Higher Abel-Jacobi Maps

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For a smooth projective variety X, the structure of the Chow group $CH^p(X)$ representing codimension p algebraic cycles modulo rational equivalence, is still basically a mystery when $p > 1$, even for 0-cycles on a surface. For any p, one has the (rational) *cycle class map*

$$\psi_0 : CH^p(X) \otimes \mathbf{Q} \to \mathrm{Hdg}^p(X) \otimes \mathbf{Q} \subseteq H^{2p}(X, \mathbf{Q}),$$

conjecturally surjective by the *Hodge conjecture*. By the work of Griffiths, we have the (rational) *Abel-Jacobi map*

$$\psi_1 = \mathrm{AJ}^p_X : \ker(\psi_0) \to J^p(X) \otimes \mathbf{Q}.$$

A number of beautiful results have been proved using this invariant (e.g. [Gri]), but through the work of Mumford-Roitman ([Mu],[Ro]) it was realized that the kernel of ψ_1 can be infinite-dimensional (for 0-cycles on a surface with $H^{2,0}(X) \neq 0$), while through the work of Griffiths and Clemens the image of ψ_1 may fail to be surjective [Gri] or even finitely generated [Cl] (for 1-cycles on a general quintic 3-fold) or yet for not dissimilar geometric situations, by work of Voisin [Vo1] and myself [Gre1], the image of ψ_1 may be 0 (for 1-cycles on a general 3-fold of degree ≥ 6). At present, there is no explicit description, even conjecturally, for what $\ker(\psi_1)$ and $\mathrm{im}(\psi_1)$ look like. Eventually it came to be understood through the work of Beilinson, Bloch, Deligne, and Murre, among others (see [Ja] for a discussion) that there ought to be a filtration

$$CH^p(X) \otimes \mathbf{Q} = F^0 CH^p(X) \otimes \mathbf{Q} \supseteq F^1 CH^p(X) \otimes \mathbf{Q} \supseteq \cdots \supseteq F^{p+1} CH^p(X) \otimes \mathbf{Q} = 0$$

with

$$F^1 CH^p(X) \otimes \mathbf{Q} = \ker(\psi_0)$$

and

$$F^2 CH^p(X) \otimes \mathbf{Q} = \ker(\psi_1).$$

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This filtration has been constructed in some cases and various geometric candidates for it have been put forward, for example, by S. Saito [Sa] and Jannsen. One suggestion as to what the graded pieces of this filtration should look like is given by *Beilinson's conjectural formula* (see [Ja])

$$Gr^m CH^p(X) \otimes \mathbf{Q} \cong \mathrm{Ext}^m_{\mathcal{MM}}(1, h^{2p-m}(X)(p)),$$

where $\mathcal{MM}$ is the conjectural category of mixed motives.

One case that stands out as being well-understood is the case of the relative Chow group $CH^2(\mathbf{P}^2, T)$, where $T \subset \mathbf{P}^2$ is the triangle $z_0 z_1 z_2 = 0$, roughly described as 0-cycles on $\mathbf{P}^2 - T = \mathbf{C}^* \times \mathbf{C}^*$, modulo divisors of meromorphic functions f on curves $C \subset \mathbf{P}^2$ such that $f = 1$ on $C \cap T$. There is a series of maps

$$\begin{aligned} &\psi_0 \colon CH^2(\mathbf{P}^2, T) \to \mathbf{Z};\\ &\psi_1 \colon \ker(\psi_0) \to \mathbf{C}^* \oplus \mathbf{C}^*;\\ &\psi_2 \colon \ker(\psi_1) \to K_2(\mathbf{C}).\end{aligned}$$

Recall

$$K_2(\mathbf{C}) = \frac{\mathbf{C}^* \otimes_{\mathbf{Z}} \mathbf{C}^*}{\{\text{Steinberg relations}\}},$$

where the Steinberg relations are generated by $\{a \otimes (1-a) \mid a \in \mathbf{C} - \{0,1\}\}$. It is known (Bloch [Bl], Suslin [Su]) that these are all surjective and ψ_2 is an isomorphism. These all have simple algebraic descriptions—ψ_0 is degree; $\psi_1(a,b) = a \oplus b$; $\psi_2(a,b) = \{a,b\}$. The essential tool in proving this is the Suslin reciprocity theorem ([Su], see also [To]).

