p-Adic Analytic Spaces

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Abstract. This report is a review of results in p-adic analytic geometry based on a new notion of analytic spaces. I'll explain the definition of analytic spaces, basic ideas of étale cohomology for them, an application to a conjecture of Deligne on vanishing cycles, the homotopy description of certain analytic spaces, and a relation between the étale cohomology of an algebraic variety and the topological cohomology of the associated analytic space.

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§1. Introduction. At the beginning of the 1960's, J. Tate discovered p-adic uniformization of elliptic curves with totally degenerate reduction. This led him to introduce rigid analytic spaces in the framework of which the above uniformization actually takes place. Basics of rigid analytic geometry were developed by him in the paper [Ta] (released in 1961) and completed by R. Kiehl in [Ki1]-[Ki2] and L. Gerritzen and H. Grauert in [GG]. Rigid analytic spaces over a field k complete with respect to a non-trivial non-Archimedean valuation are glued from local objects, affinoid spaces, which are the maximal spectra of affinoid algebras, the algebras of topologically finite type over k. The natural topology on these spaces is totally disconnected, and one has to work with a certain Grothendieck topology instead. The framework of rigid analytic geometry enables one to construct an analog of the complex analytic theory of coherent sheaves and their cohomology, but does not allow a direct application of the intuitive idea of continuity and, in particular, of the homotopy and singular homology notions.

At the beginning of the 1970's, M. Raynaud introduced a new point of view to rigid analytic spaces. Namely, they can be considered as the generic fibres of formal schemes locally finitely presented over the ring of integers k° of k, and the category of quasi-compact quasi-separated rigid spaces is equivalent to the localization of the category of formal schemes finitely presented over k° with respect to the family of formal blow-ups (see [Ra], [BL1]-[BL2]). This provided additional algebraic tools to rigid analytic geometry, but did not make it more geometric.

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In 1986, I found that p-adic analytic spaces, to which the homotopy and singular homology notions can be directly applied, do exist. They are retrieved through spectra of affinoid algebras, where the spectrum is a generalization of the Gelfand spectrum of a complex commutative Banach algebra and, in general, is different from the space of maximal ideals. The new definition is simpler than that of rigid analytic spaces, does not require the use of Grothendieck topologies, works over fields with trivial valuation as well, and is in a sense a natural generalization of the definition of complex analytic spaces. The main advantage of the new analytic spaces is their nice topology which makes geometrical considerations relevant and useful over p-adic fields too.

In [Hu1]-[Hu4], R. Huber develops another approach to rigid analytic (and more general adic) spaces. It is based on a different notion of the spectrum of an affinoid algebra which coincides, a posteriori, with the space of points of the topos generated by the corresponding rigid affinoid space, and whose maximal Hausdorff quotient is the spectrum we consider. The relation between various approaches to p-adic analytic geometry is explained in simple terms at the end of §2.

§2. ANALYTIC SPACES. First of all, let $\mathcal{A}$ be a commutative Banach ring with unity. (Besides affinoid algebras we are going to consider, a good example is the ring of integers $\mathbf{Z}$ endowed with the absolute value $|\ |_\infty$.) The *spectrum* $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ is the set of all bounded multiplicative seminorms on $\mathcal{A}$, i.e., functions $|\ | : \mathcal{A} \to \mathbf{R}_+$ with $|1| = 1$, $|f+g| \le |f|+|g|$, $|fg| = |f|\cdot|g|$ and $|f| \le ||f||$. Each point $x \in \mathcal{M}(\mathcal{A})$ gives rise to a bounded character $\chi_x : \mathcal{A} \to \mathcal{H}(x)$, where $\mathcal{H}(x)$ is the completion of the fraction field of the quotient ring of $\mathcal{A}$ by the kernel of the corresponding seminorm. The image of an element $f \in \mathcal{A}$ under χ_x is denoted by $f(x)$. The spectrum $\mathcal{M}(\mathcal{A})$ is endowed with the weakest topology with respect to which all real valued functions of the form $x \mapsto |f(x)|$ are continuous. For example, if the algebra $\mathcal{A}$ contains the field of complex numbers $\mathbf{C}$ then, by Gelfand-Mazur's theorem, all of the fields $\mathcal{H}(x)$ coincide with $\mathbf{C}$ and, therefore, the spectrum $\mathcal{M}(\mathcal{A})$ is the Gelfand space of maximal ideals. A basic fact is that $\mathcal{M}(\mathcal{A})$ is always a non-empty compact space.

