K-Theory and the Enriched Tits Building

To A. A. Suslin with admiration, on his sixtieth birthday.

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Abstract. Motivated by the splitting principle, we define certain simplicial complexes associated to an associative ring A, which have an action of the general linear group $GL(A)$. This leads to an exact sequence, involving Quillen's algebraic K-groups of A and the symbol map. Computations in low degrees lead to another view on Suslin's theorem on the Bloch group, and perhaps show a way towards possible generalizations.

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The homology of $GL_n(A)$ has been studied in great depth by A.A. Suslin. In some of his works ([20] and [21] for example), the action of $GL_n(A)$ on certain simplicial complexes facilitated his homology computations.

We introduce three simplicial complexes in this paper. They are motivated by the splitting principle. The description of these spaces is given below. This is followed by the little information we possess on their homology. After that comes the connection with K-theory.

These objects are defined quickly in the context of affine algebraic groups as follows. Let G be a connected algebraic group[1] defined over a field k. The collection of minimal parabolic subgroups $P \subset G$ is denoted by $FL(G)$ and the collection of maximal k-split tori $T \subset G$ is denoted by $SPL(G)$. The simplicial complex $\mathbb{FL}(G)$ has $FL(G)$ as its set of vertices. Minimal parabolics $P_0, P_1, ..., P_r$ of G form an r-simplex if their intersection contains a maximal k-split torus[2]. The dimension of $\mathbb{FL}(G)$ is one less than the order of the Weyl

[1] Gopal Prasad informed us that we should take G reductive or k perfect. The standard classification, via root systems, of all parabolic subgroups containing a maximal split torus, requires this hypothesis.

[2] John Rognes has an analogous construction with maximal parabolics replacing minimal parabolics. His spaces, different homotopy types from ours, are connected with K-theory as well, see [15] .

group of any $T \in SPL(G)$. Dually, we define $\mathbb{SPL}(G)$ as the simplicial complex with $SPL(G)$ as its set of vertices, and $T_0, T_1, ..., T_r$ forming an r-simplex if they are all contained in a minimal parabolic. In general, $\mathbb{SPL}(G)$ is infinite dimensional.

That both $\mathbb{SPL}(G)$ and $\mathbb{FL}(G)$ have the same homotopy type can be deduced from corollary 7, which is a general principle. A third simplicial complex, denoted by $\mathbb{ET}(G)$, which we refer to as the *enriched Tits building*, is better suited for homology computations. This is the simplicial complex whose simplices are (nonempty) chains of the partially ordered set $\mathcal{E}(G)$ whose definition follows. For a parabolic subgroup $P \subset G$, we denote by $U(P)$ its unipotent radical and by $j(P) : P \to P/U(P)$ the given morphism. Then $\mathcal{E}(G)$ is the set of pairs (P, T) where $P \subset G$ is a parabolic subgroup and $T \subset P/U(P)$ is a maximal k-split torus. We say $(P', T') \leq (P, T)$ in $\mathcal{E}(G)$ if $P' \subset P$ and $j(P)^{-1}(T) \subset j(P')^{-1}T'$. Note that $\dim \mathcal{E}(G)$ is the split rank of the quotient of $G/U(G)$ by its center. Assume for the moment that this quotient is a simple algebraic group. Then $(P, T) \mapsto P$ gives a map to the cone of the Tits building. The topology of $\mathbb{ET}(G)$ is more complex than the topology of the Tits building, which is well known to be a bouquet of spheres.

When $G = GL(V)$, we denote the above three simplicial complexes by $\mathbb{FL}(V)$, $\mathbb{SPL}(V)$ and $\mathbb{ET}(V)$. These constructions have simple analogues even when one is working over an arbitrary associative ring A. Their precise definition is given with some motivation in section 2. Some basic properties of these spaces are also established in section 2. Amongst them is Proposition 11 which shows that $\mathbb{ET}(V)$ and $\mathbb{FL}(V)$ have the same homotopy type.

$\mathbb{ET}(A^n)$ has a polyhedral decomposition (see lemma 21). This produces a spectral sequence (see Theorem 2) that computes its homology. The $E^1_{p,q}$ terms and the differentials $d^1_{p,q}$ are recognisable since they involve only the homology groups of $\mathbb{ET}(A^a)$ for $a < n$. The differentials $d^r_{p,q}$ for $r > 1$ are not understood well enough, however.

There are natural inclusions $\mathbb{ET}(A^n) \hookrightarrow \mathbb{ET}(A^d)$ for $d > n$, and the induced map on homology factors through

$$H_m(\mathbb{ET}(A^n)) \to H_0(E_n(A), H_m(\mathbb{ET}(A^n))) \to H_m(\mathbb{ET}(A^d)).$$

where $E_n(A)$ is the group of elementary matrices (see Corollary 9).

For the remaining statements on the homology of $\mathbb{ET}(A^n)$, we assume that A is a commutative ring with many units, in the sense of Van der Kallen. See [12] for a nice exposition of the definition and its consequences. Commutative local rings A with infinite residue fields are examples of such rings. Under this assumption, $E_n(A)$ can be replaced by $GL_n(A)$ in the above statement.

We have observed that $\mathbb{ET}(A^{m+1})$ has dimension m. Thus it is natural question to ask whether

$$H_0(GL_{m+1}(A), H_m(\mathbb{ET}(A^{m+1}))) \otimes \mathbb{Q} \to H_m(\mathbb{ET}(A^d)) \otimes \mathbb{Q}$$

is an isomorphism when $d > m + 1$. Theorem 3 asserts that this is true for $m = 1, 2, 3$. The statement is true in general (see Proposition 29) if a certain

Compatible Homotopy Question has an affirmative answer. The higher differentials of the spectral sequence can be dealt with if this is true. Proposition 22 shows that this holds in some limited situations.
The computation of $H_0(GL_3(A), H_2(\mathbb{ET}(A^3))) \otimes \mathbb{Q}$ is carried out at in the last lemma of the paper. This is intimately connected with Suslin's result (see [21]) connecting K_3 and the Bloch group. A closed form for $H_0(GL_4(A), H_3(\mathbb{ET}(A^4))$ is awaited. This should impact on the study of $K_4(A)$.
We now come to the connection with the Quillen K-groups $K_i(A)$ as obtained by his plus construction.
$GL(A)$ acts on the geometric realisation $|\mathbb{SPL}(A^\infty)|$ and thus we have the Borel construction, namely the quotient of $|\mathbb{SPL}(A^\infty)| \times EGL(A)$ by $GL(A)$, a familiar object in the study of equivariant homotopy. We denote this space by $\mathbb{SPL}(A^\infty)//GL(A)$. We apply Quillen's plus construction to $\mathbb{SPL}(A^\infty)//GL(A)$ and a suitable perfect subgroup of its fundamental group to obtain a space $Y(A)$. Proposition 17 shows that $Y(A)$ is an H-space and that the natural map $Y(A) \to BGL(A)^+$ is an H-map. Its homotopy fiber, denoted by $\mathbb{SPL}(A^\infty)^+$, is thus also a H-space. The n-th homotopy group of $\mathbb{SPL}(A^\infty)^+$ at its canonical base point is denoted by $L_n(A)$. There is of course a natural map $\mathbb{SPL}(A^\infty) \to \mathbb{SPL}(A^\infty)^+$. That this map is a homology isomorphism is shown in lemma 16. This assertion is easy, but not tautological: it relies once again on the triviality of the action of $E(A)$ on $H_*(\mathbb{SPL}(A^\infty))$. As a consequence of this lemma, $L_n(A) \otimes \mathbb{Q}$ is identified with the primitive rational homology of $\mathbb{SPL}(A^\infty)$, or equivalently, that of $\mathbb{ET}(A^\infty)$.
We have the inclusion $N_n(A) \hookrightarrow GL_n(A)$, where $N_n(A)$ is the semidirect product of the permutation group S_n with $(A^\times)^n$. Taking direct limits over $n \in \mathbb{N}$, we obtain $N(A) \subset GL(A)$. Let $H' \subset N(A)$ be the infinite alternating group and let H be the normal subgroup generated by H'. Applying Quillen's plus construction to the space $BN(A)$ with respect to H, we obtain $BN(A)^+$. Its n-th homotopy group is defined to be $\mathcal{H}_n(A^\times)$. From the Dold-Thom theorem, it is easy to see that $\mathcal{H}_n(A^\times) \otimes \mathbb{Q}$ is isomorphic to the group homology $H_n(A^\times) \otimes \mathbb{Q}$. When A is commutative, this is simply $\wedge^n_{\mathbb{Q}}(A^\times \otimes \mathbb{Q})$. Proposition 20 identifies the groups $\mathcal{H}_n(A^\times)$ with certain stable homotopy groups. Its proof was shown to us by J. Peter May. It is sketched in the text of the paper after the proof of the Theorem below.

Theorem 1. *Let A be a Nesterenko-Suslin ring. Then there is a long exact sequence, functorial in A:*

$$\cdots \to L_2(A) \to \mathcal{H}_2(A^\times) \to K_2(A) \to L_1(A) \to \mathcal{H}_1(A^\times) \to K_1(A) \to L_0(A) \to 0.$$

We call a ring A Nesterenko-Suslin if it satisfies the hypothesis of Remark 1.13 of their paper [13]. The precise requirement is that for every finite set F, there is a function $f_F : F \to$ the *center* of A so that the sum $\Sigma\{f_F(s) : s \in S\}$ is a *unit* of A for every nonempty $S \subset F$. If k is an infinite field, every associative k-algebra is Nesterenko-Suslin, and so is every commutative ring with many units in the sense of Van der Kallen. Remark 1.13 of Nesterenko-Suslin [13]

permits us to ignore unipotent radicals. This is used crucially in the proof of Theorem 1 (see also Proposition 12).
In the first draft of the paper, we conjectured that this theorem is true without any hypothesis on A. Sasha Beilinson then brought to our attention Suslin's paper [22] on the equivalence of Volodin's K-groups and Quillen's. From Suslin's description of Volodin's spaces, it is possible to show that these spaces are homotopy equivalent to the total space of the $N_n(A)$-torsor on $\mathbb{FL}(A^n)$ given in section 2 of this paper. This requires proposition 1 and a little organisation. Once this is done, Corollary 9 can also be obtained from Suslin's set-up. The statement "$X(R)$ is acyclic" stated and proved by Suslin in [22] now validates Proposition 12 at the infinite level, thus showing that Theorem 1 is true without any hypothesis on A. The details have not been included here.
R. Kottwitz informed us that the maximal simplices of $\mathbb{FL}(V)$ are referred to as "regular stars" in the work of Langlands(see [5]).
We hope that this paper will eventually connect with mixed Tate motives (see[3],[1]).
The arrangement of the paper is a follows. Section 1 has some topological preliminaries used through most of the paper. The proofs of Corollary 7 and Proposition 11 rely on Quillen's Theorem A. Alternatively, they can both be proved directly by repeated applications of Proposition 1. The definitions of $\mathbb{SPL}(A^n), \mathbb{FL}(A^n), \mathbb{ET}(A^n)$ and first properties are given in section two. The next four sections are devoted to the proof of Theorem 1. The last four sections are concerned with the homology of $\mathbb{ET}(A^n)$.
The lemmas, corollaries and propositions are labelled sequentially. For instance, corollary 9 is followed by lemma 10 and later by proposition 11; there is no proposition 10 or corollary 10. The other numbered statements are the three theorems. Theorems 2 and 3 are stated and proved in sections 7 and 9 respectively. Section 0 records some assumptions and notation, some perhaps non-standard, that are used in the paper. The reader might find it helpful to glance at this section for notation regarding elementary matrices and the Borel construction and the use of "simplicial complexes".

0. Assumptions and Notation

Rings, Elementary matrices, $\mathrm{Elem}(W \hookrightarrow V), \mathrm{Elem}(V, q)$, $\mathcal{L}(V)$ *and* $\mathcal{L}_p(V)$

We are concerned with the Quillen K-groups of a ring A.
We assume that A has the following property: if $A^m \cong A^n$ as left A-modules, then $m = n$. The phrase "A-module" always means left A-module.
For a finitely generated free A-module V, the collection of A-submodules $L \subset V$ so that (i) V/L is free and (ii) L is free of rank one , is denoted by $\mathcal{L}(V)$.
$\mathcal{L}_p(V)$ is the collection of subsets q of cardinality $(p+1)$ of $\mathcal{L}(V)$ so that $\oplus\{L : L \in q\} \to V$ is a monomorphism whose cokernel is free.
Given an A-submodule W of a A-module V so that the short exact sequence

$$0 \to W \to V \to V/W \to 0$$

is split, we have the subgroup $\mathrm{Elem}(W \hookrightarrow V) \subset Aut_A(V)$, defined as follows. Let $H(W)$ be the group of automorphisms h of V so that
$(id_V - h)V \subset W \subset ker(id_V - h)$. Let $W' \subset V$ be a submodule that is complementary to W. Define $H(W')$ in the same manner. The subgroup of $Aut_A(V)$ generated by $H(W)$ and $H(W')$ is $\mathrm{Elem}(W \hookrightarrow V)$. It does not depend on the choice of W' because $H(W)$ acts transitively on the collection of such W'.
For example, if $V = A^n$, and W is the A-submodule generated by any r members of the given basis of A^n, then $\mathrm{Elem}(W \hookrightarrow A^n)$ equals $E_n(A)$, the subgroup of elementary matrices in $GL_n(A)$, provided of course that $0 < r < n$.
If V is finitely generated free and if $q \in \mathcal{L}_p(V)$, the above statement implies that $\mathrm{Elem}(L \hookrightarrow V)$ does not depend on the choice of $L \in q$. Thus we denote this subgroup by $\mathrm{Elem}(V, q) \subset GL(V)$.

The Borel Construction

Let X be a topological space equipped with the action of group G. Let EG be the principal G-bundle on BG (as in [14]). The Borel construction, namely the quotient of $X \times EG$ by the G-action, is denoted by $X//G$ throughout the paper.

Categories, Geometric realisations, Posets

Every category $\mathcal{C}$ gives rise to a simplicial set, namely its nerve (see [14]). Its geometric realisation is denoted by $B\mathcal{C}$.
A poset (partially ordered set) P gives rise to a category. The B-construction of this category, by abuse of notation, is denoted by BP. Associated to P is the simplicial complex with P as its set of vertices; the simplices are finite non-empty chains in P. The geometric realisation of this simplicial complex coincides with BP.

Simplicial Complexes, Products and Internal Hom, Barycentric subdivision

Simplicial complexes crop up throughout this paper. We refer to Chapter 3,[18], for the definition of a simplicial complex and its barycentric subdivision. $\mathcal{S}(K)$ and $\mathcal{V}(K)$ denote the sets of vertices and simplices respectively of a simplicial complex K. The geometric realisation of K is denoted by $|K|$. The set $\mathcal{S}(K)$ is a partially ordered set (with respect to inclusion of subsets). Note that $B\mathcal{S}(K)$ is simply the (geometric realisation of) the barycentric subdivision $sd(K)$. The geometric realisations of K and $sd(K)$ are canonically homeomorphic to each other, but not by a simplicial map.
Given simplicial complexes K_1 and K_2, the product $|K_1| \times |K_2|$ (in the compactly generated topology) is canonically homeomorphic to $B(\mathcal{S}(K_1) \times \mathcal{S}(K_2))$.

The category of simplicial complexes and simplicial maps has a *categorical product*:
$\mathcal{V}(K_1 \times K_2) = \mathcal{V}(K_1) \times \mathcal{V}(K_2)$. A non-empty subset of $\mathcal{V}(K_1 \times K_2)$ is a simplex of $K_1 \times K_2$ if and only if it is contained in $S_1 \times S_2$ for some $S_i \in \mathcal{S}(K_i)$ for $i = 1, 2$. The geometric realisation of the product is not homeomorphic to the product of the geometric realisations, but they do have the same homotopy

type. In fact Proposition 1 of section 1 provides a contractible collection of homotopy equivalences $|K_1| \times |K_2| \to |K_1 \times K_2|$. For most purposes, it suffices to note that there is a *canonical* map $P(K_1, K_2) : |K_1| \times |K_2| \to |K_1 \times K_2|$. This is obtained in the following manner. Let $C(K)$ denote the $\mathbb{R}$-vector space with basis $\mathcal{V}(K)$ for a simplicial complex K. Recall that $|K|$ is a subset of $C(K)$. For simplicial complexes K_1 and K_2, we have the evident isomorphism

$$j : C(K_1) \otimes_{\mathbb{R}} C(K_2) \to C(K_1 \times K_2).$$

For $c_i \in |K_i|$ for $i = 1, 2$ we put $P(K_1, K_2)(c_1, c_2) = j(c_1 \otimes c_2) \in C(K_1 \times K_2)$. We note that $j(c_1 \otimes c_2)$ belongs to the subset $|K_1 \times K_2|$. This gives the canonical $P(K_1, K_2)$.

Given simplicial complexes K, L there is a simplicial complex $\mathcal{H}om(K, L)$ with the following property: if M is a simplicial complex, then the set of simplicial maps $K \times M \to L$ is naturally identified with the set of simplicial maps $M \to \mathcal{H}om(K, L)$. This simple verification is left to the reader.

Simplicial maps $f : K_1 \times K_2 \to K_3$ occur in sections 2 and 5 of this paper.

$$|f| \circ P(K_1, K_2) : |K_1| \times |K_2| \to |K_3|$$

is the map we employ on geometric realisations. Maps $|K_1| \times |K_2| \to |K_3|$ associated to simplicial maps f_1 and f_2 are seen (by contiguity) to be homotopic to each other if $\{f_1, f_2\}$ is a simplex of $\mathcal{H}om(K_1 \times K_2, K_3)$. This fact is employed in Lemma 8.
Simplicial maps $f : K_1 \times K_2 \to K_3$ are in reality maps $\mathcal{V}(f) : \mathcal{V}(K_1) \times \mathcal{V}(K_2) \to \mathcal{V}(K_3)$ with the property that $\mathcal{V}(f)(S_1 \times S_2)$ is a simplex of K_3 whenever S_1 and S_2 are simplices of K_1 and K_2 respectively. One should note that such an f induces a map of posets $\mathcal{S}(K_1) \times \mathcal{S}(K_2) \to \mathcal{S}(K_3)$, which in turn induces a continuous map $B(\mathcal{S}(K_1) \times \mathcal{S}(K_2)) \to B\mathcal{S}(K_3)$. In view of the natural identifications, this is the same as giving a map $|K_1| \times |K_2| \to |K_3|$. This map coincides with the $|f| \circ P(K_1, K_2)$ considered above.
The homotopy assertion of maps $|K_1| \times |K_2| \to |K_3|$ associated to f_1, f_2 where $\{f_1, f_2\}$ is an edge of $\mathcal{H}om(K_1 \times K_2, K_3)$ cannot be proved by the quick poset definition of the maps (for $|K_3|$ has been subdivided and contiguity is not available any more). This explains our preference for the longwinded $|f| \circ P(K_1, K_2)$ definition.

1. Some preliminaries from topology

We work with the category of compactly generated weakly Hausdorff spaces. A good source is Chapter 5 of [11]. This category possesses products. It also possesses an internal Hom in the following sense: for compactly generated Hausdorff X, Y, Z, continuous maps $Z \to \mathcal{H}om(X, Y)$ are the same as continuous maps $Z \times X \to Y$, where $Z \times X$ denotes the product in this category. This internal Hom property is required in the proof of Proposition 1 stated below.

$\mathcal{H}om(X,Y)$ is the space of continuous maps from X to Y. This space of maps has the compact-open topology, which is then replaced by the inherited compactly generated topology. This space $\mathcal{H}om(X,Y)$ is referred to frequently as $\mathrm{Map}(X,Y)$, and some times even as $\mathrm{Maps}(X,Y)$, in the text.
Now consider the following set-up. Let Λ be a partially ordered set assumed to be Artinian: (i) every non-empty subset in Λ has a minimal element with respect to the partial order, or equivalently (ii) there are no infinite strictly descending chains $\lambda_1 > \lambda_2 > \cdots$ in Λ. The poset Λ will remain fixed throughout the discussion below.
We consider topological spaces X equipped with a family of closed subsets $X_\lambda, \lambda \in \Lambda$ with the property that $X_\mu \subset X_\lambda$ whenever $\mu \leq \lambda$.
Given another $Y, Y_\lambda, \lambda \in \Lambda$ as above, the collection of Λ-compatible continuous $f : X \to Y$ (i.e. satisfying $f(X_\lambda) \subset Y_\lambda, \forall \lambda \in \Lambda$) will be denoted by $Map_\Lambda(X,Y)$. $Map_\Lambda(X,Y)$ is a closed subset of $\mathcal{H}om(X,Y)$, and this topologises $Map_\Lambda(X,Y)$.
We say that $\{X_\lambda\}$ is a *weakly admissible covering* of X if the three conditions listed below are satisfied. It is an *admissible covering* if in addition, each X_λ is contractible.

(1) For each pair of indices $\lambda, \mu \in \Lambda$, we have
$$X_\lambda \cap X_\mu = \bigcup_{\nu \leq \lambda, \nu \leq \mu} X_\nu$$
(2) If
$$\partial X_\lambda = \bigcup_{\nu < \lambda} X_\nu,$$
then $\partial X_\lambda \hookrightarrow X_\lambda$ is a cofibration
(3) The topology on X is coherent with respect to the family of subsets $\{X_\lambda\}_{\lambda \in \Lambda}$, that is, $X = \cup_\lambda X_\lambda$, and a subset $Z \subset X$ is closed precisely when $Z \cap X_\lambda$ is closed in X_λ in the relative topology, for all λ.

Proposition 1. *Assume that $\{X_\lambda\}$ is a weakly admissible covering of X. Assume also that each Y_λ is contractible.*
Then the space $\mathrm{Map}_\Lambda(X,Y)$ of Λ-compatible maps $f : X \to Y$ is contractible. In particular, it is non-empty and path-connected.

Corollary 2. *If both $\{X_\lambda\}$ and $\{Y_\lambda\}$ are admissible, then X and Y are homotopy equivalent.*

With assumptions as in the above corollary, the proposition yields the existence of Λ-compatible maps $f : X \to Y$ and $g : Y \to X$. Because $g \circ f$ and $f \circ g$ are also Λ-compatible, that they are homotopic to id_X and id_Y respectively is deduced from the path-connectivity of $\mathrm{Map}_\Lambda(X,X)$ and $\mathrm{Map}_\Lambda(Y,Y)$.

Corollary 3. *If $\{X_\lambda\}$ is admissible, then there is a homotopy equivalence $X \to B\Lambda$.*

Here, recall that $B\Lambda$ is the geometric realization of the simplicial complex associated to the set of nonempty finite chains (totally ordered subsets) in Λ; equivalently, regarding Λ as a category, $B\Lambda$ is the geometric realization of its

nerve. We put $Y = B\Lambda$ and $Y_\lambda = B\{\mu \in \Lambda : \mu \leq \lambda\}$ in Corollary 2 to deduce Corollary 3.
The proof of Proposition 1 is easily reduced to the following extension lemma.

Lemma 4. *Let $\{X_\lambda\}, \{Y_\lambda\}$ etc. be as in the above proposition. Let $\Lambda' \subset \Lambda$ be a subset, with induced partial order, so that for any $\lambda \in \Lambda'$, $\mu \in \Lambda$ with $\mu \leq \lambda$, we have $\mu \in \Lambda'$. Let $X' = \cup_{\lambda \in \Lambda'} X_\lambda$, $Y' = \cup_{\lambda \in \Lambda'} Y_\lambda$. Assume given a continuous map $f' : X' \to Y'$ with $f'(X_\lambda) \subset Y_\lambda$ for all $\lambda \in \Lambda'$. Then f' extends to a continuous map $f : X \to Y$ with $f(X_\lambda) \subset Y_\lambda$ for all $\lambda \in \Lambda$.*

Proof. Consider the collection of pairs (Λ'', f'') satisfying:
(a) $\Lambda' \subset \Lambda'' \subset \Lambda$
(b) $\mu \in \Lambda, \lambda \in \Lambda'', \mu \leq \lambda$ implies $\mu \in \Lambda''$
(c) $f'' : \cup\{X_\mu | \mu \in \Lambda''\} \to Y$ is a continuous map
(d) $f''(X_\mu) \subset Y_\mu$ for all $\mu \in \Lambda''$
(e) $f'|X_\mu = f''|X_\mu$ for all $\mu \in \Lambda'$
This collection is partially ordered in a natural manner. The coherence condition on the topology of X ensures that every chain in this collection has an upper bound. The presence of (Λ', f') shows that it is non-empty. By Zorn's lemma, there is a maximal element (Λ'', f'') in this collection. The Artinian hypothesis on Λ shows that if $\Lambda'' \neq \Lambda$, then its complement possesses a minimal element μ. Let D'' be the domain of f''. The minimality of μ shows that $D'' \cap X_\mu = \partial X_\mu$. By condition (d) above, we see that $f''(\partial X_\mu)$ is contained in the contractible space Y_μ. Because $\partial X_\mu \hookrightarrow X_\mu$ is a cofibration, it follows that $f''|\partial X_\mu$ extends to a map $g : X_\mu \to Y_\mu$. The f'' and g patch together to give a continuous map $h : D'' \cup X_\mu \to Y$. Since the pair $(\Lambda'' \cup \{\mu\}, h)$ evidently belongs to this collection, the maximality of (Λ'', f'') is contradicted. Thus $\Lambda'' = \Lambda$ and this completes the proof.