Another illustrative example (see [Gre2]) is $CH^2(\mathbf{P}^2, E)$, where E is a smooth plane cubic. Here we have a series of maps

$$\begin{aligned} &\psi_0 \colon CH^2(\mathbf{P}^2, E) \to \mathbf{Z};\\ &\psi_1 \colon \ker(\psi_0) \to 0;\\ &\psi_2 \colon \ker(\psi_1) \to \frac{\mathbf{C}^* \otimes_{\mathbf{Z}} E}{\tilde{\theta}(J^4)}.\end{aligned}$$

If $a, b \in \mathbf{P}^2 - E$, and L is the line through a and b, which meets E in $\{p_1, p_2, p_3\}$, then

$$\psi_2((a) - (b)) = \sum_{i=1}^{3} (\frac{a - p_i}{b - p_i} \otimes p_i) \in \mathbf{C}^* \otimes_{\mathbf{Z}} E.$$

Using $p_1 + p_2 + p_3 = 0$ on E (having taken 0 to be an inflection point), this has an alternative expression

$$\psi_2((a)-(b)) = \frac{(a-p_2)(b-p_1)}{(a-p_1)(b-p_2)} \otimes p_2 + \frac{(a-p_3)(b-p_1)}{(a-p_1)(b-p_3)} \otimes p_3,$$

which involves cross-ratios on L and is more clearly coordinate-independent. Once again, all three maps are surjective, and ψ_2 is an isomorphism. $\psi_0 = \deg$, $\psi_1 = 0$, inserted to preserve the pattern. To explain the notation in ψ_2, for $a \in E - \operatorname{div}(\theta)$, let

$$\tilde{\theta}(a) = \theta(a) \otimes a \in \mathbf{C}^* \otimes_{\mathbf{Z}} E.$$

Extend the definition of $\tilde{\theta}$ to $\mathbf{Z}_E$, the group ring of E, by linearity. In $\mathbf{Z}_E$, let J be the augmentation ideal $\{\sum_i n_i(a_i) \mid n_i \in \mathbf{Z}, a_i \in E, \sum_i n_i = 0\}$. Although $\tilde{\theta}(a)$ depends on the lifting of a to $\mathbf{C}$, on J^4 it does not depend on the choice of lifting of the elements. Thus $\tilde{\theta}(J^4)$ is well-defined and constitutes a generalization of the Steinberg relations; this group has been given a motivic intepretation by Goncharov and Levin [GL].

These examples provide a model for the general case—one should think of $(\mathbf{P}^2, T)$ and $(\mathbf{P}^2, E)$ as analogous to a complete surface with $h^{2,0} = 1$. For a general X, one has

$$\psi_0 \colon CH^2(X) \to \operatorname{Hdg}^2(X);$$
$$\psi_1 \colon \ker(\psi_1) \to J^2(X);$$

where ψ_0 is the cycle class map to the Hodge classes on X, and ψ_1 is the Abel-Jacobi map. We have constructed part of the missing map ψ_2 in the case of 0-cycles on a surface, using a construction that has the potential to work more generally.