Let k be a non-Archimedean field, i.e., a field complete with respect to a non-Archimedean valuation which is not assumed to be non-trivial. Given positive numbers $r_1, \dots, r_n$, one sets $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} = \{f = \sum_\nu a_\nu T^\nu \big| |a_\nu| r^\nu \to 0$ as $|\nu| \to \infty\}$. It is a commutative Banach k-algebra with the norm $||f|| = \max |a_\nu| r^\nu$. A *$k$-affinoid algebra* is a commutative Banach k-algebra $\mathcal{A}$ for which there exists an epimorphism $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathcal{A}$ which is admissible in the sense that the norm on $\mathcal{A}$ is equivalent to the quotient norm. The algebras which are affinoid in the usual sense, i.e., for which such an epimorphism can be found with $r_i = 1$, $1 \le i \le n$, are said to be *strictly k-affinoid*. One shows that k-affinoid algebras are Noetherian, and all their ideals are closed. The category of *k-affinoid spaces* is, by definition, the category anti-equivalent to that of k-affinoid algebras (with bounded homomorphisms between them). To define global objects, k-analytic spaces, one uses the classical language of charts and atlases (which, by the way, can also be used to define schemes and formal schemes).

Given a k-affinoid space $X = \mathcal{M}(\mathcal{A})$, a closed subset $V \subset X$ is an *affinoid* *domain* if there exists a bounded homomorphism of k-affinoid algebras $\mathcal{A} \to \mathcal{A}_V$ such that the image of $\mathcal{M}(\mathcal{A}_V)$ in X is contained in V and any bounded homomorphism $\mathcal{A} \to \mathcal{B}$ with the same property, where $\mathcal{B}$ is a K-affinoid algebra for some bigger non-Archimedean field K, factors through a unique bounded homomorphism $\mathcal{A}_V \to \mathcal{B}$. One shows that $\mathcal{A}_V$ is flat over $\mathcal{A}$, $\mathcal{M}(\mathcal{A}_V) \stackrel{\sim}{\to} V$, $\mathcal{H}(x) \stackrel{\sim}{\to} \mathcal{H}_V(x)$ for any affinoid domain V that contains a point x, and affinoid neighborhoods of x form a fundamental system of its compact neighborhoods. Affinoid domains possess other nice properties (similar to those of open affine subschemes of affine schemes) which justify the following definitions.

A family τ of subsets of a topological space X is said to be a *quasi-net* if, for each point $x \in X$, there exist $V_1, \ldots, V_n \in \tau$ such that $x \in V_1 \cap \ldots \cap V_n$ and the set $V_1 \cup \ldots \cup V_n$ is a neighborhood of x. A quasi-net τ is said to be a *net* if, for any pair $U, V \in \tau$, the family $\tau\big|_{U \cap V}$ is a quasi-net on $U \cap V$.

Let X be a locally Hausdorff topological space, and let τ be a net of compact subsets on X. A *k-affinoid atlas $\mathcal{A}$ on X with the net τ* is a map which assigns, to each $V \in \tau$, a k-affinoid algebra $\mathcal{A}_V$ and a homeomorphism $V \stackrel{\sim}{\to} \mathcal{M}(\mathcal{A}_V)$ and, to each pair $U, V \in \tau$ with $U \subset V$, a bounded homomorphism of k-affinoid algebras $\mathcal{A}_V \to \mathcal{A}_U$ that identifies $(U, \mathcal{A}_U)$ with an affinoid domain in $(V, \mathcal{A}_U)$. A *k-analytic space* is a triple $(X, \mathcal{A}, \tau)$ of the above form. A *strong morphism* of k-analytic spaces $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is a pair which consists of a continuous map $\varphi : X \to X'$, such that for each $V \in \tau$ there exists $V' \in \tau'$ with $\varphi(V) \subset V'$, and of a system of compatible morphisms of k-affinoid spaces $\varphi_{V/V'} : (V, \mathcal{A}_V) \to (V', \mathcal{A}'_{V'})$ for all pairs $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subset V'$. One gets a category k-$\widetilde{\mathcal{A}n}$. Furthermore, a strong morphism $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is said to be a *quasi-isomorphism* if φ induces a homeomorphism between X and X' and, for any pair $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subset V'$, $\varphi_{V/V'}$ identifies V with an affinoid domain in V'. One shows that the family of quasi-isomorphisms admits calculus of right fractions. The *category of k-analytic spaces k-$\mathcal{A}n$* is the category of fractions of k-$\widetilde{\mathcal{A}n}$ with respect to the system of quasi-isomorphisms. If one assumes that all of the k-affinoid spaces used in the definition of k-$\mathcal{A}n$ are strictly k-affinoid, one gets the *category of strictly k-analytic spaces*. We mention several properties of k-analytic spaces.

(1) The functor $X = \mathcal{M}(\mathcal{A}) \mapsto (X, \mathcal{A}, \{X\})$ from the category of k-affinoid spaces to k-$\mathcal{A}n$ is fully faithful.