□

The proof of the Proposition follows in three standard steps.
Step 1: Taking $\Lambda' = \emptyset$ in Lemma 4 we deduce that $\mathrm{Map}_\Lambda(X, Y)$ is nonempty.
Step 2: For the path-connectivity of $\mathrm{Map}_\Lambda(X, Y)$, we replace X by $X \times [0, 1]$ and replace the original poset Λ by the product $\Lambda \times \{\{0\}, \{1\}, \{0, 1\}\}$, with the product partial order, where the second factor is partially ordered by inclusion. The subsets of $X \times I$ (resp.Y) indexed by $(\lambda, 0), (\lambda, 1), (\lambda, \{0, 1\})$ are $X_\lambda \times \{0\}, X_\lambda \times \{1\}$ and $X_\lambda \times [0, 1]$ (resp. Y_λ in all three cases).
We then apply the lemma to the sub-poset $\Lambda \times \{\{0\}, \{1\}\}$.
Step 3: Finally, for the contractibility of $\mathrm{Map}_\Lambda(X, Y)$, we first choose $f_0 \in \mathrm{Map}_\Lambda(X, Y)$ and then consider the two maps $\mathrm{Map}_\Lambda(X, Y) \times X \to Y$ given by $(f, x) \mapsto f(x)$ and $(f, x) \mapsto f_0(x)$. Putting $(\mathrm{Map}_\Lambda(X, Y) \times X)_\lambda = \mathrm{Map}_\Lambda(X, Y) \times X_\lambda$ for all $\lambda \in \Lambda$, we see that both the above maps are Λ-compatible. The path-connectivity assertion in Step 2 now gives a homotopy between the identity map of $\mathrm{Map}_\Lambda(X, Y)$ and the constant map $f \mapsto f_0$. This completes the proof of Proposition 1.
We now want to make some remarks about equivariant versions of the above statements.

Given $X, \{X_\lambda; \lambda \in \Lambda\}$ as above, an action of a group G on X is called Λ-compatible if G also acts on the poset Λ so that for all $g \in G$, $\lambda \in \Lambda$, we have $g(X_\lambda) = X_{g\lambda}$.

Under the conditions of Proposition 1, suppose $\{X_\lambda\}$, $\{Y_\lambda\}$ admit Λ-compatible G-actions. There is no G-equivariant $f \in Map_\Lambda(X,Y)$ in general. However, if $f \in Map_\Lambda(X,Y)$ and g_X, g_Y denote the actions of $g \in G$ on X and Y respectively, we see that , then $g_Y \circ f \circ g_X^{-1}$ is also a Λ-compatible map. By Proposition 1, we see that this map is homotopic to f. Thus $g_Y \circ f$ and $f \circ g_X$ are homotopic to each other. *In particular,* $H_n(f) : H_n(X) \to H_n(Y)$ *is a homomorphism of* G*-modules.*

In the sequel a better version of this involving the Borel construction is needed. We recall the *Borel construction* of equivariant homotopy quotient spaces. Let EG denote a contractible CW complex on which G has a proper free cellular action; for our purposes, it suffices to fix a choice of this space EG to be the geometric realization of the nerve of the translation category of G (the category with vertices $[g]$ indexed by the elements of G, and unique morphisms between ordered pairs of vertices $([g],[h])$, thought of as given by the left action of hg^{-1}). The classifying space BG is the quotient space EG/G.

If X is any G-space, let $X//G$ denote the *homotopy quotient of* X *by* G, obtained using the Borel construction, i.e.,

$$X//G = (X \times EG)/G, \tag{1}$$

where EG is as above, and G acts diagonally. Note that the natural quotient map

$$q_X : X \times EG \to X//G$$

is a Galois covering space, with covering group G.

If X and Y are G-spaces, then considering G-equivariant maps $\widetilde{f} : X \times EG \to Y \times EG$ compatible with the projections to EG, giving a commutative diagram

$$\begin{tikzcd} X \times EG \arrow[rr, "\widetilde{f}"] \arrow[dr] & & Y \times EG \arrow[dl] \\ & EG & \end{tikzcd}$$

is equivalent to considering maps $\overline{f} : X//G \to Y//G$ compatible with the maps $q_X : X//G \to BG$, $q_Y : Y//G \to BG$, giving a commutative diagram

$$\begin{tikzcd} X//G \arrow[rr, "\overline{f}"] \arrow[dr, "q_X"'] & & Y//G \arrow[dl, "q_Y"] \\ & BG & \end{tikzcd}$$

Proposition 5. *Assume that, in the situation of proposition 1, there are* Λ*-compatible* G*-actions on* X *and* Y*. Let* EG *be as above, and consider the* Λ*-compatible families* $\{X_\lambda \times EG\}$*, which is a weakly admissible covering family for* $X \times EG$*, and* $\{Y_\lambda \times EG\}$*, which is an admissible covering family for* $Y \times EG$*.*

Then there is a G-equivariant map $\widetilde{f} : X \times EG \to Y \times EG$, compatible with the projections to EG, such that

(i) $\widetilde{f}(X_\lambda \times EG) \subset (Y_\lambda \times EG)$ *for all* $\lambda \in \Lambda$

(ii) *if* $\widetilde{g} : X \times EG \to Y \times EG$ *is another such equivariant map, then there is a G-equivariant homotopy between $\widetilde{f}$ and $\widetilde{g}$, compatible with the projections to EG*

(iii) *The space of such equivariant maps $X \times EG \to Y \times EG$, as in (i), is contractible.*

Proof. We show the existence of the desired map, and leave the proof of other properties, by similar arguments, to the reader.

Let $\mathrm{Map}_\Lambda(X, Y)$ be the contractible space of Λ-compatible maps from X to Y; note that it comes equipped with a natural G-action, so that the canonical evaluation map $X \times \mathrm{Map}_\Lambda(X, Y) \to Y$ is equivariant. This induces $X \times \mathrm{Map}_\Lambda(X, Y) \times EG \to Y \times EG$. There is also a natural G-equivariant map $\pi : X \times \mathrm{Map}_\Lambda(X, Y) \times EG \to X \times EG$. This map π has equivariant sections, since the projection $\mathrm{Map}_\Lambda(X, Y) \times EG \to EG$ is a G-equivariant map between weakly contractible spaces, so that the map on quotients modulo G is a weak homotopy equivalence (i.e., $(\mathrm{Map}_\Lambda(X, Y) \times EG)/G$ is another "model" for the classifying space $BG = EG/G$). However BG is a CW complex, so the map has a section. □

As another preliminary, we note some facts (see lemma 6 below) which are essentially corollaries of Quillen's Theorem A (these are presumably well-known to experts, though we do not have a specific reference).

If P is any poset, let $C(P)$ be the poset consisting of non-empty finite chains (totally ordered subsets) of P. If $f : P \to Q$ is a morphism between posets (an order preserving map) there is an induced morphism $C(f) : C(P) \to C(Q)$. If S is a simplicial complex (literally, a collection of finite non-empty subsets of the vertex set), we may regard S as a poset, partially ordered with respect to inclusion; then the classifying space BS is naturally homeomorphic to the geometric realisation $|S|$ (and gives the barycentric subdivision of $|S|$). A simplicial map $f : S \to T$ between simplicial complexes (that is, a map on vertex sets which sends simplices to simplices, not necessarily preserving dimension) is also then a morphism of posets. We say that a poset P is contractible if its classifying space BP is contractible.

Lemma 6. *(i) Let $f : P \to Q$ be a morphism between posets. Suppose that for each $X \in C(Q)$, the fiber poset $C(f)^{-1}(X)$ is contractible. Then $Bf : BP \to BQ$ is a homotopy equivalence.*

(ii) Let $f : S \to T$ be a simplicial map between simplicial complexes. Suppose that for any simplex $\sigma \in T$, the fiber $f^{-1}(\sigma)$, considered as a poset, is contractible. Then $|f| : |S| \to |T|$ is a homotopy equivalence.

Proof. We first prove (i). For any poset P, there is morphism of posets $\varphi_P : C(P) \to P$, sending a chain to its first (smallest) element. If $a, b \in P$ with $a \leq b$, and C is a chain in $\varphi_P^{-1}(b)$, then $\{a\} \cup C$ is a chain in $\varphi_P^{-1}(a)$. This gives an order preserving map of posets $\varphi_P^{-1}(b) \to \varphi_P^{-1}(a)$ (i.e., a "base-change" functor). This makes $C(P)$ prefibred over P, in the sense of Quillen (see page 96 in [19], for example). Also, $\varphi_P^{-1}(a)$ has the minimal element (initial object) $\{a\}$, and so its classifying space is contractible.

Hence Quillen's Theorem A (see [19], page 96) implies that $B(\varphi_P)$ is a homotopy equivalence, for any P.

Now let $f : P \to Q$ be a morphism between posets. Let $C(f) : C(P) \to C(Q)$ be the corresponding morphism on the posets of (finite, nonempty) chains. If $A \subset B$ are two chains in $C(Q)$, there is an obvious order preserving map $C(f)^{-1}(B) \to C(f)^{-1}(A)$. Again, this makes $C(f) : C(P) \to C(Q)$ prefibred. Since we assumed that $BC(f)^{-1}(A)$ is contractible, for all $A \in C(Q)$, Quillen's Theorem A implies that $BC(f)$ is a homotopy equivalence.

We thus have a commutative diagram of posets and order preserving maps

$$\begin{array}{ccc} C(P) & \stackrel{C(f)}{\to} & C(Q) \\ \varphi_P \downarrow & & \downarrow \varphi_Q \\ P & \stackrel{f}{\to} & Q \end{array}$$

where three of the four sides yield homotopy equivalences on passing to classifying spaces. Hence $Bf : BP \to BQ$ is a homotopy equivalence, proving (i).

The proof of (ii) is similar. This is equivalent to showing that $Bf : BS \to BT$ is a homotopy equivalence. Since $f : S \to T$, regarded as a morphism of posets, is naturally prefibered, and by assumption, $Bf^{-1}(\sigma)$ is contractible for each $\sigma \in T$, Quillen's Theorem A implies that Bf is a homotopy equivalence. $\square$

We make use of Propositions 1 and 5 in the following way.

Let A, B be sets, $Z \subset A \times B$ a subset such that the projections $p : Z \to A$, $q : Z \to B$ are both surjective. Consider simplicial complexes $S_Z(A)$, $S_Z(B)$ on vertex sets A, B respectively, with simplices in $S_Z(A)$ being finite nonempty subsets of fibers $q^{-1}(b)$, for any $b \in B$, and simplices in $S_Z(B)$ being finite, nonempty subsets of fibers $p^{-1}(a)$, for any $a \in A$.

Consider also a third simplicial complex $S_Z(A,B)$ with vertex set Z, where a finite non-empty subset $Z' \subset Z$ is a simplex if and only it satisfies the following condition:

$$(a_1,b_1),(a_2,b_2) \in Z' \;\Rightarrow (a_1,b_2) \in Z.$$

Note that the natural maps on vertex sets $p : Z \to A$, $q : Z \to B$ induce canonical simplicial maps on geometric realizations

$$|p| : |S_Z(A,B)| \to |S_Z(A)|, \;\; |q| : |S_Z(A,B)| \to |S_Z(B)|.$$

Corollary 7. (1) *With the above notation, the simplicial maps*

$$|p| : |S_Z(A,B)| \to |S_Z(A)|, \;\; |q| : |S_Z(A,B)| \to |S_Z(B)|$$

are homotopy equivalences. In particular, $|S_Z(A)|$, $|S_Z(B)|$ are homotopy equivalent.

(2) *If a group G acts on A and on B, so that Z is stable under the diagonal G action on $A \times B$, then the homotopy equivalences $|p|$, $|q|$ are G-equivariant homotopy equivalences. Hence there exists a natural G-equivariant homotopy equivalence between $|S_Z(A)| \times EG$ and $|S_Z(B)| \times EG$.*

Proof. Since the situation is symmetric with respect to the sets A, B, it suffices to show $|p|$ is a homotopy equivalence. Note that in the context of a G-action as stated, the G-equivariance of $|p|$ is clear.
Let Λ be the poset of all simplices of $S_Z(A)$, thought of as subsets of A, and ordered by inclusion. Clearly Λ is Artinian.
Apply Corollary 2 with $X = |S_Z(A, B)|$, $Y = |S_Z(A)|$, Λ as above, and the following Λ-admissible coverings: for $\sigma \in \Lambda$, let Y_σ be the (closed) simplex in $Y = |S_Z(A)|$ determined by σ (clearly $\{Y_\sigma\}$ is admissible); take $X_\sigma = |p|^{-1}(Y_\sigma)$ (this is evidently weakly admissible). For admissibility of $\{X_\sigma\}$, we need to show that each X_σ is contractible.
In fact, regarding the sets of simplices $S_Z(A, B)$ and $S_Z(A)$ as posets, and $S_Z(A, B) \to S_Z(A)$ as a morphism of posets, X_σ is the geometric realization of the simplicial complex determined by $\cup_{\tau \leq \sigma} p^{-1}(\tau)$.
The corresponding map of posets

$$p^{-1}(\{\tau | \tau \leq \sigma\}) \to \{\tau | \tau \leq \sigma\}$$

has contractible fiber posets – if we fix an element $x \in p^{-1}(\tau)$, and $p^{-1}(\tau)(\geq x)$ is the sub-poset of elements bounded below by x, then $y \mapsto y \cup x$ is a morphism of posets $r_x : p^{-1}(\tau) \to p^{-1}(\tau)(\geq x)$ which gives a homotopy equivalence on geometric realizations (it is left adjoint to the inclusion of the sub-poset). But the sub-poset has a minimal element, and so its realization is contractible.
The poset $\{\tau | \tau \leq \sigma\}$ is obviously contractible, since it has a maximal element. Hence, applying lemma 6(ii), X_σ is contractible.
Since the map $|p| : X \to Y$ is Λ-compatible, it is the unique such map upto Λ-compatible homotopy, and is a homotopy equivalence. $\square$

We note that the argument with Quillen's Theorem A in fact implies directly that $S_Z(A, B) \to S_Z(A)$ is a homotopy equivalence; the uniqueness assertion is not, apparently, a formal consequence of Quillen's Theorem A.

2. Flag Spaces

In this section, we discuss various constructions of spaces (generally simplicial complexes) defined using flags of free modules, and various maps, and homotopy equivalences, between these. These are used as building blocks in the proof of Theorem 1.
Let A be a ring, and let V be a free (left) A-module of rank n. Define a simplicial complex $\mathbb{FL}(V)$ as follows.
Its vertex set is

$FL(V) =$

$$= \{F = (F_0, F_1, \ldots, F_n) \mid 0 = F_0 \subset F_1 \subset \cdots \subset F_n = V \text{ are } A\text{-submodules}, \\ \text{and each quotient } F_i/F_{i-1} \text{ is a free } A\text{-module of rank } 1\}.$$

We think of this vertex set as the set of "full flags" in V.
To describe the simplices in $\mathbb{FL}(V)$, we need another definition. Let

$$SPL(V) = \{\{L_1, \ldots, L_n\} \mid L_i \subset V \text{ is a free } A\text{-submodule of rank 1, and} \\ \text{the induced map } \oplus_{i=1}^n L_i \to V \text{ is an isomorphism}\}.$$

Note that $\{L_1, \ldots, L_n\}$ is regarded as an *unordered* set of free A-submodules of rank 1 of L (i.e., as a subset of cardinality n in the set of all free A-submodules of rank 1 of V). We think of $SPL(V)$ as the "set of unordered splittings of V into direct sums of free rank 1 modules".
Given $\alpha \in SPL(V)$, say $\alpha = \{L_1, \ldots, L_n\}$, we may choose some ordering $(L_1, \ldots, L_n)$ of its elements, and thus obtain a full flag in V (i.e., an element in $FL(V)$), given by

$$(0, L_1, L_1 \oplus L_2, \cdots, L_1 \oplus \cdots \oplus L_n = V) \in FL(V).$$

Let

$$[\alpha] \subset FL(V)$$

be the set of $n!$ such full flags obtained from α.
We now define a simplex in $\mathbb{FL}(V)$ to be any subset of such a set $[\alpha]$ of vertices, for any $\alpha \in SPL(V)$. Thus, $\mathbb{FL}(V)$ becomes a simplicial complex of dimension $n! - 1$, with the sets $[\alpha]$ as above corresponding to maximal dimensional simplices.
Clearly $\mathrm{Aut}\,(V) \cong \mathrm{GL}_n(A)$ acts on the simplical complex $\mathbb{FL}(V)$ through simplicial automorphisms, and thus acts on the homology groups $H_*(\mathbb{FL}(V), \mathbb{Z})$ (and other similar invariants of $\mathbb{FL}(V)$).
Next, remark that if $F \in FL(V)$ is any vertex of $\mathbb{FL}(V)$, we may associate to it the free A-module $\mathrm{gr}_F(V) = \oplus_{i=1}^n F_i/F_{i-1}$. If (F, F') is an ordered pair of distinct vertices, which are joined by an edge in $\mathbb{FL}(V)$, then we obtain a *canonical* isomorphism (determined by the edge)

$$\varphi_{F,F'} : \mathrm{gr}_F(V) \xrightarrow{\cong} \mathrm{gr}_{F'}(V).$$

One way to describe it is by considering the edge as lying in a simplex $[\alpha]$, for some $\alpha = \{L_1, \ldots, L_n\} \in SPL(V)$; this determines an identification of $\mathrm{gr}_F(V)$ with $\oplus_i L_i$, and a similar identification of $\mathrm{gr}_{F'}(V)$, and thereby an identification between $\mathrm{gr}_F(V)$ and $\mathrm{gr}_{F'}(V)$. Note that from this description of the maps $\varphi_{F.F'}$, it follows that if F, F', F'' form vertices of a 2-simplex in $\mathbb{FL}(V)$, i.e., there exists some $\alpha \in SPL(V)$ such that $F, F', F'' \in [\alpha]$, then we also have

$$\varphi_{F,F''} = \varphi_{F',F''} \circ \varphi_{F,F'}.$$

The isomorphism $\varphi_{F,F'}$ depends only on the (oriented) edge in $\mathbb{FL}(V)$ determined by (F, F'), and not on the choice of the simplex $[\alpha]$ in which it lies. One way to see this is to use that, for any two such filtrations F, F' of V there is a canonical isomorphism $\mathrm{gr}_F^p \mathrm{gr}_{F'}^q(V) \cong \mathrm{gr}_{F'}^q \mathrm{gr}_F^p(V)$ (Schur-Zassenhaus lemma)

for each p, q. But in case F, F' are flags which are connected by an edge, then there is also a canonical isomorphism $\mathrm{gr}_F \mathrm{gr}_{F'}(V) \cong \mathrm{gr}_{F'}(V)$ (in fact the F-filtration induced on $\mathrm{gr}^p_{F'}(V)$ has only 1 non-trivial step, for each p), and similarly there is a canonical isomorphism $\mathrm{gr}_{F'} \mathrm{gr}_F(V) \cong \mathrm{gr}_F(V)$. These three canonical isomorphisms combine to give the isomorphism $\varphi_{F,F'}$.
Hence there is a well-defined *local system* $\mathbf{gr}(V)$ of A-modules on the geometric realization $|\mathbb{FL}(V)|$ of the simplicial complex $\mathbb{FL}(V)$, whose fibre over a vertex F is $\mathrm{gr}_F(V)$.
Notice further that this local system $\mathbf{gr}(V)$ comes equipped with a natural $\mathrm{Aut}(V)$ action, compatible with the natural actions on $FL(V)$ and $\mathbb{FL}(V)$. Indeed, any element $g \in \mathrm{Aut}(V)$ gives a bijection on the set of full flags $FL(V)$, with

$$F = (0 = F_0, F_1, \ldots, F_n = V) \in FL(V))$$

mapping to

$$gF = (0 = gF_0, gF_1, \ldots, gF_n = V).$$

This clearly gives an induced isomorphism $\oplus_i F_i/F_{i-1} \cong \oplus_i gF_i/gF_{i-1}$, identifying the fibers of the local system over F and gF in a specific way. It is easy to see that if $\alpha = \{L_1, \ldots, L_n\} \in SPL(V)$, then $g\alpha = \{gL_1, \ldots, gL_n\} \in SPL(V)$, giving the action of $\mathrm{Aut}\,(V)$ on $SPL(V)$, so that if a pair of vertices F, F' of $FL(V)$ lie on an edge contained in $[\alpha]$, then gF, gF' lie on an edge contained in $[g\alpha]$, and so the induced identification $\varphi_{F,F'}$ is compatible with $\varphi_{gF,gF'}$. This induces the desired action of $\mathrm{Aut}(V)$ on the local system.
Further, note that if $F, F' \in FL(V)$ are connected by an edge in $\mathbb{FL}(V)$, then we may realize $\varphi_{F,F'}$ by the action of a suitable element of $\mathrm{Aut}\,(V)$, which preserves a simplex $[\alpha]$ in which the edge lies, and permutes the lines in the splitting $\alpha \in SPL(V)$. Thus, given any edge-path joining vertices F, F' in $\mathbb{FL}(V)$, the induced composite isomorphism $\mathrm{gr}_F(V) \to \mathrm{gr}_{F'}(V)$ is again realized by the action of an element of $\mathrm{Aut}\,(V)$. In particular, given an edge-path loop based as $F \in FL(V)$, the induced automorphism of $\mathrm{gr}_F(V)$ is induced by the action on $\mathrm{gr}_F(V)$ of an element of the isotropy group of F in $\mathrm{Aut}\,(V)$, which is the "Borel subgroup" corresponding to the flag F.
Hence, the monodromy group of the local system $\mathbf{gr}(V)$ is clearly contained in $N_n(A)$, defined as a semidirect product

$$N_n(A) = (A^\times \times \cdots \times A^\times) \ltimes S_n \tag{2}$$

where S_n is the permutation group; we regard $N_n(A)$ as a subgroup of $\mathrm{Aut}\,(\oplus_i L_i)$ in an obvious way.
Now we make infinite versions of the above constructions.
Let A^∞ be the set of sequences $(a_1, a_2, \ldots, a_n, \ldots)$ of elements of A, all but finitely many of which are 0, considered as a free A-module of countable rank. There is a standard inclusion $i_n : A^n \hookrightarrow A^\infty$ of the standard free A-module of rank n as the submodule of sequences with $a_m = 0$ for all $m > n$. The induced inclusion $i : A^n \to A^{n+1}$ is the usual one, given by $i(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0)$.

We may thus view A^∞ as being given with a tautological flag, consisting of the A-submodules $i_n(A^n)$. We define a simplicial complex $\mathbb{FL}(A^\infty)$, with vertex set $FL(A^\infty)$ equal to the set of flags $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset A^\infty$ where V_i/V_{i-1} is a free A-module of rank 1, for each $i \geq 1$, and with $V_n = i(A^n)$ for all sufficiently large n. Thus $FL(A^\infty)$ is naturally the union of subsets bijective with $FL(A^n)$. To make $\mathbb{FL}(A^\infty)$ into a simplicial complex, we define a simplex to be a finite set of vertices in some subset $FL(A^n)$ which determines a simplex in the simplicial complex $\mathbb{FL}(A^n)$; this property does not depend on the choice of n, since the natural inclusion $FL(A^n) \hookrightarrow FL(A^{n+1})$, regarded as a map on vertex sets, identifies $\mathbb{FL}(A^n)$ with a subcomplex of $\mathbb{FL}(A^{n+1})$, such that any simplex of $\mathbb{FL}(A^{n+1})$ with vertices in $FL(A^n)$ is already in the subcomplex $\mathbb{FL}(A^n)$.

We consider $GL(A) \subset \mathrm{Aut}(A^\infty)$ as the union of the images of the obvious maps $i_n : GL_n(A) \hookrightarrow \mathrm{Aut}\,(A^\infty)$, obtained by automorphisms which fix all the basis elements of A^∞ beyond the first n. We clearly have an induced action of $GL(A)$ on the simplicial complex $\mathbb{FL}(A^\infty)$, and hence on its geometric realisation $|\mathbb{FL}(A^\infty)|$ through homeomorphisms preserving the simplicial structure. The inclusion $\mathbb{FL}(A^n) \hookrightarrow \mathbb{FL}(A^\infty)$ as a subcomplex is clearly $GL_n(A)$-equivariant.