The regulator map for a curve

$$X - D \xrightarrow{(f,g)} \mathbf{C}^* \times \mathbf{C}^*$$

is a homomorphism $r \colon \pi_1(X - D) \to \mathbf{C}/(2\pi i)^2\mathbf{Z} = \mathbf{C}/\mathbf{Z}(2)$ given by

$$r(\gamma) = \int_\gamma \log(f)\frac{dg}{g} - \log(g(p)) \int_\gamma \frac{df}{f},$$

where p is a base-point on γ; the answer does not depend on p. If $\gamma = \partial U$ for U a disc in $\mathbf{C}^* \times \mathbf{C}^*$, then

$$r(\gamma) = \int_U \frac{df}{f} \wedge \frac{dg}{g}.$$

This formula generalizes to a definition in the more general situation of a non-singular curve C and a map $f \colon C \to X$ to a smooth projective surface X. If

$$\mu \in \ker(H^2(X, \mathbf{Z}) \xrightarrow{f^*} H^2(C, \mathbf{Z})),$$

then $f^*\mu = dd^c g$ for $g \in A^0(C)$, unique up to adding a constant. If $\gamma \in \ker(H_1(C, \mathbf{Z}) \to H_1(X, \mathbf{Z}))$, so that $\gamma = \partial\Gamma$ in X, then we define

$$e_{X,C}(\mu, \gamma) = \int_\Gamma \mu - \int_\gamma f^*(d^c g) \in \mathbf{C}/\mathbf{Z},$$

which does not depend on any of the choices. These quantities are known as *membrane integrals*. More intrinsically, $e_{X,C}$ is the extension class of the extension of mixed Hodge structures (see [Ca])

$$0 \to \operatorname{coker}(H^1(X) \to H^1(C)) \to H^2(X, C) \to \ker(H^2(X) \to H^2(C)) \to 0.$$

Denote the term on the left $H^1(C)_{\rm new}$ and the term on the right $H^2(X)_C$; now

$$e_{X,C} \in \frac{\mathrm{Hom}_{\mathbf{C}}(H^2(X)_C, H^1(C)_{\rm new})}{\mathrm{Hom}_{\mathbf{Z}}(H^2(X)_C, H^1(C)_{\rm new}) + F^0\mathrm{Hom}_{\mathbf{C}}(H^2(X)_C, H^1(C)_{\rm new})}.$$

The class $e_{X,C}$ may also be obtained from the image under $AJ_{X\times C}$ of the graph of f minus some terms to make it homologous to 0 on $X \times C$.

We may write

$$H^2(X) = \ker(NS(X) \to H^2(C)) \oplus H^2(X)_{\rm tr},$$

which decomposes

$$e_{X,C} = (e_{X,C})_{\rm alg} \oplus (e_{X,C})_{\rm tr}.$$

The class $(e_{X,C})_{\rm alg}$ contains the same information as the map

$$\ker(NS(X) \to H^2(C)) \to \frac{J^1(C)}{\mathrm{Alb}(X)}$$

given by

$$L \mapsto f^*L.$$

If $Z \in Z^2(X)$ and $\psi_0(Z) = 0$, $\psi_1(Z) = 0$, then if we lift Z to $\tilde{Z} \in Z^1(C)$ such that $f_*\tilde{Z} = Z$ and $\deg(Z) = 0$ on each component of C, then $AJ_C(\tilde{Z})$ is represented by the extension class $e_{C,\tilde{Z}}$ of the extension of mixed Hodge structures

$$0 \to \mathrm{coker}(H^0(C) \to H^0(|\tilde{Z}|)) \to H^1(C, |\tilde{Z}|) \to H^1(C)_{\rm new} \to 0;$$

and the divisor $\tilde{Z}$ gives a map $\mathrm{coker}(H^0(C) \to H^0(|\tilde{Z}|)) \to 1$ and then

$$e_{C,\tilde{Z}} \in \frac{\mathrm{Hom}_{\mathbf{C}}(H^1(C)_{\rm new}, 1)}{\mathrm{Hom}_{\mathbf{Z}}(H^1(C)_{\rm new}, 1) + F^0\mathrm{Hom}_{\mathbf{C}}(H^1(C)_{\rm new}, 1)}.$$