(2) Each k-analytic space X has a maximal k-affinoid atlas whose elements are called *affinoid domains* in X.

(3) A subset Y of a k-analytic space X is said to be an *analytic domain* if, for any point $y \in Y$, there exist affinoid domains $V_1, \ldots, V_n$ that are contained in Y and such that $y \in V_1 \cap \ldots \cap V_n$ and the set $V_1 \cup \ldots \cup V_n$ is a neighborhood of y in Y. An analytic domain Y has a natural structure of a k-analytic space, and the family of analytic domains gives rise to a Grothendieck topology on X, called the G*-topology*.

(4) The category k-$\mathcal{A}n$ admits fibre products and, for each non-Archimedean field K over k, there is the *ground field extension functor* $X \mapsto X \widehat{\otimes} K$.

(5) Given a point $x \in X$, there is an associated non-Archimedean field $\mathcal{H}(x)$ over k and, for for each morphism $\varphi : Y \to X$, there is a *fibre Y_x of φ at x* which

is an $\mathcal{H}(x)$-analytic space. The field $\mathcal{H}(x)$ is the completion (with respect to a valuation) of a field of transcendence degree at most $\dim(X)$ over k.

(6) For each morphism $\varphi : Y \to X$, one can define its *interior* $\mathrm{Int}(Y/X)$ and the *boundary* $\partial(Y/X) = Y \backslash \mathrm{Int}(Y/X)$ so that, if Y is an analytic domain in X, then $\mathrm{Int}(Y/X)$ coincides with the topological interior of Y in X. φ is said to be *closed* if $\partial(Y/X) = \emptyset$. φ is said to be *proper* if it is proper in the topological sense and closed in the above sense.

(7) Each point of a k-analytic space has a fundamental system of open neighborhoods which are locally compact, countable at infinity and arc-wise connected. The topological dimension of a paracompact k-analytic space is at most its dimension and, if the space is strictly k-analytic, both numbers are equal. The projective space and all its Zariski open subsets are contractible, and Tate's elliptic curve is homotopy equivalent to a circle (see also §5).

(8) One can associate with each scheme $\mathcal{X}$ of locally finite type over k a closed k-analytic space $\mathcal{X}^{\mathrm{an}}$. The scheme $\mathcal{X}$ is separated (resp. proper, resp. connected) if and only if the underlying topological space of $\mathcal{X}^{\mathrm{an}}$ is Hausdorff (resp. compact, resp. arc-wise connected) and, if $\mathcal{X}$ is separated, its dimension is equal to the topological dimension of $\mathcal{X}^{\mathrm{an}}$.

(9) Given a formal scheme $\mathfrak{X}$ locally finitely presented over k°, i.e., $\mathfrak{X}$ is a locally finite union of formal schemes of the form $\mathrm{Spf}(k^\circ\{T_1, \ldots, T_n\}/(f_1, \ldots, f_m))$, one can associate with it the generic fibre $\mathfrak{X}_\eta$, which is a paracompact strictly k-analytic space, and construct a reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$, where $\mathfrak{X}_s$ is the closed fibre of $\mathfrak{X}$.

(10) Assume that the valuation on k is non-trivial. For each Hausdorff strictly k-analytic space X, one can provide the subset $X_0 = \{x \in X \big| [\mathcal{H}(x) : k] < \infty\}$ with the structure of a rigid analytic space, and one can construct a morphism of topoi $X_0^\sim \to X^\sim$. The functor $X \mapsto X_0$ is fully faithful and induces an equivalence between the category of paracompact strictly k-analytic spaces and that of quasi-separated rigid analytic spaces which have an admissible affinoid covering of finite type. (Both categories contain all the spaces needed in practice.)

Remark. One can represent the relation between different approaches to p-adic analytic geometry in a metaphoric way on the model of real numbers as follows. In rigid analytic geometry, one does not know about the existence of irrational numbers, but is given functions on $\mathbf{Q}$ which are restrictions of continuous functions from $\mathbf{R}$. To work in such a situation one is led to provide $\mathbf{Q}$ with a Grothendieck topology (generated by the closed intervals with rational ends). In the approach of R. Huber (and essentially in that of M. Raynaud), one works with the space of points of the topos of sheaves in the above Grothendieck topology. In the approach described here, one works with the space of real numbers $\mathbf{R}$ itself.

§3. Étale cohomology for analytic spaces. The necessity of constructing étale cohomology theory for p-adic analytic spaces arose in V. Drinfeld's work ([Dr1], [Dr2]) for needs of problems related to the local Langlands conjecture (see the end of §4), and one of the main requirements was to extend étale cohomology theory of schemes. Such a theory was developed in [Ber2]. In this section we explain some basic ideas that use the nice topology of analytic spaces and make the whole theory easier than that for schemes.

