Rationality of Integral Cycles

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Abstract. In this article we provide the sufficient condition for a Chow group element to be defined over the ground field. This is an integral version of the result known for $\mathbb{Z}/2$-coefficients. We also show that modulo 2 and degree r cohomological invariants of algebraic varieties can not affect rationality of cycles of codimensions up to $2^{r-1}-2$.

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1 Introduction

In many situations it is important to know, if the element of the Chow group of some variety which exists over algebraic closure is actually defined over k. In particular, this question arises while studying various discrete invariants of quadrics. An effective tool here is the method introduced in [9] which, in particular, gives that for an element of codimension m with $\mathbb{Z}/2$-coefficients, it is sufficient to check that it is defined over the function field of a sufficiently large quadric (of dimension $> 2m$). This approach was successfully applied to various questions from quadratic form theory. In particular, to the construction of fields with the new values of the u-invariant (see [11]). The above core result can be extended in various directions. One of them due to K.Zainoulline (see [13]) establishes similar statement for the quadric substituted by a norm-variety of Rost for the pure symbol in $K_n^M(k)/p$ (p-prime) and for Chow groups with $\mathbb{Z}/p$-coefficients. This is a generalisation of the case of a Pfister quadric, which is a norm-variety for the pure symbol modulo 2.
But all the mentioned results are dealing with Chow groups with torsion coefficients. In the current article I would like to address similar question for

integral Chow groups. The results obtained are similar to the $\mathbb{Z}/2$-case, but one requires an additional condition on Q (aside from its size) saying that Q has a projective line defined over the generic point of Q. Although, the "generic quadric" does not have this property, such quadrics are quite widespread. If one imposes stronger conditions on a quadric, one can show that the map

$$\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{k(Q)})$$

is surjective. In particular, this happens for a Pfister quadric, and $m < 2^{r-1}-1$. This latter result can be used to show that (modulo 2) and degree r cohomological invariants of algebraic varieties do not effect rationality of Chow group elements of codimension up to $2^{r-1}-2$.

The main tool we use is "Symmetric Operations" in Algebraic Cobordism (see [10]). These are "formal halves" of the "negative parts" of Steenrod operations ($mod\, 2$) there. If one does not care about 2-torsion effects, one can use more simple Landweber-Novikov operations instead. But the symmetric operations provide the only (known) way to get "clean" results on rationality.

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2 Symmetric operations

For any field k of characteristic 0, M.Levine and F.Morel have defined the Algebraic Cobordism theory Ω^*, which is the universal generalised oriented cohomology theory on the category Sm/k of smooth quasi-projective varieties over k (see [3, Theorem 1.2.6]), which means that for any other such theory A^* there is unique map of theories $\Omega^* \to A^*$. For a given smooth quasi-projective X, the ring $\Omega^*(X)$ is additively generated by the classes $[v : V \to X]$ with V-smooth and v-projective, modulo some relations. The value of Ω^* on $\mathrm{Spec}(k)$ coincides with $MU^{2*}(pt) = \mathbb{L}$ - the Lazard ring - see [3, Theorem 1.2.7]. Since Chow groups form a generalised oriented cohomology theory, one has a canonical map $pr : \Omega^* \to \mathrm{CH}^*$ (given by $[v : V \to X] \mapsto v_*(1_V) \in \mathrm{CH}_{\dim(V)}(X)$) which is surjective, and moreover, $\mathrm{CH}^*(X) = \Omega^*(X)/\mathbb{L}_{>0} \cdot \Omega^*(X)$ - see [3, Theorem 1.2.19]. Thus, one can reconstruct Chow groups if the Algebraic Cobordism is known. On Ω^* we have the action of Landweber-Novikov operations - [3, Example 4.1.25]. Such operations can be parametrised by the polynomials $g \in \mathbb{L}[\sigma_1, \sigma_2, \ldots]$, with