The two extensions of MHS fit together to give a 2-step extension of MHS of $H^2(X)_{\rm tr}$ by 1, which unfortunately cannot be used directly. By standard identifications, we may think of

$$(e_{X,C})_{\rm tr} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(H^2(X)_{\rm tr}, H^1(C)_{\rm new})$$

and

$$e_{C,\tilde{Z}} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(H^1(C)_{\rm new}, 1).$$

The tensor product followed by contraction gives an element

$$e_{X,C,\tilde{Z}} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(H^2(X)_{\rm tr}, 1).$$

If we let $U_2^2(X) = \{e_{X,C,\tilde{Z}} \mid f_*\tilde{Z} = 0 \text{ as a } 0\text{−cycle on } X\}$ and

$$J_2^2(X) = \frac{(\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(H^2(X)_{\rm tr}, 1)}{U_2^2(X)},$$

then $Z \mapsto [e_{X,C,\tilde{Z}}]$ gives a well-defined invariant

$$\psi_2^2 \colon \ker(\psi_1) \to J_2^2(X)$$

that is independent of the choices of C and $\tilde{Z}$, and which depends only on the rational equivalence class of Z on X. It is necessary to allow reducible curves C. Claire Voisin [Vo2] has shown that for surfaces with $h^{2,0} \neq 0$, the map ψ_2 has infinite-dimensional image, and also that it need not be injective, so that our ψ_2^2 is only part of the story.

An explanation of the role played by the extension class $(e_{X,C})_{\mathrm{alg}}$ comes from *Beilinson's conjectural formula*:

$$Gr^m CH^p(X) \otimes \mathbf{Q} \cong \mathrm{Ext}^m_{\mathcal{MM}}(1, h^{2p-m}(X)(p)),$$

where $\mathcal{MM}$ is the conjectural category of mixed motives. The map

$$f_* \colon Gr^1 CH^1(C) \to Gr^1 CH^1(X)$$

is followed by a map

$$f_*^{+1} \colon \ker(f_*) \to Gr^2 CH^2(X).$$

In terms of Beilinson's formula, this is a map

$$\mathrm{Ext}^1_{\mathcal{MM}}(1, \ker(H^1(C) \to H^3(X))) \to \mathrm{Ext}^2_{\mathcal{MM}}(1, \mathrm{coker}(H^0(C) \to H^2(X)))$$

which (see [Ja]) factors through a map

$$f_*^{+1} \colon \mathrm{Ext}^1_{\mathcal{MM}}(1, \ker(H^1(C) \to H^3(X))) \to \mathrm{Ext}^2_{\mathcal{MM}}(1, H^2(X)_{\mathrm{tr}})).$$

It is reasonable to expect that it is given by Yoneda product with an element

$$e \in \mathrm{Ext}^1_{\mathcal{MM}}(\ker(H^1(C) \to H^3(X)), H^2(X)_{\mathrm{tr}}).$$

The philosophical point here is that e should come from

$$(e_{X,C})_{\mathrm{tr}} \in \mathrm{Ext}^1_{MHS}(\ker(H^1(C) \to H^3(X)), H^2(X)_{\mathrm{tr}}).$$

In fact, one would conjecture that the map

$$\ker(J^1(C) \to J^2(X)) \to CH^2(X)$$

is zero if and only if $(e_{X,C})_{\mathrm{tr}}$ is torsion—the only if direction has been shown [Gre2].

The question then becomes how to use $(e_{X,C})_{\mathrm{tr}}$. One answer is given by ψ_2^2 above. Another piece of the puzzle is to apply the arithmetic Gauss-Manin connection ∇ to $(e_{X,C})_{\mathrm{tr}}$.

