Operator Spaces and Similarity Problems

Gilles Pisier

Abstract. We present an overview of the theory of "Operator Spaces" (sometimes called "non-commutative Banach spaces"), recently developed by Effros, Ruan, Blecher, Paulsen and others. We describe several applications of this new ideology to operator algebras and to various similarity problems.

1991 Mathematics Subject Classification: 46L05, 47D15, 46M05, 46B70
Keywords and Phrases: Operator space, operator algebra, completely bounded, similar to a contraction, uniform algebra, amenable group

1 The Theory of Operator Spaces

The notion of "operator space" is intermediate between "Banach space" and "C^*-algebra". An operator space (o.s. in short) is simply a Banach space E (or a normed space before completion) given with an isometric embedding $j\colon\ E \to B(H)$ into the space $B(H)$ of bounded operators on some Hilbert space H. By a slight abuse, we will often identify E with $j(E)$. We can then say that an o.s. is simply a (closed) subspace of $B(H)$, or equivalently a (closed) subspace of a C^*-algebra (since, by Gelfand's theorem, C^*-algebras are themselves embedded into some $B(H)$).

Although this notion had appeared earlier, the theory itself really took off only after Z.J. Ruan's thesis [Ru1] circulated. His "abstract" characterization of operator spaces (see below) plays a crucial rôle to construct new operator spaces from known ones. In particular, immediately after this, Blecher-Paulsen [BP1] and Effros-Ruan [ER3] independently discovered that the latter characterization allows to introduce a duality in the category of operator spaces (see §1.4 below) and they developed the theory much further (cf. [ER2]–[ER6], [B1, B3, P2]).

The definition of operator spaces is a bit disappointing: every Banach space E embeds isometrically into $B(H)$ for a suitable H, hence every such space can be viewed (in at least one way, and actually in many) as an operator space. But the novelty is in the morphisms (or the isomorphisms) which are no longer those of the category of Banach spaces: instead of bounded linear maps, we use completely bounded (in short c.b.) ones, defined in §1.1 below. Those emerged as a powerful tool in the early 80's in works of Haagerup, Wittstock, Paulsen (see [P1]). Their definition was somewhat implicit in the earlier works of Stinespring (1955) and Arveson (1969) on completely positive maps.

Actually, the theory has been also considerably influenced by several contributions made before Ruan's thesis, as will be seen below. We should also mention that operator spaces were preceded by "operator systems" (these are self-adjoint operator spaces containing the unit): inspired by Kadison's ideas on "function systems" and by Arveson's extension theorem (1969), Choi and Effros developed in the 70's an extensive program to study operator systems with unital completely positive maps as morphisms. In particular, the ideas of duality and quotient spaces already appeared in this context (see [CE]). There, the additional order structure dims the parallelism with Banach spaces, but the overall influence of this program can still be seen throughout the theory.

One of the great advantages of operator spaces over C^*-algebras is that they allow the use of finite dimensional tools and isomorphic invariants (as in the so-called "local theory" of Banach spaces) in operator algebra theory (see §1.9 below): we can work with a distance $d_{cb}(E,F)$ which measures the degree of isomorphism of two isomorphic operator spaces E,F (see §1.1 below). In sharp contrast, C^*-algebras are much more rigid: there, all morphisms are *automatically* contractive, all isomorphisms are *isometric* and C^*-algebras have *unique* C^*-norms. As illustrated below, operator space theory has opened the door to a massive transfer of technology coming from Banach space theory. This process (the "quantization" of Banach space theory, according to the terminology in [E]) is bound to find applications for Banach spaces too. Up to now however, this has mostly benefitted operator algebra theory by leading to the solutions of some old problems (for instance, the Halmos similarity problem for polynomially bounded operators, see Example 2.1) while opening broad new directions of research, making many points of contact with other fields.

The main motivation for operator space theory is roughly this: very often, a C^*-algebra A comes equipped with a distinguished system of generators, sometimes finite. Call E the *linear* span of these generators. Then, while the normed space structure of E reveals little about A, it turns out that the operator space structure of E carries a lot of information about A, and the specific morphisms of o.s. theory allow to keep track of the correspondence $E \leftrightarrow A$. However, many constructions which are natural within operator spaces (such as duality or interpolation) do not make sense for C^*-algebras, yet the systematic investigation of properties of E leads to a "new" frame of mind, say a new intuition which ultimately can be applied to A. A good illustration of the fruitfulness of this approach is furnished by the main result in [JP]: by producing an uncountable collection of finite dimensional operator spaces (E_i) which are mutually separated, *i.e.* such that $\inf\{d_{cb}(E_i,E_j) \mid i \neq j\} > 1$, one obtains as a corollary that the tensor product $B(\ell_2)\otimes B(\ell_2)$ admits more than one C^*-norm, thus answering a long standing open question (see §1.10). This is a good case study: an investigation that the "new ideology" would surely pursue *for its own sake* (whether the set of finite dimensional operator spaces is separable), for which the best estimates turn out to depend on deep results of number theory ("Ramanujan graphs") and which happens to lead to the solution of a well known C^*-algebraic problem, a priori not involving operator spaces. Of course, it is the firm belief that more situations like this one will come up which keeps the field blooming.

Much of the research from the intensive development of the last ten years is surveyed below. However, we found the directions currently being explored too diverse to be all duly recorded here, for lack of space. For instance, the reader should consult other sources for an account of Effros and Ruan's work on quantum groups (see [ER8]) and Ruan's work on amenability and Kac algebras ([Ru3, Ru4]).

Notation. We denote by ℓ_2^n the n-dimensional complex Hilbert space. The space $B(\ell_2^n)$ can be identified with the space M_n of all $n \times n$ matrices with complex entries. Let H_1, H_2 be two Hilbert spaces. We denote by $H_1 \otimes_2 H_2$ their Hilbertian tensor product. We denote by $B(H_1, H_2)$ the space of all bounded operators $T\colon\ H_1 \to H_2$, equipped with its usual norm. When $H_1 = H_2 = H$ we denote it simply by $B(H)$. The same notation is used below when H_1, H_2 or H are Banach spaces. When T has $\|T\| \le 1$, we call it "contractive" and refer to it as "a contraction". Given two vector spaces V_1, V_2, we denote by $V_1 \otimes V_2$ their algebraic tensor product. All the vector spaces considered here are over the *complex* scalars. We will denote by $\overline{H}$ the complex conjugate of a (complex) Hilbert space H and by $h \to \overline{h}$ the canonical antilinear isometry from H to $\overline{H}$. We will use the abbreviation o.s. either for "operator space" or for "operator spaces" depending on the context.

1.1 The "norm" of an operator space. Complete boundedness

Let $E \subset B(H)$ be an operator space. Then $M_n \otimes E$ can be identified with the space of all $n \times n$ matrices with entries in E, which we will denote by $M_n(E)$. Clearly $M_n(E)$ can be viewed as an operator space naturally embedded into $B(H^n)$, where $H^n = H \oplus \cdots \oplus H$ (n times). Let us denote by $\|\ \ \|_n$ the norm of $M_n(E)$ (*i.e.* the norm induced by $B(H^n)$). Of course, when $n = 1$, we recover the ordinary norm of E. We have a natural embedding $M_n(E) \to M_{n+1}(E)$ taking x to $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with which we can view $M_n(E)$ as included in $M_{n+1}(E)$, and $\|\ \ \|_n$ as induced by $\|\ \ \|_{n+1}$. Thus, we may consider the union $\bigcup_n M_n(E)$ as a normed space equipped with its natural norm denoted by $\|\ \ \|_\infty$ and we denote by $\mathcal{K}[E]$ its completion. We also denote $\mathcal{K}_0 = \bigcup M_n$. Our notation here is motivated by the fact that if $E = \mathbb{C}$, the completion of $\mathcal{K}_0 = \bigcup M_n$ coincides isometrically with the C^*-algebra $\mathcal{K}$ of all compact operators on the Hilbert space ℓ_2. It is easy to check that the union $\bigcup_n M_n(E)$ can be identified isometrically with $\mathcal{K}_0 \otimes E$, and if we denote by $\{e_{ij}\}$ the classical system of matrix units in $\mathcal{K}$, then any matrix $x = (x_{ij})$ in $M_n(E)$ can be identified with $\sum_{ij=1}^n e_{ij} \otimes x_{ij} \in \mathcal{K} \otimes E \subset \mathcal{K}[E]$. The basic idea of o.s. theory is that the Banach space norm on E should be replaced by the sequence of norms $(\|\ \ \|_n)$ on the spaces $(M_n(E))$, or better by the single norm $\|\ \ \|_\infty$ on the space $\mathcal{K}[E]$ (we sometimes refer to the latter as the o.s.-norm of E), so that the unit ball of E should be replaced by that of $\mathcal{K}[E]$, as illustrated in the following.

Definition. Let $E_1 \subset B(H_1)$ and $E_2 \subset B(H_2)$ be operator spaces, let $u\colon\ E_1 \to E_2$ be a linear map, and let $u_n\colon\ M_n(E_1) \to M_n(E_2)$ be the mapping taking (x_{ij}) to $(u(x_{ij}))$. Then u is called completely bounded (c.b. in short) if $\sup_n \|u_n\| < \infty$ and we define $\|u\|_{cb} = \sup_n \|u_n\|$. Equivalently, u is c.b. iff the mappings u_n extend to a single bounded map $u_\infty\colon\ \mathcal{K}[E_1] \to \mathcal{K}[E_2]$ and we have $\|u\|_{cb} = \|u_\infty\|$. We denote by $cb(E_1, E_2)$ the space of all c.b. maps $u\colon\ E_1 \to E_2$, equipped with the cb-norm. Thus, the o.s. analog of the identity $\|u\| = \sup\{\|u(x)\| \mid \|x\| \le 1\}$ can be written as

$$\|u\|_{cb} = \sup\{\|u_\infty(x)\| \ \mid \ x \in \mathcal{K}[E], \ \|x\| \le 1\}.$$

Now that we have the "right" morphisms, of course we also have the isomorphisms: we say that two operator spaces E_1, E_2 are completely isomorphic (resp. completely isometric) if there is an isomorphism $u\colon\ E_1 \to E_2$ which is c.b. as well as its inverse (resp. and moreover such that $\|u\|_{cb} = \|u^{-1}\|_{cb} = 1$). We say that an isometry $u\colon\ E_1 \to E_2$ is a complete isometry if $\|u\|_{cb} = \|u^{-1}_{|u(E)}\|_{cb} = 1$. We say that u is a complete contraction (or is completely contractive) if $\|u\|_{cb} \le 1$. Note that the preceding properties correspond respectively to the cases when u_∞ is an isomorphism, an isometry or a contraction.