Next, observe that there is a local system $\mathbf{gr}(A^\infty)$ on $\mathbb{FL}(A^\infty)$ whose fiber over a vertex $F = (F_0 = 0, F_1, \ldots, F_n, \ldots)$ is $\mathrm{gr}_F(V) = \oplus_i F_i/F_{i-1}$. This has monodromy contained in

$$N(A) = \cup_n N_n(A) \subset GL(A),$$

where we may also view $N(A)$ as the semidirect product of

$$(A^\times)^\infty = \text{diagonal matrices in } GL(A)$$

by the infinite permutation group S_∞. This local system also carries a natural $GL(A)$-action, compatible with the $GL(A)$-action on $\mathbb{FL}(A^\infty)$.

Next, we prove a property (Corollary 9) about the action of elementary matrices on homology, which is needed later. The corollary follows immediately from the lemma below.

For the statement and proof of the lemma, we suggest that the reader browse the remarks on $\mathcal{H}om(K_1 \times K_2, K_3)$ in section 0, given simplicial complexes K_i for $i = 1, 2, 3$. The notation $\mathrm{Elem}(V' \hookrightarrow V' \oplus V'')$ that appears in the lemma has also been introduced in section 0 under the heading "elementary matrices".

Lemma 8. *Let V', V'' be two free A-modules of finite rank, $i' : V' \to V' \oplus V''$, $i'' : V'' \to V' \oplus V''$ the inclusions of the direct summands. Consider the two natural maps*

$$\alpha, \beta : FL(V') \times FL(V'') \to FL(V' \oplus V'') \tag{3}$$

given by

$$\begin{aligned}\alpha : ((F'_1, \ldots, F'_r = V'), (F''_1, \ldots, F''_s)) \mapsto \\ (i'(F'_1), \ldots, i'(F'_r) = i'(V'), i'(V') + i''(F''_1), \ldots, i'(V') + i''(F''_s) = V' \oplus V''),\end{aligned}$$

$\beta : ((F'_1, \ldots, F'_r = V'), (F''_1, \ldots, F''_s)) \mapsto$
$(i''(F''_1), \ldots, i''(F''_s) = i''(V''), i''(V'') + i'(F'_1), \ldots, i''(V'') + i'(F'_r) = V' \oplus V'')$.

(A) α and β are vertices of a one-simplex of $\mathcal{H}om(\mathbb{FL}(V') \times \mathbb{FL}(V''), \mathbb{FL}(V' \oplus V''))$.
(B) The maps $|\mathbb{FL}(V')| \times |\mathbb{FL}(V'')| \to |\mathbb{FL}(V' \oplus V'')|$ induced by α, β are homotopic to each other.
(C) Let $c : |\mathbb{FL}(V')| \times |\mathbb{FL}(V'')| \to |\mathbb{FL}(V' \oplus V'')|$ denote the map produced by α. Denote the action of $g \in GL(V' \oplus V'')$ on $|\mathbb{FL}(V' \oplus V'')|$ by $|\mathbb{FL}(g)|$. Then c and $|\mathbb{FL}(g)| \circ c$ are homotopic to each other, if $g \in \mathrm{Elem}(V' \hookrightarrow V' \oplus V'') \subset GL(V' \oplus V'')$.

Proof. Part (A). By the definition of $\mathcal{H}om(K_1 \times K_2, K_3)$ in section 0, we only have to check that $\alpha(\sigma' \times \sigma'') \cup \beta(\sigma' \times \sigma'')$ is a simplex of $FL(V' \oplus V'')$ for all simplices σ' of $\mathbb{FL}(V')$ and all simplices σ'' of $\mathbb{FL}(V'')$. Clearly it suffices to prove this for maximal simplices, so we assume that both σ' and σ'' are maximal.
Note that if we consider any maximal simplex σ' in $\mathbb{FL}(V')$, it corresponds to a splitting $\{L'_1, \ldots, L'_r\} \in SPL(V')$. Similarly any maximal simplex σ'' of $\mathbb{FL}(V'')$ corresponds to a splitting $\{L''_1, \ldots, L''_s\} \in SPL(V'')$. This determines the splitting $\{i'(L'_1), \ldots, i'(L'_r), i''(L''_1), \ldots, i''(L''_s)\}$ of $V' \oplus V''$, giving rise to a maximal simplex τ of $\mathbb{FL}(V' \oplus V'')$, and clearly $\alpha(\sigma' \times \sigma'')$ and $\beta(\sigma' \times \sigma'')$ are both contained in τ. Thus their union is a simplex.
(B) follows from (A). We now address (C). We note that $c = g \circ c$ for all $g \in id + Hom_A(V'', V')$. Denoting by d the map produced by β we see that $d = g \circ d$ for all $g \in id + Hom_A(V', V'')$. Because c, d are homotopic to each other, we see that c and $g \circ c$ are in the same homotopy class when g is in either of the two groups above. These groups generate $\mathrm{Elem}(V' \hookrightarrow V' \oplus V'')$, and so this proves (C). $\square$

Corollary 9. *(i) The group $E_{n+1}(A)$ of elementary matrices acts trivially on the image of the natural map*

$$i_* : H_*(\mathbb{FL}(A^{\oplus n}), \mathbb{Z}) \to H_*(\mathbb{FL}(A^{\oplus n+1}), \mathbb{Z}).$$

(ii) The action of the group $E(A)$ of elementary matrices on $H_*(\mathbb{FL}(A^\infty), \mathbb{Z})$ is trivial.

Proof. We put $V' = A^n$ and $V'' = A$ in the previous lemma. The c in part (B) of the lemma is precisely the i being considered here. By (C) of the lemma, $g \circ i$ is homotopic to i for all $g \in \mathrm{Elem}(A^n \hookrightarrow A^{n+1}) = E_{n+1}(A)$. This proves (i). The direct limit of the r-homology of $|\mathbb{FL}(A^n)|$, taken over all n, is the r-th homology of $|\mathbb{FL}(A^\infty|$. Thus (i) implies (ii).

$\square$

We will find it useful below to have other "equivalent models" of the spaces $\mathbb{FL}(V)$, $\mathbb{FL}(A^\infty)$, by which we mean other simplicial complexes, also defined using collections of appropriate A-submodules, such that there are natural

homotopy equivalences between the different models of the same homotopy type, compatible with the appropriate group actions, etc.
We apply corollary 7 as follows. Let $V \cong A^n$. We put $A = SPL(V), B = FL(V)$ and $Z = \{\alpha, F) : F \in [\alpha]\}$. The simplicial complex $S_Z(B)$ of corollary 7 is $\mathbb{FL}(V)$ by its definition. The simplicial complex $S_Z(A)$ is our definition of $\mathbb{SPL}(V)$. The homotopy equivalence of $\mathbb{SPL}(V)$ and $\mathbb{FL}(V)$ follows from this corollary.
We define $SPL(A^\infty)$ to be the collection of sets S satisfying
(a) $L \in S$ implies that L is a free rank one A-submodule of A^∞,
(b)$\oplus\{L : L \in S\} \to A^\infty$ is an isomorphism, and
(c) the symmetric difference of S and the standard collection:
$\{A(1, 0, 0, ...), A(0, 1, 0, ...), \cdots\}$ is a finite set.
Corollary 7 is then applied to the subset $Z \subset SPL(A^\infty) \times FL(A^\infty)$ consisting of the pairs (S, F) so that there is a bijection $h : S \to \mathbb{N}$ so that for every $L \in S$,
$L \subset F_{h(L)}$ and $L \to gr^F_{h(L)}$ is an isomorphism.
The above Z defines $\mathbb{SPL}(A^\infty)$. The desired homotopy equivalence of the geometric realisations of $\mathbb{SPL}(A^\infty)$ and $\mathbb{FL}(A^\infty)$ comes from the same corollary.
We also find it useful to introduce a third model of the homotopy types of $\mathbb{FL}(V)$ and $\mathbb{FL}(A^\infty)$, the "enriched Tits buildings" $\mathbb{ET}(V)$ and $\mathbb{ET}(A^\infty)$. The latter is defined in the last remark of this section.
Let $V \cong A^n$ as a left A-module. Let $\mathcal{E}(V)$ be the set consisting of ordered pairs

$$(F, S) = ((0 = F_0 \subset F_1 \subset \cdots \subset F_r = V), (S_1, S_2, \ldots, S_r)),$$

where F is a *partial flag* in V, which means that $F_i \subset V$ is an A-submodule, such that F_i/F_{i-1} is a nonzero free module for each i, and $S_i \in SPL(F_i/F_{i-1})$ is an unordered collection of free A-submodules of F_i/F_{i-1} giving rise to a direct sum decomposition $\oplus_{L\in S_i} L \cong F_i/F_{i-1}$. Thus S is a collection of splittings of the quotients F_i/F_{i-1} for each i.
We may put a partial order on the set $\mathcal{E}(V)$ in the following way: $(F, S) \le (F', T)$ if the filtration F is a refinement of F', and the data S, T of direct sum decompositions of quotients are compatible, in the following natural sense — if $F'_{i-1} = F_{j-1} \subset F_j \subset \cdots F_l = F'_i$, then T_i must be partitioned into subsets, which map to the sets $S_j, S_{j+1}, \ldots, S_l$ under the appropriate quotient maps. In particular, (F', T) has only finitely many possible predecessors (F, S) in the partial order.
We have a simplicial complex $ET(V) := N\mathcal{E}(V)$, the nerve of the partially ordered set $\mathcal{E}(V)$ considered as a category, so that simplices are just nonempty finite chains of elements of the vertex (po)set $\mathcal{E}(V)$.
Note that maximal elements of $\mathcal{E}(V)$ are naturally identified with elements of $SPL(V)$, while minimal elements are naturally identified with elements of $FL(V)$. Simplices in $\mathbb{FL}(V)$ are nonempty finite subsets of $FL(V)$ which have a common upper bound in $\mathcal{E}(V)$, and similarly simplices in $\mathbb{SPL}(V)$ are nonempty finite subsets of $SPL(V)$ which have a common lower bound in $\mathcal{E}(V)$.

We now show that $\mathbb{ET}(V) = B\mathcal{E}(V)$, the classifying space of the poset $\mathcal{E}(V)$, is another model of the homotopy type of $|\mathbb{FL}(V)|$.
In a similar fashion, we may define a poset $\mathcal{E}(A^\infty)$, and a space $\mathbb{ET}(A^\infty)$, giving another model of the homotopy type of $|\mathbb{FL}(A^\infty)|$.
We first have a lemma on classifying spaces of certain posets. For any poset $(P, \leq)$, and any $S \subset P$, let

$$L(S) = \{x \in P |\ x \leq s \ \forall\ s \in S\},\ \ U(S) = \{x \in P |\ s \leq x \ \forall\ s \in S\}$$

be the upper and lower sets of S in P, respectively. Let $\mathcal{P}_{min}$ denote the simplicial complex with vertex set P_{min} given by minimal elements of P, and where a nonempty finite subset $S \subset P_{min}$ is a simplex if $U(S) \neq \emptyset$. Let $|\mathcal{P}_{min}|$ denote the geometric realisation of $\mathcal{P}_{min}$.

Lemma 10. *Let $(P, \leq)$ be a poset such that*
(a) $\forall\ \ s \in P$, the set $L(\{s\})$ is finite
(b) if $\emptyset \neq S \subset P$ with $L(S) \neq \emptyset$, then the classifying space $BL(S)$ of $L(S)$ (as a subposet) is contractible.

Then $|\mathcal{P}_{min}|$ is naturally homotopy equivalent to BP.

Proof. We apply Proposition 1. Take

$$\Lambda = \{L(S) |\ \emptyset \neq S \subset P \text{ and } L(S) \neq \emptyset\}.$$

This is a poset with respect to inclusion. All $\lambda \in \Lambda$ are finite subsets of P, so Λ is Artinian. By assumption, the subsets $B(\lambda) \subset BP$, for $\lambda \in \Lambda$, are contractible. On the other hand, the sets $\lambda \cap P_{min}$ give simplices in $|\mathcal{P}_{min}|$. Thus, both the spaces BP and $|\mathcal{P}_{min}|$ have Λ-admissible coverings, and are thus homotopy equivalent. $\square$

Remark. If a poset P has g.c.d. in the sense that $\emptyset \neq S \subset P$ and $\emptyset \neq L(S)$ implies $L(S) = L(t)$ for some $t \in P$, then condition (b) of the lemma is immediately satisfied. However $\mathcal{E}(V)$ does not enjoy the latter property.
For example, if $V = A^3$ with basis e_1, e_2, e_3, let $s = \{Ae_1, Ae_2, Ae_3\}$ and $t = \{Ae_1, A(e_1 + e_2), Ae_3\}$ and let $S = \{s, t\} \subset SPL(V) \subset \mathcal{E}(V)$. Then $L(S)$ has three minimal elements and two maximal elements. In particular, g.c.d. (s, t) does not exist. In this example, $B(L(S))$ is an oriented graph in the shape of the letter M.

Proposition 11. *If V is a free A module of finite rank, the poset $\mathcal{E}(V)$ satisfies the hypotheses of lemma 10. Thus, $|\mathbb{FL}(V)|$ is naturally homotopy equivalent to $\mathbb{ET}(V) = B(\mathcal{E}(V))$.*

Proof. Clearly the condition (a) of lemma 10 holds, so it suffices to prove (b). We now make a series of observations.

(i) Regard $SPL(V)$ as the set of maximal elements of the poset $\mathcal{E}(V)$. We observe that for any $s \in \mathcal{E}(V)$, if $H(s) = SLP(V) \cap U(\{s\})$, then we have that $L(\{s\}) = L(H(s))$. This is easy to see, once one has unravelled the definitions.

Thus, it suffices to show that for sets S of the type $\emptyset \neq S \subset SPL(V) \subset \mathcal{E}(V)$, we have that $B(L(S)$ is contractible. We assume henceforth that $S \subset SPL(V)$.

(ii) Given a submodule $W \subset V$ which determines a partial flag $0 \subset W \subset V$, we have a natural inclusion of posets

$$\mathcal{E}(W) \times \mathcal{E}(V/W) \subset \mathcal{E}(V),$$

where on the product, we take the partial order

$$(a_1, a_2) \leq (b_1, b_2) \Leftrightarrow a_1 \leq b_1 \in \mathcal{E}(W) \text{ and } a_2 \leq b_2 \in \mathcal{E}(V/W).$$

Note that $\alpha \in \mathcal{E}(V)$ lies in the sub-poset $\mathcal{E}(W) \times \mathcal{E}(V/W)$ precisely when W is one of the terms in the partial flag associated to α. Hence, if α lies in the sub-poset, so does the entire set $L(\{\alpha\})$.

(iii) With notation as above, if $\emptyset \neq S \subset SPL(V) \subset \mathcal{E}(V)$, and $L(S)$ has nonempty intersection with the image of $\mathcal{E}(W) \times \mathcal{E}(V/W) \subset \mathcal{E}(V)$, then clearly there exist nonempty subsets $S'(W) \subset SPL(W)$, $S''(W) \subset \mathbb{SPL}(V/W)$ such that

$$L(S) \cap (\mathcal{E}(W) \times \mathcal{E}(V/W)) = L(S'(W)) \times L(S''(W)) \tag{4}$$

(iv) If $\emptyset \neq S \subset SPL(V)$, then each $s \in S$ is a subset of

$$\mathcal{L}(V) = \{L \subset V | L \text{ is a free direct summand of rank 1 of } V\},$$

the set of lines in V. Let

$$T(S) = \cap_{s \in S} s = \text{ lines common to all members of } S,$$

so that $T(S) \subset \mathcal{L}(V)$. Let $M(S)$ denote the direct sum of the elements of $T(S)$, so that $M(S)$ is a free A-module of finite rank, and $0 \subset M(S) \subset V$ is a partial flag, in the sense explained earlier; further, $T(S)$ may be regarded also as an element of $SPL(M(S)) \subset \mathcal{E}(M(S))$.

(v) We now claim the following: if $\emptyset \neq S \subset SPL(V)$ and $b \in L(S)$, then there exists a unique subset $f(b) \subset T(S)$ such that if $M(b)$ is the (direct) sum of the lines in $f(b)$, then

$$b = (f(b), b') \in SPL(M(b)) \times \mathcal{E}(V/M(b)) \subset \mathcal{E}(M(b)) \times \mathcal{E}(V/M(b)) \subset \mathcal{E}(V).$$

Indeed, if

$$b = ((0 = W_0 \subset W_1 \subset \cdots \subset W_h = V), (t_1, t_2, \ldots, t_h)\}$$

where $t_i \in SPL(W_i/W_{i-1})$, then since $b \in L(S)$, we must have that $t_1 \subset s$ for all $s \in S$, which implies that $t_1 \subset T(S)$. Take $M(b) = W_1$, $t_1 = f(b) \in SPL(M(b))$.

Let $\mathcal{P}(T(S))$ be the poset of nonempty subsets of $T(S)$, with respect to inclusion. Then $b \mapsto f(b)$ gives an order-preserving map $f : L(S) \to \mathcal{P}(T(S))$.

(vi) For any $b \in L(S)$, we have

$$L(b) = L(\{f(b)\}) \times L(b') \subset \mathcal{E}(M(b)) \times \mathcal{E}(V/M(b)),$$

so that if $b_1 \in L(b)$, then $\emptyset \neq f(b_1) \subset f(b)$.

We will now complete the proof of Proposition 11. We proceed by induction on the rank of V. Suppose $S \subset SPL(V)$ is nonempty, and $L(S) \neq \emptyset$.
If $M(S) = V$, then $S = \{s\}$ for some s, and $L(S) = L(\{s\})$ is a cone, hence contractible. So assume $M(S) \neq V$.
If $T \subset T(S)$ is non-empty, and $M(T) \subset V$ the (direct) sum of the lines in T, then in the notation of (4) above, with $W = M(T)$, we have $S'(W) = \{T\}$, and so $f^{-1}(T) = \{T\} \times L(S''(W)))$ for some $S''(W) \subset SPL(V/W)$.
Now by induction, we have that $L(S''(W))$ is contractible, provided it is non-empty. Hence the non-empty fiber posets of f are contractible. If $\emptyset \neq T \subset T' \subset T(S)$, then there is a morphism of posets $f^{-1}(T) \to f^{-1}(T')$ given as follows: if $b \in f^{-1}(T)$, and

$$b = ((0 = W_0 \subset W_1 \subset \cdots \subset W_h = V), (t_1, t_2, \ldots, t_h)\}$$

where $t_i \in SPL(W_i/W_{i-1})$, then since $b \in f^{-1}(T)$, we must have $t_1 = T$, $W_1 = M(T)$. Now define $b' \in f^{-1}(T')$ using the partial flag

$$0 = W' \subset W_1 + M(T') \subset W_2 + M(T') \subset \cdots \subset W_h + M(T') = V$$

and elements $t'_i \in SPL(W_i + M(T')/W_{i-1} + M(T'))$ induced by the t_i. This is easily seen to be well-defined, and gives a morphism of posets $f^{-1}(T) \to f^{-1}(T')$. In particular, if $T \subset T' \subset T(S)$ and $f^{-1}(T)$ is non-empty, then so is $f^{-1}(T')$.
Now take any $b \in L(S)$ and put $T = f(b), T' = T(S)$ in the above to deduce that $f^{-1}(T(S)) \neq \emptyset$. By (iii) above, we see that every $f^{-1}X$ is nonempty (and therefore contractible as well) for every nonempty $X \subset T(S)$.
We see that all the fiber posets $f^{-1}(T)$ considered above are nonempty.
This makes f pre-cofibered, in the sense of Quillen (see [19], page 96), with contractible fibers. Hence by Quillen's Theorem A, f induces a homotopy equivalence on classifying spaces. But $\mathcal{P}(T(S))$ is contractible (for example, since $T(S)$ is the unique maximal element). □

Remark. Proposition 5 and the remarks preceding it apply to the above Proposition. In particular, we obtain homotopy equivalences $f : \mathbb{ET}(V) \to \mathbb{FL}(V)$ so that the induced maps on homology are $GL(V)$-equivariant.

Remark. We now define the poset $\mathcal{E}(A^\infty)$ and show that $\mathbb{ET}(A^\infty) = B\mathcal{E}(A^\infty)$ is homotopy equivalent to $|\mathbb{FL}(A^\infty|$.
We have already observed that a short exact sequence of free modules of finite rank

$$0 \to V' \to V \to V'' \to 0$$

induces a natural inclusion $\mathcal{E}(V') \times \mathcal{E}(V'') \hookrightarrow \mathcal{E}(V)$ of posets. In particular, when $V'' \cong A$, this yields an inclusion $\mathcal{E}(V') \hookrightarrow \mathcal{E}(V)$.
We have $... \subset A^n \subset A^{n+1} \subset ... \subset A^\infty$ as in the definition of $\mathbb{FL}(A^\infty)$. From the above, we obtain a direct system of posets

$$...\mathcal{E}(A^n) \hookrightarrow \mathcal{E}(A^{n+1}) \hookrightarrow ...$$

and we define $\mathcal{E}(A^\infty)$ to be the direct limit of this system of posets.

We put $P = \mathcal{E}(A^\infty)$ in lemma 10. We note that $\alpha \leq \beta, \alpha \in \mathcal{E}(A^\infty), \beta \in \mathcal{E}(A^n)$ implies that $\alpha \in \mathcal{E}(A^n)$. It follows that $P = \mathcal{E}(A^\infty)$ satisfies the requirements of the lemma because each $\mathcal{E}(A^n)$ does. It is clear that $P_{min} = FL(A^\infty)$, and furthermore that $\mathcal{P}_{min} = \mathbb{FL}(A^\infty)$. This yields the homotopy equivalence of $|\mathbb{FL}(A^\infty)|$ with $\mathbb{ET}(A^\infty)$.
By Proposition 5, it follows that $\mathbb{ET}(A^\infty)//GL(A)$ and $|\mathbb{FL}(A^\infty)|//GL(A)$ are also homotopy equivalent to each other.
It has already been remarked that Corollary 7 gives the homotopy equivalence of $|\mathbb{SPL}(A^\infty|$ with $|\mathbb{FL}(A^\infty)|$. Combined with Proposition 5, this gives the homotopy equivalence of $|\mathbb{SPL}(A^\infty|//GL(A)$ with $|\mathbb{FL}(A^\infty)|//GL(A)$. The remarks preceding that proposition, combined with corollary 9, show that the action of $E(A)$ on the homology groups of $|\mathbb{SPL}(A^\infty)|$ is trivial.

3. homology of the Borel constrcuction

Let V be a free A-module of rank n. Fix $\beta \in SPL(V)$ and let $N(\beta) \subset GL(V)$ be the stabiliser of β (when $V = A^n$ and β is the standard splitting, then $N(\beta)$ is the subgroup $N_n(A)$ of the last section). That there is a $GL(V)$-equivariant $N(\beta)$-torsor on $|\mathbb{FL}(V)|$ has been observed in the previous section. In a similar manner, one may construct a $GL(V)$-equivariant $N(\beta)$-torsor on $\mathbb{ET}(V)$. This gives rise to a $N(\beta)$-torsor on $\mathbb{ET}(V)//GL(V)$. Because $BN(\beta)$ is a classifying space for such torsors, we obtain a map $\mathbb{ET}(V)//GL(V) \to BN(\beta)$, well defined up to homotopy.
On the other hand, the inclusion of β in $\mathbb{ET}(V)$ gives rise to an inclusion $BN(\beta) = \{\beta\}//N(\beta) \hookrightarrow \mathbb{ET}(V)//GL(V)$. It is clear that the composite $BN(\beta) \hookrightarrow \mathbb{ET}(V)//GL(V) \to BN(\beta)$ is homotopic to the identity. Thus $BN(\beta)$ is a homotopy retract of $\mathbb{ET}(V)//GL(V)$, but not homotopy equivalent to $\mathbb{ET}(V)//GL(V)$. Nevertheless we have the following statement:

Proposition 12. *The map $BN(\beta) \to \mathbb{ET}(V)//GL(V)$ induces an isomorphism on integral homology, provided A is as in theorem 1.*

Proof. Fix a basis for V, identifying $GL(V)$ with $GL_n(A)$. Let $\beta \in SPL(V)$ be the element naturally determined by this basis. Regarded as a vertex of $\mathbb{ET}(V)$, let $(\beta, *) \mapsto \overline{\beta}$ under the natural map

$$\pi : \mathbb{ET}(V)//GL(V) \to \mathbb{ET}(V)/GL(V)$$

from the homotopy quotient to the geometric quotient, where $* \in EGL(V)$ is the base point (corresponding to the vertex labelled by the identity element of $GL(V)$).
For any $x \in \mathbb{ET}(V)$, let $\mathcal{H}(x) \subset GL(V)$ be the isotropy group of x for the $GL(V)$-action on $\mathbb{ET}(V)$. Note that since

$$\mathbb{ET}(V)//GL(V) = (\mathbb{ET}(V) \times EGL(V))\,/GL(V),$$

the fiber $\pi^{-1}(\pi((x,*))$ may be identified with $EGL(V)/\mathcal{H}(x)$, which has the homotopy type of $B\mathcal{H}(x)$.

