [Du5] by universal formulae assuming $\det(S + S^T) \neq 0$. In the quantum cohomology of a d-fold X the last condition is valid for $d =$ even. For $d =$ odd one has $\det(S - S^T) \neq 0$. In this case one can represent the deformed flat coordinates by oscillatory integrals with the phase function $\lambda(p,q;u) = \nu(p;u) + q^2$ depending on two variables p, q. The details will be published elsewhere.

7. GRAVITATIONAL DESCENDENTS is a physical name for intersection numbers $< \tau_{m_1}(a_1) \otimes \ldots \otimes \tau_{m_l}(a_l) >$ of the pull-back cocycles $p_1^*(a_1), \ldots, p_l^*(a_l)$ with the Mumford - Morita - Miller cocycles $\psi_1^{m_1}, \ldots, \psi_l^{m_l} \in H^*(X_{[\beta],l})$ [W2], [DW], [KM]. We will describe first their genus $g = 0$ generating function $\mathcal{F}_0(T) = \sum_{[\beta]} \sum_l \langle e^{\sum_{\alpha=1}^n \sum_{p=0}^\infty T^{\alpha,p}\tau_p(e_\alpha)} \rangle_{[\beta],l,g=0}$. Here $T = (T^{\alpha,p})$ are indeterminates (the coordinates on the "big phase space", according to the physical terminology). This function has the form $\mathcal{F}_0(T) = 1/2 \sum \Omega_{\alpha,p;\,\beta,q}\left(t(T)\right) \tilde{T}^{\alpha,p}\tilde{T}^{\beta,q}$ where $\tilde{T}^{\alpha,p} = T^{\alpha,p}$ for $(\alpha,p) \neq (1,1)$, $\tilde{T}^{1,1} = T^{1,1} - 1$, the functions $\Omega_{\alpha,p;\,\beta,q}(t)$ on M are the coefficients of the expansion of the matrix valued function $\Omega_{\alpha\beta}(z,w;t) := (z+w)^{-1}\left[\left(\Phi_0^T(w;t)\Phi_0(z;t)\right)_{\alpha\beta} - \eta_{\alpha\beta}\right] = \sum_{p,\,q\geq 0} \Omega_{\alpha,p;\,\beta,q}(t) z^p w^q$, the vector function $t(T) = \left(t^1(T), \ldots, t^n(T)\right)$

$$t^\alpha(T) = T^{\alpha,0} + \sum_{q>0} T^{\beta,q} \nabla^\alpha \Omega_{\beta,q;\,1,0}(t)|_{t^\alpha = T^{\alpha,0}} + \ldots \tag{7.1}$$

is defined as the unique solution of the following fixed point equation $t = \nabla \sum_{\alpha,p} T^{\alpha,p}\Omega_{\alpha,p;\,1,0}(t)$.

The generating function $\mathcal{F}_1(T)$ of the genus $g = 1$ descendents has the form [DZ2], [DW], [Ge] $\mathcal{F}_1(T) = \left[G(t) + \frac{1}{24}\log\det M_{\alpha\beta}(t,\dot{t})\right]_{t=t(T),\, \dot{t}=\partial_{T^{1,0}}t(T)}$ where $G(t)$ is the G-function of the FM, the matrix $M_{\alpha\beta}(t,\dot{t})$ has the form $M_{\alpha\beta}(t,\dot{t}) = \partial_\alpha\partial_\beta\partial_\gamma F(t)\dot{t}^\gamma$, the vector function $t(T)$ is the same as above. The structure of the genus $g = 2$ corrections is still unclear, although there are some interesting conjectures [EX] related, in the case of quantum cohomology, to the Virasoro constraints for the full partition function

$$Z(T;\varepsilon) = \exp \sum_{g=0}^\infty \varepsilon^{2g-2}\mathcal{F}_g(T) \tag{7.2}$$

ε is a formal small parameter called *string coupling constant.*

8. Integrable hierarchies of PDEs of the KdV type and FMs. The idea that FMs may serve as moduli of integrable hierarchies of evolutionary equations (see [W2], [Du2], [Du3]) is based on

(1) the theorem of Kontsevich - Witten identifying the partition function (7.2) in the case $X =$ point as the tau-function of a particular solution of the KdV hierarchy.

(2) The construction [Du2, Du3] of bihamiltonian integrable hierarchy of the Whitham type $\partial_{T^{\alpha,p}} t = \left\{t(X), H_{\alpha,p}\right\}_1 = K^{(0)}_{\alpha,p}(t, t_X)$ (the vector function in the r.h.s. depends linearly on the derivatives t_X) such that the full genus zero partition function is the tau-function of a particular solution (7.1) to the hierarchy. The solution is specified by the symmetry constraint $t_X - \sum T^{\alpha,p}\partial_{T^{\alpha,p-1}} t = 1$. The phase space of the hierarchy is the loopspace $\mathcal{L}(M) = \left\{\left(t^1(X), \dots, t^n(X)\right) \mid X \in S^1\right\}$, the first Hamiltonian structure is $\left\{t^\alpha(X), t^\beta(Y)\right\}_1 = \eta^{\alpha\beta}\delta'(X - Y)$, the second one $\{\ ,\ \}_2$ is determined [Du3] by the flat metric $(\ ,\ )$ according to the general scheme of [DN]. The Hamiltonians are $H_{\alpha,p} = \int \Omega_{\alpha,p;\,1,0}(t)\, dX$. Actually, any linear combination $\{\ ,\ \}_2 - \lambda\{\ ,\ \}_1$ with an arbitrary λ is again a Poisson bracket on the loop space since $(\ ,\ )$ and $<\ ,\ >$ form *a flat pencil of metrics* on T^*M [Du3, Du4] (this *bihamiltonian property* is a manifestation of integrability of the hierarchy, see [Mag], [Du4]).

What we want to construct is a deformation of the hierarchy of the form $\partial_{T^{\alpha,p}} t = K^{(0)}_{\alpha,p}(t, t_X) + \sum_{g\geq 1} \varepsilon^{2g} K^{(g)}_{\alpha,p}(t, t_X, \dots, t^{(2g+1)})$ where $K^{(g)}_{\alpha,p}$ are some vector valued polynomials in $t_X, \dots, t^{(2g+1)}$ with the coefficients depending on $t \in M$. All the equations of the hierachy must commute pairwise. The full partition function must be the tau-function of a particular solution to the hierachy. The first $g = 1$ correction for an arbitrary semisimple FM was constructed in [DZ]. Its bihamiltonian structure is described, for $d \neq 1$, by a nonlinear deformation of the Virasoro algebra with the central charge $c = 6\varepsilon^2(1 - d)^{-2}[n - 4\mathrm{tr}\,\mu^2]$. For the FMs corresponding to the ADE Coxeter groups this formula gives the known result [FL] for the central charge of the classical W-algebra of the ADE-type $c = 12\varepsilon^2\rho^2$, where ρ is half of the sum of positive roots of the corresponding root system.

More recently it has been proved [DZ3] for a semisimple FM that the partition function (7.2) is annihilated, within the genus one approximation, by half of a Virasoro algebra described in terms of the monodromy data of the FM.

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