An invariant which complements the one above was obtained in joint work with Phillip Griffiths [GG]. By work of Katz [Ka2] and Grothendieck [Gro], there is for any smooth projective variety X defined over $\mathbf{C}$ the arithmetic Gauss-Manin connection

$$\nabla_{X/\mathbf{Q}}: H^k(X,\mathbf{C}) \to \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} H^k(X,\mathbf{C}).$$

To capture this abstractly, we define an *arithmetic Hodge structure (AHS)* to be a complex vector space V with a finite descending filtration $F^\bullet V$ and a $\mathbf{Q}$-linear connection $\nabla: V \to \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} V$ satisfying $\nabla^2 = 0$ (flatness) and $\nabla F^p V \subseteq \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{p-1} V$ (Griffiths transversality) for all p.

A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of AHS (exact on each F^p) has extension class

$$e \in \mathrm{Ext}^1_{\mathrm{AHS}}(C,A) = H^1(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(C,A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(C,A)}).$$

To obtain this, let $\phi \in F^0\mathrm{Hom}_{\mathbf{C}}(C,B)$ be a lifting of g. Now

$$g \circ \nabla_{\mathrm{Hom}_{\mathbf{C}}(C,B)}\phi = 0,$$

so

$$\nabla_{\mathrm{Hom}_{\mathbf{C}}(C,B)}\phi = f \circ e$$

for a unique $e \in F^{-1}\mathrm{Hom}_{\mathbf{C}}(C,A)$. The class of e in

$$H^1(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(C,A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(C,A)})$$

is independent of the choice of ϕ.

A 2-step exact sequence $0 \to A \to B \to C \to D \to 0$ of AHS has a well-defined injective map from the Yoneda Ext

$$\mathrm{Ext}^2_{\mathrm{AHS}}(D,A) \to H^2(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(D,A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(D,A)}).$$

This is obtained by composing the extension class of the two 1-step extensions $0 \to A \to B \to E \to 0$, $0 \to E \to C \to D \to 0$ it breaks into, and then using the natural map

$$\begin{gathered} H^1(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(E,A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(E,A)}) \otimes_{\mathbf{C}} \\ H^1(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(D,E), \nabla_{\mathrm{Hom}_{\mathbf{C}}(D,E)}) \to \\ H^2(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(D,A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(D,A)}). \end{gathered}$$

This is exactly the obstruction to finding an AHS V with an additional increasing filtration $W_\bullet V$ by sub-AHS with $W_0 = 0$, $W_1 \cong A$, $W_2 \cong B$, $W_3 = V$, $V/W_1 \cong C$, and $V/W_2 \cong D$ and realizing the given 1-step extensions. The extension class is injective on $\mathrm{Ext}^2_{\mathrm{AHS}}(D,A)$, but it is not clear that all extension classes can occur.

This approach has much in common with the work of Carlson and Hain [CH].

This extension class theory fits in well with the pre-existing *arithmetic cycle class map* (see [EP] and work of Srinivas [Sr])

$$\eta: CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathbf{H}^{2p}(\Omega^{\geq p}_{X/\mathbf{Q}})$$

whose graded pieces are

$$\eta_m\colon \ker(\eta_{m-1}) \to H^m(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{p-\bullet}H^{2p-m}(X,\mathbf{C}), \nabla_{X/\mathbf{Q}});$$

one expects these to be consistent with the conjectural Bloch-Beilinson-Deligne-Murre filtration on $CH^p(X)\otimes_{\mathbf{Z}}\mathbf{Q}$. For 0-cycles on a surface, we are able to show that η_2 is the element of $\mathrm{Ext}^2_{\mathrm{AHS}}(H^2(X),1)$ coming from the 2-step extension of AHS using $Z\subset C\subset X$ analogous to the construction above in the mixed Hodge structure case. A parallel construction was found independently by Asakura and Saito [AS], who have used it for some interesting geometric applications.

I would like to close by listing a few open problems that particularly appeal to me and which seem especially relevant to the next phase of the study of algebraic cycles.