In particular, the fiber $\pi^{-1}(\overline{\beta})$ has the homotopy type of $BN_n(A)$, Further, the principal $N_n(A)$ bundle on $EGL(V)/\mathcal{H}(\beta)$ is naturally identified with the universal $N_n(A)$-bundle on $BN_n(A)$ – its pullback to $\{\beta\}\times EGL(V)$ is the trivial $N_n(A)$-bundle, regarded as an $N_n(A)$-equivariant principal bundle, where $N_n(A)$ acts on itself (the fiber of the trivial bundle) by translation. This means that the composite

$$\pi^{-1}(\overline{\beta}) \to \mathbb{ET}(V)/GL(V) \to BN_n(A)$$

is a homotopy equivalence, which is homotopic to the identity, if we identify $EGL(V)/\mathcal{H}(\beta)$ with $BN_n(A)$.
Thus, the lemma amounts to the assertion that $\pi^{-1}(\overline{\beta}) \to \mathbb{ET}(V)//GL(V)$ *induces an isomorphism in integral homology.*
Fix $\alpha \in FL(V)$ with $\alpha \leq \beta$ in the poset $\mathcal{E}(V)$. Let

$$P = \{\lambda \in \mathcal{E}(V) | \alpha \leq \lambda \leq \beta\}.$$

One sees easily that (i) BP is contractible, and (ii) the map $BP \to \mathbb{ET}(V)/GL(V)$ is a homoemorphism. The first assertion is obvious, since P has a maximal (as well as a minimal) element, so that BP is a cone. For the second assertion, we first note that an element $b \in P \subset \mathcal{E}(V)$ is uniquely determined by the ranks of the modules in the partial flag in V associated to b. Conversely, given any increasing sequence of numbers $n_1 < \ldots < n_h = \operatorname{rank} V$, there does exist an element of P whose partial flag module ranks are these integers. Given any element $b \in \mathcal{E}(V)$, there exists an element $g \in GL(V)$ so that $g(b) = b' \in P$; the element b' is the unique one determined by the sequence of ranks associated to b. Finally, one observes that if $b \in P$, and $g \in GL(V)$ such that $g(b) \in P$, then in fact $g(b) = b$: this is a consequence of the uniqueness of the element of P with a given sequence of ranks. These observations imply that $BP \to \mathbb{ET}(V)/GL(V)$ is bijective; it is now easy to see that it is a homeomorphism.
We may view $\mathbb{ET}(V)//GL(V)$ as the quotient of $BP \times EGL(V)$ by the equivalence relation

$$(x, y) \sim (x', y') \Leftrightarrow x = x', \text{ and } y' = g(y) \text{ for some } g \in \mathcal{H}(x). \tag{5}$$

The earlier map $\pi : \mathbb{ET}(V)//GL(V) \to \mathbb{ET}(V)/GL(V)$ may be viewed now as the map induced by the projection $BP \times EGL(V) \to BP$. We may, with this identification, also identify $\overline{\beta}$ with β.
Next, we construct a "good" fundamental system of open neighbourhoods of an arbitrary point $x \in BP$, which we need below. Such a point x lies in the relative interior of a unique simplex $\sigma(x)$ (called the *carrier* of x) corresponding to a chain $\lambda_0 < \lambda_1 < \cdots < \lambda_r$. Then one sees that the stabiliser $\mathcal{H}(x) \subset GL(V)$ is given by

$$\mathcal{H}(x) = \bigcap_{i=0}^{r} \mathcal{H}(\lambda_i),$$

since any element of $GL(V)$ which stabilizes the simplex $\sigma(x)$ must stabilize each of the vertices (for example, since the $GL(V)$ action preserves the partial order).

Let star$\,(x)$ be the union of the relative interiors of all simplices in BP containing $\sigma(x)$ (this includes the relative interior of $\sigma(x)$ as well, so it contains x). It is a standard property of simplicial complexes that star$\,(x)$ is an open neighbourhood of x in BP. Then if $z \in$ star$\,(x)$, clearly $\sigma(z)$ contains $\sigma(x)$, and so $\mathcal{H}(z) \subset \mathcal{H}(x)$.

Next, for such a point z, and any $y \in EGL(V)$, it makes sense to consider the path

$$t \mapsto (tz + (1-t)x, y) \in \sigma(z) \times EGL(V) \subset BP \times EGL(V)$$

(where we view the expression $tz+(1-t)x$ as a point of $\sigma(z)$, using the standard barycentric coordinates). In fact this path is contained in star$\,(x) \times \{y\}$, and gives a continuous map

$$H(x) : \mathrm{star}\,(x) \times EGL(V) \times I \to \mathrm{star}\,(x) \times EGL(V)$$

which exhibits $\{x\} \times EGL(V)$ as a strong deformation retract of star$\,(x) \times EGL(V)$. Further, this is compatible with the equivalence relation $\sim$ in (5) above, so that we obtain a strong deformation retraction

$$\overline{H(x)} : \pi^{-1}(\mathrm{star}\,(x)) \times I \to \pi^{-1}(\mathrm{star}\,(x)).$$

In a similar fashion, we can construct a fundamental sequence of open neighbourhoods $U_n(x)$ of x in BP, with $U_1(x) = \mathrm{star}\,(x)$, and set

$$U_n(x) = H(x)(\mathrm{star}\,(x) \times EGL(V) \times [0, 1/n)).$$

The same deformation retraction H determines, by reparametrization, a deformation retraction

$$H_n(x) : \pi^{-1}(U_n(x)) \times I \to \pi^{-1}(U_n(x))$$

of $\pi^{-1}(U_n(x)$ onto $\pi^{-1}(x)$.

Thus, if $P' = P \setminus \beta$, then

$$\pi^{-1}(\mathrm{star}\,(\beta)) = \mathbb{ET}(V)//GL(V) \setminus \pi^{-1}(BP'),$$

and from what we have just shown above, the inclusion

$$\pi^{-1}(\beta) \to \pi^{-1}(\mathrm{star}\,(\beta)) = \mathbb{ET}(V)//GL(V) \setminus \pi^{-1}(BP')$$

is a homotopy equivalence. To simplify notation, we let $X = \mathbb{ET}(V)//GL(V)$, so that we have the map $\pi : X \to BP$, and $X^0 = X \setminus \pi^{-1}(BP')$. Let $\pi^0 = \pi\mid_{X^0} : X^0 \to BP$.

We are reduced to showing, with this notation, that the inclusion of the (dense) open subset

$$X^0 \to X$$

induces an isomorphism in integral homology. Equivalently, it suffices to show that this inclusion induces an isomorphism on cohomology with arbitrary constant coefficients M. By the Leray spectral sequence, this is a consequence of showing that the maps of sheaves

$$R^i\pi_*M_X \to R^i\pi_*^0 M_{X^0}$$

is an isomorphism, which is clear on stalks $x \in BP \setminus BP'$. Now consider stalks at a point $x \in BP'$. For any point $x' \in \mathrm{star}\,(x)$, note that x lies in some face of $\sigma(x')$ (the carrier of x'). We had defined a fundamental system of neighbourhoods $U_n(x)$ of x in BP; explicitly we have

$$U_n(x) = \{tx' + (1-t)x | 0 \leq t < 1/n \text{ and } x' \in \mathrm{star}\,(x)\}.$$

Here, as before, we make sense of the above expression $tx' + (1-t)x$ using barycentric coordinates in $\sigma(x')$.
Define

$$z_n(x) = \frac{1}{2n}\beta + (1 - \frac{1}{2n})x.$$

Note that $z \in BP \setminus BP' = \mathrm{star}\,(\beta)$. Further, observe that $U_n(x) \cap BP \setminus BP'$ is contractible, contains the point z, and for any $w \in U_n(x) \cap BP \setminus BP'$, contains the line segment joining z and w (this makes sense, in terms of barycentric coordinates of any simplex containing both $z_n(x)$ and w; this simplex is either the carrier of w, or the cone over it with vertex β, of which $\sigma(w)$ is a face).
This implies $\mathcal{H}(w) \subset \mathcal{H}(z_n(x)) = \mathcal{H}(x) \cap \mathcal{H}(\beta)$, for all $w \in U_n(x)$. A minor modification of the proof (indicated above) that $\pi^{-1}(x) \subset \pi^{-1}(U_n(x))$ is a strong deforamtion retract, yields the statement that

$$\pi^{-1}(z_n(x)) \to \pi^{-1}(U_n(x) \setminus BP')$$

is a strong deformation retract. Hence, the desired isomorphism on stalks follows from:

(6) $B(\mathcal{H}(x) \cap \mathcal{H}(\beta)) \to B(\mathcal{H}(x))$ induces isomorphisms in integral homology.

We now show how this statement, for the appropriate rings A, is reduced to results of [13].
First, we discuss the structure of the isotropy groups $\mathcal{H}(x)$ encountered above. Let $\lambda \in P$, given by

$$\lambda = (F, S) = ((0 = F_0 \subset F_1 \subset \cdots \subset F_r = V), (S_1, S_2, \ldots, S_r)),$$

where we also have $\alpha \leq \lambda \leq \beta$ for our chosen elements $\alpha \in FL(V)$ and $\beta \in SPL(V)$. We may choose a basis for each of the lines in the splitting β; then $\alpha \in FL(V)$ uniquely determines an order among these basis elements, and thus a basis for the underlying free A-module V, such that the i-th submodule in the full flag α is the submodule generated by the first i elements in β. Now the stabilizer $\mathcal{H}(\alpha)$ may be viewed as the group of upper triangular matrices in $GL_n(A)$, while $\mathcal{H}(\beta)$ is the group generated by the diagonal subgroup in $GL_n(A)$ and the group of permutation matrices, identified with the permutation group S_n.
In these terms, $\mathcal{H}(\lambda)$ has the following structure. The filtration $F = (0 = F_0 \subset F_1 \subset \cdots \subset F_r = V)$ is a sub-filtration of the full flag α, and so determines a "unipotent subgroup" $U(\lambda)$ of elements fixing the elements of this partial flag,

and acting trivially on the graded quotients F_i/F_{i-1}. These are represented as matrices of the form

$$\begin{bmatrix} I_{n_1} & * & * & \cdots & * \\ 0 & I_{n_2} & * & \cdots & * \\ 0 & 0 & I_{n_3} & \cdots & * \\ & \vdots & \ddots & & \vdots \\ 0 & \cdots & & & I_{n_r} \end{bmatrix}$$

where $n_i = \operatorname{rank}(W_i/W_{i-1})$, I_{n_i} is the identity matrix of size n_i; these are the matrices which are strictly upper triangular with respect to a certain "ladder". Next, we may consider the group $S(\lambda) \subset S_n$ of permutation matrices, supported within the corresponding diagonal blocks, of the form

$$\begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ & \vdots & \ddots & & \vdots \\ 0 & \cdots & & & A_r \end{bmatrix}$$

where each A_j is a permutation matrix. Finally, we have the diagonal matrices $T_n(A) \subset GL_n(A)$, which are contained in $\mathcal{H}(\lambda)$ for any such λ. In fact $\mathcal{H}(\lambda) = U(\lambda)T_n(A)S(\lambda)$, where the group $T_n(A)S(\lambda)$ normalizes the subgroup $U(\lambda)$, making $\mathcal{H}(\lambda)$ a semidirect product of $U(\lambda)$ and $T_n(A)S(\lambda)$. We also have that $S(\lambda)$ normalizes $U(\lambda)T_n(A)$.

In particular, $\mathcal{H}(\alpha)$ has trivial associated permutation group $S(\alpha) = \{I_n\}$, while $\mathcal{H}(\beta)$ has trivial unipotent group $U(\beta) = \{I_n\}$ associated to it.

Now if $x \in BP$, and $\sigma(x)$ is the simplex associated to the chain $\lambda_0 < \cdots < \lambda_r$ in the poset P, then it is easy to see that $\mathcal{H}(x)$ is the semidirect product of $U(x) := U(\lambda_r)$ and $T_n(A)S(x)$, with $S(x) := S(\lambda_0)$, since as seen earlier, $\mathcal{H}(x)$ is the intersection of the $\mathcal{H}(\lambda_i)$. In other words, the "unipotent part" and the "permutation group" associated to $\mathcal{H}(x)$ are each the smallest possible ones from among the corresponding groups attached to the vertices of the carrier of x. Again we have that $S(x)$ normalizes $U(x)T_n(A)$.

We return now to the situation in (6). We see that the groups $\mathcal{H}(x) = U(x)T_n(A)S(x)$ and $\mathcal{H}(x) \cap \mathcal{H}(\beta) = T_n(A)S(x)$ both have the same associated permutation group $S(x)$, which normalizes $U(x)T_n(A)$ as well as $T_n(A)$. By comparing the spectral sequences

$$E^2_{p,q} = H_p(S(x), H_q(U(x)T_n(A), \mathbb{Z})) \Rightarrow H_{p+q}(\mathcal{H}(x), \mathbb{Z}),$$

$$E^2_{p,q} = H_p(S(x), H_q(T_n(A), \mathbb{Z})) \Rightarrow H_{p+q}(\mathcal{H}(x) \cap \mathcal{H}(\beta), \mathbb{Z})$$

we see that it thus suffices to show that the inclusion

$$T_n(A) \subset U(x)T_n(A) \tag{7}$$

induces an isomorphism on integral homology.

Now lemma 13 below finishes the proof. □

To state lemma 13 we use the following notation. Let $I = \{i_0 = 0 < i_1 < i_2 < \cdots < i_r = n\}$ be a subsequence of $\{0, 1, \ldots, n\}$, so that I determines a partial flag

$$0 \subset A^{i_1} \subset A^{i_2} \subset \cdots \subset A^{i_r} = A^n,$$

where $A^j \subset A^n$ as the submodule generated by the first j basis vectors. Let $U(I)$ be the "unipotent" subgroup of $GL_n(A)$ stabilising this flag, and acting trivially on the associated graded A-module, and let $G(I) \subset GL_n(A)$ be the subgroup generated by $U(I)$ and $T_n(A) = (A^\times)^n$, the subgroup of diagonal matrices. Then $T_n(A)$ normalises $U(I)$, and $G(I)$ is the semidirect product of $U(I)$ and $T_n(A)$.

Lemma 13. *Let A be a Nesterenko-Suslin ring. For any I as above, the homomorphism $G(I) \to G(I)/U(I) \cong T_n(A)$ induces an isomorphism on integral homology $H_*(G(I), \mathbb{Z}) \to H_*(T_n(A), \mathbb{Z})$.*

Proof. We work by induction on n, where there is nothing to prove when $n = 1$, since we must have $G(I) = T_1(A) = A^\times = GL_1(A)$. Next, if $n > 1$, and $I = \{0 < n\}$, then $U(I)$ is the trivial group, so there is nothing to prove.
Hence we may assume $n > 1$, $r \geq 2$, and thus $0 < i_1 < n$. There is then a natural homomorphism $G(I) \to G(I')$, where $I' = \{0 < i_2 - i_1 < \cdots < i_r - i_1 = n - i_1\}$, and $G(I') \subset GL_{n-i_1}(A)$. Let $n' = n - i_1$. The induced homomorphism $T_n(A) \to T_{n'}(A)$ is naturally split, with kernel $T_{i_1}(A) \subset GL_{i_1}(A) \subset GL_n(A)$. Let

$$U_1(I) = \ker\left(U(I) \to U(I')\right) = \ker\left(G(I) \to GL_{i_1}(A) \times GL_{n'}(A)\right).$$

Then $U_1(I)$ is a normal subgroup of $G(I)$, from the last description, and

$$G(I)/U_1(I) \cong U(I') \cdot T_n(A) = T_{i_1}(A) \times G(I').$$

Now $U_1(I)$ may be identified with $M_{i_1,n'}(A)$, the additive group of matrices of size $i_1 \times n'$ over A; this matrix group has a natural action of $GL_{i_1}(A)$, and thus of the diagonal matrix group $T_{i_1}(A)$, and the resulting semidirect product of $T_{i_1}(A)$ with $U_1(A)$ is a subgroup of $G(I)$ (in fact, it is the kernel of $G(I) \to G(I')$). This matrix group $M_{i_1,n'}(A)$ is isomorphic, as $T_{i_1}(A)$-modules, to the direct sum

$$\oplus_{i=1}^{i_1} A^{n'}(i),$$

where $A^{n'}(i)$ is the free A-module of rank n', with a $T_{i_1}(A)$-action given by the the "i-th diagonal entry" character $T_{i_1}(A) \to A^\times$. Thus, the semidirect product $T_{i_1}(A)U_1(I)$ has a description as a direct product

$$T_{i_1}(A)U_1(i) \cong H \times H \times \cdots \times H = H^{n_i}$$

with $H = A^{n'} \cdot A^\times$ equal to the naturally defined semidirect product of the free A module $A^{n'}$ with $A^\times$, where $A^\times$ operates by scalar multiplication.
Proposition 1.10 and Remark 1.13 in the paper [13] of Nesterenko and Suslin implies immediately that $H \to H/A^{n'} \cong A^\times$ induces an isomorphism on integral homology.
We now use the following facts.

(i) If $H \subset K \subset G$ are groups, with H, K normal in G, and if $K \to K/H$ induces an isomorphism in integral homology, so does $G \to G/H$; this follows at once from a comparison of the two spectral sequences

$$E^2_{r,s} = H_r(G/K, H_s(K, \mathbb{Z})) \Rightarrow H_{r+s}(G, \mathbb{Z}),$$

$$E^2_{r,s} = H_r(G/K, H_s(K/H, \mathbb{Z})) \Rightarrow H_{r+s}(G/H, \mathbb{Z}).$$

(ii) If $H_i \subset G_i$ are normal subgroups, for $i = 1, \ldots, n$, such that $G_i \to G_i/H_i$ induce isomorphisms on integral homology, then for $G = \prod_{i=1}^n G_i$, $H = \prod_{i=1}^n H_i$, the map $G \to G/H$ induces an isomorphism on integral homology. This follows from the Kunneth formula.

The fact (ii) implies that $T_{i_1}(A)U_1(I) \to T_{i_1}(A)$ induces an isomorphism on integral homology. Then (i) implies that $G(I) \to T_{i_1}(A) \times G(I')$ induces an isomorphism on integral homology. By induction, we have that $G(I') \to G(I')/U(I')$ induces an isomorphism on integral homology. Hence $T_{i_1}(A) \times G(I') \to T_{i_1} \times G(I')/U(I')$ also induces an isomorphism on integral homology. Thus, we have shown that the composition $G(I) \to G(I)/U(I) = T_n(A)$ induces an isomorphism on integral homology. □

4. $\mathbb{SPL}(A^\infty)^+$ and the groups $L_n(A)$

We first note that there is a small variation of Quillen's plus construction.
Let (X, x) be a pointed CW complex, (X_0, x) a contractible pointed subcomplex, G a group of homeomorphisms of X which acts transitively on the path components of X, and let H be a perfect subgroup of G, such that H stabilizes X_0.
Then $X//G$ is clearly path connected, and comes equipped with
(i) a natural map $\theta : X//G \to BG = EG/G$, induced by the projection $X \times EG \to EG$
(ii) a map $(X_0 \times EG)/H \to X//G$, induced by the H-stable contractible set $X_0 \subset X$
(iii) a homotopy equivalence $BH \to (X_0 \times EG)/H$, such that the composition $BH \to X//G \xrightarrow{\theta} BG$ is homotopic to the natural map $BH \to BG$
(iv) a natural map $(X, x) \hookrightarrow (X//G, x_0)$ determined by the base point of EG.
Note that, in particular, there is a natural inclusion $H \hookrightarrow \pi_1(X//G, x_0)$, which gives a section over $H \subset G$ of the surjection $\theta_* : \pi_1(X//G, x_0) \to \pi_1(BG, *) = G$.

Lemma 14. *In the above situation, there is a pointed CW complex (Y, y), together with a map $f : (X//G, x_0) \to (Y, y)$ such that*
(i) the natural composite map

$$H \hookrightarrow \pi_1(X//G, x_0) \xrightarrow{f} \pi_1(Y, y)$$

is trivial
(ii) if $g : (X//G, x_0) \to (Z, z)$ such that H is in the kernel of

$$\pi_1(X//G, x_0) \to \pi_1(Z, z)$$

then g factors through f, uniquely upto a pointed homotopy
(iii) f induces isomorphisms on integral homology; more generally, if L is any local system on Y, the map on homology with coefficients $H_*(X//G, f^*L) \to H_*(Y, L)$ *is an isomorphism*
(iv) $h : (X, x) \to (X', x')$ *is a pointed map of such CW complexes with G-actions, such that h is G-equivariant, then there is a map* $(Y, y) \to (Y', y')$*, making* $(X, x) \mapsto (Y, y)$ *is functorial (on the category of pointed CW complexes with suitable G actions, and equivariant maps), and f yields a natural transformation of functors.*

The pair (Y, y) is obtained by applying Quillen's plus construction to $(X//G, x_0)$ with respect to the perfect normal subgroup $\widetilde{H}$ of $\pi_1(X//G, x_0)$ which is generated by H. Part (ii) of the lemma is in fact the universal property of the plus construction. As is well-known, this may be done in a functorial way. *We sometimes write* $(Y, y) = (X//G, x_0)^+$ *to denote the above relationship.*
In what follows, the pair (G, H) is invariably $(GL_n(A), A_n)$ for $5 \leq n \leq \infty$. Here A_n is the alternating group contained in $N_n(A)$. The normal subgroups of $GL_n(A)$ generated by A_n and $E_n(A)$ coincide with each other. It follows that if we take $X = X_0$ to be a point, the Y given by the above lemma is just the "original" $BGL_n(A)^+$.
Recall that there is a natural action of $GL(A)$ on the simplicial complex $\mathbb{SPL}(A^\infty)$, and hence on its geometric realization $|\mathbb{SPL}(A^\infty)|$. We apply lemma 14 with $G = GL(A)$, $H = A_\infty$ the infinite alternating group, $X = |\mathbb{SPL}(A^\infty)|$, and $X_0 = \{x_0\}$ is the vertex of X fixed by $N(A)$ and obtain the pointed space

$$(Y(A), y) = (|\mathbb{SPL}(A^\infty)|//G), x_0)^+.$$

Taking X' to be a singleton in (iii) of the above lemma, we get a canonical map

$$\varphi : (Y(A), y) \to (BGL(A)^+, *)$$

of pointed spaces.
Let $(\mathbb{SPL}(A^\infty)^+, z)$ denote the homotopy fibre of φ. We define

$$L_n(A) = \pi_n(\mathbb{SPL}(A^\infty)^+, z) \ \forall \ n \geq 0.$$

The homotopy sequence of the fibration $\mathbb{SPL}(A^\infty)^+ \to Y(A) \to BGL(A)^+$ combined with the path-connectedness of $Y(A)$ yields:

Corollary 15. *There is an exact sequence*

$$\begin{aligned} \cdots \to K_{n+1}(A) \to L_n(A) \to \pi_n(Y(A), y) \to K_n(A) \cdots \\ \cdots \to L_1(A) \to \pi_1(Y(A), y) \to K_1(A) \to L_0(A) \to 0 \end{aligned}$$

where $L_0(A)$ *is regarded as a pointed set.*

Lemma 16. *The natural map* $|\mathbb{SPL}(A^\infty)| \to \mathbb{SPL}(A^\infty)^+$ *induces an isomorphism on integral homology.*

Proof. We may identify the universal covering of $BGL(A)^+$ with $BE(A)^+$, where $BE(A)^+$ is the plus construction (see lemma 14) applied to $BE(A)$ with respect to the infinite alternating group (or, what is the same thing, with respect to $E(A)$ itself). Let $\widetilde{\varphi} : \widetilde{Y} \to BE(A)^+$ be the corresponding pullback map obtained from φ.

We first note that $\mathbb{SPL}(A^\infty)^+$ is also naturally identified with the homotopy fiber of $\widetilde{\varphi}$. There is then a homotopy pullback $\widehat{\varphi} : \widehat{Y} \to BE(A)$ of $\widetilde{\varphi}$ with respect to $BE(A) \to BE(A)^+$. Thus, our map $\mathbb{SPL}(A^\infty) \to \mathbb{SPL}(A^\infty)^+$ may be viewed as the natural map on fibers associated to a map

$$\mathbb{SPL}(A^\infty)//E(A) \to \widehat{Y} \tag{8}$$

of Serre fibrations over $BE(A)$.