Let E_1, E_2 be two completely isomorphic operator spaces, we define

$$d_{cb}(E_1, E_2) = \inf\{\|u\|_{cb}\ \|u^{-1}\|_{cb}\}$$

where the infimum runs over all possible complete isomorphisms $u\colon\ E_1 \to E_2$. This is of course analogous to the "Banach-Mazur distance" between two Banach spaces E_1, E_2 defined classically by $d(E_1, E_2) = \inf\{\|u\|\ \|u^{-1}\|\}$, the infimum being this time over all isomorphisms $u\colon\ E_1 \to E_2$. By convention, we set $d(E_1, E_2) = \infty$ (or $d_{cb}(E_1, E_2) = \infty$) when no (complete) isomorphism exists.

We take this opportunity to correct a slight abuse in the definition of an operator space: consider two (isometric) embeddings $j_1\colon\ E \to B(H_1)$ and $j_2\colon\ E \to B(H_2)$ of the same Banach space into some $B(H)$. We will say (actually we rarely use this) that these are "equivalent" (or define "equivalent o.s. structures") if $j_2(j_1)^{-1}\colon\ j_1(E) \to j_2(E)$ is a complete isometry. Then, by an o.s. structure on a Banach space B what we really mean is an equivalence class with respect to this relation. As often, we will frequently abusively identify an equivalence class with one of its representative, *i.e.* with a "concrete" operator subspace $E \subset B(H)$.

Consider for instance a C^*-algebra A. Then any two isometric $*$-representations $j_1\colon\ A \to B(H_1)$ and $j_2\colon\ A \to B(H_2)$ are necessarily "equivalent" in the above sense. (Recall that C^*-algebras such as A and $M_n(A)$ have *unique* C^*-norms). We will call the resulting operator space structure on A the "natural" one, (this applies a fortiori to von Neumann algebras). Note that, throughout this text, whenever a C^*-algebra is viewed as an o.s., it *always* means in the "natural" way (unless explicitly stated otherwise).

We end this section by a brief review of the factorization properties of c.b. (or c.p.) maps. The following statement (due to Wittstock, Haagerup and Paulsen independently) plays a very important role throughout the theory.

Fundamental Factorization Theorem of c.b. maps. *For any c.b. map $u\colon\ E_1 \to E_2$ $(E_i \subset B(H_i), i = 1, 2)$ between operator spaces, there are a Hilbert space H, a $*$-representation $\pi\colon\ B(H_1) \to B(H)$ and operators $V\colon\ H \to H_2$ and $W\colon\ H_2 \to H$ with $\|V\|\ \|W\| \le \|u\|_{cb}$ such that, for any x in E_1, we have $u(x) = V\pi(x)W$.*

We say that $u\colon\ E_1 \to E_2$ is completely positive (c.p. in short) if for any n and any x in $M_n(E_1) \cap M_n(B(H_1))_+$ we have $u_n(x) \in M_n(E_2) \cap M_n(B(H_2))_+$. (Here

$M_n(B(H))_+$ denotes the positive cone of the C^*-algebra $M_n(B(H))$.) Actually, c.p. maps are of interest only when E_1 is a C^*-algebra or an operator system. When, say, E_1 is a C^*-algebra, $E_2 = B(H_2)$, then u is c.p. iff the above factorization actually holds with $V = W^*$ (Stinespring). In that case, it is known (Hadwin-Wittstock) that a map $u\colon\ E_1 \to B(H_2)$ is c.b. iff it is a linear combination of c.p. maps. Moreover, when $E_2 = B(H_2)$, any c.b. map $u\colon\ E_1 \to E_2$, defined on an arbitrary o.s. $E_1 \subset B(H_1)$, extends with the same c.b. norm to the whole of $B(H_1)$. This property of $B(H)$ plays the same role for o.s. as the Hahn-Banach extension theorem for Banach spaces. The o.s. which possess this extension property (like $E_2 = B(H_2)$ above) are called *injective*, they are all of the form $E = pAq$, where A is an injective C^*-algebra and p, q are two projections in A ([Ru2]), moreover (R. Smith, unpublished) when E is finite dimensional, A also can be chosen finite dimensional. In the isomorphic theory of Banach spaces, the separable injectivity of c_0 is classical (Sobczyk), and Zippin proved the deep fact that this characterizes c_0 up to *isomorphism*; the analogous o.s. questions are studied in [Ro]. Of course, there is a parallel notion of *projective* o.s. in terms of lifting property, see [B2, ER9] for more on this theme.

We refer the reader to [P1] for more information and for precise references on c.b. maps. See the last chapter in [Pi7] for the notion of p-complete boundedness in the case when H_1, H_2 are replaced by two Banach spaces; see also [LM1] for the multilinear case.

1.2 Minimal tensor product. Examples

Let $E_1 \subset B(H_1)$ and $E_2 \subset B(H_2)$ be two operator spaces. There is an obvious embedding $j\colon\ E_1 \otimes E_2 \to B(H_1 \otimes_2 H_2)$ characterized by the identity $j(x_1 \otimes x_2)(h_1 \otimes h_2) = x_1(h_1) \otimes x_2(h_2)$. We denote by $E_1 \otimes_{\min} E_2$ the completion of $E_1 \otimes E_2$ for the norm $x \to \|j(x)\|$. Clearly j extends to an isometric embedding, which allows us to view $E_1 \otimes_{\min} E_2$ as an operator space embedded into $B(H_1 \otimes_2 H_2)$. This is called the minimal (= spatial) tensor product of E_1 and E_2. For example, let $E \subset B(H)$ be an operator space. Then $M_n \otimes_{\min} E$ can be identified with the space $M_n(E)$, and $\mathcal{K}[E]$ can be identified isometrically with $\mathcal{K} \otimes_{\min} E$. Thus, for any linear map $u\colon\ E_1 \to E_2$, we have $\|u\|_{cb} = \|I \otimes u\colon\ \mathcal{K} \otimes_{\min} E_1 \to \mathcal{K} \otimes_{\min} E_2\| = \|I \otimes u\colon\ \mathcal{K} \otimes_{\min} E_1 \to \mathcal{K} \otimes_{\min} E_2\|_{cb}$. More generally, it can be shown that, for any operator space $F \subset B(K)$ (K Hilbert), we have $\|I_F \otimes u\colon\ F \otimes_{\min} E_1 \to F \otimes_{\min} E_2\| \le \|u\|_{cb}$. Consequently, if $v\colon\ F_1 \to F_2$ is another c.b. map between operator spaces, we have $\|v \otimes u\colon\ F_1 \otimes_{\min} E_1 \to F_2 \otimes_{\min} E_2\|_{cb} \le \|v\|_{cb}\|u\|_{cb}$. Thus c.b. maps can also be characterized as the ones which "tensorize" with respect to the minimal tensor product.

Remark. When E_1, E_2 are C^*-subalgebras in $B(H_1)$ and $B(H_2)$, then $E_1 \otimes_{\min} E_2$ is a C^*-subalgebra of $B(H_1 \otimes_2 H_2)$. By a classical theorem of Takesaki (see also §1.10 below), the norm $\|\quad\|_{\min}$ is the smallest C^*-norm on the tensor product of two C^*-algebras (and it does not depend on the particular realizations $E_i \subset B(H_i)$, $i = 1, 2$). For Banach spaces, Grothendieck [G] showed that the injective tensor product of two Banach spaces corresponds to the smallest reasonable tensor norm on $B_1 \otimes B_2$. The analogous result for operator spaces is proved in [BP1]: $E_1 \otimes_{\min} E_2$

is indeed characterized by a certain minimality among the "reasonable" operator space structures on $E_1 \otimes E_2$.

Just like in the C^*-case, the minimal tensor product is "*injective*" in the o.s. category: this means that, given o.s. E_1, E_2, if $F_i \subset E_i$ $(i = 1, 2)$ are further closed subspaces, then $F_1 \otimes_{\min} F_2$ can be identified with a closed subspace of $E_1 \otimes_{\min} E_2$. Moreover, this tensor product is "commutative" (this means that $E_1 \otimes_{\min} E_2$ can be identified with $E_2 \otimes_{\min} E_1$) and "*associative*" (this means that given E_i $i = 1, 2, 3$, we have

$$(E_1 \otimes_{\min} E_2) \otimes_{\min} E_3 \simeq E_1 \otimes_{\min} (E_2 \otimes_{\min} E_3)).$$

We will meet below several other tensor products enjoying these properties.

1.3 Ruan's theorem. Examples

It is customary to describe a Banach space before completion, simply as a vector space equipped with a norm. Ruan's theorem allows to take a similar viewpoint for operator spaces. Let V be a (complex) vector space and, for each $n \geq 1$, let $\| \quad \|_n$ be a norm on $M_n(V) = M_n \otimes V$. For convenience, if $x \in M_n(V)$, $a, b \in M_n$ we denote by $a \cdot x \cdot b$ the "matrix product" defined in the obvious way. Consider the following two properties:

$$(R_1) \quad \forall n \geq 1 \quad \forall a, b \in M_n\ \forall x \in M_n(V) \quad \|a \cdot x \cdot b\|_n \leq \|a\|_{M_n} \|x\|_n \|b\|_{M_n}$$

$$(R_2)\ \forall n, m \geq 1 \forall x \in M_n(V) \forall y \in M_m(V) \ \left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}.$$

It is easy to check that the sequence of norms associated to any operator space structure on V does satisfy this. We can now state Ruan's theorem, which is precisely the converse (a simplified proof appears in [ER5]).

Theorem ([Ru1]). *Let V be a complex vector space equipped with a sequence of norms $(\| \quad \|_n)_{n \geq 1}$, where, for each $n \geq 1$, $\| \quad \|_n$ is a norm on $M_n(V)$. Then this sequence of norms satisfies (R_1) and (R_2) iff there is a Hilbert space H and a linear embedding $j\colon\ V \to B(H)$ such that, for each $n \geq 1$, the map $I_{M_n} \otimes j\colon\ M_n(V) \to M_n(B(H))$ is isometric, in other words, iff the sequence $(\| \quad \|_n)$ "comes" from an operator space structure on V.*

Some examples. Let $C \subset B(\ell_2)$ and $R \subset B(\ell_2)$ be the "column" and "row" Hilbert spaces defined by $C = \overline{\text{span}}[e_{i1} \mid i \geq 1]$ and $R = \text{span}[e_{1j} \mid j \geq 1]$. Then, we have (completely isometrically) $\mathcal{K} \simeq C \otimes_{\min} R$. Moreover, the o.s.-norm for these two examples can be easily computed as follows: for any finitely supported sequence $(a_i)_{i \geq 1}$ of elements of $\mathcal{K}$ we have:

$$\left\| \sum a_i \otimes e_{i1} \right\|_{\mathcal{K}[C]} = \left\| \sum a_i^* a_i \right\|_{\mathcal{K}}^{1/2} \quad \text{and} \quad \left\| \sum a_j \otimes e_{1j} \right\|_{\mathcal{K}[R]} = \left\| \sum a_j a_j^* \right\|_{\mathcal{K}}^{1/2}.$$

Thus even though these spaces are clearly isometric (as Banach spaces) to ℓ_2, their (o.s. sense)-norm is quite different, and actually it can be shown that R and C are not completely isomorphic. More precisely, let $C_n = \text{span}[e_{i1} \mid 1 \leq i \leq n]$

and $R_n = \mathrm{span}[e_{1j} \mid 1 \le j \le n]$. Then it can be shown that $d_{cb}(R_n, C_n) = n$, which is the maximum value of $d_{cb}(E,F)$ over all pairs E, F of n-dimensional operator spaces (see §1.9 below). Thus R_n, C_n (although they are mutually isometric and isometric to ℓ_2^n) are "extremally" far apart as operator spaces. Some simple questions about them can be quite tricky. For instance, consider the direct sum $R \oplus C \subset B(\ell_2 \oplus \ell_2)$ (with the induced o.s. structure) and an operator subspace $E \subset R \oplus C$ such that there is a c.b. projection from $R \oplus C$ onto E. By [Oi], we have then $E \simeq E_1 \oplus E_2$ (completely isomorphically) with $E_1 \subset R$ and $E_2 \subset C$.