$$S^g_{L.-N.}([v : V \to X]) := v_*(g(c_1, c_2, \ldots) \cdot 1_V) \in \Omega^*(X),$$

where $c_i = c_i(-T_V + v^*T_X)$ is the i-th Chern class of the virtual normal bundle of v. If one does not mind moding out the 2-torsion, then all the results of the next section can be obtained using the Landweber-Novikov operations only. But to obtain precise statements one needs more subtle "Symmetric operations". These operations were introduced in [8] and [10]. It is convenient

to parametrise them by $q(s) \in \mathbb{L}[[s]]$, where $\Phi^{s^r} : \Omega^d(X) \to \Omega^{2d+r}(X)$. The operation $\Phi^{q(s)}$ is constructed as follows. For a smooth morphism $W \to U$, let $\widetilde{\square}(W/U)$ denotes the blow-up of $W \times_U W$ at the diagonal W. For a smooth variety W denote: $\widetilde{\square}(W) := \widetilde{\square}(W/\operatorname{Spec}(k))$. Denote as $\widetilde{C}^2(W)$ and $\widetilde{C}^2(W/U)$ the quotient variety of $\widetilde{\square}(W)$, respectively, $\widetilde{\square}(W/U)$ by the natural $\mathbb{Z}/2$-action. These are smooth varieties. Notice that they have natural line bundle $\mathcal{L}$, which lifted to $\widetilde{\square}$ becomes $\mathcal{O}(1)$ - see [4], or [10, p.492]. Let $\tilde{\rho} := c_1(\mathcal{L}) \in \Omega^1(\widetilde{C}^2)$.
If $[v] \in \Omega^d(X)$ is represented by $v : V \to X$, then v can be decomposed as $V \xrightarrow{g} W \xrightarrow{f} X$, where g is a regular embedding, and f is smooth projective. One gets natural morphisms:

$$\widetilde{C}^2(V) \overset{\alpha}{\hookrightarrow} \widetilde{C}^2(W) \overset{\beta}{\hookleftarrow} \widetilde{C}^2(W/X) \xrightarrow{\gamma} X.$$

Now, $\Phi^{q(s)}([v]) := \gamma_* \beta^* \alpha_*(q(\tilde{\rho}))$. Denote as $\phi^{q(s)}([v])$ the composition $pr \circ \Phi^{q(s)}([v])$. As was proven in [10, Theorem 2.24], $\Phi^{q(s)}$ gives a well-defined operation $\Omega^*(X) \to \Omega^*(X)$. I should note that in [10] and [9] we use slightly different parametrisation for symmetric operations. To stress this difference I used the different name for the uniformiser. In [10] the parameter is t and its relation to our s is given by: $t = [-1]_\Omega(s)$, where $[-1]_\Omega(s) \in \mathbb{L}[[s]]$ is the inverse in terms of the universal formal group law. The difference basically amounts to signs, and with the new choice the formulae are just a little bit nicer.
It was proven in [8] that the Chow-trace of Φ is the half of the Chow-trace of certain Landweber-Novikov operation.

Proposition 2.1 ([8, Propositions 3.8, 3.9], [10, Proposition 3.14]) *For* $[v] \in \Omega^d(X)$,

(1) $2\phi^{s^r}([v]) = pr(-S^{r+d}_{L-N}([v]))$, *for* $r > 0$;

(2) $2\phi^{s^0}([v]) = pr([v]^2 - S^d_{L-N}([v]))$,

where we denote $S^{\sigma_r}_{L-N}$ *as* S^r_{L-N}.

The additive properties of ϕ are given by the following:

Proposition 2.2 ([10, Proposition 2.8])

(1) *Operation* ϕ^{s^r} *is additive for* $r > 0$;

(2) $\phi^{s^0}(x + y) = \phi^{s^0}(x) + \phi^{s^0}(y) + pr(x \cdot y)$.