From a Leray-Serre spectral sequence argument, we see that since (from lemma 14) $BE(A) \to BE(A)^+$ induces a isomorphism on integral homology, so does $\widehat{Y} \to \widetilde{Y}$. Since also $\mathbb{SPL}(A^\infty)//E(A) \to \widetilde{Y}$ is a homology isomorphism (from lemma 14 again), we see that $\mathbb{SPL}(A^\infty)//E(A) \to \widehat{Y}$ induces an isomorphism on integral homology.

Now we use that the map (8) is a map between two total spaces of Serre fibrations over a common base, inducing a homology isomorphism on these total spaces. We also know that the monodromy representation of $\pi_1(BE(A)) = E(A)$ on the homology of the fibers is trivial, in both cases: for $\widehat{Y}$ this is because it is a pullback from a Serre fibration over a simply connected base, while for $\mathbb{SPL}(A^\infty)$, this is one of the key properties we have already established (see the finishing sentence of section 2). The proof is now complete modulo the remark below, which is a straightforward consequence of the Leray-Serre spectral sequence of a fibration. □

Remark. Let $p : E \to B$ and $p' : E' \to B$ be fibrations with fibers F and F' respectively over the base-point $b \in B$. Let $v : E \to E'$ be a map so that $p' \circ v = p$. Assume that B is path-connected. Then $E \to E'$ is a homology isomorphism implies $F \to F'$ is a homology isomorphism under the following additional assumption:

$M \neq 0$ implies $H_0(\pi_1(B, b), M) \neq 0$ for every $\pi_1(B, b)$-subquotient M of $H_i(F), H_j(F')$ for all i, j.

5. The H-space structure

Recall that $BGL(A)^+$ has an H-space structure in a standard way, obtained from the direct sum operation on free modules of finte rank; this was constructed in [6].

The aim of this section is to prove the proposition below.

Proposition 17. *The space $Y(A)$ has an H-space structure, such that $Y(A) \to BGL(A)^+$ is homotopic to an H-map, for the standard H-space structure on $BGL(A)^+$.*

We first remark that if V is a free A-module of finite rank, then $|\mathbb{SPL}(V)|//GL(V)$ is homeomorphic to the classifying space of the following category $\mathcal{SPL}(V)$: its objects are simplices in $\mathbb{SPL}(V)$ (thus, certain finite nonempty subsets of $SPL(V)$), and morphisms $\sigma \to \tau$ are defined to be elements $g \in GL(V)$ such that $g(\sigma) \subset \tau$, that is, such that $g(\sigma)$ is a face of the simplex τ of $\mathbb{SPL}(V)$.
Let $\mathcal{A}ut(V)$ be the category with a single object $*$, with morphisms given by elements of $GL(V)$, so that the classifying space $B\mathcal{A}ut(V)$ is the standard model for $BGL(V)$. There is a functor $F_V : \mathcal{SPL}(V) \to \mathcal{A}ut(V)$, mapping every obeject σ to $*$, and mappng an arrow $\sigma \to \tau$ in $\mathcal{SPL}(V)$ to the corresponding element $g \in GLV()$. The fiber $F_V^{-1}(*)$ is the poset of simplices of $\mathbb{SPL}(V)$, whose classifying space is thus homeomorphic to $|\mathbb{SPL}(V)|$.
It is fairly straightforward to verify that $B\mathcal{SPL}(V)$ is homeomorphic to $|\mathbb{SPL}(V)|//GL(V)$ (where we have used the classifying space of the translation category of $GL(V)$ as the model for the contractible space $E(GL(V))$). One way to think of this is to consider the category $\widetilde{\mathcal{SPL}}(V)$, whose objects are pairs (σ, h) with σ a simplex of $\mathbb{SPL}(V)$, and $h \in GL(V)$, with a unique morphism $(\sigma, h) \to (\tau, g)$ precisely when $g^{-1}h(\sigma) \subset \tau$. It is clear that by considering the full subcategories of objects of the form (σ, g), where $g \in GL(V)$ is a fixed element, each of which is naturally equivalent to the poset of simplices in $\mathbb{SPL}(V)$, that the classifying space of $\widetilde{\mathcal{SPL}}(V)$ is homeomorphic to $|\mathbb{SPL}(V)| \times E(GL(V)))$. Now it is a simple matter to see (e.g., use the criterion of Quillen, given in [19], lemma 6.1, page 89) that $B\widetilde{\mathcal{SPL}}(V) \to B\mathcal{SPL}(V)$, given by $(\sigma, h) \mapsto h^{-1}(\sigma)$, is a covering space which is a principal $GL(V)$-bundle, where the deck transformations are given by the natural action of $GL(V)$ on $|\mathbb{SPL}(V)| \times E(GL(V))$.
$\mathcal{L}(V)$ denotes the collection of A-submodules $L \subset V$ so that L is free of rank one and V/L is a free module. Now we note that if V', V'' are free A-modules of finite rank, we note that there is a natural inclusion $\mathcal{L}(V') \sqcup \mathcal{L}(V'') \hookrightarrow \mathcal{L}(V' \oplus V'')$. This in turn yields a natural map

$$\varphi_{V',V''} : SPL(V') \times SPL(V'') \to SPL(V' \oplus V''),$$

given by $\varphi_{V',V''}(s,t) = s \sqcup t$.
It follows easily from the definition of $\mathbb{SPL}$ that the above map on vertices induces a simplicial map

$$\Phi_{V',V''} : \mathbb{SPL}(V') \times \mathbb{SPL}(V'') \to \mathbb{SPL}(V' \oplus V'').$$

As explained in section 0, at the level of geometric realisations, this has two descriptions. The first description may be used to show that the counterpart of lemma 8(C) is valid for $\mathbb{SPL}$, namely the homotopy class of the inclusion

$$|\mathbb{SPL}(V')| \times |\mathbb{SPL}(V'')| \to |\mathbb{SPL}(V' \oplus V'')|$$

remains unaffected by composition with the action of $g \in \mathrm{Elem}(V' \hookrightarrow V' \oplus V'')$ on $|\mathbb{SPL}(V' \oplus V'')|$.
The second description however is more useful in this context. Let us abbreviate notation and denote the (partially ordered) set of simplices of $\mathbb{SPL}(V)$ simply

by $\mathcal{S}(V)$. The desired map $\mathcal{S}(V') \times \mathcal{S}(V'') \to \mathcal{S}(V' \oplus V'')$ is given simply by $(\sigma, \tau) \mapsto \varphi_{V',V''}(\sigma \times \tau)$. The resulting map $B(\mathcal{S}(V') \times \mathcal{S}(V'')) \to B\mathcal{S}(V' \oplus V'')$ is the second description of

$$|\mathbb{SPL}(V')| \times |\mathbb{SPL}(V'')| \to |\mathbb{SPL}(V' \oplus V'')|$$

for (a) $B\mathcal{C}' \times B\mathcal{C}'' \cong B(\mathcal{C}' \times \mathcal{C}'')$ and (b) $B\mathcal{S}(V)$ is simply the barycentric subdivision of $|\mathbb{SPL}(V)|$.

This latter description also allows us to go a step further and define the functor $\mathcal{SPL}(V') \times \mathcal{SPL}(V'') \to \mathcal{SPL}(V' \oplus V'')$, given on objects by $(\sigma, \tau) \mapsto \varphi_{V',V''}(\sigma \times \tau)$ as before; on morphisms, it is given by the natural map $\mathrm{GL}\,(V') \times \mathrm{GL}\,(V'') \to GL(V' \oplus V'')$. Hence on classifying spaces, it induces a product

$$|\mathbb{SPL}(V')|//GL(V') \times |\mathbb{SPL}(V'')|//GL(V'') \to |\mathbb{SPL}(V' \oplus V'')|//GL(V' \oplus V'')|.$$

This is clearly compatible with the product

$$BGL(V') \times BGL(V'') \to BGL(V' \oplus V'')$$

under the natural maps induced by the functors $\mathcal{SPL} \to \mathcal{A}ut$ for the three free modules.

One verifies that $\mathbf{SPL}(A) = \coprod_V \mathcal{SPL}(V)$, with respect to the bifunctor

$$+ : \mathbf{SPL}(A) \times \mathbf{SPL}(A) \to \mathbf{SPL}(A)$$

induced by direct sums on free modules, and the functors $\Phi_{V',V''}$, form a symmetric monoidal category.

An equivalent category, also denoted $\mathbf{SPL}(A)$ by abuse of notation, is that whose objects are pairs (V, σ), where V is a free A-module of finite rank, and $\sigma \in \mathbb{SPL}(V)$ a simplex, and where morphisms $(V, \sigma) \to (W, \tau)$ are isomorphisms $f : V \to W$ of A-modules such that $f(\sigma)$ is a face of τ.

For the purposes of stabilization, we slightly modify the above to consider the related maps

$$\varphi_{m,n} : SPL(A^m) \times SPL(A^n) \to SPL(A^\infty)$$

given by mapping the basis vector $e_i \in A^m$ in the first factor to the basis vector $e_{2i-1} \in A^\infty$, for each $1 \leq i \leq m$, and the basis vector $e_j \in A^n$ in the second factor to the basis vector $e_{2j} \in A^\infty$. A pair of splittings of A^m, A^n determine one for the free module spanned by the images of the two sets of basis vectors; now one extends this to a splitting of A^∞ by adjoining the remaining basis vectors of A^∞ (that is, adjoining those vectors not in the span of the earlier images). If our first two splittings are those given by the basis vectors, which correspond to the base points in $|\mathbb{SPL}(A^m)|$ and $|\mathbb{SPL}(A^n)|$, the resulting point in $SPL(A^\infty)$ is again the base point of $|\mathbb{SPL}(A)|$.

The corresponding functors

$$\Phi_{m,n} : \mathcal{SPL}(A^m) \times \mathcal{SPL}(A^n) \to \mathcal{SPL}(A^\infty)$$

are compatible with similar functors

$$\mathcal{A}ut(A^m) \times \mathcal{A}ut(A^n) \to \mathcal{A}ut(A^\infty)$$

which, on classifying spaces, yield the diagram of product maps, preserving base points,

$$\begin{array}{ccc} |\mathbb{SPL}(A^m)//GL_n(A)| \times |\mathbb{SPL}(A^n)|//GL_n(A) & \to & |SPL(A^\infty|//GL(A) \\ \downarrow & & \downarrow \\ BGL_m(A) \times BGL_n(A) & \to & BGL(A) \end{array}$$

where the bottom arrow is the one used in [6] to define the H-space structure on $BGL(A)^+$.

As we increase m, n, the corresponding diagrams are compatible with respect to the obvious stabilization maps $|\mathbb{SPL}(A^m)| \hookrightarrow |\mathbb{SPL}(A^{m+1})|$, $|\mathbb{SPL}(A^n) \hookrightarrow |\mathbb{SPL}(A^{n+1})|$. Hence we obtain on the direct limits a diagram

$$\begin{array}{ccc} |\mathbb{SPL}(A^\infty)//GL(A)| \times |\mathbb{SPL}(A^\infty)|//GL(A) & \to & |SPL(A^\infty|//GL(A) \\ \downarrow & & \downarrow \\ BGL(A) \times BGL(A) & \to & BGL(A) \end{array}$$

From lemma 14, it follows that there is an induced diagram at the level of plus constructions

$$\begin{array}{ccc} Y(A) \times Y(A) & \to & Y(A) \\ \downarrow & & \downarrow \\ BGL(A)^+ \times BGL(A)^+ & \to & BGL(A)^+ \end{array}$$

Here we have, as remarked above, taken the homotopy equivalent model $|\mathbb{SPL}(A^\infty|//GL(A)$ for the homotopy type earlier denoted $Y(A)$. We abuse notation and use the same symbol to denote this model as well.

It is shown in [6] that the botton arrow defines an H-space structure on $BGL(A)$. We claim that, by analogous arguments, the top arrow also defines an H-space structure on $Y(A)$. Granting this, the map $Y(A) \to BGL(A)^+$ is then an H-map between path connected H-spaces, and so the homotopy fiber $Z(A)$ has the homotopy type of an H-group as well (and this was what we set out to prove here).

To show that the product $Y(A) \times Y(A) \to Y(A)$ defines an H-space structure, we need to show that left or right translation on $Y(A)$ (with respect to this product) by the base point is homotopic to the identity. This is also the main point in [6], for the case of $BGL(A)^+$. We first show:

Lemma 18. *An arbitrary inclusion $j : \{1, 2, \ldots, n\} \hookrightarrow \mathbb{N}$ determines an inclusion of A-modules $A^n \to A^\infty$, given on basis vectors by $e_i \mapsto e_{j(i)}$, which induces a map*

$$|\mathbb{SPL}(A^n)|//GL_n(A) \to Y(A)$$

which is homotopic (preserving the base point) to the map induced by standard inclusion $i_n : A^n \to A^\infty$.

Proof. We can find an automorphism g of A^∞ contained in the infinite alternating group A_∞, such that $g \circ j = i_n$, where g acts on A^∞ by permuting the basis vectors (note that the induced self-map of $|SPL(A^\infty)| \times EGL(A)$ fixes the base point). Regarding g as an element of $\pi_1(|\mathbb{SPL}(A^\infty)|//GL(A))$, this implies that the maps $(i_n)_*$ and j_*, considered as elements of the set of pointed homotopy classes of maps

$$[|\mathbb{SPL}(A^n)|//GL_n(A), |\mathbb{SPL}(A^\infty)|//GL(A)]\,,$$

are related by $g_*(j_*) = (i_n)_*$, where g_* denotes the action of the fundamental group of the target on the set of pointed homotopy classes of maps. However, g is in the kernel of the map on fundamental groups associated to the map

$$|\mathbb{SPL}(A^\infty)|//GL(A) \to (|\mathbb{SPL}(A^\infty|//GL(A))^+.$$

Hence the induced maps

$$|\mathbb{SPL}(A^n)|//GL(A) \to (|\mathbb{SPL}(A^\infty)|//GL(A))^+$$

determined by i_n and j are homotopic. □

Corollary 19. *The map $Y(A) \to Y(A)$ defined by an arbitrary injective map $\alpha : \mathbb{N} \hookrightarrow \mathbb{N}$ is homotopic, preserving the base point, to the identity.*

Proof. We first note that if for $n \geq 5$, we let $Y_n(A) = (|\mathbb{SPL}(A^n)|//GL_n(A))^+$ be the result of applying lemma 14 to $|\mathbb{SPL}(A^n)|//GL_n(A)$ for the alternating group A_n, then there are natural maps $Y_n(A) \to Y(A)$, preserving base points, and inducing an isomorphism $\varinjlim_n \pi_*(Y_n(A)) = \pi_*(Y(A))$.

We claim that if $\alpha_n : \{1, 2 \ldots, n\} \hookrightarrow \mathbb{N}$ is the inclusion induced by restricting α, then the induced map $(\alpha_n)_* : Y_n(A) \to Y(A)$ is homotopic, preserving the base points, to the natural map $Y_n(A) \to Y(A)$. This follows from lemma 18, combined with the defining universal property of the plus construction, given in lemma 14.

This implies that the map $\alpha : Y(A) \to Y(A)$ must then induce isomorphisms on homotopy groups, and hence is a homotopy equivalence, by Whitehead's theorem.

Thus, we have a map from the set of such injective maps α to the group of base-point preserving homotopy classes of self-maps of $Y(A)$. This is in fact a homomorphism of monoids, where the operation on the injective self-maps of $\mathbb{N}$ is given by composition of maps.

Now we use a trick from [6]: any homomorphism of monoids from the monoid of injective self-maps of $\mathbb{N}$ to a group is a trivial homomorphism, mapping all elements of the monoid to the identity. This is left to the reader to verify (or see [6]). □

We note that the above monoidal category $\mathbf{SPL}(A)$ can be used to give another, perhaps more insightful construction of the homotopy type $Y(A)$, analogous to Quillen's $\mathcal{S}^{-1}\mathcal{S}$ construction for $BGL(A)^+$. We sketch the argument below.

We first take $\mathbf{SPL}_0(A)$ to be the full subcategory of $\mathbf{SPL}(A)$ consisting of pairs (V, σ) where $\sigma \in SPL(V)$, i.e.,σ is a 0-simplex in $\mathbb{SPL}(V)$. This full subcategory is in fact a monoidal subcategory, which is a groupoid (all arrow are isomorphisms). Also, $\mathbf{SPL}(A)$ is a symmetric monoidal category, in that the sum operation is commutative upto coherent natural isomorphisms.

Then, using Quillen's results (see Chapter 7 in [19], particularly Theorem 7.2), one can see that $\mathbf{SPL}_0(A)^{-1}\mathbf{SPL}(A)$ is a monoidal category whose classifying space is a connected H-space, which is naturally homology equivalent to $|\mathbb{SPL}(A^\infty)|//GL(A)$. This then forces this classifying space to be homotopy equivalent to $Y(A)$, such that the H-space operations are compatible upto homotopy. This is analogous to the identification made in Theorem 7.4 in [19] of $\mathcal{S}^{-1}\mathcal{S}$ with $K_0(R) \times BGL(R)^+$ for a ring R, and appropriate $\mathcal{S}$. (We do not get the factor K_0 appearing in our situation since we work only with free modules).

6. Theorem 1 and the groups $\mathcal{H}_n(A^\times)$

Proof of Theorem 1. In view of Proposition 17, we see that $\mathbb{SPL}(A^\infty)^+$, the homotopy fiber of the H-map $Y(A) \to BGL(A)^+$, is a H-space as well. It follows that $L_0(A) = \pi_0(\mathbb{SPL}(A^\infty)^+)$ is a monoid. Furthermore, the arrow $K_1(A) \to L_0(A)$ in Corollary 15 is a monoid homomorphism. Thus this corollary produces an exact sequence of Abelian groups.

$\mathbb{SPL}(A^n)$ has a canonical base point fixed under the action of $N_n(A)$. As in sections 3 and 4, this gives a natural inclusion $BN_n(A) \to \mathbb{SPL}_n(A)//GL_n(A)$. This is a homology isomorphism by lemma 12. Taking direct limits over all $n \in \mathbb{N}$, we see that $BN(A) \to \mathbb{SPL}(A^\infty)//GL(A)$ is a homology isomorphism. Applying Quillen's plus construction with respect to the normal subgroup of $N(A)$ generated by the infinite alternating group, we obtain a space $BN(A)^+$. That $BN(A)^+$ has a canonical H-space structure follows easily by the method of the previous section. Now the map $BN(A)^+ \to Y(A)$ obtained by lemma 14(ii) is a homology isomorphism of simple path-connected CW complexes and is therefore a homotopy equivalence (see [4]) Theorem 4.37, page 371 and Theorem 4.5, page 346). This gives the isomorphism $\mathcal{H}_n(A^\times) \to L_n(A)$. The theorem now follows from corollary 15.

We now turn to the description of the groups. Let $X = B(A^\times)$. Let $X_+ = X \sqcup \{*\}$ be the pointed space with $*$ as its base-point. Let QX_+ be the direct limit of $\Omega^n\Sigma^n X_+$ where Σ denotes reduced suspension.

Proposition 20. $\mathcal{H}_n(A^\times) \cong \pi_n(QX_+)$.

This statement was suggested to us by Proposition 3.6 of [17].

A complete proof of the proposition was shown us by Peter May. A condensed version of what we learnt from him is given below.

Theorem 2.2, page 67 of [8] asserts that $\alpha_\infty : C_\infty X_+ \to QX_+$ is a group completion. This is proved in pages 50-59, [10]. The C_∞ here is a particular case of the construction 2.4, page 13 of [9], given for any operad. For $C_\infty(Y)$, where Y is a pointed space, the easiest definition to work with is found in May's review of [16]. It runs as follows. Let $V = \cup_{n=0}^\infty \mathbb{R}^n$. Let $C_k(Y)$ be the collection of ordered pairs (c, f) where $c \subset V$ has cardinality k and $f : c \to Y$ is any function. We identify (c, f) with (c', f') if

(i) $c' \subset c$, (ii) $f|c' = f'$, and (iii) $f(a) = *$ for all $a \in c, a \notin c'$. Here $*$ stands for the base-point of Y. Then $C_\infty Y$ is the space obtained from the disjoint union

of the $C_k(Y), k \geq 0$ by performing these identifications. This is a H-space. In our case, when $Y = X \sqcup \{*\}$, it is clear that $C_\infty Y$ is the disjoint union of all the $C_k(X)$ as a topological space. Thanks to "infinite codimension" one gets easily the homotopy equivalence of $C_k(X)$ with $X^k//S_k$ where S_k is the permutation group of $\{1, 2, .., k\}$. Now assume that X is any path-connected space equipped with a nondegenerate base-point $x \in X$. This x gives an inclusion of $X^n \hookrightarrow X^{n+1}$. Denote by X^∞ the direct limit of the X^n. Thus X^∞ is a pointed space equipped with the action of the infinite permutation group $S_\infty = \cup n S_n$. Put $Z = X^\infty//S_\infty$. As in section 4, we obtain Z^+ by the use of the infinite alternating group. As in section 5, we see that this is a H-space. It is an easy matter to check that the group completion of $\sqcup_k C_k X$ is homotopy equivalent to $\mathbb{Z} \times Z^+$. This shows that $\pi_n(QX_+) \cong \pi_n(Z^+)$ for all $n > 0$.
The proposition is the particular case: $X = B(A^\times)$.