Another source of basic but very useful examples is given by the operator spaces $\min(B)$ and $\max(B)$ associated to a given Banach space B (cf. [BP1, P3]). These can be described as follows: consider the set of all norms α on $\mathcal{K}_0 \otimes B$ satisfying (R_1) and (R_2) and respecting the norm of B, *i.e.* such that $\alpha(e_{11} \otimes x) = \|x\|\ \forall x \in B$. Then this set admits a minimal element $\alpha_{\min}$ and a maximal one $\alpha_{\max}$, corresponding to the two o.s. $\min(B)$ and $\max(B)$. If B is given to us as an operator space, then $\min(B)$ or $\max(B)$ is the same Banach space but in general a different o.s. The space $\min(B)$ can be realized completely isometrically by any isometric embedding of B into a *commutative* C^*-algebra. While the spaces $\min(B)$ are rather simple, they explain why operator spaces are viewed as "non-commutative Banach spaces".

1.4 Duality. Quotient. Interpolation

Let $E \subset B(H)$ be an operator space. The dual E^* is a quotient of $B(H)^*$, so, a priori, it does not seem to be an o.s. However, it admits a very fruitful o.s. structure introduced (independently) in [BP1] and [ER3] as follows.

Let F be another operator space and let $V = cb(E,F)$. By identifying $M_n(V)$ with $cb(E, M_n(F))$ equipped with its c.b. norm, we obtain a sequence of norms satisfying (R_1) and (R_2). Therefore there is a specific operator space structure on $cb(E,F)$ for which the identification $M_n(cb(E,F)) = cb(E, M_n(F))$ becomes isometric for all $n \ge 1$. We call this the "natural" o.s. structure on $cb(E,F)$. In particular, when $F = \mathbb{C}$ we obtain an operator space structure on $E^* = cb(E, \mathbb{C})$ (it is easy to see that for any linear form $\xi \in E^*$ we have $\|\xi\| = \|\xi\|_{cb}$ and as mentioned above there is only *one* reasonable way to equip $\mathbb{C}$ with an operator space structure). Thus, the dual Banach space E^* is now equipped with an o.s. structure which we call the "dual o.s. structure" (the resulting o.s. is called the standard dual in [BP1]). It is characterized by the property that for any o.s. F, the natural mapping $u \to \tilde{u}$ from $F \otimes E^*$ (resp. $E^* \otimes F$) into $cb(E,F)$ defines an isometry from $F \otimes_{\min} E^*$ (resp. $E^* \otimes_{\min} F$) into $cb(E,F)$. When $\dim(F) < \infty$, this is onto, whence an isometric identity $F \otimes_{\min} E^* = cb(E,F) (= E^* \otimes_{\min} F)$. Note that, for any o.s. F and any $u\colon\ E \to F$ we have $\|u\|_{cb} = \|u^*\|_{cb}$. Moreover, the inclusion $E \subset E^{**} = (E^*)^*$ is completely isometric ([B2]) and E is the o.s. dual of an o.s. iff it admits a completely isometric "realization" as a weak-$*$ closed subspace of $B(H)$ (cf. [ER2, B2]). To illustrate this with some examples, we have completely isometric identities (cf. [BP1, ER4, B2]) $R^* \simeq C$, $C^* \simeq R$ and $\min(B)^* \simeq \max(B^*)$, $\max(B)^* \simeq \min(B^*)$ for any Banach space B.

Let M be a von Neumann algebra with predual M_*. The "natural" o.s. structure just defined on M^* induces a fortiori an o.s. structure on $M_* \subset M^*$ which, once more, we call the "natural" one. At this point, a problem of "coherence" of the various duals of M_* arises, but (fortunately) Blecher [B2] showed that everything "ticks": if we equip M_* with the o.s. structure just defined, its o.s. dual coincides completely isometrically with M equipped with its natural o.s. structure. In sharp contrast, this is no longer true for general operator spaces: Le Merdy (cf. [LM1]) has shown that there is an o.s. structure on $B(H)^*$ which is not the dual of any o.s. structure on $B(H)$.

The principle we just used to define the o.s. duality is valid in numerous other situations, such as quotients ([Ru1]) or interpolation spaces [Pi1]. Let $E_2 \subset E_1 \subset B(H)$ be operator spaces and let $\| \quad \|_n$ be the norm on $M_n(E_1/E_2)$ naturally associated to $M_n(E_1)/M_n(E_2)$ equipped with the quotient norm. Again it turns out that these norms verify (R_1) and (R_2), whence they yield an o.s. structure on E_1/E_2, characterized by the isometric identity $\mathcal{K}[E_1/E_2] = \mathcal{K}[E_1]/\mathcal{K}[E_2]$. We thus obtain a notion of quotient of operator spaces satisfying the usual rules of the Banach space duality, namely $(E_1/E_2)^* \simeq E_2^{\perp}$ and $E_2^* \simeq E_1^*/E_2^{\perp}$ (completely isometrically). We will say that a surjective linear map $u\colon\ E \to F$ is a complete surjection (resp. a complete metric surjection) if the associated map $E/\ker(u) \to F$ is a complete (resp. a completely isometric) isomorphism. Equivalently, that means that u^* is a completely isomorphic (resp. completely isometric) embedding of F^* into E^*.

We now turn briefly to the complex interpolation method, introduced for Banach spaces around 1960 by A. Calderón and J. L. Lions independently, cf. [BL]. Assume given a pair of operator spaces E_0, E_1 together with continuous linear injections $E_0 \to \mathcal{X}$, $E_1 \to \mathcal{X}$ into a topological vector space (actually a Banach space if we wish). Then, for any $0 < \theta < 1$, the complex interpolation method produces an "intermediate Banach space" $(E_0, E_1)_\theta$. Then again Ruan's theorem allows us to equip $(E_0, E_1)_\theta$ with an o.s. structure characterized by the isometric identity $\mathcal{K}[(E_0, E_1)_\theta] = (\mathcal{K}[E_0], \mathcal{K}[E_1])_\theta$. The fact that the functor of interpolation essentially commutes with duality, which is well known for Banach spaces, is extended to o.s. in [Pi1], but the proof requires rather delicate factorization properties of operator valued analytic functions.

1.5 Projective tensor product. Approximation property (OAP)

Since the minimal tensor product is the o.s. analog of Grothendieck's injective tensor product, it is tempting to look for the o.s. analog of the projective tensor product. This question is treated independently in [BP1] and [ER3]. Effros and Ruan pursued further: they introduced analogs of Grothendieck's approximation property ([ER2]), of integral or nuclear operators, of the Dvoretzky-Rogers theorem (characterizing finite dimensional spaces by the coincidence of unconditional and absolute convergence of series) and more. Their program meets several interesting obstacles (due mainly to the lack of local reflexivity, see §1.11 below), but roughly goes through (see [ER6, ER7]). For related work, see also [EWi] on "non-commutative convexity" and a paper by E. Effros and C. Webster in [Ka] devoted to "Operator analogues of locally convex spaces".

For lack of space, we refer to the original papers for precise definitions, and merely summarize the main results. Let us denote by $E_1 \otimes^\wedge E_2$ the o.s. version of the projective tensor product. Note that the norm of this o.s. is *different* from Grothendieck's projective tensor norm $\| \quad \|_\wedge$ and the Banach space projective tensor product $E_1 \widehat{\otimes} E_2$ is *not* the underlying Banach space to $E_1 \otimes^\wedge E_2$. Nevertheless, it is shown in [BP1] that, in some sense, this corresponds to the largest o.s.-norm on $\mathcal{K}_0 \otimes E_1 \otimes E_2$.
The projective operator space tensor product $E_1 \otimes^\wedge E_2$ is characterized by the isometric (actually completely isometric) identities $(E_1 \otimes^\wedge E_2)^* \simeq cb(E_1, E_2^*) \simeq cb(E_2, E_1^*)$. Moreover, the natural map $E_1 \otimes^\wedge E_2 \to E_1 \otimes_{\min} E_2$ is a complete contraction. The projective tensor product is commutative and associative, but in general not injective. However, it is, of course "projective", *i.e.* if $u_1\colon\ E_1 \to F_1$ and $u_2\colon\ E_2 \to F_2$ are complete metric surjections then $u_1 \otimes u_2$ also defines a complete metric surjection from $E_1 \otimes^\wedge E_2$ onto $F_1 \otimes^\wedge F_2$. Another important property from [ER2] is as follows: let M, N be two von Neumann algebras with preduals M_*, N_*. Let $M \overline{\otimes} N$ denote their von Neumann algebra tensor product. Then we have a completely isometric identity $(M \overline{\otimes} N)_* \simeq M_* \otimes^\wedge N_*$. This is a non-commutative analog of Grothendieck's classical isometric identity $L_1(\mu') \widehat{\otimes} L_1(\mu'') \simeq L_1(\mu' \times \mu'')$ relative to a pair of measure spaces $(\Omega', \mu'), (\Omega'', \mu'')$.

Following [ER2], an o.s. E is said to have the OAP if there is a net of finite rank (c.b.) maps $u_i\colon\ E \to E$ such that the net $I \otimes u_i$ converges pointwise to the identity on $\mathcal{K}[E]$. This is the o.s. analog of Grothendieck's approximation property (AP) for Banach spaces. When the net (u_i) is bounded in $cb(E, E)$, we say that E has the CBAP (this is analogous to the BAP for Banach spaces). To quote a sample result from [ER2]: E has the OAP iff the natural map $E^* \otimes^\wedge E \to E^* \otimes_{\min} E$ is injective. The class of groups G for which the reduced C^*-algebra of G has the OAP is studied in [HK] (see also §9 in [Ki1]). The ideas revolving around the OAP or the CBAP are likely to lead to a simpler and more conceptual proof of the main result of [Sz], but unfortunately this challenge has resisted all attempts so far.