Let $[v] \in \Omega^*(X)$ be some cobordism class, and $[u] \in \mathbb{L}$ be the class of a smooth projective variety U over k of positive dimension. We will use the notation $\eta_2(U)$ for the (minus) Rost invariant $\frac{deg(c_{\dim(U)}(-T_U))}{2} \in \mathbb{Z}$ (see [4]).

Proposition 2.3 ([10, Proposition 3.15]) *In the above notations, let* $r = (\operatorname{codim}(v) - 2\dim(u))$. *Then, for any* $i \geqslant \max(r; 0)$,

$$\phi^{s^{i-r}}([v] \cdot [u]) = -\eta_2(U) \cdot (pr \circ S^i_{L-N})([v]).$$

The following proposition describes the behaviour of Φ with respect to pull-backs and regular push-forwards. For $q(s) = \sum_{i\geqslant 0} q_i s^i \in CH^*(X)[[s]]$ let us define $\phi^{q(s)} := \sum_{i\geqslant 0} q_i \phi^{s^i}$. For a vector bundle $\mathcal{V}$ denote $c(\mathcal{V})(s) := \prod_i (s+\lambda_i)$, where $\lambda_i \in CH^1$ are the *roots* of $\mathcal{V}$. This is the usual total Chern class of $\mathcal{V}$.

Proposition 2.4 ([10, Propositions 3.1, 3.4]) *Let $f : Y \to X$ be some morphism of smooth quasi-projective varieties, and $q(s) \in CH^*(X)[[s]]$. Then*

(1) $f^*\phi^{q(s)}([v]) = \phi^{f^*q(s)}(f^*[v])$;

(2) *If f is a regular embedding, then $\phi^{q(s)}(f_*([w])) = f_*(\phi^{f^*q(s)\cdot c(\mathcal{N}_f)(s)}([w]))$, where $\mathcal{N}_f$ is the normal bundle of the embedding.*

And, consequently, for f - a regular embedding:

(3) $\phi^{q(s)}(f_*([1_Y])\cdot[v]) = \phi^{q(s)\cdot f_*(c(\mathcal{N}_f)(s))}([v])$

3 Rationality of cycles over function fields of quadrics

Let k be a field of characteristic 0, Y be a smooth quasi-projective variety, and Q be a smooth projective quadric defined over k. For $\overline{y} \in \mathrm{CH}^m(Y_{\overline{k}})$ we say that $\overline{y}$ is *k-rational*, if it belongs to the image of the natural restriction map:

$$\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{\overline{k}}).$$

The following result shows that rationality of $\overline{y}$ can be checked over the function field of Q provided Q is sufficiently large and a little bit "special".

Theorem 3.1 *In the above notations, suppose that $m < \dim(Q)/2$, and $\mathfrak{i}_1(Q) > 1$. Then*

$$\overline{y} \text{ is defined over } k \;\Leftrightarrow\; \overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q).$$

With stronger conditions on Q we can prove more subtle result.

Proposition 3.2 *Let Q be smooth projective quadric with $\mathfrak{i}_1(Q) > m < \dim(Q)/2$. Then the map $\mathrm{CH}^m(Y) \twoheadrightarrow \mathrm{CH}^m(Y_{k(Q)})$ is surjective, for all smooth quasi-projective Y.*

Applying it to the case of a Pfister quadric, we get:

Corollary 3.3 *Let Q_α be an r-fold Pfister quadric. Then the map*

$$\mathrm{CH}^m(Y) \twoheadrightarrow \mathrm{CH}^m(Y_{k(Q_\alpha)})$$

is surjective, for all smooth quasi-projective Y, and all $m < 2^{r-1}-1$.