7. Polyhedral structure of the enriched Tits building

From what has been shown so far, we see that it is of interest to determine the stable rational homology of the flag complexes $|\mathbb{FL}(A^n)|$ (or equivalently, of $|\mathbb{SPL}(A^n)|$, or $\mathbb{ET}(A^n)$). We will construct a spectral sequence that, in principle, gives an inductive procedure to do so.
But first we introduce some notation and a definition for posets.
Let P be a poset. For $p \in P$, we put $e(p) = BL(p)$ where $L(p) = \{q \in P | q \leq p\}$ and $\partial e(p) = BL'(p)$ where $L'(p) = L(p) \setminus \{p\}$. If $\partial e(p)$ is homeomorphic to a sphere for every $p \in P$, we say the poset P is *polyhedral*. We denote by $d(p)$ the dimension of $e(p)$. When P is polyhedral, the space BP gets the structure of a CW complex with $\{e(p) : p \in P\}$ as the closed cells. Its r-skeleton is BP_r where $P_r = \{p \in P : d(p) \leq r\}$. The homology of BP is then computed by the associated complex of cellular chains $Cell_\bullet(BP)$, where

$$Cell_r(BP) = \bigoplus_{\{p | \dim e(p) = r\}} H_r(e(p), \partial e(p), \mathbb{Z}).$$

Lemma 21. *$\mathcal{E}(A^n)$ is a polyhedral poset in the above sense. Its dimension is $n - 1$.*

Proof. First consider the case when $p \in SPL(A^n)$ is a maximal element in $\mathcal{E}(A^n)$. Then p is an unordered collection of n lines in A^n (here, as in §2, a "line" denotes a free A-submodule of rank 1 which is a direct summand, and the set of lines in A^n is denoted by $\mathcal{L}(A^n)$). Note that the subset $p \subset \mathcal{L}(A^n)$ of cardinality n determines a poset $\widetilde{p}$, whose elements are chains $q_\bullet = \{q_1 \subset q_2 \subset \cdots \subset q_r = p\}$ of nonempty subsets, where $r_\bullet \leq q_\bullet$ if each q_i is an r_j for some j, i.e., the "filtration" $r_\bullet$ "refines" $q_\bullet$. We claim that, from the definition of the partial order on $\mathcal{E}(A^n)$, the poset $\widetilde{p}$ is naturally isomorphic to the poset $L(p)$. Indeed, an element $q \in \mathcal{E}(A^n)$ consists of a pair, consisting of a partial flag

$$0 = W_0 \subset W_1 \subset \cdots \subset W_r = A^n$$

such that W_i/W_{i-1} is free, and a sequence $t_1, \dots, t_r$ with $t_i \in SPL(W_i/W_{i-1})$. The condition that this element of $\mathcal{E}(A^n)$ lies in $L(p)$ is that each W_i is a direct sum of a subset of the lines in p, say $q_i \subset p$, giving the chain of subsets $q_1 \subset q_2 ... \subset q_r = p$; the splitting t_i is uniquely determined by the lines in $q_i \setminus q_{i-1}$.
Let $\Delta(p)$ be the $(n-1)$-simplex with p as its set of vertices. Now the chains of non-empty subsets of p correspond to simplices in the barycentric subdivison $sd\Delta(p)$, where the barycentre b corresponds to the chain $\{p\}$. Hence, from the definition of $\widetilde{p}$, it is clear that it is isomorphic to the poset whose elements are simplices in the barycentric subdivision of Δ_n with b as a vertex, with partial order given by reverse inclusion. Hence $B\widetilde{p}$ is naturally identified with the subcomplex of the second barycentric subdivision $sd^2\Delta(p)$ which is the union of all simplices containing the barycentre. This explicit description implies in particular that $BL'(p)$ is homeomorphic to S^{n-2} (with a specific triangulation).
Before proceeding to the general case, we set up the relevant notation for orientations. For a set q of cardinality r, we put $\det(q) = \wedge^r \mathbb{Z}[q]$ where $\mathbb{Z}[q]$ denotes the free Abelian group with q as basis. we observe that there is a natural isomorphism:

$$H_{n-1}(e(p), \partial e(p)) \cong H_{n-1}(\Delta(p), \partial\Delta(p)) = \det(p).$$

Now let $p \in \mathcal{E}(A^n)$ be arbitrary, corresponding to a partial flag

$$0 = W_0 \subset W_1 \subset \cdots \subset W_r = A^n.$$

and splittings $t_i \in SPL(W_i/W_{i-1})$. Then the natural map

$$\prod_{i=1}^{r} \mathcal{E}(W_i/W_{i-1}) \to \mathcal{E}(A^n)$$

is an embedding of posets, where the product has the ordering given by $(a_1, \dots, a_r) \leq (b_1, \dots, b_r)$ precisely when $a_i \leq b_i$ in $\mathcal{E}(W_i/W_{i-1})$ for each i. One sees that, by the definition of the partial order in $\mathcal{E}(A^n)$, the induced map

$$\prod_{i=1}^{r} L(t_i) \to L(p)$$

is bijective. Hence there is a homeomorphism of pairs

$$(BL(p), BL'(p)) = \prod_{i=1}^{r} (BL(t_i), BL'(t_i)),$$

and so $BL'(p) \cong S^{n-r-1}$, and $BL(p)$ is an $n-r$-cell. □

We now proceed to construct the desired spectral sequence. We use the following notation: if $p \in \mathcal{E}(V)$, where V is a free A-module of finite rank, and $W_1 \subset V$ is the smallest non-zero submodule in the partial flag associated to p, define $t(p) = \operatorname{rank} W_1 - 1$. Clearly $t : \mathcal{E}(V) \to \mathbb{Z}$ is monotonic. Hence $F_r\mathcal{E}(V) = \{p \in \mathcal{E}(V) | t(p) \leq r\}$ is a sub-poset. Define

$$F_r\mathbb{ET}(V) = BF_r\mathcal{E}(V) = \cup\{e(p) | t(p) \leq r\},$$

so that

$$F_0\mathbb{ET}(V) \subset F_1\mathbb{ET}(V) \subset \cdots F_{n-1}\mathbb{ET}(V) = \mathbb{ET}(V)$$

is an increasing finite filtration of the CW complex $\mathbb{ET}(V)$ by subcomplexes. Hence there is an associated spectral sequence

$$E^1_{r,s} = H_{r+s}(F_r\mathbb{ET}(V), F_{r-1}\mathbb{ET}(V), \mathbb{Z}) \Rightarrow H_{r+s}(\mathbb{ET}(V), \mathbb{Z}).$$

Our objective now is to recognise the above E^1 terms.
It is convenient to use the complexes of cellular chains for these sub CW-complexes, which are thus sub-chain complexes of $Cell_\bullet(\mathbb{ET}(V))$. For simplicity of notation, we write $Cell_\bullet(V)$ for $Cell_\bullet(\mathbb{ET}(V))$. We have the description

$$E^1_{r,s} = H_{r+s}(\mathrm{gr}^F_r Cell_\bullet(V)).$$

We will now exhibit $\mathrm{gr}^F_r Cell_\bullet(V)$ as a direct sum of complexes. Let $W \subset V$ be a submodule such that $W, V/W$ are both free, and $\mathrm{rank}\, W = r+1$. Let $q \in SPL(W)$. The we have an inclusion of chain complexes

$$Cell_\bullet(e(q)) \otimes Cell_\bullet(V/W) \subset Cell_\bullet(W) \otimes Cell_\bullet(V/W) \subset F_r Cell_\bullet(V).$$

It is clear that

$$\mathrm{image}\, Cell_\bullet(\partial e(q)) \otimes Cell_\bullet(V/W) \subset F_{r-1}Cell_\bullet(V),$$

so that we have an induced homomorphism of complexes

$$(Cell_\bullet(e(q))/Cell_\bullet(\partial e(q))) \otimes Cell_\bullet(V/W) \to \mathrm{gr}^F_r Cell_\bullet(V).$$

Composing with the natural chain homomorphism

$$H_r(e(q), \partial e(q), \mathbb{Z})[r] \to (Cell_\bullet(e(q))/Cell_\bullet(\partial e(q)))$$

for each q, we finally obtain a chain map

$$I : \bigoplus_{(W, q \in SPL(W))} H_r(e(q), \partial e(q), \mathbb{Z})[r] \otimes Cell_\bullet(V/W) \to \mathrm{gr}^F_r Cell_\bullet(V).$$

Finally, it is fairly straightforward to verify that I is an isomorphism of complexes.
We deduce that the E^1 terms have the following description:

$$E^1_{r,s} = \bigoplus_{\substack{\mathrm{rank}\, W = r+1 \\ q \in SPL(W)}} \det(q) \otimes H_s(Cell_\bullet(V/W), \mathbb{Z}).$$

We define $\mathcal{L}_r(V)$ to be the collection of $q \subset \mathcal{L}(V)$ of cardinality $(r+1)$ for which (a) and (b) below hold:
(a) $\oplus\{L : L \in q\} \to V$ is injective. Its image will be denoted by $W(q)$
(b) $V/W(q)$ is free of rank $(n-r-1)$.
Summarising the above, we obtain:

Theorem 2. *There is a spectral sequence with E^1 terms*

$$E^1_{r,s} = \bigoplus_{q \in \mathcal{L}_r(V)} \det(q) \otimes H_s(\mathbb{ET}(V/W(q)), \mathbb{Z}).$$

that converges to $H_{r+s}(\mathbb{ET}(V))$. We note that $E^1_{r,s} = 0$ whenever $(r+s) \geq (n-1)$ with one exception: $(r,s) = (n-1, 0)$. Here $V \cong A^n$.

8. Compatible homotopy

It is true [3] that $i : \mathbb{ET}(W) \times \mathbb{ET}(V/W) \hookrightarrow \mathbb{ET}(V)$ has the property that $g \circ i$ is freely homotopic (not preserving base points) to i whenever $g \in \mathrm{Elem}(W \hookrightarrow V)$. There are several closed subsets of $\mathbb{ET}(A^n)$ with the property that homotopy class of the inclusion morphism into $\mathbb{ET}(A^n)$ remains unaffected by composition with the action of $g \in E_n(A)$. To prove that the union of a finite collection of such closed subsets has the same property, one would require the homotopies provided for any two members of the collection to agree on their intersection. This is the problem we are concerned with in this section.
We proceed to set up the notation for the problem.
With $q \in \mathcal{L}_r(V)$ as in theorem 2, we shall define the subspaces $U(q) \subset \mathbb{ET}(V)$ as follows. Let $W(q) = \oplus\{L | L \in q\}$. We regard q as an element of $SPL(W(q))$ and thus obtain the cell $e(q) = BL(q) \subset \mathbb{ET}(W(q))$. This gives the inclusion $\mathbb{ET}'(q) = e(q) \times \mathbb{ET}(V/W(q)) \subset \mathbb{ET}(W(q)) \times \mathbb{ET}(V/W(q)) \subset \mathbb{ET}(V)$.
We put $U(q) = \cup\{\mathbb{ET}'(t) | \emptyset \neq t \subset q\}$.

Main Question: Let $i : U(q) \hookrightarrow \mathbb{ET}(V)$ denote the inclusion. Is it true that $g \circ i$ is homotopic to i for every $g \in \mathrm{Elem}(V, q)$?
We focus on the apparently weaker question below.
Compatible Homotopy Question: Let $M \subset V$ be a submodule complementary to $W(q)$. Let $g' \in GL((W(q))$ be elementary, i.e. $g' \in \mathrm{Elem}(W(q), q)$. Define $g \in GL(V)$ by $gm = m$ for all $m \in M$ and $gw = g'w$ for all $w \in W(q)$. Is it true that $g \circ i$ is homotopic to i?
Assume that the second question has an affirmative answer in all cases. In particular, this holds when $M = 0$. Here $V = W(q)$ and $g = g'$ is an arbitrary element of $\mathrm{Elem}(V, q)$. Let t be a non-empty subset of q. Then $U(t) \subset U(q)$. We deduce that $j : U(t) \hookrightarrow \mathbb{ET}(V)$ is homotopic to $g \circ j$ for all $g \in \mathrm{Elem}(V, q)$. But $\mathrm{Elem}(V, t) = \mathrm{Elem}(V, q)$. Thus the Main Question has an affirmative answer for (q, i) replaced by (t, j), which of course, up to a change of notation, covers the general case.

Proposition 22. *The compatible homotopy question has an affirmative answer if $q \in \mathcal{L}_r(V)$ and $r \leq 2$.*

The rest of this section is devoted to the proof of this proposition. To proceed, we will require to introduce the class $\mathcal{C}$.

[3] this only requires the analogue of lemma 8(A) for the enriched Tits building. More general statements are contained in lemmas 23 and 24 .

This is our set-up. Let X be a finite set, let V_x be a finitely generated free module for each $x \in X$ and let $V = \oplus\{V_x : x \in X\}$.
Let $s = \Pi_{x\in X}s(x) \in \Pi_{x\in X}SPL(V_x)$. For each $x \in X$, we regard $s(x)$ as a subset of $\mathcal{L}(V)$ and put $Fs = \cup\{s(x)|x \in X\}$. Thus $Fs \in SPL(V)$. *The collection of maps* $f : \Pi_{x\in X}\mathbb{ET}(V_x) \to \mathbb{ET}(V)$ *with the property that* $f(\Pi_{x\in X}BL(s(x))) \subset BL(Fs)$ *for all* $s \in \Pi_{x\in X}SPL(V_x)$ *is denoted by* $\mathcal{C}$. See lemma 10 and proposition 11 and its proof for relevant notation.
Every maximal chain C of subsets of X (equivalently every total ordering of X) gives a member $i(C) \in \mathcal{C}$. For instance, if $X = \{1, 2, ..., n\}$ and C consists of the sets $\{1, 2, ..., k\}$ for $1 \le k \le n$, we put
$D_k = \oplus_{i=1}^k V_i$ and $E = \Pi_{i=1}^n \mathbb{ET}(D_i/D_{i-1})$, denote by
$u : \Pi_{x\in X}\mathbb{ET}(V_x) \to E$ and $v : E \to \mathbb{ET}(V)$ the natural isomorphism and natural inclusion respectively, and put $i(C) = v \circ u$.

Lemma 23. *The above space* $\mathcal{C}$ *is contractible.*

Proof. The aim is to realise $\mathcal{C}$ as the space of Λ-compatible maps for a suitable Λ and appeal to Proposition 1. Let $\Lambda(x) = \{L(S)|\emptyset \ne S \subset SPL(V_x), \emptyset \ne L(S)\}$. Proposition 11 asserts that the subspaces $\{B\lambda(x) : \lambda(x) \in \Lambda(x)\}$ give an admissible cover of $\mathbb{ET}(V_x)$.
For $\lambda = \Pi_{x\in X}\lambda(x) \in \Lambda = \Pi_{x\in X}\Lambda(x)$, we put $I(\lambda) = \Pi_{x\in X}B\lambda(x)$ and deduce that $\{I(\lambda) : \lambda \in \Lambda\}$ gives an admissible cover of $\Pi_{x\in X}\mathbb{ET}(V_x)$.
We define next a closed $J(\lambda) \subset \mathbb{ET}(V)$ for every $\lambda \in \Lambda$ with the properties:
(A): $J(\lambda) \subset J(\mu)$ whenever $\lambda \le \mu$ and
(B): $J(\lambda)$ is contractible for every $\lambda \in \Lambda$.
For each $\lambda(x) \in \Lambda(x)$, let $U\lambda(x)$ be its set of upper bounds in $SPL(V_x)$. It follows that that $LU\lambda(x)) = \lambda(x)$. As observed before, we have
$F : \Pi_{x\in X}SPL(V_x) \to SPL(V)$. Thus given $\lambda = \Pi_x\lambda(x) \in \Lambda$,
we set $H(\lambda) = F(\Pi_{x\in X}U\lambda(x)) \subset SPL(V)$ and then put $J(\lambda) = BLH(\lambda) \subset \mathbb{ET}(V)$.
The space of Λ-compatible maps $\Pi_{x\in X}\mathbb{ET}(V_x) \to \mathbb{ET}(V)$ is seen to coincide with $\mathcal{C}$. That the $J(\lambda)$ satisfy property (A) stated above is straightforward.
The contractibility of $J(\lambda)$ for all $\lambda \in \Lambda$ is guaranteed by proposition 11 once it is checked that these sets are nonempty. But we have already noted that $\mathcal{C}$ is nonempty. Let $f \in \mathcal{C}$. Now $I(\lambda) \ne \emptyset$ and $f(I(\lambda)) \subset J(\lambda)$ implies $J(\lambda) \ne \emptyset$. Thus the $J(\lambda)$ are contractible, and as said earlier, an application of Proposition 1 completes the proof of the lemma.

□

We remark that the class $\mathcal{C}$ of maps $\Pi_{i=1}^n\mathbb{ET}(W_i) \to \mathbb{ET}(\oplus_{i=1}^n W_i)$ has been defined in general.
We will continue to employ the notation: $V = \oplus\{V_x : x \in X\}$ all through this section. Let P be a partition of X. Each $p \in P$ is a subset of X and we put

$$V_p = \oplus\{V_x|x \in p\} \text{ and } \mathbb{ET}(P) = \Pi\{\mathbb{ET}(V_p)|p \in P\}.$$

When $Q \le P$ is a partition of X (i.e. Q is finer than P), we shall define the contractible collection $\mathcal{C}(Q, P)$ of maps $f : \mathbb{ET}(Q) \to \mathbb{ET}(P)$ by demanding (a) that f is the product of maps $f(p)$

$$f(p) : \Pi\{\mathbb{ET}(V_q) : q \subset p \text{ and } q \in Q\} \to \mathbb{ET}(V_p)$$

and also (b) each $f(p)$ is in the class $\mathcal{C}$. For this one should note that $V_p = \oplus\{V_q : q \in Q \text{ and } q \subset p\}$.

We observe next that there is a distinguished collection $\mathcal{D}(Q,P) \subset \mathcal{C}(Q,P)$. To see this, recall that we had the embedding $i(C)$ for every maximal chain C of subsets of X (alternatively, for every total ordering of X). Given $Q \leq P$, denote the set of total orderings of $\{q \in Q : q \subset p\}$ by $T(p)$, for every $p \in P$. The earlier $C \mapsto i(C)$ now yields, after taking a product over $p \in P$,
$i : \Pi\{T(p) : p \in P\} \to \mathcal{C}(Q,P)$, and we denote by $\mathcal{D}(Q,P) \subset \mathcal{C}(Q,P)$ the image of i.

The lemma below is immediate from the definitions.

Lemma 24. *Given partitions $R \leq Q \leq P$ of X, if f is in $\mathcal{C}(R,Q)$ (resp. in $\mathcal{D}(R,Q)$)and g is in $\mathcal{C}(Q,P)$ (resp. in $\mathcal{D}(Q,P)$), then it follows that $g \circ f$ is in $\mathcal{C}(R,P)$ (resp. $\mathcal{D}(R,P)$).*

We will soon have to focus on the fixed points of certain unipotent $g \in GL(V)$ on $\mathbb{ET}(V)$. For instance, if $x, y \in X$ and $x \neq y$, we may consider $g = id_V + h$ where $h(V) \subset V_y$ and $h(V_z) = 0$ for all $z \in X, z \neq x$. Let C be a chain of subsets of X, so that $X \in C$. This chain C gives rise to a partition $P(C)$ of X and also $i(C) \in \mathcal{D}(P(C), \{X\})$ in a natural manner. Let $C_x = \cap\{S \in C : x \in S\}$. Then $C_x \in C$ because C is a chain. Define C_y in a similar manner. *We say the chain C is (x,y)-compatible if $C_y \subset C_x$ and $C_x \neq C_y$* . This condition on C ensures that the embedding $i(C) : \mathbb{ET}(P(C)) \to \mathbb{ET}(V)$ has its image within the fixed points of the above $g \in GL(V)$.

Now let Q be a partition of X so that $q \in Q, x \in q$ implies $y \notin q$. We shall define next the class of (x,y)-compatible $\mathcal{C}$ maps $\mathbb{ET}(Q) \to \mathbb{ET}(V)$ in the following manner. Let Λ be the set of chains C of subsets of X so that $X \in C$ and $Q \leq P(C)$ (i.e. Q is finer than the partition $P(C)$). For each $C \in \Lambda$, let $Z(C)$ be the collection of $i(C) \circ f$ where $f \in \mathcal{C}(Q, P(C))$. Finally, let $Z = \cup\{Z(C) : C \in \Lambda\}$. This set Z is defined to be the collection of (x,y)-compatible maps of class $\mathcal{C}$ from $\mathbb{ET}(Q)$ to $\mathbb{ET}(V)$. Every $z \in Z$ is a map $z : \mathbb{ET}(Q) \to \mathbb{ET}(V)$ whose image is contained in the fixed points of the above g on $\mathbb{ET}(V)$. Furthermore, in view of lemma 24, this collection of maps is contained in $\mathcal{C}(Q, \{X\})$.

Lemma 25. *Let Q be a partition of X that separates x and y. Then the collection of (x,y)-compatible class $\mathcal{C}$ maps $\mathbb{ET}(Q) \to \mathbb{ET}(V)$ is contractible.*

Proof. In view of the fact that each $\mathcal{C}(Q,P)$ is contractible, by cor 3, it follows that the space of (x,y)-compatible chains is homotopy equivalent to $B\Lambda$, where Λ is the poset of chains C in the previous paragraph. It remains to show that $B\Lambda$ is contractible.

We first consider the case where Q is the set of all singletons of X. Let $\mathcal{S}$ be the collection of subsets $S \subset X$ so that $y \in S$ and $x \notin S$. For $S \in \mathcal{S}$, let

$\mathcal{F}(S)$ be the collection of chains C of subsets of X so that $S \in C$ and $X \in C$. We see that Λ is precisely the union of $\mathcal{F}(S)$ taken over all $S \in \mathcal{S}$. Let D be a finite subset of $\mathcal{S}$. We see that the intersection of the $B\mathcal{F}(S)$, taken over $S \in D$, is nonempty if and only if D is a chain. Furthermore, when D is a chain, this intersection is clearly a cone, and therefore contractible. By cor 3, we see that $B\Lambda$ has the same homotopy type as the classifying space of the poset of chains of $\mathcal{S}$. But this is simply the barycentric subdivision of $B\mathcal{S}$. But the latter is a cone as well, with $\{y\}$ as vertex. This completes the proof that $B\Lambda$ is contractible, when Q is the finest possible partition of X.
We now come to the general case, when Q is an arbitrary partition of X that separates x, y. So we have $x', y' \in Q$ with $x \in x', y \in y'$ and $x' \neq y'$. The set Λ is identified with the collection of chains C' of subsets of Q so that
(a) $Q \in C'$, and (b) there is some $L \in C'$ so that $x' \notin L$ and $y' \in L$.
Thus the general case follows from the case considered first: one replaces (X, x, y) by (Q, x', y'). $\square$

In a similar manner, we may define, for every ordered r-tuple $(x_1, x_2, ..., x_r)$ of distinct elements of X, the set of $(x_1, x_2, ..., x_r)$-compatible chains C–we demand that for each $0 < i < r$, there is a member S of the chain so that $x_i \notin S$ and $x_{i+1} \in S$. Let Q be a partition of X that separates $x_1, x_2, ..., x_r$. Then the poset of chains C , compatible with respect to this ordered r-tuple, and for which $Q \leq i(P)$, is also contractible. One may see this through an inductive version of the proof of the above lemma. A corollary is that the collection of $(x_1, ..., x_r)$-compatible class $\mathcal{C}$ maps $\mathbb{ET}(Q) \to \mathbb{ET}(V)$ is also contractible. We skip the proof. This result is employed in the proof of Proposition 22 for $r = 2$ (which has already been verified in the above lemma), and for $r = 3$, with $\#(Q) \leq 4$. Here it is a simple verification that the poset of chains that arises as above has its classifying space homeomorphic to a point or a closed interval.

We are now ready to address the proposition. For this purpose, we assume that there is $c \in X$ so that $V_x \cong A$ for all $x \in X \setminus \{c\}$. To obtain consistency with the notation of the proposition, we set $q = X \setminus \{c\}$. The closed subset $U(q) \subset \mathbb{ET}(V)$ in the proposition is the union of $\mathbb{ET}'(t)$ taken over all $\emptyset \neq t \subset q$. For such t, we have $W(t) = V(t) = \oplus\{V_x : x \in t\}$. Recall that $\mathbb{ET}'(t)$ is the product of the cell $e(t) \subset \mathbb{ET}(V(t))$ with $\mathbb{ET}(V/V(t))$. To proceed, it will be necessary to give a contractible class of maps $D \to \mathbb{ET}(V)$ for certain closed subsets $D \subset U(q)$.
The closed subsets $D \subset U(q)$ we consider have the following shape. For each $\emptyset \neq t \subset q$, we first select a closed subset $D(t) \subset e(t)$ and then take D to be the union of the $D(t) \times \mathbb{ET}(V/V(t))$, taken over all such t. This D remains unaffected if $D(t)$ is replaced by its saturation $sD(t)$. Here $sD(t)$ is the collection of $a \in e(t)$ for which $\{a\} \times \mathbb{ET}(V/V(t))$ is contained in D.
When $\emptyset \neq t \subset q$, we denote by $p(t)$ the partition of X consisting of all the singletons contained in t, and in addition, the complement $X \setminus t$. Then there is a canonical identification $j(t) : \mathbb{ET}(p(t)) \to \mathbb{ET}(V/V(t))$.

A map $f : D \to \mathbb{ET}(V)$ is said to be in class $\mathcal{C}$ if for every $\emptyset \neq t \subset q$ and for every $a \in sD(t)$, the map $\mathbb{ET}(p(t)) \to \mathbb{ET}(V)$ given by $b \mapsto f(a, j(t)b)$ belongs to $\mathcal{C}(p(t), \{X\})$. By lemma 24, we see that it suffices to impose this condition on all $a \in D(t)$, rather than all $a \in sD(t)$.

We observe that for every $a \in e(t)$, the map $\mathbb{ET}(p(t)) \to \mathbb{ET}(V)$ given by $b \mapsto (a, j(t)b)$ belongs to $\mathcal{C}(p(t), \{X\})$. As a consequence, we see that the inclusion $D \hookrightarrow \mathbb{ET}(V)$ is of class $\mathcal{C}$.

When concerned with (x, y)-compatible maps, we will assume that $D(t) = \emptyset$ whenever t and $\{x, y\}$ are disjoint. Under this assumption, a map $f : D \to \mathbb{ET}(V)$ is said to be (x, y)-compatible of class $\mathcal{C}$ if $\mathbb{ET}(p(t)) \to \mathbb{ET}(V)$ given by $b \mapsto f(a, j(t)b)$ is a (x, y)-compatible map in $\mathcal{C}(p(t), \{X\})$ for all pairs (a, t) such that $a \in sD(t)$.

In a similar manner, we define (x, y, z)-compatible maps of class $\mathcal{C}$ as well. For this, it is necessary to assume that $D(t)$ is empty whenever the partition $p(t)$ does not separate (x, y, z), equivalently if $\{x, y, z\} \setminus t$ has at least two elements.

Lemma 26. *Assume furthermore that $D(t)$ is a simplicial subcomplex of $e(t)$. Then the space of maps $D \to \mathbb{ET}(V)$ in class $\mathcal{C}$ is contractible. The same is true of the space of such maps that are (x, y)-compatible, or (x, y, z)-compatible.*

Proof. We denote by d the cardinality of $\{t : D(t) \neq \emptyset\}$. We proceed by induction on d, beginning with $d = 0$ where the space of maps is just one point.

We choose t_0 of maximum cardinality so that $D(t_0) \neq \emptyset$. Let D' be the union of $D(t) \times \mathbb{ET}(V/V(t))$ taken over all $t \neq t_0$. Let $\mathcal{C}(D')$ and $\mathcal{C}(D)$ denote the space of class $\mathcal{C}$ maps $D' \to \mathbb{ET}(V)$ and $D \to \mathbb{ET}(V)$ respectively. By the induction hypothesis, $\mathcal{C}(D')$ is contractible. We observe that the intersection of D' and $e(t_0) \times \mathbb{ET}(V/V(t_0))$ has the form $G \times \mathbb{ET}(V/V(t_0)$ where $G \subset e(t_0)$ is a subcomplex. Furthermore, $G \cup D(t_0)$ is the saturated set $sD(t_0)$ described earlier.

