Slices and Transfers

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Abstract. We study Voevodsky's slice tower for S^1-spectra, and raise a number of questions regarding its properties. We show that the 0th slice does not in general admit transfers, although it does for a $\mathbb{P}^1$-loop-spectrum. We define a new tower for each of the higher slices, and show that the layers in these towers have the structure of Eilenberg-Maclane spectra on effective motives.

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Contents

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Introduction

Voevodsky [22] has defined an analog of the classical Postnikov tower in the setting of motivic stable homotopy theory by replacing the simplicial suspension $\Sigma_s := - \wedge S^1$ with $\mathbb{P}^1$-suspension $\Sigma_{\mathbb{P}^1} := - \wedge \mathbb{P}^1$; we call this construction the *motivic Postnikov tower.*
Let $\mathcal{SH}(k)$ denote the motivic stable homotopy category of $\mathbb{P}^1$-spectra. One of the main results on motivic Postnikov tower in this setting is

Theorem 1. *Let k be a field of characteristic zero. For $E \in \mathcal{SH}(k)$, the slices s_nE have the natural structure of an $\mathcal{H}\mathbb{Z}$-module, and hence determine objects in the category of motives $DM(k)$.*

The statement is a bit imprecise, as the following expansion will make clear: Röndigs-Østvær [19, 20] have shown that the homotopy category of strict $\mathcal{H}\mathbb{Z}$-modules is equivalent to the category of motives, $DM(k)$. Additionally, Voevodsky [22] and the author [11] have shown that the 0th slice of the sphere spectrum $\mathbb{S}$ in $\mathcal{SH}(k)$ is isomorphic to $\mathcal{H}\mathbb{Z}$. Each $E \in \mathcal{SH}(k)$ has a canonical structure of a module over the sphere spectrum $\mathbb{S}$, and thus the slices s_nE acquire an $\mathcal{H}\mathbb{Z}$-module structure, in $\mathcal{SH}_{S^1}(k)$. This has been refined to the model category level by Pelaez [17], showing that the slices of a $\mathbb{P}^1$-spectrum E have a natural structure of a strict $\mathcal{H}\mathbb{Z}$-module, hence are motives.
Let $\mathbf{Spt}_{S^1}(k)$ denote the category of S^1-spectra, with its homotopy category (for the $\mathbb{A}^1$ model structure) $\mathcal{SH}_{S^1}(k)$. The analog for motives is the category complexes of presheaves with transfer and its $\mathbb{A}^1$-homotopy category $DM^{eff}(k)$, the category of effective motives over k. We consider the motivic Postnikov tower in$\mathcal{SH}_{S^1}(k)$, and ask the questions:

(1) Is there a ring object in $\mathbf{Spt}_{S^1}(k)$, $\mathcal{H}\mathbb{Z}^{eff}$, such that the homotopy category of $\mathcal{H}\mathbb{Z}^{eff}$ modules is equivalent to the category of effective motives $DM^{eff}(k)$?
(2) What properties (if any) need an S^1-spectrum E have so that the slices s_nE have a natural structure of Eilenberg-Maclane spectra of a homotopy invariant complex of presheaves with transfer?

Naturally, if $\mathcal{H}\mathbb{Z}^{eff}$ exists as in (1), we are asking the slices in (2) to be (strict) $\mathcal{H}\mathbb{Z}^{eff}$ modules. Of course, a natural candidate for $\mathcal{H}\mathbb{Z}^{eff}$ would be the 0-S^1-spectrum of $\mathcal{H}\mathbb{Z}$, $\Omega^\infty_{\mathbb{P}^1}\mathcal{H}\mathbb{Z}$, but as far as I know, this property has not yet been investigated.
As we shall see, the 0-S^1-spectrum of a $\mathbb{P}^1$-spectrum does have the property that its (S^1) slices are motives, while one can give examples of S^1-spectra for which the 0th slice does not have this property. This suggests a relation of the question of the structure of the slices of an S^1-spectrum with a motivic version of the recognition problem:

(3) How can one tell if a given S^1-spectrum is an n-fold $\mathbb{P}^1$-loop spectrum?