Proof: In this case, $\mathrm{i}_1(Q_\alpha) = 2^{r-1}$, and $\dim(Q_\alpha)/2 = 2^{r-1} - 1$ (actually, the case of a Pfister quadric is the only one where the second inequality in Proposition 3.2 is needed). □

This immediately implies:

Theorem 3.4 *Let k be any field of characteristic* 0*, and $r \in \mathbb{N}$, then there exists a field extension F/k such that:*

- $\mathrm{K}_r^M(F)/2 = 0$*;*
- *The map $\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_F)$ is surjective, for all $m < 2^{r-1} - 1$, for all smooth quasi-projective Y defined over k.*

Proof: Take F - the standard *Merkurjev's tower of fields*, that is, $F := \varinjlim F_i$, where $F_{i+1} = F_i'$, and for any field G, the extension G' of G is defined as $\varinjlim G(\times_{i\in I} Q_i)$, where the latter limit is taken over all finite sets I of r-fold Pfister quadrics defined over G. Then $\mathrm{K}_r^M(F)/2 = 0$, and it is sufficient to prove the respective property for the map $\mathrm{CH}^m(Y_E) \to \mathrm{CH}^m(Y_{E(Q_\alpha)})$, where Q_α is an r-fold Pfister quadric. It remains to apply Corollary 3.3. □

This result shows that ($mod\,2$) and degree $\geqslant r$ cohomological invariants of smooth algebraic varieties could not affect rationality of cycles of codimension up to $2^{r-1} - 2$. As the example of a Pfister quadric itself shows, this boundary is sharp.

Remark: The above Corollary can also be proven by other means. Namely, the computations of M.Rost ([6, Theorem 5], see also [2, Theorem 8.1], or [10, Theorem 4.1]) show:

Proposition 3.5 (M.Rost) *Let Q_α be r-fold Pfister quadric over the field k. Then, for any field extension F/k, the map*

$$\mathrm{CH}^n(Q_\alpha) \twoheadrightarrow \mathrm{CH}^n(Q_\alpha|_F)$$

is surjective, for any $n < \dim(Q_\alpha)/2 = 2^{r-1} - 1$.

It is remarkable that the respective map

$$\Omega^n(Q_\alpha) \hookrightarrow \Omega^n(Q_\alpha|_F)$$

on algebraic cobordism groups is instead injective, for any n - see [10, Theorem 4.1].

Combined with the following general result Proposition 3.5 gives our Corollary 3.3.

Proposition 3.6 (R.Elman, N.Karpenko, A. Merkurjev, [1, Lemma 88.5]) *Let X, Y be smooth varieties over k, such that, for any field extension F/k, and for any $n \leqslant m$, the map:*

$$\mathrm{CH}^n(X) \twoheadrightarrow \mathrm{CH}^n(X_F)$$

is surjective. Then, the map:

$$\mathrm{CH}^m(Y) \twoheadrightarrow \mathrm{CH}^m(Y_{k(X)})$$

is surjective as well.

Here I should point out, that although one gets a different "elementary" proof of Corollary 3.3, it involves a quite non-trivial ingredient - the Rost computation of the Chow groups of Pfister quadrics. The Pfister quadrics and also few small-dimensional quadrics are the only ones for which such computation is known. In particular, to prove the whole Proposition 3.2 using this method one needs an analogue of Proposition 3.5.

Both Theorem 3.1 and Proposition 3.2 are consequences of the following statement.

Proposition 3.7 *Let Q be a smooth projective quadric of dimension $> 2m$ with $\mathfrak{i}_1(Q) > 1$, E/k be field extension such that $i_W(q_E) > m$, and Y be a smooth quasi-projective k-variety. Then, for any $y \in \mathrm{CH}^m(Y_{k(Q)})$ there exists $x \in \mathrm{CH}^m(Y)$ such that $x_{E(Q)} = y_{E(Q)}$.*