(i) Hodge-theoretic formula for $\nabla_{X/\mathbf{Q}}$

In cases of smooth projective varieties over $\mathbf{C}$ where Torelli's theorem holds (i.e. X is determined by Hodge-theoretic data on $H^*(X)$, at least theoretically $\nabla_{X/\mathbf{Q}}$ is determined on $H^i(X)$ by the Hodge structure of $H^i(X)$. It would be helpful to have a formula for this. Such a formula, involving Eisenstein series, was found by Katz [Ka1] for elliptic curves. For abelian varieties and K-3 surfaces, it would be very revealing to have a formula for $\nabla_{X/\mathbf{Q}}$. It would also be interesting to have an example where Torelli's theorem fails and $\nabla_{X/\mathbf{Q}}$ is different for two X's with the same Hodge structure; the alternative to this is the very attractive prospect that there is a general Hodge-theoretic formula for $\nabla_{X/\mathbf{Q}}$.

One facet of this question is the conjecture of Deligne (see [DMOS]), subordinate to the Hodge conjecture, that for X defined over $\mathbf{C}$, a Hodge class ξ necessarily satisfies

$$\nabla_{X/\mathbf{Q}}\xi = 0.$$

One possible "explanation" why this might be true is that a formula as alluded to above exists—such a formula would be expected to have the property that if $H^i(X)=H_1\oplus H_2$ as Hodge structures, then H_1, H_2 would be $\nabla_{X/\mathbf{Q}}$-stable.

A related question is to ask whether $Gr^mCH^p(X)\otimes\mathbf{Q}$ is determined by the Hodge structure of $H^{2p-m}(X)$, or whether one definitely needs further information contained in the motive $h^{2p-m}(X)$, e.g $\nabla_{X/\mathbf{Q}}$.

(ii) $Gr^2CH^2(A)$ for an abelian surface A

If $\mathbf{Z}_A$ is the group ring of A with augmentation ideal J, then

$$S^2_{\mathbf{Z}}A \cong \frac{J^2}{J^3}$$

maps surjectively to $Gr^2CH^2(A)$ by

$$a\otimes b \mapsto ((a)-(0)) * ((b)-(0)).$$

Thus

$$Gr^2CH^2(A) = \frac{S^2_{\mathbf{Z}}A}{U}$$

for some subgroup $U \subset S^2_{\mathbf{Z}}A$. Describe U in terms of the Hodge structure of A. For $A = E_1 \times E_2$ a product of elliptic curves,

$$Gr^2CH^2(E_1 \times E_2) = \frac{E_1 \otimes_{\mathbf{Z}} E_2}{U'}$$

for some subgroup $U' \subseteq E_1 \otimes E_2$. Somekawa has given a description of U', but not in explicit Hodge-theoretic terms. The subgroups U, U' may be thought of as generalized Steinberg relations, as in the example given earlier of $Gr^2CH^2(\mathbf{P}^2, E)$. This is an excellent test case.

(iii) Higher regulators for K-groups

The *Borel regulator map*

$$r: K_3(\mathbf{C})^{\text{ind}} = Gr_2K_3(\mathbf{C}) \to \mathbf{C}^*$$

is, by the work of Goncharov [Go] the Abel-Jacobi map

$$CH^2(\mathbf{P}^3, T_2)_{\text{hom}} \to J^2(\mathbf{P}^3, T_2),$$

where T_2 is the tetrahedron $\{z_0z_1z_2z_3 = 0\}$. One should think of $(\mathbf{P}^3, T_2)$ as the analogue of a 3-fold with trivial canonical bundle. Conjecturally, r is injective when tensored with $\mathbf{Q}$. Its image has the same qualitative properties that Clemens showed the image of AJ^2_X possesses for the general quintic 3-fold—zero-dimensional, but not finitely generated even over $\mathbf{Q}$. One should think of r as a "toy model" model for the Abel-Jacobi map for codimension 2 cycles, in much the same was as $K_2(\mathbf{C})$ is the toy model for $Gr^2CH^2(X)$ for a surface with $H^{2,0} \neq 0$. The toy model for the higher Abel-Jacobi maps