In this paper, we prove two main results about the "motivic" structure on the slices of S^1-spectra:

THEOREM 2. *Suppose* $\mathrm{char}\, k = 0$. *Let E be an S^1-spectrum. Then for each $n \geq 1$, there is a tower*

$$\ldots \to \rho_{\geq p+1} s_n E \to \rho_{\geq p} s_n E \to \ldots \to s_n E$$

in $\mathcal{SH}_{S^1}(k)$ with the following properties:

(1) *the tower is natural in E.*

(2) *Let $s_{p,n}E$ be the cofiber of $\rho_{\geq p+1} s_n E \to \rho_{\geq p} s_n E$. Then there is a homotopy invariant complex of presheaves with transfers $\hat{\pi}_p((s_n E)^{(n)})^* \in DM^{eff}_-(k)$ and a natural isomorphism in $\mathcal{SH}_{S^1}(k)$,*

$$EM_{\mathbb{A}^1}(\hat{\pi}_p((s_n E)^{(n)})^*) \cong s_{p,n} E,$$

where $EM_{\mathbb{A}^1} : DM^{eff}_-(k) \to \mathcal{SH}_{S^1}(k)$ is the Eilenberg-Maclane spectrum functor.

This result is proven in section 9.

One can say a bit more about the tower appearing in theorem 2. For instance, $\mathrm{holim}_p \, \mathrm{fib}(\rho_{\geq p} s_n E \to s_n E)$ is weakly equivalent to zero, so the spectral sequence associated to this tower is weakly convergent. If $s_n E$ is globally N-connected (i.e., there is an N such that $s_n E(X)$ is N-connected for all $X \in \mathbf{Sm}/k$) then the spectral sequence is strongly convergent. The "$\hat{\pi}_p$" appears in the notation due to the construction of $\hat{\pi}_p((s_n E)^{(n)})^*(X)$ arising from a "Bloch cycle complex" of codimension n cycles on $X \times \Delta^*$ with coefficients in $\pi_p(\Omega^n s_n E)$.

In other words, the *higher* slices of an arbitrary S^1-spectrum have some sort of transfers "up to filtration". The situation for the 0th slice appears to be more complicated, but for a $\mathbb{P}^1$-loop spectrum we have at least the following result:

THEOREM 3. *Suppose* $\mathrm{char}\, k = 0$. *Take $E \in \mathcal{SH}_{\mathbb{P}^1}(k)$. Then for all m, the homotopy sheaf $\pi_m(s_0 \Omega_{\mathbb{P}^1} E)$ has a natural structure of a homotopy invariant sheaf with transfers.*

We actually prove a more precise result (corollary 8.5) which states that the 0th slice $s_0 \Omega_{\mathbb{P}^1} E$ is itself a presheaf with transfers, with values in the stable homotopy category $\mathcal{SH}$, i.e., $s_0 \Omega_{\mathbb{P}^1} E$ has "transfers up to homotopy". This raises the question:

(4) Is there an operad acting on $s_0 \Omega^n_{\mathbb{P}^1} E$ which shows that $s_0 \Omega^n_{\mathbb{P}^1} E$ admits transfers up to homotopy and higher homotopies up to some level?

Part of the motivation for this paper came out of discussions with Hélène Esnault concerning the (admittedly vague) question: Given a smooth projective variety X over some field k, that admits a 0-cycle of degree 1, are there "motivic" properties of X that lead to the existence of a k-point, or conversely, that give obstructions to the existence of a k-point? The fact that the existence of 0-cycles of degree 1 has something to do with the transfer maps from 0-cycles on X_L to 0-cycles on X, as L runs over finite field extensions of k, while the lack of a transfer map in general appears to be closely related to the subtlety of the existence of k-points led to our inquiry into the "motivic" nature of the spaces $\Omega^n_{\mathbb{P}^1}\Sigma^n_{\mathbb{P}^1}X_+$, or rather, their associated S^1-spectra.