Proof: Consider $y \in \mathrm{CH}^m(Y_{k(Q)})$. Using the surjections

$$\Omega^m(Y \times Q) \stackrel{pr}{\twoheadrightarrow} \mathrm{CH}^m(Y \times Q) \twoheadrightarrow \mathrm{CH}^m(Y_{k(Q)}),$$

we can lift y to some element $v \in \Omega^m(Y \times Q)$.
Since $i_W(Q_E) > m$, $q_E = (\perp_{i=0}^m \mathbb{H}) \perp q'$, for some quadratic form q' defined over E. Consider the cobordism motive of our quadric Q_E (see [5], or [12]) By [7, Proposition 2] and [12, Corollary 2.8], we have that $M^{\Omega}(Q_E) = (\oplus_{i=0}^m \mathbf{L}(i)[2i]) \oplus M'$, where

$$M' = M^{\Omega}(Q')(m+1)[2m+2] \oplus (\oplus_{i=0}^m \mathbf{L}(\dim(Q)-i)[2\dim(Q)-2i]),$$

where $\mathbf{L}(j)[2j]$ is the cobordism Tate-motive (see [12]). Moreover, we can always choose the generator of $\mathbb{L} \cong \Omega^*(\mathbf{L}(i)[2i]) \subset \Omega^*(Q_E)$ to be h^i. Let us denote the passage $k \to E$ as $\bar{\ }$. Then, our element $v \in \Omega^m(Y \times Q)$ restricted to E can be presented as $\overline{v} = \sum_{i=0}^m \overline{v}^i \cdot h^i + \overline{v}'$, where $\overline{v}^i \in \Omega^{m-i}(Y_E)$, and $\overline{v}' \in \Omega^m(M^{\Omega}(Y) \otimes M')$. Applying the composition

$$\Omega^m(Y \times Q_E) \to \mathrm{CH}^m(Y \times Q_E) \to \mathrm{CH}^m(Y_{E(Q)})$$

to $\overline{v}$ we get $pr(\overline{v}^0)_{E(Q)}$. On the other hand, from commutativity of the diagram

$$\begin{tikzcd}
\Omega^m(Y\times Q_E) \arrow[r] & \mathrm{CH}^m(Y\times Q_E) \arrow[r] & \mathrm{CH}^m(Y_{E(Q)}) \\
\Omega^m(Y\times Q) \arrow[r] \arrow[u] & \mathrm{CH}^m(Y\times Q) \arrow[r] \arrow[u] & \mathrm{CH}^m(Y_{k(Q)}) \arrow[u]
\end{tikzcd}$$

it must coincide with $y_{E(Q)}$. Thus, it is sufficient to show that $pr(\overline{v}^0)\in\mathrm{CH}^m(Y_E)$ is defined over k.

Lemma 3.8 *In the above situation, let $e: P\hookrightarrow Q$ be a linear embedding of smooth quadrics, with $\dim(P)=m$. Let $\rho: M^{\Omega}(Q)\to M^{\Omega}(Q)$ be cobordism-motivic endomorphism of Q. Let $v\in\Omega^m(Y\times Q)$. Then*

$$(id\times e)^*(id\times\rho)^*(\overline{v})=\sum_{i=0}^{m}\left(\sum_{j=0}^{i}\alpha_{i,j}\cdot\overline{v}^j\right)\cdot h^i,$$

where $\alpha_{i,j}\in\mathbb{L}_{i-j}$. Moreover, $\alpha_{i,i}\in\mathbb{Z}$ is visible on the level of Chow groups: $\rho_{CH}(h^i)=\alpha_{i,i}\cdot h^i$.