1.6 The Haagerup tensor product

Curiously, the category of operator spaces admits a tensor product which (at least in the author's opinion) has no true counterpart for Banach spaces, namely the Haagerup tensor product introduced by Effros and Kishimoto (inspired by some unpublished work of Haagerup). But, while these authors originally considered only the resulting Banach space, it is the operator space case which turned out to be the most fruitful, through the fundamental works of Christensen and Sinclair [CS1] (see also [CS2]) and its extension by Paulsen and Smith [PS]. See also [BS] for the "weak-$*$ Haagerup tensor product" of dual o.s.

Let E_1, E_2 be two operator spaces. Consider $x_i \in \mathcal{K} \otimes E_i$ $(i = 1, 2)$. We denote by $(x_1, x_2) \to x_1 \odot x_2$ the bilinear form from $\mathcal{K} \otimes E_1 \times \mathcal{K} \otimes E_2$ to $\mathcal{K} \otimes (E_1 \otimes E_2)$ defined on elementary tensors by setting $(k_1 \otimes e_1) \odot (k_2 \otimes e_2) = (k_1 k_2) \otimes (e_1 \otimes e_2)$. We set $\alpha_i(x_i) = \|x_i\|_{\mathcal{K} \otimes_{\min} E_i}$ $(i = 1, 2)$. Then, for any $x \in \mathcal{K} \otimes E_1 \otimes E_2$, we define $\alpha_h(x) = \inf\{\alpha_1(x_1)\alpha_2(x_2)\}$, where the infimum runs over all possible decompositions of x of the form $x = x_1 \odot x_2$ with $x_1 \in \mathcal{K} \otimes E_1$, $x_2 \in \mathcal{K} \otimes E_2$. Once again it can be shown (by Ruan's theorem) that this defines an o.s. structure on $E_1 \otimes E_2$, so that we obtain, after completion, an operator space denoted by $E_1 \otimes_h E_2$ and called the Haagerup tensor product.

This definition can be extended to an arbitrary number of factors $E_1, E_2, \dots, E_N$ and the result is denoted by $E_1 \otimes_h \cdots \otimes_h E_N$. In [CES], the following very useful "realization" of $E_1 \otimes_h \cdots \otimes_h E_N$ is presented: assume E_i given as a subspace of a C^*-algebra A_i, then $E_1 \otimes_h \cdots \otimes_h E_N$ can be identified with a subspace of the (C^*-algebraic) "free product" $A_1 * A_2 * \cdots * A_N$. More precisely the linear mapping $j\colon\ E_1 \otimes_h \cdots \otimes_h E_N \to A_1 * \cdots * A_N$ defined by $j(x_1 \otimes \cdots \otimes x_N) = x_1 x_2 \dots x_N$ is a completely isometric embedding. This is closely related to the fundamental factorization of c.b. multilinear maps, obtained in [CS1] for C^*-algebras and in [PS] in full generality, as follows:

An N-linear map $\varphi\colon\ E_1 \times E_2 \times \cdots \times E_N \to B(H)$ defines a complete contraction from $E_1 \otimes_h \cdots \otimes_h E_N$ to $B(H)$ iff there are a Hilbert space $\widehat{H}$, completely contractive maps $\sigma_i\colon\ E_i \to B(\widehat{H})$ and operators $V\colon\ \widehat{H} \to\ H$ and $W\colon\ H \to \widehat{H}$ with $\|V\|\ \|W\| \le 1$ such that $\varphi(x_1, \dots, x_N) = V\sigma_1(x_1) \dots \sigma_N(x_N)W$.

The preceding result has many important applications notably to the Hochschild cohomology of operator algebras (see [E] [CES] and [SSm]).

The Haagerup tensor product enjoys unusually nice properties: it is associative, and *both* injective and projective (which is quite rare!), but it is *not* commutative: the spaces $E_1 \otimes_h E_2$ and $E_2 \otimes_h E_1$ can be very different. However, there is a symmetrized version of the Haagerup tensor product, introduced recently in [OiP] and denoted there by $E_1 \otimes_\mu E_2$, which has proved fruitful. For instance, in the situation of the preceding theorem, the paper [OiP] contains a characterization (up to a numerical factor when $N > 2$) of the N-linear maps $\varphi\colon\ E_1 \times \cdots \times E_N \to B(H)$ which admit a factorization as above but with the additional condition that the ranges of $\sigma_1, \dots, \sigma_N$ mutually commute.

Another very striking property of the Haagerup tensor product is its *self-duality* (which explains of course its being both injective and projective), for which we refer to [ER4] (according to [ER4], the first point below is due to Blecher):

Let E_1, E_2 be operator spaces. Then if E_1 and E_2 are finite dimensional we have $(E_1 \otimes_h E_2)^* \simeq E_1^* \otimes_h E_2^*$ completely isometrically. Moreover, in the general case we have a completely isometric embedding $E_1^* \otimes_h E_2^* \subset (E_1 \otimes_h E_2)^*$.

Here are sample results from [ER4] or [B1]. For every operator space E, we have a completely isometric isomorphism $M_n(E) \simeq C_n \otimes_h E \otimes_h R_n$ taking (x_{ij}) to $\sum e_{i1} \otimes x_{ij} \otimes e_{1j}$. In particular $C_n \otimes_h R_n \simeq M_n$ and $C \otimes_h R \simeq \mathcal{K}$, $R \otimes_h C \simeq \mathcal{K}^*$. If H is an arbitrary Hilbert space, let H_r and H_c be the o.s. defined by setting $H_r = B(\overline{H}, \mathbb{C})$ and $H_c = B(\mathbb{C}, H)$. Then if K is another Hilbert space, we have (completely isometrically) $H_c \otimes_h K_c = (H \otimes_2 K)_c$ and $H_r \otimes_h K_r = (H \otimes_2 K)_r$.

1.7 Characterizations of operator algebras and operator modules

In the Banach algebra literature, an operator algebra is defined as a closed subalgebra of $B(H)$, for some Hilbert space H, or equivalently a closed subalgebra of a C^*-algebra $C \subset B(H)$. When C is commutative, A is called a uniform algebra. Now consider an operator algebra $A \subset B(H)$ and let $I \subset A$ be a closed (twosided) ideal. Then, curiously, the quotient A/I is still an operator algebra (due to B. Cole for uniform algebras and to G. Lumer and A. Bernard in general): there is (for some suitable $\mathcal{H}$) an isometric homomorphism $j\colon\ A/I \to B(\mathcal{H})$. In the 70's several authors (Craw, Davie, Varopoulos, Charpentier, Tonge, Carne) tried to characterize operator algebras by certain continuity properties of the product map $p\colon\ A \otimes A \to A$. Although this chain of thoughts lead to a negative result (see [Ca]), it turns out that, in the operator space framework, the same things work! More precisely:

THEOREM ([BRS]). *Let A be a Banach algebra with a normalized unit element and equipped with an o.s. structure. Then the product map $p\colon\ A \otimes A \to A$ extends completely contractively to $A \otimes_h A$ iff there exists, for H suitable, a unital and completely isometric homomorphism $j\colon\ A \to B(H)$. Equivalently, this holds iff the natural matrix product $f.g$ of any two elements f, g in $\mathcal{K}[A]$ satisfies $\|f.g\| \le \|f\|\|g\|$. In other words, A is an operator algebra (completely isometrically) iff $\mathcal{K}[A]$ is a Banach algebra.*

Of course it is natural to wonder whether the mere complete *boundedness* of the product $p\colon\ A \otimes_h A \to\ A$ characterizes operator algebras up to complete *isomorphism*. This resisted for a few years, until Blecher [B3] proved that indeed this is true. The original proofs of [BRS, B3] did not use the earlier Cole-Lumer-Bernard results (and actually obtained them as corollaries), but it is also possible to go in the converse direction, with some extra work (see [Pi5]). We refer the reader to [LM2] for an extension of the Cole-Lumer-Bernard theorem to quotients of subalgebras of $B(X)$ when X is a Banach space, and to [BLM] and [LM5] for a detailed study of the operator algebra structures on ℓ_p, or the Schatten p-classes. See also [LM6] for a version of the above theorem adapted to dual operator algebras.

Operator spaces which are also modules over an operator algebra (in other words "operator modules") can also be characterized in a similar way (see [CES] and [ER1], see also [Ma] for dual modules) and suitably modified versions of the Haagerup tensor product are available for them. Operator modules play a central rôle in [BMP] where the foundations of a Morita theory for non self-adjoint operator algebras are laid. There Blecher, Muhly and Paulsen show that operator modules are an appropriate "metric" context for the C^*-algebraic theory of strong Morita equivalence, and the related theory of C^*-modules. For example, Rieffel's C^*-module tensor product is exactly the Haagerup module tensor product of the C^*-modules with their natural operator space structures. See [BMP], [B4], Blecher's survey in [Ka] and references contained therein for more on this.

1.8 THE OPERATOR HILBERT SPACE OH AND NON-COMMUTATIVE L_p-SPACES

Let us say that an operator space is Hilbertian if the underlying Banach space is isometric to a Hilbert space. Examples of this are in abundance, but apparently none of them is self-dual, which induces one to believe that operator spaces do not admit a true analog of Hilbert spaces. Therefore, the next result which contradicts this impression, comes somewhat as a surprise. (Notation: if E is an operator space, say $E \subset B(H)$, then $\overline{E}$ is the complex conjugate of E equipped with the o.s. structure corresponding to the embedding $\overline{E} \subset \overline{B(H)} = B(\overline{H})$.)

THEOREM ([Pi1]). *Let H be an arbitrary Hilbert space. There exists, for a suitable $\mathcal{H}$, a Hilbertian operator space $E_H \subset B(\mathcal{H})$ such that the canonical identification (derived from the scalar product) $E_H^* \to \overline{E}_H$ is completely isometric. Moreover, the space E_H is unique up to complete isometry. Let $(T_i)_{i\in I}$ be an orthonormal basis in E_H. Then, for any finitely supported family $(a_i)_{i\in I}$ in $\mathcal{K}$, we have*

$$\|\sum a_i \otimes T_i\|_{\mathcal{K}[E_H]} = \|\sum a_i \otimes \overline{a}_i\|_{\min}^{1/2}.$$

When $H = \ell_2$, we denote the space E_H by OH and we call it the "operator Hilbert space". Similarly, we denote it by OH_n when $H = \ell_2^n$ and by $OH(I)$ when $H = \ell_2(I)$. The preceding result suggests to systematically explore all the situations of Banach space theory where Hilbert space plays a central rôle (there are many!) and to investigate their analog for operator spaces. This program is pursued in [Pi1, Pi6]. The space OH has rather striking complex interpolation properties (see [Pi1]). For instance, we have completely isometric identities $(\min(\ell_2), \max(\ell_2))_{\frac{1}{2}} \simeq OH$ and $(R, C)_{\frac{1}{2}} \simeq OH$. (In the latter case, we should mention that the pair (R, C) is viewed as "compatible" using the transposition map $x \to {}^t x$ from R to C which allows to view both R and C as continuously injected into $\mathcal{X} = C$.) Concerning the Haagerup tensor product, for any sets I and J, we have a completely isometric identity $OH(I) \otimes_h OH(J) \simeq OH(I \times J)$.