NOTATION AND CONVENTIONS. Throughout this paper the base-field k will be a field of characteristic zero. $\mathbf{Sm}/k$ is the category of smooth finite type k-schemes. We let $\mathbf{Spc}_\bullet$ denote the category of pointed space, i.e., pointed simplicial sets, and $\mathcal{H}_\bullet$ the homotopy category of $\mathbf{Spc}_\bullet$, the *unstable homotopy category*. Similarly, we let $\mathbf{Spt}$ denote the category of spectra and $\mathcal{SH}$ its homotopy category, the *stable homotopy category*. We let $\mathbf{Spc}_\bullet(k)$ denote the category of *pointed spaces over* k, that is, the category of $\mathbf{Spc}_\bullet$-valued presheaves on $\mathbf{Sm}/k$, and $\mathbf{Spt}_{S^1}(k)$ the category of S^1*-spectra over* k, that is, the category of $\mathbf{Spt}$-valued presheaves on $\mathbf{Sm}/k$. We let $\mathbf{Spt}_{\mathbb{P}^1}(k)$ denote the category of $\mathbb{P}^1$*-spectra over* k, which we take to mean the category of $\Sigma_{\mathbb{P}^1}$-spectrum objects over $\mathbf{Spt}_{S^1}(k)$. Concretely, an object is a sequence $(E_0, E_1, \ldots)$, $E_n \in \mathbf{Spt}_{S^1}(k)$, together with bonding maps $\epsilon_n : \Sigma_{\mathbb{P}^1}E_n \to E_{n+1}$. Regarding the categories $\mathbf{Spt}_{S^1}(k)$, $\mathcal{SH}_{S^1}(k)$ and $\mathcal{SH}(k)$, we will use the notation spelled out in [11]. In addition to this source, we refer the reader to [8, 14, 15, 20, 22]. Relying on these sources for details, we remind the reader that $\mathcal{H}_\bullet(k)$ is the homotopy category of the category of $\mathbf{Spc}_\bullet(k)$, for the so-called $\mathbb{A}^1$-model structure. Similarly $\mathcal{SH}_{S^1}(k)$ and $\mathbf{Spt}_{\mathbb{P}^1}(k)$ have model structures, which we call the $\mathbb{A}^1$-model structures, and $\mathcal{SH}_{S^1}(k)$, $\mathcal{SH}(k)$ are the respective homotopy categories. For details on the category $DM^{eff}(k)$, we refer the reader to [3, 5].

We will be passing from the unstable motivic (pointed) homotopy category over k, $\mathcal{H}_\bullet(k)$, to the motivic homotopy category of S^1-spectra over k, $\mathcal{SH}_{S^1}(k)$, via the infinite (simplicial) suspension functor

$$\Sigma^\infty_s : \mathcal{H}_\bullet(k) \to \mathcal{SH}_{S^1}(k)$$

For a smooth k-scheme $X \in \mathbf{Sm}/k$ and a subscheme Y of X (sometimes closed, sometimes open), we let (X, Y) denote the homotopy push-out in the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \\ \operatorname{Spec} k & \end{tikzcd}$$

and as usual write X_+ for $(X \amalg \operatorname{Spec} k, \operatorname{Spec} k)$. We often denote $\operatorname{Spec} k$ by $*$. For an object S of $\mathcal{H}_\bullet(k)$, we often use S to denote $\Sigma^\infty_s S \in \mathcal{SH}_{S^1}(k)$ when the context makes the meaning clear; we also use this convention when passing to various localizations of $\mathcal{SH}_{S^1}(k)$.

We let $[n]$ denote the set $\{0, \ldots, n\}$ with the standard total order, and let $\mathbf{Ord}$ denote the category with objects $[n]$, $n = 0, 1, \ldots$ and morphisms the order-preserving maps of sets. Let Δ^n denote the algebraic n-simplex $\operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1$, with vertices $v^n_0, \ldots, v^n_n$, where v^n_i is defined by $t_j = 0$ for $j \neq i$. As is well-known, sending $g : [n] \to [m]$ to the affine-linear extension $\Delta(g) : \Delta^n \to \Delta^m$ of the map on the set of vertices, $v_j^n \mapsto v_{g(j)}^m$ defines the cosimplicial k-scheme $n \mapsto \Delta^n$.