Recall that the analog of étale cohomology for complex analytic spaces is the usual topological cohomology with coefficients in sheaves, and the reason is that an étale morphism between complex analytic spaces is a local isomorphism. The topological cohomology of k-analytic spaces cannot be a good analog of étale cohomology since, for example, the projective space is contractible, and so one may try to work with a class of étale morphisms which naturally generalizes that for complex analytic spaces and coincides with it over $\mathbf{C}$. Having in mind the nice topology of analytic spaces, one is easily led to the following definition.

A morphism of k-analytic spaces $\varphi : Y \to X$ is said to be *étale* if for each point $y \in Y$ there exist open neighborhoods $\mathcal{V}$ of y and $\mathcal{U}$ of $\varphi(y)$ such that φ induces a finite étale morphism $\mathcal{V} \to \mathcal{U}$. (The latter means that, for each affinoid domain $U = \mathcal{M}(\mathcal{A})$ in $\mathcal{U}$, the preimage $\varphi^{-1}(U) = \mathcal{M}(\mathcal{B})$ is an affinoid domain and $\mathcal{B}$ is a finite étale $\mathcal{A}$-algebra.) An important fact is that a morphism $\mathcal{Y} \to \mathcal{X}$ between schemes of locally finite type over k is étale if and only if the induced morphism $\mathcal{Y}^{\rm an} \to \mathcal{X}^{\rm an}$ is étale. Another important fact is the following. For a k-analytic space X and a point $x \in X$, let $\mathrm{Fét}(X, x)$ be the category of germs of k-analytic spaces finite étale over the germ of X at x, and let $\mathrm{Fét}(\mathcal{H}(x))$ be the category of schemes finite étale over the spectrum of $\mathcal{H}(x)$ (the latter is anti-equivalent to the category of finite separable $\mathcal{H}(x)$-algebras). The remarkable fact is that there is an equivalence of categories $\mathrm{Fét}(X, x) \xrightarrow{\sim} \mathrm{Fét}(\mathcal{H}(x))$. In other words, locally over the point x étale morphisms to X correspond to finite separable extensions of $\mathcal{H}(x)$. Notice that over $\mathbf{C}$ the latter means that an étale morphism is a local isomorphism.

The *étale topology* $X_{\rm ét}$ on a k-analytic space X is the Grothendieck topology on the category of étale morphisms $U \to X$ generated by the pretopology for which the set of coverings of $U \to X$ is formed by the families $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ with $U = \cup_{i \in I} f_i(U_i)$. This topology gives rise to the étale cohomology groups $H^q(X, F)$ with coefficients in an abelian étale sheaf F. A global section of F over X has the support which is a closed subset of X and, if X is Hausdorff, the étale cohomology groups with compact support $H^q_c(X, F)$ are defined as the right derived functors of the functor of global sections with compact support. In the same way one defines, for a Hausdorff morphism $\varphi : Y \to X$, the functors $F \mapsto R^q\varphi_! F$.

Consider the morphism of sites $\pi : X_{\rm ét} \to |X|$, where $|X|$ is the underlying topological space of X. The equivalence of categories $\mathrm{Fét}(X, x) \xrightarrow{\sim} \mathrm{Fét}(\mathcal{H}(x))$ easily implies that, for any abelian étale sheaf F, the stalk $(R^q\pi_* F)_x$ coincides with the cohomology group $H^q(G_{\mathcal{H}(x)}, F_x)$, where $G_{\mathcal{H}(x)}$ is the Galois group of $\mathcal{H}(x)$. Assume that F is torsion. By property (5) from §2, $H^q(G_{\mathcal{H}(x)}, F_x) = 0$ for q bigger than $\dim(X)$ plus the cohomological dimension of k. On the other hand, if X is paracompact, the topological dimension of X is at most $\dim(X)$. Thus, the spectral sequence of the morphism π implies that $H^q(X, F) = 0$ for q bigger than $2 \cdot \dim(X)$ plus the cohomological dimension of k. In a similar way, using properties of cohomology of topological spaces and of profinite groups one describes, for a Hausdorff morphism $\varphi : Y \to X$, the stalks of the sheaves $R^q\varphi_! F$ in terms of the cohomology groups with compact support of the fibres of φ. The proof of the corresponding fact for schemes is highly non-trivial.