$$Gr^mCH^p(X) \otimes \mathbf{Q} \to J^p_m(X)$$

should be maps, injective when tensored with $\mathbf{Q}$,

$$r^p_m: Gr_pK_{2p-m}(\mathbf{C}) \to \frac{\otimes^m_{\mathbf{Z}}\mathbf{C}^*}{U^p_m}$$

for some subgroup $U^p_m \subset \otimes^m_{\mathbf{Z}}\mathbf{C}^*$. For $m = 1$ these are the Borel regulators, while for $m = p$ they are the isomorphisms to Milnor K-theory. Can these maps, or something like them, be constructed?

(iv) Explicit Suslin Reciprocity Theorem

The Suslin Reciprocity Theorem [Su], used for example to compute $CH^2(\mathbf{P}^2, T)$ above, gives the vanishing of certain elements of $\wedge^m_{\mathbf{Z}}\mathbf{C}^*$ in $K_m(\mathbf{C})$, but does not explicitly produce the elements of the Steinberg ideal that makes them vanish. On some level, these are produced by the proof, but the *transfer map* or *norm map* N is not geometrically explicit. For example, given a rational curve $Y \subset \mathbf{P}^2$ and $f \in \mathbf{C}(Y)$ such that $f|_{Y\cap T} = 1$, we know not only that $\mathrm{div}(f) \in Z_0(\mathbf{C}^* \times \mathbf{C}^*)$ maps to 1 in $K_2(\mathbf{C})$, but in fact if we map it to $\wedge^2_{\mathbf{Z}}\mathbf{C}^*$, if $\mathrm{div}(f) = \sum_i n_ip_i$, it maps to the product of the Steinberg symbols $(CR(p_i) \wedge (1 - CR(p_i)))^{n_i}$, where $CR(p_i)$ is the cross-ratio of p_i and one point each from the intersections of Y with each of the lines in T, with the product taken over all choices and all i (Goncharov,[Gre2]). For Y of higher genus, there is no comparably satisfying formula. In general, it would be nice to have as simple a version of N as possible for this type of geometric situation.

(v) Definition of $F^2CH^p(X) \otimes \mathbf{Q}$

Nori showed [No] that, for $p \geq 3$, $Z \equiv_{\mathrm{AJ}} 0$ does not imply $NZ \equiv_{\mathrm{alg}} 0$ for some $N > 0$. However, one might hope that for any p, $Z \equiv_{\mathrm{AJ}} 0$ implies that there exists a codimension 1 subvariety $Y \subset X$ such that $Z \subset Y$ and $NZ \equiv_{\mathrm{hom}} 0$ on Y for some $N > 0$. In many ways, a natural definition for $F^2CH^p(X) \otimes \mathbf{Q}$ is those Z such that both $NZ \equiv_{\mathrm{AJ}} 0$ and there exists a codimension 1 subvariety $Y \subset X$ such that $Z \subset Y$ and $NZ \equiv_{\mathrm{hom}} 0$ on Y for some $N > 0$. This conjecture would make the two plausible definitions of F^2 the same. It also fits in with what is needed to make the construction of ψ_2^2 go through for codimension 2 cycles in general. In particular, it would imply that for $p = 2$, $Z \equiv_{\mathrm{AJ}} 0$ implies $NZ \equiv_{\mathrm{alg}} 0$ for some $N > 0$.

The general conjecture would be that if $Z \in F^mCH^p(X) \otimes \mathbf{Q}$, then there exists a codimension 1 subvariety $Y \subset X$ such that $Z \subset Y$ and $Z \in F^{m-1}CH^{p-1}(Y) \otimes \mathbf{Q}$. This is what is needed to carry out the construction of higher Abel-Jacobi maps for arbitrary codimension.

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