Proof: By dimensional reasons, any map from $M^{\Omega}(P)$ to M' is zero. Thus, $(id\times e)^*(id\times\rho)^*(\overline{v})=\sum_{j=0}^{m}\overline{v}^j\cdot(e\circ\rho)^*(h^j)$. Clearly, $\rho^*(h^j)=\sum_{i\geqslant j}\alpha_{i,j}\cdot h^i+\beta'_j$, where $\alpha_{i,j}\in\mathbb{L}_{i-j}$, and $\beta'_j\in\Omega^j(M')$. Again, by the same reasons, $e^*(\beta'_j)=0$, and we get the first statement. Projecting to CH^* we get the description of $\alpha_{i,i}$. □

Lemma 3.9 *In the situation of Proposition 3.7, let $e: P\hookrightarrow Q$ be a linear embedding of smooth quadrics, where $\dim(P)=m$, and $v\in\Omega^m(Y\times Q)$. Then there exist $w,z\in\Omega^m(Y\times P)$ such that: $\overline{w}=\sum_{i=0}^m\overline{w}^i\cdot h^i$, $\overline{z}=\sum_{i=0}^m\overline{z}^i\cdot h^i$, and*

1) $\overline{w}^i=\overline{v}^i$;
2) $\overline{z}^i=\sum_{j=0}^i\alpha_{i,j}\cdot\overline{v}^j$, *where* $\alpha_{i,j}\in\mathbb{L}_{i-j}$;
3) $\alpha_{0,0}=1$, *and* $\alpha_{i,i}=0$, *for all odd* i.

Proof: Since $\mathfrak{i}_1(Q)>1$, the (Chow) motive of Q is decomposable, and if N is indecomposable direct summand containing $\mathbf{Z}$ (when restricted to $\overline{k}$), then N does not contain $\mathbf{Z}(i)[2i]$, for any odd i, by the result of R.Elman, N.Karpenko, A.Merkurjev - [1, Proposition 83.2] (here $\mathbf{Z}(j)[2j]$ is the Tate-motive in the Chow-motivic category - see [5], or [12]).
Let $\rho_{CH}\in\mathrm{End}_{Chow(k)}(M^{CH}(Q))$ be the projector corresponding to N. Then $\rho^*_{CH}(1)=1$, and $\rho^*_{CH}(h^i)=0$, for any odd i. Using the surjective map $pr:\Omega^*\twoheadrightarrow\mathrm{CH}^*$, we can lift ρ_{CH} to a cobordism-motivic morphism $\rho: M^{\Omega}(Q)\to M^{\Omega}(Q)$. Take $z:=(id\times(e\circ\rho))^*(v)$, and $w:=(id\times e)^*(v)$. □

In the above notations, let $\overline{u} = \sum_{i=0}^{m} \alpha_{i,i}\overline{v}^i \cdot h^i \in \Omega^m(Y \times P_E)$. Then:

Lemma 3.10 *For any* $0 \leqslant k \leqslant [m/2]$, *the element* $pr(\Phi^{s^{m-2k}}(\pi_*(h^k \cdot (\overline{z} - \overline{u}))))$ *is a linear combination of* $pr(S^j_{L-N}(\overline{v}^j))$ *with* *even* *coefficients.*

Proof: We have: $\pi_*(h^k \cdot (\overline{z} - \overline{u})) = \sum_{i=1}^{m} \sum_{0 \leqslant j < i} [P_{m-k-i}] \cdot \alpha_{i,j} \cdot \overline{v}^j$, where $[P_l] \in \mathbb{L}$ is the class of an l-dimensional quadric. Thus, by Proposition 2.3,

$$pr(\Phi^{s^{m-2k}}(\pi_*(h^k \cdot (\overline{z} - \overline{u})))) = -\sum_{i=1}^{m} \sum_{0 \leqslant j < i} \eta([P_{m-k-i}] \cdot \alpha_{i,j}) \cdot pr(S^j_{L-N}(\overline{v}^j)).$$

But $2 \cdot \eta$ is multiplicative, P_{m-k-i} is a quadric (possibly, zero-dimensional), and $\dim(\alpha_{i,j}) > 0$. Thus, all the coefficients are even. □

Lemma 3.11 *If* $w \in \Omega^m(Y \times P)$ *decomposes over* E *as* $\overline{w} = \sum_{i=0}^{m} (\overline{w}^i \cdot h^i)$, *then any linear combination of* $pr(S^i_{L-N}(\overline{w}^i))$ *with even coefficients is defined over* k.