Among results of [Ber2] (and [Ber4]) are the invariance of cohomology under algebraically closed extensions of the ground field, a Poincaré Duality theorem, a cohomological purity theorem, a base change theorem for cohomology with compact support, a smooth base change theorem, and comparison theorems. The latter state that, given a compactifiable morphism (resp. a morphism of finite type) $\varphi : \mathcal{Y} \to \mathcal{X}$ between schemes of locally finite type over k and an étale abelian sheaf $\mathcal{F}$ on $\mathcal{Y}$ which is torsion (resp. constructible with torsion orders prime to char(k)), there are canonical isomorphisms $(R^q\varphi_!\mathcal{F})^{\mathrm{an}} \tilde{\to} R^q\varphi_!^{\mathrm{an}}\mathcal{F}^{\mathrm{an}}$ (resp. $(R^q\varphi_*\mathcal{F})^{\mathrm{an}} \tilde{\to} R^q\varphi_*^{\mathrm{an}}\mathcal{F}^{\mathrm{an}}$).

By the way, the notion of a *smooth* morphism we work with is as follows. It is a morphism $Y \to X$ which factors locally through an étale morphism $Y \to X \times \mathbf{A}^d$. In [Ber3], we also proved that if k is algebraically closed and X is a compact quasi-algebraic k-analytic space, i.e., X is a finite union of affinoid domains isomorphic to affinoid domains in the analytification of a scheme, then for any integer n prime to char$(\widetilde{k})$, the characteristic of the residue field $\widetilde{k}$ of k, the cohomology groups $H^q(X, \mathbf{Z}/n\mathbf{Z})$ are finite.

In [Hu2]-[Hu4], R. Huber develops étale cohomology in the framework of his adic spaces. Besides the results mentioned above, he got finiteness results which imply, for example, that in the case, when k is of characteristic zero, the above fact is true without the assumption that X is quasi-algebraic.

§4. VANISHING CYCLES FOR FORMAL SCHEMES. In this section we describe an application of étale cohomology of analytic spaces to a conjecture of Deligne from [Del]. Let $\mathcal{X}$ be a scheme of finite type over a Henselian discrete valuation ring R, $\mathcal{Y}$ a subscheme of the closed fibre $\mathcal{X}_s$ of $\mathcal{X}$, l a prime different from the characteristic of the residue field of R. The conjecture states that (a) the restrictions of the vanishing cycles sheaves $R^q\Psi_\eta(\mathbf{Q}_l)$ of $\mathcal{X}$ to the subscheme $\mathcal{Y}$ depends only on the formal completion $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ of $\mathcal{X}$ along $\mathcal{Y}$ and, in particular, the automorphism group of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ acts on them, and (b) there exists an ideal of definition of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ such that any automorphism of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ trivial modulo this ideal acts trivially on the above sheaves.

Some partial results were obtained earlier by J.-L. Brylinski in [Bry] (the case when R is of mixed characteristic, $\mathcal{X}$ is of dimension one over R and $\mathcal{Y}$ is a closed point of $\mathcal{X}_s$), G. Laumon in [La] and the author in [Ber6] (the case when R is equicharacteristic and $\mathcal{Y}$ is a closed point of $\mathcal{X}_s$), and in [Ber3] (the case when $\mathcal{Y}$ is an open subscheme of $\mathcal{X}_s$). We describe here the results from [Ber7] which give a positive answer in the general case.

Let k be a field complete with respect to a discrete valuation (which is not assumed to be non-trivial). A formal scheme over k° is said to be *special* if it is a locally finite union of affine formal schemes of the form Spf(A), where A is a quotient of the adic ring $k^\circ\{T_1, \dots, T_n\}[[S_1, \dots, S_m]]$ by an ideal. (All ideals of that ring are closed in the adic topology.) Given a special formal scheme $\mathfrak{X}$, its *closed fibre* is the scheme of locally finite type over $\widetilde{k}$, $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$, where $\mathcal{J}$ is an ideal of definition of $\mathfrak{X}$ that contains the maximal ideal of k°. Due to P. Berthelot, one can associate with $\mathfrak{X}$ its *generic fibre* $\mathfrak{X}_\eta$, which is a paracompact strictly k-analytic space, and a *reduction map* $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$ so that, for any subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, there is a canonical isomorphism $(\mathfrak{X}_{/\mathcal{Y}})_\eta \widetilde{\to} \pi^{-1}(\mathcal{Y})$, where $\mathfrak{X}_{/\mathcal{Y}}$ is the formal completion of $\mathfrak{X}$ along $\mathcal{Y}$ (it is also a special formal scheme). In [Ber7] we constructed a *vanishing cycles functor* Ψ_η from the category of étale sheaves on $\mathfrak{X}_\eta$ to the category of étale sheaves on $\mathfrak{X}_{\overline{s}}$, where $\mathfrak{X}_{\overline{s}}$ is the lift of $\mathfrak{X}_s$ to the algebraic closure of $\widetilde{k}$, and proved the following results.