Finally, we should mention that OH is "homogeneous" (an o.s. E is called homogeneous if any linear map $u\colon\ E \to E$ satisfies $\|u\| = \|u\|_{cb}$). While OH is unique, the class of homogeneous Hilbertian operator spaces (which also includes R, C, $\min(\ell_2)$ and $\max(\ell_2)$) is very rich and provides a very fruitful source of examples (see e.g. [Pi1, Pi6, Oi, Z]).

Since operator spaces behave well under interpolation (see §1.4), it is natural to investigate what happens to L_p-spaces, either scalar or vector valued. While in classical Lebesgue-Bochner theory, the Banach space valued L_p-spaces have been around for a long time, in the non-commutative case there seemed to be no systematic analogous "vector valued" theory. It turns out that operator spaces provide apparently the "right" framework for such a theory and a large part of [Pi2] tries to demonstrate it. Note however that the space of "values" E has to be an operator space, (not "only" a Banach space) and moreover we need to assume M hyperfinite for this theory to run "smoothly".

Many natural questions arise when one tries to "transfer" the Banach space theory of L_p-spaces to the o.s. framework. For instance, it is open whether OH embeds completely isomorphically into the predual of a von Neumann algebra (*i.e.* into a so-called "non-commutative L_1-space"). The natural candidates (either Gaussian variables, Rademacher functions or free semi-circular systems in Voiculescu's sense) span in L_1 (commutative or not) an operator space denoted by $R + C$ in [Pi1, Pi2] and extensively studied there. Note that, in sharp contrast to the Banach analogue, the o.s. spanned by the Rademacher functions in $L_p([0, 1])$ (meaning classical L_p with the "interpolated" o.s. structure) depends on p and it coincides with OH only when $p = 2$. Its dependence on p is entirely elucidated by F. Lust-Piquard's non-commutative Khintchine inequalities (see [Pi2]).

In another direction, very recently Marius Junge found a notion of "non-commutative p-stable process", which allowed him to prove that *if* $1 < p < 2$ *any space* $L_p(\varphi)$ *(relative to a von Neumann algebra* M *equipped with a faithful normal semi-finite trace* φ*) embeds isometrically into a non-commutative* L_1*-space.* This striking result was clearly inspired by o.s. considerations, even though the *completely* isomorphic version is still unclear.

1.9 Local theory. Exactness. Finite dimensional operator spaces

Let E, F be two Banach (resp. operator) spaces. Recall that their "distance" $d(E, F)$ (resp. $d_{cb}(E, F)$) has been defined in §1.1. These are not really distances in the usual sense, but we can replace them if we wish by $\delta(E, F) = \text{Log}\, d(E, F)$ (resp. $\delta_{cb}(E, F) = \text{Log}\, d_{cb}(E, F)$). Still however it is customary to use d and d_{cb} instead of δ and δ_{cb}. Let $n \geq 1$. Let OS_n (resp. B_n) be the set of all n-dimensional operator (resp. Banach) spaces, in which we agree to *identify* two spaces whenever they are completely isometric (resp. isometric). Then, it is an exercise to check that OS_n (resp. B_n) equipped with the distance δ_{cb} (resp. δ) is a complete metric space. In the Banach (= normed) space case, (B_n, δ) is even *compact*, this is the celebrated "Banach-Mazur compactum"! However, (OS_n, δ_{cb}) is not compact, and furthermore (in answer to a question of Kirchberg, see [Ki2]) it was proved in [JP] that it is *not separable* if $n > 2$ ($n = 2$ remains open). The paper [JP] actually gives three different approaches to this fact. The best asymptotic estimate uses Lubotzky-Phillips-Sarnak's work (see [Lu]) on "Ramanujan graphs". (This improvement over our two other approaches was pointed out by A. Valette, see his paper [Va] for more on this theme.) To state this estimate precisely, we need the following notation: let $\delta(n)$ be the infimum of the numbers $\varepsilon > 0$ such that (OS_n, δ_{cb}) admits a countable $\text{Log}(\varepsilon)$-net. Then, the non-separability of OS_3 means that $\delta(3) > 1$. Moreover, if $n = p + 1$ with p prime ≥ 3 (or p equal to a prime power, see [Va]), we have $\delta(n) \geq n(2\sqrt{n-1})^{-1} \geq \sqrt{n}/2$. On the other hand we have $\delta(n) \leq \sqrt{n}$ for all n. Indeed, it can be shown (see [Pi1]) that for any E in OS_n we have $d_{cb}(E, OH_n) \leq \sqrt{n}$, from which $\delta(n) \leq \sqrt{n}$ follows trivially. Note that the space OH_n appears thus as a "center" for (OS_n, δ_{cb}), in analogy with ℓ_2^n in the Banach space case. As a consequence we can estimate the "diameter" of OS_n: for any pair (E, F) in OS_n, we have $d_{cb}(E, F) \leq d_{cb}(E, OH_n) d_{cb}(OH_n, F) \leq n$. These estimates are optimal since $d_{cb}(R_n, OH_n) = d_{cb}(C_n, OH_n) = n^{1/2}$ and $d_{cb}(C_n, R_n) = n$. As in the "local theory" of Banach spaces (see *e.g.* [DJT]), these ideas can be used to study an infinite dimensional C^*-algebra through the collection of its finite dimensional subspaces. To illustrate this, let X be an o.s. For any (finite dimensional) operator space E, we define $d_{SX}(E) = \inf\{d_{cb}(E, F)\}$ where the infimum runs over all the subspaces $F \subset X$ isomorphic to E (and $d_{SX}(E) = \infty$, say, if there is no such F). In the Banach space case, if we take $X = c_0$ and replace d_{cb} by d, then the resulting number is equal to 1 for any E in $\bigcup_n B_n$. In sharp contrast, there is no *separable* o.s. X such that $d_{SX}(E) = 1$ for any E in OS_3, since this would contradict the non-separability of OS_3.

Various choices of X lead to interesting estimates of the "growth" of $d_{SX}(E)$. For instance, taking $X = \mathcal{K}$ we find, for any E in OS_n, $d_{S\mathcal{K}}(E) \leq \sqrt{n}$ (see Th.

9.6 in [Pi1]), but on the other hand if $E = \ell_1^n$ (= o.s. dual of ℓ_∞^n) equipped with its "natural" structure, we have $d_{S\mathcal{K}}(\ell_1^n) \geq a_n$ where $a_n = n(2\sqrt{n-1})^{-1/2} \geq \sqrt{n}/2$. We also have $n^{1/4} \geq d_{S\mathcal{K}}(OH_n) \geq (a_n)^{1/2}$, for all $n > 1$. Inspired by Kirchberg's results on the C^*-case (cf. [Ki1, Wa]), we study in [Pi4] the notion of "exact operator space": an o.s. $Y \subset B(H)$ is called exact if $\sup\{d_{S\mathcal{K}}(E) \mid E \subset Y, \dim(E) < \infty\} < \infty$. A C^*-algebra is exact in Kirchberg's sense iff it is exact in the preceding sense, so the reader can use this as the definition of an "exact C^*-algebra" (but actually Kirchberg proved that a C^*-algebra is exact iff it embeds into a nuclear one, see [Ki1]). Exact o.s. have surprisingly strong properties: *if E, F are both exact, then any c.b. map $u\colon E \to F^*$ factors boundedly through a Hilbert space* ([JP]). Although this is reminiscent of Grothendieck's classical factorization theorem, actually such a result has no Banach space counterpart!

Another very useful choice is $X = C^*(\mathbb{F}_\infty)$ the "full" C^*-algebra of the free group on countably infinitely many generators; for lack of space, we refer the reader to [JP] for more information on $d_{SX}(.)$ in this case.

1.10 Application to tensor products of C^*-algebras

Let A_1, A_2 be C^*-algebras. By classical results due respectively to Takesaki (1958) and Guichardet (1965), there is a minimal C^*-norm and a maximal one, denoted respectively by $\|\ \|_{\min}$ and $\|\ \|_{\max}$ on $A_1 \otimes A_2$. The resulting C^*-algebras (after completion) are denoted respectively by $A_1 \otimes_{\min} A_2$ and $A_1 \otimes_{\max} A_2$. Thus, the tensor product $A_1 \otimes A_2$ admits a unique C^*-norm iff $A_1 \otimes_{\min} A_2 = A_1 \otimes_{\max} A_2$. (Note: this holds for all A_2 iff A_1 is nuclear, or iff A_1^{**} is injective, see [CE] for precise references.) Kirchberg's work [Ki2] highlights pairs A_1, A_2 satisfying this unicity. In particular, he proved this holds if $A_1 = B(\ell_2)$ and $A_2 = C^*(\mathbb{F}_\infty)$ (see [Pi3] for a simple proof using o.s. theory). However, the results of the preceding section imply that this does *not* hold when $A_1 = A_2 = B(\ell_2)$ (see [JP]), thus answering a long standing open question. Here is a brief sketch: let $(E_i)_{i\in I}$ be a family of n-dimensional operator spaces and let $u_i \in E_i^* \otimes E_i$ be associated to the identity map I_i on E_i. Using the dual o.s. structure on E_i^*, we have embeddings $E_i \subset B(\ell_2)$, $E_i^* \subset B(\ell_2)$ so that we may consider u_i as an element of $B(\ell_2) \otimes B(\ell_2)$ and (by definition of the o.s. structure of E_i^*) we have $\|u_i\|_{\min} = \|I_i\|_{cb} = 1\ \forall i \in I$. Then, (see [JP] for a proof) if $\|u_i\|_{\max} = \|u_i\|_{\min}\ \forall i \in I$, the family $\{E_i \mid i \in I\}$ is necessarily separable in (OS_n, δ_{cb}). Thus the non-separability of (say) OS_3 (see the preceding section) implies $B(\ell_2) \otimes_{\min} B(\ell_2) \neq B(\ell_2) \otimes_{\max} B(\ell_2)$. More precisely, let $\lambda(n) = \sup\{\|u\|_{\max}\}$ where the supremum runs over all $u \in B(\ell_2) \otimes B(\ell_2)$ with $\|u\|_{\min} = 1$ and rank $\leq n$. Then, the same idea (see [JP]) leads to $\delta(n) \leq \lambda(n) \leq \sqrt{n}$ for all $n \geq 1$, hence, by the estimates of $\delta(n)$ given in §1.9, $\lambda(n)$ grows like $\sqrt{n}$ (up to a constant factor) when $n \to \infty$.

In sharp contrast, the question whether there is a unique C^*-norm on $A_1 \otimes A_2$ when $A_1 = A_2 = C^*(\mathbb{F}_\infty)$ remains an outstanding open problem, equivalent to a number of fundamental questions, for instance this holds iff every separable II_1-factor embeds in a (von Neumann) ultraproduct of the hyperfinite II_1 factor or equivalently iff every non-commutative L_1-space is finitely representable (see below for the definition) in the Banach space of all trace class operators on ℓ_2.