Proof: Consider the projection $Y \times P \stackrel{\pi}{\to} Y$. It is sufficient to observe that, for all $0 \leqslant i \leqslant m$, the elements $pr(S^i_{L-N}\pi_*(h^{m-i} \cdot \overline{w})) = 2pr(S^i_{L-N}(\overline{w}^i)) + 2\sum_{0 \leqslant j < i} \eta(P_{i-j}) pr(S^j_{L-N}(\overline{w}^j))$ are defined over k. □

Denote as $\eta(x)$ the power series $\sum_{i \geqslant 0} \eta(P_i) \cdot x^i$, where $\eta(P_l) = \frac{deg(c_l(-T_{P_l}))}{2}$ is the (minus) *Rost invariant* of an l-dimensional quadric P_l.

Proposition 3.12 *Let* Y *be smooth quasi-projective variety,* P *- smooth projective quadric of dimension* m, *and* $z \in \Omega^m(Y \times P)$ *such element that* $\overline{z} = \sum_{i=0}^{m} \overline{z}^i \cdot h^i$, *where* $\overline{z}^i \in \Omega^{m-i}(Y_E)$. *Then, for any polynomial* $f \in \mathbb{Z}[x]$ *of degree* $\leqslant [m/2]$, *the linear combination*

$$\sum_{j=0}^{m} g_{m-j} \cdot pr(S^j_{L-N}(\overline{z}^j)),$$

is defined over k, *where* $g(x) = \sum_l g_l \cdot x^l$ *is "the degree* $\leqslant m$ *part" of the product* $f(x) \cdot \eta(x)$.

Proof: Let $f \in \mathbb{Z}[x]$ be some polynomial of degree $\leqslant [m/2]$, and f_i be its coefficients. Consider the element $y := pr(\sum_i f_i \cdot \Phi^{s^{m-2i}}(\pi_*(h^i \cdot z)))$. Then $\overline{y} = \sum_i f_i \sum_j pr(\Phi^{s^{m-2i}}([P_{m-i-j}] \cdot \overline{z}^j))$, where $[P_l]$ is the class of quadric of dimension l in $\mathbb{L}$. By Proposition 2.3, this expression is equal to

$$-\sum_i \sum_j f_i \cdot \eta(P_{m-i-j}) \cdot pr(S^j_{L-N}(\overline{z}^j)) = -\sum_{j=0}^{m} g_{m-j} \cdot pr(S^j_{L-N}(\overline{z}^j)),$$

where the polynomial $g(x) = \sum_l g_l \cdot x^l$ is the degree $\leqslant m$ part of the product $f(x) \cdot \eta(x)$, where $\eta(x) = \sum_{r \geqslant 0} \eta(P_r) \cdot x^r$. $\square$

It follows from Proposition 3.12, and Lemmas 3.9, 3.10 and 3.11 that, for any $f \in \mathbb{Z}[x]$ of degree $\leqslant [m/2]$, the element

$$\sum_{j=0}^{m} g_{m-j} \cdot pr(S^j_{L-N}(\overline{u}^j)) = \sum_{j=0}^{m} g_{m-j}\alpha_{j,j} \cdot pr(S^j_{L-N}(\overline{v}^j))$$

is defined over k. Now it is sufficient to find a polynomial $f \in \mathbb{Z}[x]$ of degree $\leqslant [m/2]$ such that the polynomial $g(x) := (f(x) \cdot \eta(x))_{\leqslant m}$ has an odd coefficient at x^m and even coefficients at all the smaller monomials of the same pairity. We can pass to $\mathbb{Z}/2$-coefficients, where we have:

Lemma 3.13 *There exists such polynomial $f \in \mathbb{Z}/2[x]$ of degree $\leqslant [m/2]$ that $(f(x) \cdot \eta(x))_{\leqslant m}$ $(mod 2) = x^m +$ terms of parity $(m-1)$.*

Proof: Recall that $\eta_l = \eta(P_l) = (-1)^l \frac{(2l)!}{l!(l+1)!}$, and $\eta(x)$ $(mod 2) = \sum_{i \geqslant 0} x^{2^i - 1} = \gamma^{-1}$, where $\gamma = 1 + \sum_{i \geqslant 0} x^{2^i}$. We will use some facts about these power series obtained in [11].
In the case $m = 2n+1$ - odd, consider $f(x) := (\gamma^m)_{\leqslant n}$ $(= a_m$ in the notations of [11]). Since $\gamma^m = (\gamma^m)_{\leqslant n} + (\gamma^m)_{> n}$, we have that

$$((\gamma^m)_{\leqslant n} \cdot \gamma^{-1})_{\leqslant n} = \gamma^{2n}_{\leqslant n}$$

contains only terms of even degree. But by [11, Corollary 3.10 and (1)],

$$((\gamma^m)_{\leqslant n} \cdot \gamma^{-1})_{\leqslant 2n+1} = ((\gamma^m)_{\leqslant n} \cdot \gamma^{-1})_{\leqslant n} + x^{2n} + x^{2n+1}.$$

Hence, $(f \cdot \gamma^{-1})_{<m}$ consists of terms of even degree, and $(f \cdot \gamma^{-1})_m = x^m$.
In the case $m = 2n$ - even, it remains to take $f(x) := a_{m-1} \cdot x$. $\square$

Remark: Actually, the above polynomial f is unique. Moreover, it is exactly the polynomial δ appearing in the proof of [11, Proposition 3.5], where a completely different selection criterion was used!

Proposition 3.7 is proven.

$\square$

References

[1] R.Elman, N.Karpenko, A.Merkurjev, *The Algebraic and Geometric Theory of Quadratic Forms*, AMS Colloquium Publications, 56, 2008, 435pp.

[2] N.Karpenko, A.Merkurjev, *Rost projectors and Steenrod operations*, Documenta Math., 7 (2002), 481-493.

[3] M.Levine, F.Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer-Verlag, 2007.

[4] A.Merkurjev, *Rost degree formula*, Preprint, 2000, 1-19.

[5] A.Nenashev, K.Zainoulline, *Oriented Cohomology and motivic decompositions of relative cellular spaces*, J.Pure Appl. Algebra 205 (2006), no.2, 323-340.

[6] M.Rost, *Some new results on the Chowgroups of quadrics*, Preprint, 1990, 1-5. (http://www.math.uni-bielefeld.de/ rost/data/chowqudr.pdf)

[7] M.Rost, *The motive of a Pfister form*, Preprint, 1998, 1-13. (http://www.math.uni-bielefeld.de/ rost/motive.html)

[8] A.Vishik, *Symmetric operations* (in Russian), Trudy Mat. Inst. Steklova 246 (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 92-105. English transl.: Proc. of the Steklov Institute of Math. 246 (2004), 79-92.

[9] A.Vishik, *Generic points of quadrics and Chow groups*, Manuscripta Math 122 (2007), No.3, 365-374.

[10] A.Vishik, *Symmetric operations in Algebraic Cobordism*, Adv. Math 213 (2007), 489-552.

[11] A.Vishik, *Fields of u-invariant* $2^r + 1$, *Algebra, Arithmetic and Geometry - Manin Festschrift*, Birkhauser, 2010, in press.

[12] A.Vishik, N.Yagita, *Algebraic cobordisms of a Pfister quadric*, J. of London Math. Soc., 76 (2007), n.2, 586-604.

[13] K. Zainoulline, *Special correspondences and Chow traces of Landweber-Novikov operations*, J. Reine und Angew. Math. 628 (2009), 195-204.

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