Theorem 1. *Given a scheme $\mathcal{X}$ of finite type over a local Henselian ring with the completion k°, a subscheme $\mathcal{Y} \subset \mathcal{X}_s$ and an étale abelian constructible sheaf $\mathcal{F}$ on $\mathcal{X}_\eta$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$, *there are canonical isomorphisms* $(R^q\Psi_\eta\mathcal{F})\big|_{\overline{\mathcal{Y}}} \widetilde{\to} R^q\Psi_\eta(\widehat{\mathcal{F}}_{/\mathcal{Y}})$, *where $\widehat{\mathcal{F}}_{/\mathcal{Y}}$ is the pullback of $\mathcal{F}$ on $(\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta$.*

In [Hu4], a similar result is proven for any special formal scheme (instead of $\mathcal{X}$) under the assumption that the characteristic of k is zero.

Theorem 1 gives a precise meaning to the part (a) of Deligne's conjecture and implies that, given a second scheme $\mathcal{X}'$ of finite type over k°, a subscheme $\mathcal{Y}' \subset \mathcal{X}'_s$ and an integer n prime $\mathrm{char}(\widetilde{k})$, any morphism of formal schemes $\varphi : \widehat{\mathcal{X}}'_{/\mathcal{Y}'} \to \widehat{\mathcal{X}}_{/\mathcal{Y}}$ induces a homomorphism $\theta^q_n(\varphi)$ from the pullback of $(R^q\Psi_\eta(\mathbf{Z}/n\mathbf{Z})_{\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}}$ to $(R^q\Psi_\eta(\mathbf{Z}/n\mathbf{Z})_{\mathcal{X}'_\eta})\big|_{\overline{\mathcal{Y}}'}$. In particular, given a prime l different from $\mathrm{char}(\widetilde{k})$, the automorphism group of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ acts on $(R^q\Psi_\eta(\mathbf{Q}_l)_{\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}}$.

Theorem 2. *(i) Given $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, $\widehat{\mathcal{X}}'_{/\mathcal{Y}'}$ and n as above, there exists an ideal of definition $\mathcal{J}'$ of $\widehat{\mathcal{X}}'_{/\mathcal{Y}'}$ such that for any pair of morphisms $\varphi, \psi : \widehat{\mathcal{X}}'_{/\mathcal{Y}'} \to \widehat{\mathcal{X}}_{/\mathcal{Y}}$, which coincide modulo $\mathcal{J}'$, one has $\theta^q_n(\varphi) = \theta^q_n(\psi)$.*

(ii) Given $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ and l as above, there exists an ideal of definition $\mathcal{J}$ of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ such that any automorphism of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, trivial modulo $\mathcal{J}$, acts trivially on $(R^q\Psi_\eta(\mathbf{Q}_l)_{\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}}$.

The proof of Theorem 2 uses a result from [Ber3] on the continuity of the action of a topological group on the étale cohomology groups of a k-analytic space if the original action of the group on the space is continuous.

The results from [Ber3] and [Ber7], described above, have been used by G. Faltings ([Fa]) and M. Harris ([Ha]) in their work on a conjecture of V. Drinfeld, and by M. Harris and R. Taylor ([HT]) in their work on the local Langlands conjecture over a p-adic field.

§5. The homotopy structure of analytic spaces. In this section we describe algebraic and homotopy topology results from [Ber8] obtained in an attempt to prove local contractibility of analytic spaces. To simplify the exposition, we do not formulate the results in the strongest possible form.

A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ between formal schemes locally finitely presented over k° is said to be *poly-stable* if locally in the étale topology it is of the form $\mathrm{Spf}(B_0 \widehat{\otimes}_A \dots \widehat{\otimes}_A B_p) \to \mathrm{Spf}(A)$, where each B_i is of the form $A\{T_0, \dots, T_n\}/(T_0 \cdot \ldots \cdot T_n - a)$ with $a \in A$. A *poly-stable fibration of length l over k°* is a sequence of poly-stable morphisms $\underline{\mathfrak{X}} = (\mathfrak{X}_l \to \mathfrak{X}_{l-1} \to \ldots \to \mathfrak{X}_1 \to \mathfrak{X}_0 = \mathrm{Spf}(k^\circ))$. Such objects form a category in the evident way. To take into account morphisms which are non-trivial on the ground field, we introduce a category $\mathcal{P}stf_l^{\text{ét}}$ whose objects are pairs $(k, \underline{\mathfrak{X}})$, where k is a non-Archimedean field and $\underline{\mathfrak{X}}$ is a poly-stable fibration of length l over k°, and morphisms $(K, \underline{\mathfrak{Y}}) \to (k, \underline{\mathfrak{X}})$ are pairs consisting of an isometric embedding of fields $k \hookrightarrow K$ and an étale morphism of poly-stable fibrations over K°, $\underline{\mathfrak{Y}} \to \underline{\mathfrak{X}}\widehat{\otimes}_{k^\circ} K^\circ$. (For brevity the pair $(k, \underline{\mathfrak{X}})$ is denoted by $\underline{\mathfrak{X}}$.)