(See the fascinating discussion at the end of [Ki2].)

Let X, Y be Banach spaces. We say that Y is finitely representable in X if for every $\epsilon > 0$ and every finite dimensional subspace $E \subset Y$ there is a finite dimensional subspace $F \subset X$ such that $d(E, F) < 1 + \epsilon$. This notion was used extensively by R. C. James in his theory of "super-reflexivity" (see *e.g.* [DJT]), (but actually Grothendieck already considered it explicitly in the appendix to his famous "Résumé", see [G] page 108-109; his terminology was "Y a un type métrique inférieur à celui de X"). Of course, this immediately extends to the o.s. setting: when X, Y are o.s. we say that Y is o.s.-finitely representable in X if the preceding property holds with $d_{cb}(E, F)$ instead of $d(E, F)$. Equivalently, we have $d_{SX}(E) = 1$ for any finite dimensional $E \subset Y$.

1.11 Local reflexivity

In Banach space theory, the "principle of local reflexivity" says that every Banach space B satisfies $B(F, B)^{**} = B(F, B^{**})$ isometrically for any finite dimensional (normed) space F. Consequently, B^{**} is always finitely representable in B. This useful principle goes back to Lindenstrauss-Rosenthal with roots in Grothendieck's and Schatten's early work (see [DJT] p. 178 and references there). Similarly, an o.s. E is called "locally reflexive" if we have $cb(F, E)^{**} = cb(F, E^{**})$ isometrically for any finite dimensional o.s. F (and when this holds for all F, it actually holds completely isometrically). This property was "exported" first to C^*-algebra theory by Archbold-Batty, then for operator spaces in [EH]. As the reader can guess, not every o.s. is locally reflexive, so the "principle" now fails to be universal: as shown in [EH], $C^*(\mathbb{F}_\infty)$ is not locally reflexive. Local reflexivity passes to subspaces (but not to quotients) and is trivially satisfied by all *reflexive* o.s. (a puzzling fact since reflexivity is a property of the underlying Banach space only!). It is known that all nuclear C^*-algebras are locally reflexive (essentially due to Archbold-Batty, see [EH]). More generally, by Kirchberg's results, exactness $\Rightarrow$ local reflexivity for C^*-algebras (see [Ki1] or [Wa]), but the converse remains open. Actually, it might be true that exact $\Rightarrow$ locally reflexive for all o.s. but the converse is certainly false since there are reflexive but non-exact o.s. (such as OH). All this shows that local reflexivity is a rather rare property. Therefore, it came as a big surprise (at least to the author) when, in 97, Effros, Junge and Ruan [EJR] managed to prove that every predual of a von Neumann algebra (a fortiori the dual of any C^*-algebra) is locally reflexive. This striking result is proved using a non standard application of Kaplansky's classical density theorem, together with a careful comparison of the various notions of "integral operators" relevant to o.s. theory (see a very recent preprint by M. Junge and C. Le Merdy for an alternate proof). Actually, [EJR] contains a remarkable strengthening: for any von Neumann algebra M, the dual $M^* = (M_*)^{**}$ is o.s.-finitely representable in M_*. This is already nontrivial when $M = B(H)$!

2 Similarity problems

Let A, B be unital Banach algebras. By a "morphism" $u\colon\ A \to B$, we mean a unital homomorphism (*i.e.* u is a linear map satisfying $u(1) = 1$ and $u(xy) = u(x)u(y)$ for all x, y in A). Note that, since $u(1) = 1, u$ contractive means here $\|u\| = 1$ (and of course u bounded means $1 \leq \|u\| < \infty$). We will be concerned mainly with the case $B = B(H)$ with H Hilbert. We then say that u is similar to a contractive morphism (in short s.c.) if there is an invertible operator $\xi\colon\ H \to H$ such that the "conjugate" morphism u_ξ defined by $u_\xi(x) = \xi^{-1}u(x)\xi$ is contractive. Moreover, we denote $\mathrm{Sim}(u) = \inf\{\|\xi\|\ \|\xi^{-1}\| \mid \|u_\xi\| = 1\}$. For simplicity, we discuss only the unital case, we denote by $\mathcal{K}_1$ the unitization of $\mathcal{K}$ and we set $\mathcal{K}_1[A] = \mathcal{K}_1 \otimes_{\min} A$, so that $\mathcal{K}_1[A]$ is a unital operator algebra whenever A is one. We will be interested in the following.

General problem. Which unital Banach algebras A have the following similarity property: (SP) Every bounded morphism $u\colon\ A \to B(H)$ (H being here an arbitrary Hilbert space) is similar to a contractive one (in short s.c.).

Complete boundedness is the key modern notion behind the advances made recently on several instances of this general problem, some of them formulated about fifty years ago. In most cases of interest, the above problem is equivalent to the following. When is it true that all bounded morphisms $u\colon\ A \to B(H)$ are "automatically" *completely* bounded? Before stating this precisely in Theorem 2.5, we prefer to discuss some examples.

Example 2.1 (Uniform algebras). Let A be the disc algebra $A(D)$, formed of all bounded analytic functions $f\colon\ D \to \mathbb{C}$ on the open unit disc $D \subset \mathbb{C}$ which extend continuously to $\overline{D}$, equipped with the norm $\|f\|_\infty = \sup\{|f(z)| \mid z \in D\}$. Note that the set of all polynomials is dense in $A(D)$. Let $\varphi_0 \in A(D)$ be the element such that $\varphi_0(z) = z$. Since this algebra is singly generated (by φ_0) a morphism $u\colon\ A(D) \to B(H)$ is entirely determined by the single operator $T = u(\varphi_0)$. Moreover, u is bounded iff T is "polynomially bounded" which means that there is a constant C such that for any polynomial P we have $\|P(T)\| \leq C\|P\|_\infty$, and in addition $\|u\|$ is the best possible constant C. On the other hand, by a famous 1951 inequality of von Neumann, any contraction T satisfies $\|P(T)\| \leq \|P\|_\infty$ for any P, *i.e.* we have polynomial boundedness with $C = 1$. Therefore, u is similar to a contractive morphism iff $T = u(\varphi_0)$ is similar to a contraction, *i.e.* iff there is ξ invertible such that $\|\xi^{-1}T\xi\| \leq 1$. Thus, the problem whether the disc algebra satisfies (SP) coincides with a question raised in 1970 by Halmos: is every polynomially bounded operator $T\colon\ H \to H$ similar to a contraction? A counterexample was recently given in [Pi8]. The original proof of polynomial boundedness in [Pi8] was rather technical but shortly afterwards simpler proofs have been found by Kislyakov [Kis] and Davidson-Paulsen [DP]. They lead to the same class of examples. Since the disc algebra fails (SP), it is now conceivable that the same is true for any proper uniform algebra, but this remains open in general (even though the case of the polydisc algebra or the ball algebra over $\mathbb{C}^n$ follows easily from the disc case).

Of course, the similarity problem for continuous semi-groups of operators $(T_t)_{t\geq 0}$ is also quite natural, see [LM7] and the references there for more on this topic.

EXAMPLE 2.2 (**C^*-ALGEBRAS**). Let A be a unital C^*-algebra. Then it is easy to check that a morphism $u\colon\ A \to B(H)$ is contractive (*i.e.* has $\|u\| = 1$) iff u is a $*$-representation (*i.e.* $u(x^*) = u(x)^*$ for all x). We then have automatically $\|u\|_{cb} = 1$. It is an outstanding conjecture of Kadison (1955) that *all* C^*-algebras have (SP). is equivalent to the (open) problem whether, for any C^*-subalgebra $A \subset B(H)$, every bounded derivation $\delta\colon\ A \to B(H)$ is inner. Many partial results (mainly due to E. Christensen and U. Haagerup) are known (see Remark 2.10 below). In particular, it is known ([C1, H]) that if a bounded morphism has a cyclic vector (or admits a finite cyclic set), then it is similar to a $*$-representation. However, the general case remains open (and the author doubts its validity). The reduced C^*-algebra of the free group with countably infinitely many generators might be a counterexample, but actually even the von Neumann algebra $A = \bigoplus_n M_n$ (ℓ_∞-direct sum) is not known to satisfy (SP).

EXAMPLE 2.3 (GROUP REPRESENTATIONS). Let G be a discrete group and let $A = \ell_1(G)$ be its group algebra under the convolution product. Then A has (SP) iff every uniformly bounded representation $\pi\colon\ G \to B(H)$ is unitarizable. When this holds, we will say that "G is unitarizable". Note that we mainly restrict below to the discrete case, but otherwise all representations are implicitly assumed to be continuous on G with respect to the strong operator topology on $B(H)$. Here, we allow non-unitary representations (= homomorphisms from G to $GL(H)$), and we set $|\pi| = \sup\{\|\pi(t)\| \mid t \in G\}$. We say that π is uniformly bounded (u.b. in short) if $|\pi| < \infty$, and we call π unitarizable if there is an invertible $\xi\colon\ H \to H$ such that $t \to \xi^{-1}\pi(t)\xi$ is a unitary representation. (Note: There is a one to one correspondence between the bounded morphisms $u\colon\ \ell_1(G) \to B(H)$ and the u.b. representations $\pi\colon\ G \to B(H)$. An operator T is unitary iff $T \in B(H)$ is invertible and both T, T^{-1} are contractions. Hence u is s.c. iff π is unitarizable.)

Sz.-Nagy proved in 1947 that $\mathbb{Z}$ is unitarizable. Shortly afterwards (1950), Dixmier and Day independently proved that, for any discrete (actually any locally compact) group G, amenable implies unitarizable and Dixmier [Di] asked whether the converse also holds. This is still open in full generality. However, in 1955, Ehrenpreis and Mautner showed that $SL_2(\mathbb{R})$ is not unitarizable. Since "unitarizable" passes to quotients, it follows (implicitly) that non-commutative free groups are not unitarizable, but very explicit constructions by many authors (see [MP]) are now known for this, and, by induction, the same is true for any discrete group containing a copy of $\mathbb{F}_2$ (the free group on 2 generators). This suggests there might be a counterexample to Dixmier's question (*i.e.* a unitarizable group which is *not* amenable) among the Burnside groups which are the main examples of non-amenable groups without free subgroups (see Olshanskii's book [Ol], and see also §5.5 in Gromov's [Gr] for examples of infinite discrete groups with Kazhdan's property T and without any free subgroup). Nevertheless, if one takes into account the estimate in Dixmier's argument for amenable $\Rightarrow$ unitarizable, then a converse result can be proved (see Theorem 2.11 below).