For a closed subset $H \subset e(t_0)$, denote the space of $\mathcal{C}$-maps $H \times \mathbb{ET}(V/V(t_0)) \to \mathbb{ET}(V)$ by $A(H)$. Note that $A(H) = \mathrm{Maps}(H, \mathcal{C}(p(t_0), \{X\}))$. By lemma 23, the space $\mathcal{C}(p(t_0), \{X\})$ is itself contractible. It follows that $A(H)$ is contractible. In particular, both $A(G)$ and $A(sD(t_0))$ are contractible. The natural map $A(sD(t_0)) \to A(G)$ is a fibration, because the inclusion $G \hookrightarrow sD(t_0)$ is a cofibration. The fibers of $A(sD(t_0)) \to A(G)$ are thus contractible. It follows that

$$A(sD(t_0)) \times_{A(G)} \mathcal{C}(D') \to \mathcal{C}(D')$$

which is simply $\mathcal{C}(D) \to \mathcal{C}(D')$, enjoys the same properties: it is also a fibration with contractible fibers. Because $\mathcal{C}(D')$ is contractible, we deduce that $\mathcal{C}(D)$ is itself contractible. This completes the proof of the first assertion of the lemma. The remaining assertions follow in exactly the same manner by appealing to lemma 25. □

Proof of Proposition 22. Choose $x \neq y$ with $x, y \in q$. Let $g = id_V + \alpha$ where $\alpha(V) \subset V_y$ and $\alpha(V_k) = 0$ for all $k \neq x \in X$. To prove the proposition, it suffices to show that $g \circ i$ is homotopic to i where $i : U(q) \to \mathbb{ET}(V)$ is the given inclusion. *This notation x, y, α, g will remain fixed throughout the proof.*

Case 1. Here $q = \{x, y\}$. Now x, y are separated by the partitions $p(t)$ for every non-empty $t \subset q$. By the second assertion of the above lemma, there exists $f : U(q) \to \mathbb{ET}(V)$ of class $\mathcal{C}$ and (x, y)-compatible. The given inclusion $i : U(q) \to \mathbb{ET}(V)$ is also of class $\mathcal{C}$. By the first assertion of the same lemma, f is homotopic to i. Now the image of f is contained in the fixed-points of g and so we get $g \circ f = f$. It follows that $g \circ i$ is homotopic to i. This completes the proof of the proposition when $1 = r = \#(q) - 1$.

Case 2. Here $q = \{x, y, z\}$ with x, y, z all distinct.
We take Y_1 to be the union of $e(t) \times \mathbb{ET}(V/V(t))$ taken over all $t \subset q, t \neq \{z\}, t \neq \emptyset$. We put $Y_2 = \mathbb{ET}(V_z) \times \mathbb{ET}(V/V_z)$ and $Y_3 = Y_1 \cap Y_2$. We note that $U(q) = Y_1 \cup Y_2$.
The given inclusion $i : U(q) \to \mathbb{ET}(V)$ restricts to $i_k : Y_k \to \mathbb{ET}(V)$ for $k = 1, 2, 3$. The required homotopy is a path $\gamma : I \to \mathrm{Maps}(U(q), \mathbb{ET}(V))$ so that $\gamma(0) = i$ and $\gamma(1) = g \circ i$. Equivalently we require paths γ_k in $\mathrm{Maps}(Y_k, \mathbb{ET}(V))$ for $k = 1, 2$ so that
(a) $\gamma_k(0) = i_k$ and $\gamma_k(1) = g \circ i_k$ for $k = 1, 2$ and
(b) both γ_1 and γ_2 restrict to the same path in $\mathrm{Maps}(Y_3, \mathbb{ET}(V))$.
In view of the fact that $Y_3 \hookrightarrow Y_1$ is a cofibration, the weaker conditions (a') and (b') on fundamental groupoids suffice for the existence of such a γ:
(a'): $\gamma_k \in \pi_1(\mathrm{Maps}(Y_k, \mathbb{ET}(V)); i_k, g \circ i_k)$ for $k = 1, 2$
(b'): both γ_1 and γ_2 restrict to the same element of $\pi_1((\mathrm{Maps}(Y_3, \mathbb{ET}(V)); i_3, g \circ i_3)$
We have the spaces: $Z_k = \mathrm{Maps}(Y_k, \mathbb{ET}(V))$ for $k = 1, 2, 3$.These spaces come equipped with the data below:
(A)The $GL(V)$-action on $\mathbb{ET}(V)$ induces a $GL(V)$-action on Z_k
(B)The maps of class $\mathcal{C}$ give contractible subspaces $\mathcal{C}_k \subset Z_k$ for $k = 1, 2, 3$.
(C) We have $i_k \in \mathcal{C}_k$ for $k = 1, 2, 3$.
(D) The natural maps $Z_k \to Z_3$ for $k = 1, 2$ are $GL(V)$-equivariant, they take i_k to i_3 and restrict to maps $\mathcal{C}_k \to \mathcal{C}_3$.
Note that the $GL(V)$-action on Z_k turns the disjoint union:
$\mathcal{G}_k = \sqcup\{\pi_1(Z_k; i_k, hi_k) | h \in GL(V)\}$ into a group: given ordered pairs $(h_j, v_j) \in \mathcal{G}_k$, i.e. $h_j \in GL(V)$ and $v_j \in \pi_1(Z_k; i_k, h_j i_k)$ for $j = 1, 2$, we get $h_1 v_2 \in \pi_1(Z_k; h_1 i_k, h_1 h_2 i_k)$ and obtain thereby $v = (h_1 v_2).v_1 \in \pi_1(Z_k; i_k, h_1 h_2 i_k)$ and this produces the required binary operation $(h_1, v_1) * (h_2, v_2) = (h_1 h_2, v)$.
The projection $\mathcal{G}_k \to GL(V)$ is a group homomorphism. The following elementary remark will be used in an essential manner when checking condition (b').
The data (H, F, Δ) where
(i) $H \subset GL(V)$ is a subgroup,
(ii)$F \in Z_k$ is a fixed-point of H, and
(iii)$\Delta \in \pi_1(Z_k; F, i_k)$
produces the lift $H \to \mathcal{G}_k$ of the inclusion $H \hookrightarrow GL(V)$ by $h \mapsto (h, (h\Delta).\Delta^{-1})$.

Finally we observe that there are natural homomorphisms $\mathcal{G}_k \to \mathcal{G}_3$ induced by $Z_k \to Z_3$ for $k = 1, 2$.
Construction of γ_1.
The partitions $p(t)$ for $t \neq \{z\}$ separate x, y. By lemma 26, we have a (x, y)-compatible class $\mathcal{C}$-map $f : Y_1 \to \mathbb{ET}(V)$. Both i_1 and f belong to $\mathcal{C}_1$ and thus we get $\delta \in \pi_1(\mathcal{C}_1; f, i_1)$. Now f is fixed by our $g \in GL(V)$, so we also get $g\delta \in \pi_1(g\mathcal{C}_1; f, gi_1)$. The path $(g\delta).\delta^{-1}$ is the desired $\gamma_1 \in \pi_1(Z_1; i_1, gi_1)$.
Construction of γ_2.
Recall that $g = id_V + \alpha$. We choose $m : V_x \to V_z$ and $n : V_z \to V_y$ so that $nm(a) = \alpha(a)$ for all $a \in V_x$. We extend m, n by zero to nilpotent endomorphisms of V, once again denoted by $m, n : V \to V$ and put $u = id_V + n, v = id_V + m$ and note that $g = uvu^{-1}v^{-1}$.
Note that the partition $p(\{z\})$ separates both the pairs (x, z) and (z, y). We thus obtain $f', f'' \in \mathcal{C}_2$ so that f' is (x, z)-compatible and f'' is (y, z)-compatible and also $\delta' \in \pi_1(\mathcal{C}_2; f', i_2)$ and $\delta'' \in \pi_1(\mathcal{C}_2; f'', i_2)$. Noting that f', f'' are fixed by v, u respectively, we obtain
$\epsilon' = (v\delta').\delta'^{-1} \in \pi_1(Z_2; i_2, vi_2)$ and
$\epsilon'' = (u\delta'').\delta''^{-1} \in \pi_1(Z_2; i_2, ui_2)$.
Thus $v' = (v, \epsilon')$ and $u' = (u, \epsilon'')$ both belong to $\mathcal{G}_2$. We obtain γ_2 by
$u' * v' * u'^{-1} * v'^{-1} = (g, \gamma_2) \in \mathcal{G}_2$
Checking the validity of (b').
Let $\gamma_{13}, \gamma_{23} \in \pi_1(Z_3; i_3, gi_3)$ be the images of γ_1 and γ_2 respectively. We have to show that $\gamma_{13} = \gamma_{23}$.
Consider the spaces $\mathcal{H}, \mathcal{H}', \mathcal{H}''$ consisting of ordered pairs $(f_3, \delta_3), (f_3', \delta_3'), (f_3'', \delta_3'')$ respectively, where f_3, f_3', f_3'' are all in $\mathcal{C}_3$,
f_3 is (x, y)-compatible, f_3' is (x, z)-compatible, and f_3'' is (y, z)-compatible, and $\delta_3, \delta_3', \delta_3''$ are all paths in $\mathcal{C}_3$ that originate at f_3, f_3', f_3'' respectively, and they all terminate at i_3. By lemma 26, we see that the spaces $\mathcal{H}, \mathcal{H}', \mathcal{H}''$ are all contractible.
For $t = \{x, z\}, \{y, z\}, \{x, y, z\}$, the partition $p(t)$ separates (x, y, z). Note that Y_3 is contained in the union of these three $\mathbb{ET}'(t)$. By lemma 26, there is a (x, y, z)-compatible $F \in \mathcal{C}_3$. Let Δ be a path in $\mathcal{C}_3$ that originates at F and terminates at i_3. We see that $(F, \Delta) \in \mathcal{H} \cap \mathcal{H}' \cap \mathcal{H}''$.
Note that $\mathcal{H} \to \pi_1(Z_3; i_3, gi_3)$ given by $(f_3, \delta_3) \mapsto (g\delta_3).\delta_3^{-1}$ is a constant map because $\mathcal{H}$ is contractible. The (f, δ) employed in the construction of γ_1 restricts to an element of $\mathcal{H}$. Also, (F, Δ) belongs to $\mathcal{H}$. It follows that $\gamma_{13} = (g\Delta).\Delta^{-1}$. In a similar manner, we deduce that if $\epsilon_3', \epsilon_3''$ denote the images of ϵ', ϵ'' in the fundamental groupoid of Z_3, then
$\epsilon_3' = (v\Delta).\Delta^{-1} \in \pi_1(Z_3; i_3, vi_3)$ and $\epsilon_3'' = (u\Delta).\Delta^{-1} \in \pi_1(Z_3; i_3, ui_3)$
Thus $\mathcal{G}_2 \to \mathcal{G}_3$ takes $v', u' \in \mathcal{G}_2$ to $(v, (v\Delta).\Delta^{-1}), (u, (u\Delta).\Delta^{-1}) \in \mathcal{G}_3$ respectively. It follows that their commutator $[u', v']$ maps to $(g, \gamma_{23}) \in \mathcal{G}_3$ under this homomorphism.
We apply the remark preceeding the construction of γ_1 to the subgroup H generated by u, v and F and Δ as above. We conclude that γ_{23} equals $(g\Delta).\Delta^{-1}$.

That the latter equals γ_{13} has already been shown. Thus $\gamma_{13} = \gamma_{23}$ and this completes the proof of the Proposition.

9. Low dimensional stabilisation of homology

This section contains applications of corollary 9, proposition 22 and Theorem 2 to obtain some mild information on the homology groups of $\mathbb{ET}(V)$. The notation $\mathcal{L}(V), \mathcal{L}_r(V), W(q), \det(q)$ introduced to state Theorem 2 will be freely used throughout. The spectral sequence in theorem 2 with coefficients in an Abelian group M will be denoted by $SS(V;M)$. When $V = A^n$, this is further abbreviated to $SS(n;M)$, or even to $SS(n)$ when it is clear from the context what M is.
The concept of a commutative ring with *many units* is due to Van der Kallen. An exposition of the definition and consequences of this term is given in [12]. We note that this class of rings includes semilocal rings with infinite residue fields. The three consequences of this hypothesis on A are listed as I,II,III below. These statements are followed by some elementary deductions. *Throughout this section, we will assume that our ring A has this property.*

I: $SL_n(A) = E_n(A)$.
This permits a better formulation of Lemma 8 in many instances.
Ia: Let $0 \to W \to P \to Q \to 0$ be an exact sequence of free A-modules with of ranks $a, a+b, b$. Let $d = g.c.d.(a,b)$. The group H of automorphisms of this exact sequence that induce homotheties on both W and Q may be regarded as a subgroup of $GL(P)$. This group acts trivially on the image of the embedding $i : \mathbb{ET}(W) \times \mathbb{ET}(Q) \to \mathbb{ET}(P)$. Furthermore $\{\det(g) | g \in H\}$ equals $(A^\times)^d$. Thus, if a, b are relatively prime, by lemma 8, we see that $g \circ i$ is freely homotopic to i for all $g \in GL(V)$.
We shall take $\mathrm{rank}(W) = 1$ in what follows. Here $\mathbb{ET}(W) \times \mathbb{ET}(Q)$ is canonically identified with $\mathbb{ET}(Q)$. The induced $\mathbb{ET}(Q) \to \mathbb{ET}(P)$ gives rise on homology to an arrow $H_m(\mathbb{ET}(Q)) \to H_m(\mathbb{ET}(P))$ which has a factoring:

$$H_m(\mathbb{ET}(Q)) \twoheadrightarrow H_0(PGL(Q), H_m(\mathbb{ET}(Q))) \to$$
$$\to H^0(PGL(P), H_m(\mathbb{ET}(P)) \hookrightarrow H_m(\mathbb{ET}(P)).$$

The kernel of $H_m(\mathbb{ET}(Q)) \to H_m(\mathbb{ET}(P))$ does not depend on the choice of the exact sequence. Denoting this kernel by $KH_m(Q) \subset H_m(\mathbb{ET}(Q))$ therefore gives rise to unambiguous notation. We abbreviate $H_m(\mathbb{ET}(A^n)), KH_m(A^n)$ to $H_m(n), KH_m(n)$ respectively.
Ib: In the spectral sequence $SS(n)$, we have:

1. $H_0(PGL_{n-1}(A), H_m(n-1)) \cong H_0(PGL_n(A), E^1_{0,m})$.
2. $E^\infty_{0,m}$ is the image of $H_m(n-1) \to H_m(n)$.
3. $E^1_{0,m} \to E^\infty_{0,m}$ factors as follows:
$$E^1_{0,m} \twoheadrightarrow E^2_{0,m} \twoheadrightarrow H_0(PGL_n(A), E^1_{0,m}) \twoheadrightarrow E^\infty_{0,m}.$$
4. If $H_0(PGL_n(A), E^p_{p,m+1-p}) = 0$ for all $p \geq 2$, then the given arrow $H_0(PGL_n(A), E^1_{0,m}) \to E^\infty_{0,m}$ is an isomorphism.

(5) Assume that $H_m(n-2) \to H_m(n-1)$ is surjective. Then the arrow $E^2_{0,m} \to H_0(PGL_n(A), E^1_{0,m})$ in (3) above is an isomorphism.

The factoring in part (3) above is a consequence of the factoring of $H_m(\mathbb{ET}(Q)) \to H_m(\mathbb{ET}(P))$ in part I(a).
For part (4), one notes that the composite $E^2_{2,m-1} \to E^2_{0,m} \to H_0(PGL_n(A), E^1_{0,m})$ vanishes because $H_0(PGL_n(A), E^2_{2,m-1})$ itself vanishes. Thus we obtain a factoring: $E^2_{0,m} \to E^3_{0,m} \to H_0(PGL_n(A), E^1_{0,m})$. Proceeding inductively, we obtain the factoring:
$E^2_{0,m} \to E^\infty_{0,m} \to H_0(PGL_n(A), E^1_{0,m})$. In view of (3), we see that part (4) follows.
For part (5), it suffices to note that for every $L_0, L_1 \in \mathcal{L}(A^n)$ with $n > 1$, there is some $L_2 \in \mathcal{L}(A^n)$ with the property that both $\{L_0, L_2\}$ and $\{L_1, L_2\}$ belong to $\mathcal{L}_1(A^n)$. This fact is contained in consequence III of *many units*.

II: A is a Nesterenko-Suslin ring.
Let $r > 0, p \geq 0$. Put $N = (p+1)!$ and $n = r + p + 1$. Let $\mathcal{F}_r$ be the category with free rank A-modules of rank r as objects; the morphisms in $\mathcal{F}_r$ are A-module isomorphisms. Let F be a functor from $\mathcal{F}_r$ to the category of $\mathbb{Z}[\frac{1}{N}]$ -modules. Assume that $F(a.id_D) = id_{FD}$ for every $a \in A^\times$ and for every object D of $\mathcal{F}_r$. In other words, the natural action of $GL_r(A)$ on $F(A^r)$ factors through the action of $PGL_r(A)$.
For a free A-module V of rank n, define $Ind' F(V)$ by

$$Ind' F(V) = \oplus\{\det(q) \otimes F(V/W(q)) : q \in \mathcal{L}_p(V)\}.$$

An alternative description of $Ind' F(V)$ is as follows. Fix some $q \in \mathcal{L}_p(V)$. Let $G(q)$ be the stabiliser of q in $GL(V)$. Then $\det(q)$ and $F(V/W(q))$ are $G(q)$-modules in a natural manner. We have a natural isomorphism of $\mathbb{Z}[GL(V)]$-modules:

$$Ind' F(V) \cong \mathbb{Z}[GL(V)] \otimes_{\mathbb{Z}[G(q)]} [\det(q) \otimes_{\mathbb{Z}} F(V/W(q))].$$

IIa: If $H_i(PGL_r(A), F(A^r)) = 0$ for all $i < m$, then $H_i(PGL_n(A), Ind' F(A^n)) = 0$ for all $i < p + m$. Furthermore,

$$H_{p+m}(PGL_n(A), Ind' F(A^n)) \cong H_m(PGL_r(A), F(A^r)) \otimes Sym^p(A^\times))$$

By Shapiro's lemma, the result of [13] cited earlier, and the fact that the group homology $H_i(M, C)$ is isomorphic to $C \otimes \Lambda^i(M)$ for all commutative groups M and $\mathbb{Z}[1/i!]$-modules C given trivial M-action, IIa reduces to the statement: Let $\Sigma(q)$ denote the group of permutations of a set q of $(p+1)$ elements. Let M be an Abelian group on which $(p+1)!$ acts invertibly. Then
$H_0(\Sigma(q), \det(q) \otimes \Lambda^i(M^q))$ vanishes when $i < p$ and is isomorphic to $Sym^p(M)$ when $i = p$.

III: The standard application of the *many units* hypothesis, see ([20] for instance) is that general position is available in the precise sense given below. Let $V \cong A^n$. We denote by $K(V)$ the simplicial complex whose set of vertices

is $\mathcal{L}(V)$. A subset $S \subset \mathcal{L}(V)$ of cardinality $(r+1)$ is an r-simplex of $K(V)$ if every $T \subset S$ of cardinality $t+1 \leq n$ belongs to $\mathcal{L}_t(V)$. If $L \subset K(V)$ is a finite simplicial subcomplex, then there is some $e \in \mathcal{L}(V)$ with the property that $s \cup \{e\}$ is a $(r+1)$-simplex of $K(V)$ for every r-simplex s of L. This gives an embedding $Cone(L) \hookrightarrow K(V)$ with e as the vertex of the cone. Thus $K(V)$ is contractible. The complex of oriented chains of this simplicial complex will be denoted by $C_\bullet(V)$. Thus the reduced homologies $\widetilde{H}_i(C_\bullet(V))$ vanish for all i. The group $D(V) = Z_{n-1}C_\bullet(V) = B_{n-1}C_\bullet(V)$ comes up frequently.
IIIa:

(1) Let $p < n$. Let M be a $\mathbb{Z}[1/N]$-module where $N = (p+1)!$. Then $H_j(PGL(V), C_p \otimes M)$ vanishes for $j < p$ and is isomorphic to $Sym^p(A^\times) \otimes M$ when $j = p$.
(2) $H_j(PGL(V), C_n \otimes M)$ vanishes for all $j \geq 0$ and for all $\mathbb{Z}[1/(n+1)!]$-modules M.
(3) $H_0(PGL(V), D(V) \otimes M)$ for any Abelian group M is isomorphic to $M/2M$ if n is even, and vanishes if when n is odd
(4) $H_0(PGL(V), B_pC_\bullet(V) \otimes M) = 0$ for every $\mathbb{Z}[1/2]$-module M and for every $0 \leq p < n$.
(5) $H_1(PGL(V), Z_pC_\bullet(V) \otimes M) = 0$ for every $\mathbb{Z}[1/(p+2)!]$- module M and $1 \leq p \leq n-2$.

Note that (1) above follows from IIa when F is the constant functor M.
For (2), one observes that $PGL(V)$ acts transitively on the set of n-simplices of the simplicial complex $K(V)$. The stabiliser of an n-simplex is the permutation group Σ on $(n+1)$ letters. The claim now follows from Shapiro's lemma.
The presentation $C_{p+2}(V) \otimes M \to C_{p+1}(V) \otimes M \to B_pC_\bullet(V) \otimes M \to 0$ and the observation $H_0(PGL(V), C_{p+1}(V) \otimes M) \cong M/2M$ whenever $p < n$ suffice to take care of (3) and (4).
For assertion (5), one applies the long exact sequence of group homology to the short exact sequence:
$0 \to Z_{p+1}C_\bullet(V) \otimes M \to C_{p+1}(V) \otimes M \to Z_pC_\bullet(V) \otimes M \to 0.$
One therefore obtains the exact sequence:
$H_1(PGL(V), C_{p+1}(V) \otimes M) \to H_1(PGL(V), Z_pC_\bullet(V) \otimes M) \to H_0(PGL(V), Z_{p+1}C_\bullet(V) \otimes M)$. The end terms here vanish by (1) and (4).
IIIb:

(1) $\mathbb{ET}(V)$ is connected.
(2) $E^2_{0,0} = \mathbb{Z}, E^2_{n-1,0} = D(V), E^2_{m,0} = 0$ if $m \neq 0, n-1$ for the spectral sequence $SS(V)$.
(3) $H_1(\mathbb{ET}(V)) \cong \mathbb{Z}/2\mathbb{Z}$ if $\mathrm{rank}(V) > 2$.

Note that (1) is a consequence of (2). Part (2) is deduced by induction on $\mathrm{rank}(V) = n$. The induction hypothesis enables the identification of the $E^1_{m,0}$ terms of the spectral sequence for V (together with differentials) with the $C_m(V)$ (together with boundary operators) when $m < n$. Thus (2) follows.

For part (3), consider the spectral sequence $SS(3)$. Here $E^2_{2,0} = D(A^3)$ and $H_0(PGL_3(A), D(A^3)) = 0$ by IIIa(3). Thus the hypothesis of Ib(4) holds for SS(3). Consequently,
$H_1(3) \cong E^\infty_{1,0} \cong H_0(PGL_2(A), H_1(2)) = H_0(PGL_2(A), D(A^2)) \cong \mathbb{Z}/2\mathbb{Z}$,
the last isomorphism given by IIIa(3) once again. The isomorphism $H_1(n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 3$ is contained in the lemma below for $N = 1$.

Lemma 27. *Let M be an Abelian group. Let $N \in \mathbb{N}$. For $0 < r < N$, we are given $m(r) \geq 0$ so that $H_r(d; M) \to H_r(d+1; M)$*
is a surjection if $d = r + m(r) + 1$, and
an isomorphism if $d > r + m(r) + 1$.
Let $m(N) = \max\{0, m(1)+1, m(2)+1, ..., m(N-1)+1\}$.
Then $H_N(d; M) \to H_N(d+1; M)$ is
(a) an isomorphism if $d > N + m(N) + 1$,
(b) is a surjection if $d = N + m(N) + 1$.
(c) The surjection in (b) above factors through an isomorphism $H_0(PGL_d(A), H_N(d)) \to H_N(d+1; M)$ if if M is a $\mathbb{Z}[1/2]$-module.