Consider first the case when the valuation on k is trivial, i.e., all the formal schemes considered are in fact schemes of locally finite type over k. For such a (reduced) scheme $\mathcal{X}$, we set $\mathcal{X}^{(0)} = \mathcal{X}$ and, for $i \geq 0$, denote by $\mathcal{X}^{(i+1)}$ the non-normality locus of $\mathcal{X}^{(i)}$. The irreducible components of the locally closed subsets $\mathcal{X}^{(i)} \backslash \mathcal{X}^{(i+1)}$ are called *strata of* $\mathcal{X}$. One shows that, given a poly-stable fibration $\underline{\mathcal{X}} = (\mathcal{X}_l \to \ldots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathrm{Spec}(k))$, the closure of any stratum of the scheme $\mathcal{X}_l$ is a union of strata, and one associates with $\underline{\mathcal{X}}$ a simplicial set $C(\underline{\mathcal{X}})$ which encodes combinatorics of mutual inclusions between strata. (The construction of the latter is too involved to be given here, but in the case, when $\mathcal{X}_l$ is smooth and connected, $C(\underline{\mathcal{X}})$ is a point.) In this way one gets a functor C from $\mathcal{P}stf_l^{\text{ét}}$ to the category of simplicial sets that takes a poly-stable fibration $\underline{\mathfrak{X}}$ to the simplicial set $C(\underline{\mathfrak{X}}_s)$ associated with the closed fibre of $\underline{\mathfrak{X}}$. Its composition with the geometric realization functor gives a functor $|C|$ from $\mathcal{P}stf_l^{\text{ét}}$ to the category of locally compact spaces.

THEOREM 1. *For every poly-stable fibration $\underline{\mathfrak{X}} = (\mathfrak{X}_l \stackrel{f_{l-1}}{\to} \ldots \stackrel{f_1}{\to} \mathfrak{X}_1)$ of length l, one can construct a proper strong deformation retraction $\Phi : \mathfrak{X}_{l,\eta} \times [0, l] \to \mathfrak{X}_{l,\eta} : (x, t) \mapsto x_t$ of $\mathfrak{X}_{l,\eta}$ to a closed subset $S(\underline{\mathfrak{X}})$, the* skeleton of $\underline{\mathfrak{X}}$, *so that the following holds:*

(i) $(x_t)_{t'} = x_{\max(t,t')}$ for all $0 \leq t, t' \leq l$;

(ii) $f_{l-1,\eta}(x_t) = f_{l-1,\eta}(x)_{t-1}$ for all $1 \leq t \leq l$;

(iii) the homotopy Φ induces a strong deformation retraction of each Zariski open subset $\mathcal{U}$ of $\mathfrak{X}_{l,\eta}$ to $S(\underline{\mathfrak{X}}) \cap \mathcal{U}$; if $\mathfrak{X}_{l,\eta}$ is normal and $\mathcal{U}$ is dense, the intersection coincides with $S(\underline{\mathfrak{X}})$;

(iv) given a morphism $\varphi : \underline{\mathfrak{Y}} \to \underline{\mathfrak{X}}$ in $\mathcal{P}stf_l^{\text{ét}}$, one has $\varphi_{l,\eta}(y_t) = \varphi_{l,\eta}(y)_t$.

The latter property implies that the correspondence $\underline{\mathfrak{X}} \mapsto S(\underline{\mathfrak{X}})$ is a functor from $\mathcal{P}stf_l^{\text{ét}}$ to the category of locally compact spaces.

THEOREM 2. *There is a canonical isomorphism of functors $|C| \stackrel{\sim}{\to} S$.*

The simplest consequence of Theorems 1 and 2 tells that the analytification of any Zariski open subset of a proper scheme with good reduction is contractible. In the case of the Drinfeld upper half-plane Ω^d over a local non-Archimedean field K, which is the generic fibre of a formal scheme $\widehat{\Omega}^d$ (see [Dr2]), the space $|C(\widehat{\Omega}^d)|$ is the Bruhat-Tits building of the group $\mathrm{SL}_d(K)$. The embedding of the latter in Ω^d was used in [Ber5] in the proof of the fact that the group of analytic automorphisms of Ω^d coincides with $\mathrm{PGL}_d(K)$.

Theorems 1 and 2 and results of J. de Jong on alterations from [deJ2]-[deJ3] are used to prove the following results.