We now explain the intimate connection of the property (SP) with dilation theory and complete boundedness. For convenience, we first discuss the completely contractive case. Consider a morphism $u\colon\ A \to B(H)$ on a unital operator algebra

$A \subset B(\mathcal{H})$. Then, since u is assumed unital, u is completely contractive (this means $\|u\|_{cb} = 1$) iff u extends completely positively to $B(\mathcal{H})$ (Arveson) or iff there is a Hilbert space K containing H and a C^*-representation $\pi\colon\ B(\mathcal{H}) \to B(K)$ such that, for any x in A, we have $u(x) = P_H \pi(x)_{|H}$. (One then says that π restricted to A "dilates" u, or that u is a "compression" of it.) Moreover, the subspace $H \subset K$ is necessarily semi-invariant (in Sarason's sense) for $\pi(A)$, which means that there is a pair of $\pi(A)$-invariant (closed) subspaces $E_2 \subset E_1 \subset K$ such that $H = E_1 \ominus E_2$. Thus $\|u\|_{cb} = 1$ iff u can be "dilated" to a $*$-representation. All this is well known, see Theorem 4.8 in [Pi7] for details. For convenience, we will use the following definition (we prefer to avoid the term "maximal algebras" used in [BP2], which might lead to some confusion with "maximal o.s.").

Definition 2.4. Let $A \subset B(\mathcal{H})$ be a unital operator algebra. We say that A satisfies condition (CC) if, for any morphism $u\colon\ A \to B(H)$ (H arbitrary Hilbert), the implication $\|u\| = 1 \Rightarrow \|u\|_{cb} = 1$ holds.

The precise class of algebras which satisfy (CC) is not clear (see [DoP]). However, it is satisfied by $A(D), A(D^2)$ (but not by $A(D^n)$ for $n > 2$ by an example of S. Parrott, see [P1]), by *all* C^*-algebras and also by $\mathcal{K}_1[A]$ for any unital operator algebra A. Thus the next result, provides a characterization of the morphisms which are s.c. for a broad class of algebras. The C^*-case is due to Haagerup [H] and the general one to Paulsen (see [P1]).

Theorem 2.5. *Let A be a unital operator algebra and let $u\colon\ A \to B(H)$ be a morphism. If u is c.b. then u is s.c. and, if A satisfies (CC), the converse holds. We have then $\|u\|_{cb} = Sim(u)$. Thus, assuming (CC), A satisfies (SP) iff for every morphism $u\colon\ A \to B(H)$, $\|u\| < \infty$ implies $\|u\|_{cb} < \infty$.*

Remark 2.6. Applying this to the disc algebra, we get Paulsen's useful criterion: an operator $T\colon\ H \to H$ is similar to a contraction iff it is completely polynomially bounded, which means that there is a constant C such that, for any N and any $N \times N$ matrix (P_{ij}) with polynomial entries we have $\|(P_{ij}(T))\|_{M_N(B(H))} \le C \sup_{z \in D} \|(P_{ij}(z))\|_{M_N}$. Now fix an integer N and denote by $C_N(T)$ the smallest C such that this holds for all $N \times N$ matrices (P_{ij}). Then the above question of Halmos is the same as asking whether $C_1(T) < \infty \Rightarrow \sup_{N \ge 1} C_N(T) < \infty$, and Theorem 2.5 implies that $\sup_{N \ge 1} C_N(T) = \inf\{\|\xi^{-1}\|\ \|\xi\| \mid \|\xi^{-1} T \xi\| \le 1\}$. It can be shown (see [Pi8, Bo]) that there is a numerical constant β such that $C_N(T) \le \beta\sqrt{N}\ C_1(T)$ for all T and $N \ge 1$. However, the counterexamples in [Pi8] show that this cannot be improved: there is a numerical constant $\delta > 0$ such that for any $N \ge 1$ and $\varepsilon > 0$, there is a $T = T_{N,\varepsilon}$ such that $C_1(T) < 1 + \varepsilon$ but still $C_N(T) \ge \delta\varepsilon\sqrt{N}$.

We now turn to a sufficient condition for the property (SP).

Definition 2.7. We say that an operator algebra A has length $\le d$ if there is a constant $K \ge 0$ such that, for any x in $\mathcal{K}[A]$, there are $\alpha_0, \alpha_1, \ldots, \alpha_d$ in $\mathcal{K}[\mathbb{C}]$ and $D_1, \ldots, D_d$ *diagonal* in $\mathcal{K}[A]$ such that $x = \alpha_0 D_1 \alpha_1 D_2 \ldots D_d \alpha_d$ and $\prod \|\alpha_i\| \prod \|D_i\| \le K\|x\|$.

We will denote by $\ell(A)$ the smallest d such that this holds. Equivalently, $\ell(A) \le d$ iff every x in $\mathcal{K}[A]$ can be factorized as above (the constant K then exists by the open mapping theorem).

Remark. This notion from [Pi9] was inspired by the remarkable paper [BP2]. There, Blecher and Paulsen prove that a unital operator algebra A satisfies (CC) iff any x in $\mathcal{K}[A]$ with $\|x\| \le 1$ lies in the norm closure of the set of all (*arbitrarily long*) products of the form $\alpha_0 D_1 \alpha_1 \dots D_d \alpha_d$ with $\Pi\|\alpha_i\|\Pi\|D_i\| \le 1$ and $d \ge 1$.

Proposition 2.8. *If an operator algebra A has length $\le d$, then A satisfies (SP) and more precisely any morphism $u\colon\ A \to B(H)$ satisfies (with the notation of Definitions 2.7) $\|u\|_{cb} \le K\|u\|^d$.*

Proof. Using the notation in §1.1 and Definition 2.7, we have $u_\infty(x) = \alpha_0 u_\infty(D_1)\alpha_1 \dots u_\infty(D_d)\alpha_d$ hence $\|u_\infty(x)\| \le \Pi\|\alpha_i\|\Pi\|u_\infty(D_i)\|$, but since each D_i is *diagonal*, we have $\|u_\infty(D_i)\| \le \|u\|\ \|D_i\|$, whence $\|u_\infty(x)\| \le K\|u\|^d\|x\|$, and therefore $\|u\|_{cb} \le K\|u\|^d$. ☐

Let A be a unital Banach algebra. For any $c \ge 1$, let $\Phi_A(c) = \sup\{\mathrm{Sim}(u)\}$ where the supremum runs over all morphisms $u\colon\ A \to B(H)$ (H arbitrary Hilbert) with $\|u\| \le c$, and let $d(A) = \inf\{\alpha \ge 0 \mid \exists K\ \forall c \ge 1\ \ \Phi_A(c) \le Kc^\alpha\}$.
Although the preceding criterion seems too restrictive at first glance, it turns out that bounded "length" is essentially the only way that an operator algebra can have (SP), as the next result from [Pi9] shows.

Theorem 2.9. *Let A be a unital operator algebra satisfying condition (CC). Then A satisfies (SP) iff there is a d such that A has length $\le d$. More precisely, $\ell(A) = d(A)$ and the infimum defining $d(A)$ is a minimum attained when $\alpha = \ell(A)$.*

Remark. One surprising feature of this result is that there is apparently no direct a priori argument showing that $d(A)$ is an integer. Note that even when A fails (CC), the preceding result can be applied to a suitably defined "enveloping algebra" of A satisfying (CC) (see [Pi9]).

Warning. Until progress is made, the really weak point (embarrassing for the author) of the preceding statement is that, up to now, no example is known of A with $3 < \ell(A) < \infty$. However, an analog of the equality $\ell(A) = d(A)$ is proved in [Pi9] in the more general framework of an operator space generating an operator algebra; in this generalized framework, it is easy to produce the desired examples. We refer the reader to [LM4] for a version of Theorem 2.9 adapted to dual operator algebras and weak-$*$ continuous morphisms.

Remark 2.10. Here is a short list of the C^*-algebras which are known to have (SP): if A is a nuclear C^*-algebra (due to Bunce-Christensen, see [C1]) then $d(A) \le 2$ (and actually $d(A) = 2$ unless $\dim(A) < \infty$), if $A = B(H)$ and $\dim(H) = \infty$ we have (SP) and $d(B(H)) = 3$ (see [H] for ≤ 3 and [Pi9] for ≥ 3). More generally if A has no tracial state, it has (SP) and $d(A) \le 3$, in particular this holds if $A = \mathcal{K}_1[B]$ with B an arbitrary unital C^*-algebra ([H]). (Note: if B is a non-self-adjoint unital operator algebra, $A = \mathcal{K}_1[B]$ satisfies (SP) with $d(A) \le 5$.) Let A be a C^*-algebra generating a semi-finite von Neumann algebra M, then

$d(A) \leq 2$ implies that M is injective ([Pi9]); in particular, if A is the reduced C^*-algebra of a discrete group G, we conclude that G is amenable. Finally, if A is a II_1-factor with property Γ (in particular if it is hyperfinite), it has (SP) with $d(A) \leq 44$ (see [C2], the latter estimate can presumably be improved significantly.)

Let us return to the group case (Example 2.3). Then we define $d(G) = d(\ell_1(G))$. The following partial answer to Dixmier's question holds:

Theorem 2.11 ([Pi9]). *A discrete group G is amenable iff $d(G) \leq 2$. More precisely, G is amenable iff there is a constant K and $\alpha < 3$ such that, for any u.b. representation $\pi\colon\ G \to B(H)$, there is an invertible ξ with $\|\xi^{-1}\|\ \|\xi\| \leq K|\pi|^\alpha$ such that $\xi^{-1}\pi(\cdot)\xi$ is a unitary representation. (When G is amenable, Dixmier [Di] and Day proved that the latter holds with $K = 1$ and $\alpha = 2$).*

Warning: We know of no example of G such that $2 < d(G) < \infty$!
See [Pi9] for an analog of Theorem 2.9 in the group case: the relevant notion of length is like in Definition 2.7 with $A = C^*(G)$, but the diagonal matrices D_i are now restricted to have their entries in the set of scalar multiples of elements of G viewed, as usual, as embedded into $A = C^*(G)$. The notion of length can also be studied in the more general framework of a Banach algebra B generated by a subset $\mathcal{B}$ of its unit ball, [Pi9, Pi10].

References

[B1] D. Blecher, Tensor products of operator spaces II. Canad. J. Math. 44 (1992), 75–90.

[B2] _______, The standard dual of an operator space, Pacific J. Math. 153 (1992), 15–30.

[B3] _______, A completely bounded characterization of operator algebras, Math. Ann. 303 (1995), 227–240.

[B4] _______, A new approach to Hilbert C^*-modules, Math. Ann. 307 (1997) 253-290.

[BL] J. Bergh and J. Löfström, *Interpolation spaces. An Introduction*, Springer-Verlag, New York, 1976.

[BLM] D. Blecher and C. Le Merdy, On quotients of function algebras, and operator algebra structures on ℓ_p, J. Operator Theory 34 (1995) 315–346.

[BMP] D. Blecher, P. Muhly and V. Paulsen, Categories of Operator Modules (Morita Equivalence and projective Modules). Memoirs Amer. Math. Soc. To appear.

[Bo] J. Bourgain, On the similarity problem for polynomially bounded operators on Hilbert space, Israel J. Math. 54 (1986), 227–241.

[BP1] D. Blecher and V. Paulsen, Tensor products of operator spaces, J. Funct. Anal. 99 (1991), 262–292.