Proof. Consider the spectral sequence $SS(V; M)$ that computes the homology of $\mathbb{ET}(V)$ with coefficients in M. Here V is free of rank $N + h + 2$, where $h \geq m(N)$. We make the following claim:
*Claim:*If $E^2_{s,r} \neq 0$ and $0 < s$ and $r < N$, then $s + r \geq N + 1 + h - m(N)$. Furthermore, when equality holds, $H_0(PGL(V), E^2_{s,r} \otimes \mathbb{Z}[1/2]) = 0$.
We assume the claim and prove the lemma. We take $h = m(N)$. All the $E^2_{s,r}$ with $s + r = N$ are zero except possibly for $(s, r) = (0, N)$. Part (b) of the lemma now follows from Ib(2). We consider next the $E^2_{s,r}$ with $s + r = N + 1$ and $s \geq 2$ (or equivalently with $r < N$). It follows that $E^s_{s,r}$ is a quotient of $E^2_{s,r}$. The second assertion of the claim now show that $H_0(PGL(V), E^s_{s,r}) = 0$ if M is a $\mathbb{Z}[1/2]$-module. Part (c) of the lemma now follows from Ib(4).
We take $h > m(N)$ and prove part (a) by induction on h. The inductive hypothesis implies that $H_N(N + h; M) \to H_N(N + h + 1; M)$ is surjective. By Ib(5), it follows that $H_N(N + 1 + h; M) \to E^2_{0,N}$ is an isomorphism. Now there are no nonzero $E^2_{s,r}$ with $s + r = N + 1$ and $s \geq 2$. Thus $E^2_{0,N} = E^\infty_{0,N}$. It follows that $H_N(N + 1 + h; M) \to H_N(N + 2 + h; M)$ is an isomorphism.
It only remains to prove the claim. We address this matter now.
For $r = 0$, both assertions of the claim are valid by IIIb(2)and IIIa(3).
So assume now that $0 < r < N$. Let $SH(r) = H_r(d; M)$ for $d = r + m(r) + 2$. In view of our hypothesis, the chain complex
$E^1_{0,r} \leftarrow E^1_{1,r} \leftarrow ... \leftarrow E^1_{p-1,r} \leftarrow E^1_{p,r}$
for $N + h + 1 = p + r + m(r) + 1$ is identified with
$C_0(V) \otimes SH(r) \leftarrow ... \leftarrow C_{p-1}(V) \otimes SH(r) \leftarrow \oplus\{\det(q) \otimes H_r(\mathbb{ET}(V/W(q)); M) | q \in \mathcal{L}_p(V)\}$.
As in IIIb(2), it follows that $E^2_{s,r} = 0$ whenever $0 < s < p$. Furthermore, we deduce the following exact sequence for $E^2_{p,r}$:

$\oplus\{\det(q) \otimes KH_r(V/W(q); M) | q \in \mathcal{L}_p\} \to E^2_{p,r} \to Z_pC_\bullet \otimes SH(r) \to 0$. By IIIa(4), we see that $H_0(PGL(V), E^2_{p,r}) = 0$ if M is a $\mathbb{Z}[1/2]$-module.
Note that $p + r = N + h - m(r) \geq N + 1 + h - M(N)$. This completes the proof of the claim, and therefore, the proof of the lemma as well.

□

The Proposition below is an application of Proposition 22. The notation here is that of Theorem 2. We regard $B^r_{p,q}$ and $Z^r_{p,q}$ as subgroups of $E^1_{p,q}$ for all $r > 1$. The notation $KH_m(Q)$ has been introduced in Ia, the first application of *many units*.

Proposition 28. *Let rank$(V) = n$. Let M be a $\mathbb{Z}[1/2]$-module. In the spectral sequence $SS(V; M)$, we have:*

(1) $\oplus\{\det(q) \otimes KH_m(V/W(q)) | q \in \mathcal{L}_1(V)\} \subset B^\infty_{1,m}$ *if $n > 1$.*
(2) $E^\infty_{1,m} = 0$ *if $n > 2$ and M is a $\mathbb{Z}[1/6]$-module.*
(3) *If, in addition, it is assumed that*
$H_{m+1}(n-2; M) \to H_{m+1}(n-1; M)$ *is surjective, then*
$\oplus\{\det(q) \otimes KH_m(V/W(q)) | q \in \mathcal{L}_2(V)\} \subset B^\infty_{2,m}$.

Proof. Let $q \in \mathcal{L}_r(V)$. We have $U(q) \subset \mathbb{ET}(V)$ as in Proposition 22. The spectral sequence of Theorem 2 was constructed from an increasing filtration of subspaces of $\mathbb{ET}(V)$. Intersecting this filtration with $U(q)$ we obtain a spectral sequence that computes the homology of $U(q)$. Its terms will be denoted by $E^a_{b,c}(q)$. One notes that $E^1_{b,m}(q)$ is the direct sum of $\det(u) \otimes H_m(\mathbb{ET}(V/W(u)))$ taken over all $u \subset q$ of cardinality $(b+1)$.
We denote the terms of the spectral sequence in theorem 2 by $E^a_{b,c}(V)$. The given data also provides a homomorphism $E^a_{b,c}(q) \to E^a_{b,c}(V)$ of E^1-spectral sequences. We assume that M is a $\mathbb{Z}[1/(r+1)!]$-module.
We choose a basis $e_1, e_2, ..., e_n$ of V so that
$q = \{Ae_i : 1 \leq i \leq r+1\}$. Let $G \subset GL(V)$ be the subgroup of $g \in GL(V)$ so that
(A) $g(q) = q$, (B)$g(e_i) = e_i$ for all $i > r+1$, (C), the matrix entries of g are $0, 1, -1$ and (D) $\det(g) = 1$. Now G acts on the pair $U(q) \subset \mathbb{ET}(V)$. Thus the above homomorphism of spectral sequences is one such in the category of G-modules.We observe:
(a) G is a group of order $2(r+1)!$
(b) there are no nonzero G-invariants in $E^1_{i,m}(q)$ for $i > 0$, and consequently the same holds for all G-subquotients, in particular for $E^a_{i,m}(q)$ for all $a > 0$ as well.
Proof of part 1. Take $r = 1$. Proposition 22 implies that the image of $H_m(U(q)) \to H_m(\mathbb{ET}(V))$ has trivial G-action. In view of (b) above, this shows that $E^\infty_{1,m}(q) \to E^\infty_{1,m}(V)$ is zero. But $E^\infty_{1,m}(q) = Z^\infty_{1,m}(q) = \det(q) \otimes KH_m(V/W(q))$. It follows that $\det(q) \otimes KH_m(V/W(q)) \subset B^\infty_{1,m}(V)$. Part (1) follows.
Proof of part (2). We take $r = 2$. Here we have
$Z^\infty_{1,m}(q) = Z^2_{1,m}(q)$. Appealing to Proposition 22 and observation (b) once

again, we see that the image of the homomorphism $Z^2_{1,m}(q) \to Z^2_{1,m}(V)$ is contained in $B^\infty_{1,m}$. Part (2) therefore follows from the claim below.
Claim: $\oplus\{Z^2_{1,m}(q) | q \in \mathcal{L}_2(V)\} \to Z^2_{1,m}(V)$ is surjective.
Denote the image of $H_m(n-2; M) \to H_m(n-1; M)$ by I. A simple computation produces the exact sequences:
$0 \to \oplus\{\det(u) \otimes KH_m(V/W(u) : u \in \mathcal{L}_1(V), u \subset q\} \to Z^2_{1,m}(q) \to \det(q) \otimes I \to 0$, and
$0 \to \oplus\{\det(u) \otimes KH_m(V/W(u) : u \in \mathcal{L}_1(V)\} \to Z^2_{1,m}(V) \to Z_1C_\bullet(V) \otimes I \to 0$.
The claim now follows from the above description of $Z^2_{1,m}(q)$ and $Z^2_{1,m}(V)$. Thus part (2) is proved.
*Proof of part (3).*We take $r = 2$ once again. The surjectivity of $H_{m+1}(n-2; M) \to H_{m+1}(n-1; M)$ implies that $E^2_{0,m+1}(q)$ has trivial G-action. By observation (b), we see that $d^2_{2,m} : E^2_{2,m}(q) \to E^2_{0,m+1}(q)$ is zero. It follows that $E^\infty_{2,m}(q) = Z^2_{2,m}(q)$ here. Proposition 22 and observation (b) once again show that the image of $Z^2_{2,m}(q) \to Z^2_{2,m}(V)$ is contained in $B^\infty_{2,m}(V)$. Because $Z^2_{2,m}(q) = \det(q) \otimes KH_m(V/W(q))$, part (3) follows.
This completes the proof of the Proposition. □

Theorem 3. *Let $H_m(n; M)$ denote $H_m(\mathbb{ET}(A^n); M)$ where M is a $\mathbb{Z}[1/6]$-module. We have:*
(1) $H_1(n; M) = 0$ for all $n > 2$,
(2) $H_0(GL_3(A), H_2(3; M)) \to H_2(n; M)$ is an isomorphism for all $n \geq 4$,
(3) $H_0(GL_4(A), H_3(4; M)) \to H_3(n; M)$ is an isomorphism for all $n \geq 5$,
(4) $H_0(GL_{2m-2}(A), H_m(2m-2; M)) \to H_m(n; M)$ is an isomorphism for all $n > 2m - 2$.

Proof. Part (1) has already been proved.
Proof of part 2. For this, we study $SS(V; M)$ where $\operatorname{rank}(V) = 4$. We first note that
(i) $E^2_{3,0} = D(V)$ and therefore $H_0(PGL(V), E^2_{3,0}) = 0$.
(ii) $E^2_{1,1} = E^1_{1,1} = \oplus\{\det(q) \otimes D(V/W(q)) : q \in \mathcal{L}_1(V)\}$, and therefore $H_1(PGL(V), E^2_{1,1}) = 0$ by IIa.
(iii) $E^2_{u,v} = 0$ except when $(u, v) = (0, 0), (0, 2), (1, 1), (3, 0)$. We have $E^\infty_{1,1} = 0$ by proposition 28 and thus obtain the short exact sequence:
$0 \to E^3_{3,0} \to E^2_{3,0} \to E^2_{1,1} \to 0$.
By (i) and (ii) above, we see that $H_0(PGL(V), E^3_{3,0}) = 0$. By Ib(1,4), we see that $H_0(PGL_3(A), H_2(3; M)) \to H_2(4; M)$ is an isomorphism. In particular, $H_2(4; M)$ receives the trivial $PGL_4(A)$-action. Taking $N = 2$ and $m(1) = 0$ in lemma 27, we see that $H_2(4; M) \to H_2(n; M)$ is an isomorphism for all $n \geq 4$. This proves part (2).
Proof of part 3. We inspect $SS(V; M)$ where $V = A^5$. We note that
(1) $E^\infty_{1,2}$ and $E^\infty_{2,1}$ both vanish. This follows from proposition 28, once it is noted that $KH_1(2; M) = H_1(2; M)$.
(2) $E^2_{0,2} = H_2(4; M)$ has the trival $PGL(V)$-action.

(3) $H_i(PGL(V), E^2_{2,1}) = 0$ for all $i < 3$. This follows from IIa and IIIa(3) after observing that $H_1(2; M) \cong D(A^2) \otimes M$.
(4) From (2) and (3) we see that $d^2_{2,1} = 0$.
(5) We deduce that $E^2_{2,1} \cong E^2_{0,4}/E^3_{0,4}$ and $E^2_{1,2} = E^3_{1,2} \cong E^3_{0,4}/E^4_{0,4}$ from observations (1) and (4).
(6) $H_1(PGL(V), E^2_{1,2}) = 0$.
To see this, first note the the short exact sequence:
$0 \to P \to E^2_{1,2} \to Q \to 0$, where
$P = \oplus\{\det(q) \otimes KH_2(V/W(q) : q \in \mathcal{L}_1(V)\}$ and $Q = H_2(4; M) \otimes Z_1C_\bullet(V)$.
The vanishing of $H_1(PGL_3(A), Q)$ follows from IIIa(5). By IIa, the vanishing of $H_1(PGL_3(A), P)$ is reduced to the vanishing of $H_0(PGL_3(A), KH_2(A^3)$.
Now let I be the augmentation ideal of the group algebra $R[PGL_3(A)]$ where $R = \mathbb{Z}[1/6]$. In view of the fact that $PGL_3(A)_{ab}$ is 3-torsion, we see that $I = I^2$. It follows that for all $\mathbb{Z}[1/6]$-modules N equipped with $PGL_3(A)$-action, we have $IN = I^2N$, or equivalently, $H_0(PGL_3(A), IN) = 0$. We apply this remark to $N = H_2(3; M)$. By part (2) of the proposition, we see that $KH_2(3; M) = IN$. This proves that $H_0(PGL_3(A), KH_2(A^3)) = 0$. We have completed the proof of observation 6.
(7) $H_0(PGL(V), E^4_{4,0}) = 0$.
In view of the filtration of (5), it suffices to check that $H_1(PGL(V), E^2_{a,b}) = 0$ for $(a, b) = (1, 2)$ and $(2, 1)$ (which has been seen in observations (3) and (6)) and also that $H_0(PGL(V), E^2_{4,0}) = 0$ (and this is clear because $E^2_{4,0} = D(V)$).
(8) $H_0(PGL_4(A), H_3(4; M)) \to H_3(V; M)$ is an isomorphism.
That $H_0(PGL_4(A), H_3(4; M)) \to E^\infty_{0,3}$ is an isomorphism follows from observation (7) and Ib(4). Now $E^\infty_{a,b} = 0$ whenever $a + b = 3$ and $(a, b) \neq (3, 0)$. This proves (8).
(9) $H_3(5; M) \to H_3(n; M)$ is an isomorphism for all $n \geq 5$.
This follows from lemma 27 by taking $N = 3$ and $m(1) = m(2) = 0$. This finishes the proof of part (3).
Part (4) now follows from the same lemma and induction.

□

Remark. It can be checked that parts (1,2,4) of the above theorem are valid for $\mathbb{Z}[1/2]$-modules M. In part (3), it is true that $H_3(n; M) \cong H_3(n + 1; M)$ for $n > 4$ and also that $H_0(PGL_4(A), H_3(4; M) \to H_3(5; M)$ is a surjection.

Proposition 29. *Assume that the Compatible Homotopy Question has an affirmative answer. Then, for all $\mathbb{Z}[1/r!]$-modules M and for all $d > r + 1$, $H_0(PGL_{r+1}(A), H_r(r + 1; M)) \to H_r(d; M)$ is an isomorphism.*

Proof. For $r = 1$, this statement has been checked in IIIb(3) and lemma 27. Let $N > 1$. We assume that the above statement has been proved for all $r < N$. Let M be a $\mathbb{Z}[1/N!]$-module. In lemma 27, may may now take $m(1) = m(2) = \dots = m(N - 1) = 0$. From this lemma, we obtain:

$$H_0(PGL_{N+2}(A), H_N(N + 2; M)) \to H_N(N'; M)$$

is an isomorphism for all $N' > N+2$. So the proposition is proved once it is checked that

$$H_0(PGL_{N+1}(A), H_N(N+1; M)) \to H_N(N+2; M)$$

is an isomorphism. To prove this, we consider the spectral sequence $SS(V; M)$ where $V = A^{N+2}$. We will prove:
(i) $E^2_{a,b} = 0$ or $a = 0$ or $a + b = N$ or $(a, b) = (N + 1, 0)$. Furthermore the action of $PGL(V)$ on $E^2_{0,b}$ is trivial when $b < N$.
(ii) if $a > 0$ and $b > 0$, then $H_i(PGL(V), E^2_{a,b}) = 0$ for $i = 0, 1$.
(iii) $E^\infty_{a,b} = 0$ when $a > 0$ and $b > 0$.
(iv) $E^2_{a,b} \cong E^{b+1}_{N+1,0}/E^{b+2}_{N+1,0}$ whenever $a > 0, b > 0$ and $a + b = N$.
We first observe that (iv) is true for any spectral sequence of $PGL(V)$-modules where (i),(ii) and (iii) hold. Next note that (ii) and (iv) imply that $H_0(PGL(V), E^{N+1}_{N+1,0})$ is contained in $H_0(PGL(V), E^2_{N+1,0})$. And since the latter is zero, we see that the former also vanishes.
We deduce that both arrows $H_0(PGL(V), E^1_{0,N}) \to E^\infty_{0,N} \to H_N(N+2; M)$ are isomorphisms exactly as in earlier proofs. Thus it only remains to prove (i), (ii) and (iii).
Proof of (i). This is contained in the proof of lemma 27.
Proof of (ii). $0 \to P \to E^2_{a,b} \to Q \to 0$ is exact, where
$P = \oplus\{\det(q) \otimes KH_b(V/W(q)|q \in \mathcal{L}_a(V)\}$, and $Q = Z_a C_\bullet \otimes H_b(b + 2; M)$ as in the proof of the lemma 27. The required vanishing of $H_i(PGL(V), T)$ for $i = 0, 1$ holds for $T = Q$ by IIIa(5). For $T = P$ and $a > 1$, the required vanishing follows from II(a). For $T = P$ and $a = 1$, this is deduced from the vanishing of $H_0(PGL_N(A), KH_{N-1})$ (see the proof of observation (6) in the proof of theorem 3).
Proof of (iii). We follow the steps of the proof of Proposition 28. We first choose $q \in \mathcal{L}_{N-1}(V)$ and consider the inclusion $U(q) \hookrightarrow \mathbb{ET}(V)$. As in that proof we get a homomorphism of E^1 spectral sequences of G-modules with $G \subset SL(V)$ as given there. The terms of these spectral sequences are denoted by $E^a_{b,c}(q)$ and $E^a_{b,c}(V)$ respectively. From the inductive hypothesis, we deduce:
(i′) $E^2_{a,b}(q) = 0$ or $a = 0$ or $a + b = N$ or $(a, b) = (N + 1, 0)$. Furthermore the action of G on $E^2_{0,b}(q)$ is trivial when $b < N$.
(ii′) if $a > 0, h > 0$, then $H_0(G, E^h_{a,b}(q)) = 0$.
These observations together imply
(iii′) $Z^\infty_{a,b}(q) = Z^2_{a,b}(q)$ when $a > 0$ and $b > 0$.
For $a > 0, b > 0$, we obtain $E^\infty_{a,b}(q) \to E^\infty_{a,b}(V)$ is zero, from the affirmative answer to the Compatible Homotopy Question. For such (a, b),
the image of $x(q) : Z^2_{a,b}(q) \to Z^2_{a,b}(V)$ is thus contained in $B^\infty_{a,b}(V)$. As in the proof of proposition 28, we see that the sum of the images of $x(q)$, taken over all $q \in \mathcal{L}_{N-1}(V)$, is all of $Z^2_{a,b}(V)$. It follows that $Z^2_{a,b}(V) = B^\infty_{a,b}(V)$ and thus $E^\infty_{a,b}(V) = 0$ whenever $a > 0, b > 0$. This proves assertion (iii) and this completes the proof of the Proposition.

□

10. A double complex

We will continue to assume that A is a commutative ring with many units. The paper [2] of Beilinson, Macpherson and Schechtman introduces a Grassmann complex, intersection and projection maps, and a torus action. The terms of the double-complex constructed below may be obtained from the quotients by the torus action of the objects of [2]. The arrows of the double-complex are signed sums of their intersection and projection maps.

$D(V), C_\bullet(V)$ etc. are as in the previous section. When $\mathrm{rank}(V) = n$, we have the resolution:

$0 \leftarrow D(V) \leftarrow C_n(V) \leftarrow C_{n+1}(V)....$

We put $\overline{C}_r(V) = H_0(PGL(V), C_r(V))$ when $r \geq n$ and define $\overline{C}_r(V)$ to be zero otherwise. We put $\overline{C}_r(A^n) = \overline{C}_r(n)$. We observe that the above resolution of $D(V)$ tensored with the rationals is a projective resolution in the category of $\mathbb{Q}[PGL(V)]$-modules. It follows that $H_i(PGL_n(A), D(A^n)) \otimes \mathbb{Q} \cong H_{n+i}(\overline{C}(n)_\bullet) \otimes \mathbb{Q}$. We denote by $\partial' : \overline{C}_r(n) \to \overline{C}_{r-1}(n)$ the boundary operator of $\overline{C}(n)_\bullet$. We will now define $\partial'' : \overline{C}_r(n) \to \overline{C}_r(n-1)$.

Let $V \cong A^n$. Let $(L_0, L_1, ..., L_r)$ be an ordered $(r+1)$-tuple in $\mathcal{L}(V)$ that gives rise to a r-simplex of $K(V)$ (see consequence III of *many units* for notation). We define $\partial_i(L_0, L_1, ..., L_r) \in C_{r-1}(V/L_i)$ by $\partial_i(L_0, L_1, ..., L_r) = (\overline{L}_0, .., \overline{L}_{i-1}, \overline{L}_{i+1}, ..., \overline{L}_r)$ where $\overline{L}_j = L_j + L_i/L_i \in \mathcal{L}(V/L_i)$ whenever $j \neq i$. Now let

$$g_r(L_0, L_1, ..., L_r) = \Sigma_{i=0}^r (-1)^i \partial(L_0, L_1, ..., L_r) \in \oplus\{C_{r-1}(V/L) : L \in \mathcal{L}(V)\}.$$

The above $g_r : C_r(V) \to \oplus\{C_{r-1}(V/L) : L \in \mathcal{L}(V)\}$ anti-commutes with the boundary operator. The functor $M \to H_0(PGL(V), M)$ takes g_r to $\partial'' : \overline{C}_r(n) \to \overline{C}_{r-1}(n-1)$. This defines ∂''.

We put $F_r(A) = \oplus\{\overline{C}_r(n) : n \geq 1\}$ and define $\partial : F_r(A) \to F_{r-1}(A)$ by $\partial = \partial' + \partial''$. The exact relation between the homology of $F_\bullet(A)$ and groups $L_n(A)$ is as yet unclear. However, we do have:

Lemma 30. $H_3(F_\bullet(A)) \otimes \mathbb{Q} \cong L_2(A) \otimes \mathbb{Q} \cong H_3(\overline{C}_\bullet(2)) \otimes \mathbb{Q}$.

We sketch a proof. In view of the H-space structure, $L_i(A) \otimes \mathbb{Q}$ is the primitive homology of $\mathbb{ET}(A^n)$ with $\mathbb{Q}$ coefficients for n large. The vanishing of $H_1(n; \mathbb{Q})$ for $n > 2$ implies that the primitive homology is all of $H_i(n; \mathbb{Q})$ for $i = 2, 3$ and n large. By theorem 3, we get $L_i(A) \otimes \mathbb{Q} \cong H_0(PGL_{i+1}(A), H_i(i+1; \mathbb{Q}))$ for $i = 2, 3$. For the computation of $H_0(PGL(V), H_2(\mathbb{ET}(V); \mathbb{Q}))$ where $V = A^3$, we recall the exact sequence obtained from $SS(V; \mathbb{Q})$:

$0 \to H_2(\mathbb{ET}(V); \mathbb{Q}) \to D(V) \otimes \mathbb{Q} \to D_2(V) = \oplus\{D(V/L) : L \in \mathcal{L}(V)\} \to 0.$

This identifies $L_2(A) \otimes \mathbb{Q}$ with the cokernel of $H_1(PGL(V), D(V)) \otimes \mathbb{Q} \to H_1(PGL(V), D_2(V)) \otimes \mathbb{Q}$. In view of IIa and the above remarks, this is readily identified with $H_3(F_\bullet(A)) \otimes \mathbb{Q}$.That gives the first isomorphism of the lemma. For the second isomorphism, what one needs is:

Claim: The arrow $H_4(\overline{C}_\bullet(3)) \to H_3(\overline{C}_\bullet(2))$ induced by ∂'' is zero.

The proof of this claim, which we address now, was already known to Spencer Bloch. Let $V = A^3$. Given an ordered 5-tuple $(L_0, ..., L_4)$ with the $L_i \in \mathcal{L}(V)$

as vertices of a 4-simplex in $K(V)$ (i.e. in general position), they belong to a conic C and the projection from the points L_i induces an isomorphism $p_i : C \to \mathbb{P}(V/L_i)$. We put $(M_0, ..., M_4) = (p_0L_0, p_0L_1, ..., p_0L_4)$. Let $q_i = p_i \circ p_0^{-1}$. With the ∂_i as in the definition of g_4, we see that $\partial_i(L_0, ..., L_4) \in C_3(V/L_i)$ and $q_i\partial_i(L_0, ..., L_4) \in C_3(V/L_0)$ both give rise to the same element of $\overline{C}_3(2)$. It follows that $\partial(M_0, M_1, ..., M_4) \mapsto \partial''(L_0, ..., L_4)$ under the map $C_3(V/L_0) \to \overline{C}_3(2)$. Thus $\partial''(L_0, ..., L_4) \mapsto 0 \in H_3(\overline{C}_\bullet(2)$. This proves the claim and the lemma.

Thus we have shown that
$L_2(A) \otimes \mathbb{Q} \cong \operatorname{coker}(\overline{C}_4(A^2) \to \overline{C}_3(A^2))$.
The Bloch group tensored with $\mathbb{Q}$ is the homology of

$$\overline{C}_4(A^2) \to \overline{C}_3(A^2) \to \Lambda^2(A^\times) \otimes \mathbb{Q}.$$

Thus this discussion amounts to a proof of Suslin's theorem on the Bloch group. It remains to obtain a closed form for $L_3(A) \otimes \mathbb{Q}$ by this method.

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