THEOREM 3. *Assume that the valuation on k is non-trivial. Let X be a k-analytic space* locally embeddable in a smooth space, *i.e., each point of X has* *an open neighborhood isomorphic to a strictly k-analytic domain in a smooth k-analytic space (for example, it is true if X is a smooth k-analytic space.) Then X is locally contractible.*

Let X be a separated connected k-analytic space locally embeddable in a smooth space. Theorem 3 implies that X has a universal covering, which is a strictly k-analytic space and is a Galois covering of X with the Galois group isomorphic to the fundamental group of the underlying topological space $|X|$. Furthermore, if X is paracompact, the cohomology groups $H^q(|X|, \mathbf{Z})$ (which are the same as those of the associated rigid analytic space) coincide with the singular cohomology groups.

Theorem 4. *Let $\mathcal{X}$ be a separated scheme of finite type over a non-Archimedean field k. Then*

(i) the groups $H^i(|\mathcal{X}^{\mathrm{an}}|, \mathbf{Z})$ are finitely generated;

(ii) there exists a finite separable extension k' of k such that for any non-Archimedean field K over k one has $H^i(|(\mathcal{X} \otimes k')^{\mathrm{an}}|, \mathbf{Z}) \xrightarrow{\sim} H^i(|(\mathcal{X} \otimes K)^{\mathrm{an}}|, \mathbf{Z})$.

§6. An analytic analog of Tate's conjecture over finite and local fields. This section is a report on the work in progress [Ber9]. Assume that k is a finite or a local non-Archimedean field. (Finite fields are considered as non-Archimedean ones endowed with the trivial valuation.) For a separated scheme $\mathcal{X}$ of finite type over k, we set $\overline{\mathcal{X}} = \mathcal{X} \otimes k^a$, where k^a is an algebraic closure of k, and denote by $\overline{\mathcal{X}}^{\mathrm{an}}$ the $\widehat{k^a}$-analytic space $(\mathcal{X} \widehat{\otimes} \widehat{k^a})^{\mathrm{an}}$. Let l be a prime different from $\mathrm{char}(k)$. The representation of the Galois group G of k^a on the l-adic étale cohomology groups $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ is continuous and, by Theorem 4 from §5, on the groups $H^i(|\overline{\mathcal{X}}^{\mathrm{an}}|, \mathbf{Z})$ is smooth in the sense that the stabilizer of any element is open in G.

The homomorphisms $H^i(|\overline{\mathcal{X}}^{\mathrm{an}}|, \mathbf{Z}) \to H^i(|\overline{\mathcal{X}}^{\mathrm{an}}|, \mathbf{Z}/l^n\mathbf{Z}) \to H^i(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Z}/l^n\mathbf{Z})$ and the isomorphism of the comparison theorem $H^i(\overline{\mathcal{X}}, \mathbf{Z}/l^n\mathbf{Z}) \xrightarrow{\sim} H^i(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Z}/l^n\mathbf{Z})$ give rise to a homomorphism $H^i(|\overline{\mathcal{X}}^{\mathrm{an}}|, \mathbf{Z}) \to H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$. Since it is Galois equivariant, its image is contained in $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)^{\mathrm{sm}}$, where for an l-adic representation V of G we denote by V^{sm} the subspace consisting of the elements with open stabilizer in G. The above homomorphism gives rise to a homomorphism $H^i(|\mathcal{X}^{\mathrm{an}}|, \mathbf{Z}) \to H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ whose image is contained in $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)^G$.

If k is a finite field, let F be the Frobenius automorphism of k^a. Otherwise, let F be a fixed element of G that lifts the Frobenius of the residue field of k. For an l-adic representation V of G, let V_μ denote the maximal F-invariant subspace of V, where all eigenvalues of F are roots of unity. One evidently has $V^{\mathrm{sm}} \subset V_\mu$.

Theorem. $H^i(|\overline{\mathcal{X}}^{\mathrm{an}}|, \mathbf{Z}) \otimes \mathbf{Q}_l \xrightarrow{\sim} H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)_\mu$.

The first corollary justifies the title of this section.

Corollary 1. $H^i(|\mathcal{X}^{\mathrm{an}}|, \mathbf{Z}) \otimes \mathbf{Q}_l \xrightarrow{\sim} H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)^G$.

Corollary 2. $H^i_c(|\overline{\mathcal{X}}^{\mathrm{an}}|, \mathbf{Z}) \otimes \mathbf{Q}_l \xrightarrow{\sim} H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)_\mu$ *and* $H^i_c(|\mathcal{X}^{\mathrm{an}}|, \mathbf{Z}) \otimes \mathbf{Q}_l \xrightarrow{\sim} H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)^G$.

Notice that the above results imply that $V_\mu = V^{\mathrm{sm}}$ for $V = H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ and $H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)$. Recall also that in the case of positive characteristic of k it is not yet known that the dimensions of the groups $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ and $H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)$ do not depend on l.

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