[BP2] _______, Explicit constructions of universal operator algebras and applications to polynomial factorization, Proc. Amer. Math. Soc. 112 (1991), 839–850.

[BRS] D. Blecher, Z. J. Ruan, and A. Sinclair, A characterization of operator algebras, J. Funct. Anal. 89 (1990), 188–201.

[BS] D. Blecher and R. Smith. The dual of the Haagerup tensor product. J. London Math. Soc. 45 (1992) 126-144.

[C1] E. Christensen, On non-self-adjoint representations of operator algebras , Amer. J. Math. 103 (1981), 817–834.

[C2] ______, Similarities of II_1 factors with property Γ, Journal Operator Theory 15 (1986), 281–288.

[CE] M. D. Choi and E. Effros, Injectivity and operator spaces, J. Funct. Anal. 24 (1977), 156–209.

[CES] E. Christensen, E. Effros, and A. Sinclair, Completely bounded multilinear maps and C^*-algebraic cohomology, Invent. Math. 90 (1987), 279–296.

[CS1] E. Christensen and A. Sinclair, Representations of completely bounded multilinear operators, J. Funct. Anal. 72 (1987), 151–181.

[CS2] ______, A survey of completely bounded operators, Bull. London Math. Soc. 21 (1989), 417–448.

[Ca] T. K. Carne, Not all H'-algebras are operator algebras, Proc. Camb. Phil. Soc. 86 (1979), 243–249.

[DP] K. Davidson and V. Paulsen, On polynomially bounded operators, J. für die reine und angewandte Math. 487 (1997), 153–170.

[Di] J. Dixmier, Les moyennes invariantes dans les semi-groupes et leurs applications, Acta Sci. Math. Szeged 12 (1950), 213–227.

[DJT] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge University press, 1995.

[DoP] R. Douglas and V. Paulsen, *Hilbert Modules Over Function Algebras*, Research Notes 217, Pitman-Longman, 1989.

[E] E. Effros, Advances in Quantized Functional Analysis, Proceedings International Congress of Mathematicians, Berkeley, 1986, pp. 906–916.

[EH] E. Effros and U. Haagerup, Lifting problems and local reflexivity for C^*-algebras, Duke Math. J. 52 (1985), 103–128.

[EJR] E. Effros, M. Junge, and Z. J. Ruan, Integral mappings and the property of local reflexivity for non-commutative L_1-spaces, 1998, preprint.

[ER1] E. Effros and Z. J. Ruan, Representations of operator bimodules and their applications, J. Operator theory, 19 (1988) 137-157.

[ER2] ______, On approximation properties for operator spaces, International J. Math. 1 (1990), 163–187.

[ER3] ______, A new approach to operator spaces, Canadian Math. Bull. 34 (1991), 329–337.

[ER4] ______, Self duality for the Haagerup tensor product and Hilbert space factorization, J. Funct. Anal. 100 (1991), 257–284.

[ER5] _______, On the abstract characterization of operator spaces, Proc. Amer. Math. Soc. 119 (1993), 579–584.

[ER6] _______, Mapping spaces and liftings for operator spaces, Proc. London Math. Soc. 69 (1994), 171–197.

[ER7] _______, The Grothendieck-Pietsch and Dvoretzky-Rogers Theorems for operator spaces, J. Funct. Anal. 122 (1994), 428–450.

[ER8] _______, Discrete quantum groups I. The Haar measure, Internat. J. Math. 5 (1994) 681-723.

[ER9] _______, On the $O\mathcal{L}_p$-spaces. Preprint, 1998.

[EWi] E. Effros and S. Winkler, Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems, J. Funct. Anal. 144 (1997) 117-152.

[G] A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques, Boll. Soc. Mat. São-Paulo 8 (1956), 1–79 and 80–110.

[Gr] M Gromov, Hyperbolic groups, p. 75-265 in *Essays in Group Theory* (edited by S. Gersten), Springer-verlag, New-York, 1987.

[H] U. Haagerup, Solution of the similarity problem for cyclic representations of C^*-algebras, Annals of Math. 118 (1983), 215–240.

[HK] U. Haagerup and J. Kraus, Approximation properties for group C^*-algebras and group von Neumann algebras, Trans. Amer. Math. Soc. 344 (1994) 667–699.

[JP] M. Junge and G. Pisier, Bilinear forms on exact operator spaces and $B(H) \otimes B(H)$, Geometric and Functional Analysis (GAFA Journal) 5 (1995), 329–363.

[Ju1] M. Junge, *Factorization Theory for Spaces of Operators*, Habilitationsschrift , Kiel University, 1996.

[Ju2] _______, Embeddings of non-commutative L_p-spaces in non-commutative L_1-spaces, $1 < p < 2$. Preprint Kiel University, June 1998.

[K] R. Kadison, On the orthogonalization of operator representations, Amer. J. Math. 77 (1955), 600–620.

[Ka] A. Katavolos (editor), *Operator algebras and applications*, (Proc. Conf. Samos, August 19–28, 1996), NATO ASIC, 495, Kluwer, Dordrecht, 1997.

[Ki1] E. Kirchberg, Exact C^*-Algebras, Tensor Products, and Classification of Purely Infinite Algebras, Proceedings ICM 94, Zürich, vol.2, 943-954, Birkhauser, 1995.

[Ki2] _______, On non-semisplit extensions, tensor products and exactness of group C^*-algebras, Invent. Math. 112 (1993), 449–489.

[Ki3] _______, The derivation and the similarity problem are equivalent, J. Operator Theory, 36 (1996) 59-62.

[Kis] S. V. Kislyakov. Operators (not) similar to a contraction: Pisier's counterexample via singular integrals. Zapiski Nauchnykh Seminarov POMI, 247 (1997), p.79-95 (Russian). English translation in J. Math.Sci.

[LM1] C. Le Merdy, Factorization of p-completely bounded multilinear maps, Pacific J. Math., 172 (1996) 187-213.

[LM2] ______, Representations of a quotient of a subalgebra of $B(X)$, Math. Proc. Cambridge Phil. Soc. 119 (1996), 83–90.

[LM3] ______, On the duality of operator spaces, Canad. Math. Bull. 38 (1995), 334–346.

[LM4] ______, The weak-$*$ similarity property on dual operator algebras, Preprint, 1998.

[LM5] ______, The Schatten space S_4 is a Q-algebra, Proc. Amer. Math. Soc. 126 (1998) 715-719.

[LM6] ______, An operator space characterization of dual operator algebras, Amer. J. Math. (To appear).

[LM7] ______, The similarity problem for bounded analytic semigroups on Hilbert space, Semigroup Forum, 56 (1998) 205–224.

[Lu] A. Lubotzky, *Discrete groups, Expanding graphs and invariant measures*, Birkhauser 1994.

[Ma] B. Magajna, Strong operator modules and the Haagerup tensor product, Proc. London Math. Soc. 74 (1997), 201–240.

[MP] A. M. Mantero, J. Pytlik, R. Szwarc, and A. Zappa, Equivalence of two series of spherical representations of a free group. Ann. Mat. Pura Appl. 165 (1993) 23–28.

[Oi] T. Oikhberg, Geometry of operator spaces and products of orthogonal projections, Ph. D. Thesis, Texas A&M University, 1998.

[OiP] T. Oikhberg and G. Pisier, The "maximal" tensor product of operator spaces, Proc. Edinburgh Math. Soc., to appear.

[Ol] A. Yu. Olshanskii, *The geometry of defining relations in groups*, Kluwer, Dordrecht, 1991.

[P1] V. Paulsen, *Completely Bounded Maps and Dilations*, Pitman Research Notes 146, Pitman-Longman, Wiley, 1986.

[P2] ______, Representation of function algebras, abstract operator spaces and Banach space geometry, J. Funct. Anal. 109 (1992), 113–129.

[P3] ______, The maximal operator space of a normed space, Proc. Edinburgh Math. Soc. 39 (1996), 309–323.

[Pi1] G. Pisier, The operator Hilbert space OH, complex interpolation and tensor norms, Memoirs Amer. Math. Soc. 122, 585 (1996), 1–103.

[Pi2] ______, Non-commutative vector valued L_p-spaces and completely p-summing maps, Astérisque (Soc. Math. France), 247 (1998), 1–111.

[Pi3] ______, A simple proof of a theorem of Kirchberg and related results on C^*-norms, J. Operator Theory 35 (1996), 317–335.

[Pi4] _______, Exact operator spaces. Colloque sur les algèbres d'opérateurs, in *Recent Advances in Operator Algebras*, Astérisque 232 (1995), 159–186.

[Pi5] _______, An introduction to the theory of operator spaces, to appear.

[Pi6] _______, Dvoretzky's theorem for operator spaces and applications, Houston J. Math. 22 (1996), 399–416.

[Pi7] _______, *Similarity Problems and Completely Bounded Maps*, Springer Lecture Notes 1618, 1995.

[Pi8] _______, A polynomially bounded operator on Hilbert space which is not similar to a contraction, J. Amer. Math. Soc. 10 (1997), 351–369.

[Pi9] _______, The similarity degree of an operator algebra, St. Petersburg Math. J. (1998), to appear.

[Pi10] _______, Joint similarity problems and the generation of operator algebras with bounded length, Integr. Equ. Op. Th., to appear.

[PS] V. Paulsen and R. Smith, Multilinear maps and tensor norms on operator systems, J. Funct. Anal. 73 (1987), 258–276.

[Ro] H. P. Rosenthal, The complete separable extension property, Preprint 1998.

[Ru1] Z. J. Ruan, Subspaces of C^*-algebras, J. Funct. Anal. 76 (1988), 217–230.

[Ru2] _______, Injectivity and operator spaces, Trans. Amer. Math. Soc. 315 (1989) 89-104.

[Ru3] _______, The operator amenability of $A(G)$, Amer. J. Math. 117 (1995) 1449-1474.

[Ru4] _______, Amenability of Hopf von Neumann algebras and Kac algebras, J. Funct. Anal. 139 (1996) 466-499.

[SSm] A. M. Sinclair and R. R. Smith, *Hochschild cohomology of von Neumann algebras*, Cambridge Univ. Press, LMS lecture notes 203, Cambridge 1995.

[Sz] A. Szankowski, $B(H)$ does not have the approximation property, Acta Math. 147 (1981), 89–108.

[Va] A. Valette, An application of Ramanujan graphs to C^*-algebra tensor products, Discrete Math. 167/168 (1997) 597-603.

[Wa] S. Wassermann, *Exact C^*-Algebras and Related Topics*, Lecture Notes Series, Seoul Nat. Univ., 1994.

[Z] C. Zhang, *Representation and Geometry of Operator Spaces*, Ph.D. Thesis, University of Houston, 1995.

Gilles Pisier
Texas A&M University,
College Station, TX 77843, USA
and Université Paris 6,
Equipe d'Analyse, Case 186,
4 Place Jussieu,
75252 Paris Cedex 05, France