We recall that, for a category $\mathcal{C}$, the category of pro-objects in $\mathcal{C}$, pro-$\mathcal{C}$, has as objects functors $f : I \to \mathcal{C}$, $i \in I$, where I is a small left-filtering category, a morphism $(f : I \to \mathcal{C}) \to (g : J \to \mathcal{C})$ is a pair $(\rho : I \to J, \theta : f \to g \circ \rho)$, with the evident composition, and we invert morphisms of the form

$$(\rho : I \to J, \mathrm{id} : f := g \circ \rho \to g \circ \rho)$$

if $\rho : I \to J$ has image a left co-final subcategory of J. In this paper we use categories of pro-objects to allow us to use various localizations of smooth finite type k-schemes. This is a convenience rather than a necessity, as all maps and relations lift to the level of finite type k-schemes.

I am very grateful to the referee for making a number of perceptive and useful comments, which led to the correction of some errors and an improvement of the exposition.

Dedication. This paper is warmly dedicated to Andrei Suslin, who has given me more inspiration than I can hope to tell.

1. Infinite $\mathbb{P}^1$-loop spectra

We first consider the case of the 0-S^1-spectrum of a $\mathbb{P}^1$-spectrum. We recall some constructions and results from [20]. We let

$$\Omega^\infty_{\mathbb{P}^1} : \mathcal{SH}(k) \to \mathcal{SH}_{S^1}(k)$$
$$\Omega^\infty_{\mathbb{P}^1,mot} : DM(k) \to DM^{eff}(k)$$

be the (derived) 0-spectrum (resp. 0-complex) functor, let

$$EM_{\mathbb{A}^1} : DM(k) \to \mathcal{SH}(k)$$
$$EM^{eff}_{\mathbb{A}^1} : DM^{eff}(k) \to \mathcal{SH}_{S^1}(k)$$

the respective Eilenberg-Maclane spectrum functors. The functors $\Omega^\infty_{\mathbb{P}^1}$, $\Omega^\infty_{\mathbb{P}^1,mot}$ are right adjoints to the respective infinite suspension functors

$$\Sigma^\infty_{\mathbb{P}^1} : \mathcal{SH}_{S^1}(k) \to \mathcal{SH}(k)$$
$$\Sigma^\infty_{\mathbb{P}^1,mot} : DM^{eff}(k) \to DM(k)$$

and the functors $EM_{\mathbb{A}^1}$, $EM^{eff}_{\mathbb{A}^1}$ are similarly right adjoints to the "linearization" functors

$$\mathbb{Z}^{tr} : \mathcal{SH}(k) \to DM(k)$$
$$\mathbb{Z}^{tr} : \mathcal{SH}_{S^1}(k) \to DM^{eff}(k)$$

induced by the functor $\mathbb{Z}^{tr}$ from simplicial presheaves on $\mathbf{Sm}/k$ to presheaves with transfer on $\mathbf{Sm}/k$ sending the representable presheaf $\mathrm{Hom}_{\mathbf{Sm}/k}(-, X)$ to the free presheaf with transfers $\mathbb{Z}^{tr}_X := \mathrm{Hom}_{SmCor(k)}(-X)$, and taking the Kan extension. The discussion in [20, §2.2.1] show that both these adjoint pairs arise from Quillen adjunctions on suitable model categories (followed by a chain of Quillen equivalences), where on the model categories, the functors $EM_{\mathbb{A}^1}$,

$EM_{\mathbb{A}^1}^{eff}$ are just forgetful functors and the functors Ω^∞ just take a sequence $E_0, E_1, \ldots$ to E_0. Thus one has

$$EM_{\mathbb{A}^1}^{eff} \circ \Omega^\infty_{\mathbb{P}^1, mot} \cong \Omega^\infty_{\mathbb{P}^1} \circ EM_{\mathbb{A}^1} \tag{1.1}$$

as one has an identity of the two functors on the model categories.

THEOREM 1.1. *Fix an integer $n \geq 0$. Then there is a functor*

$$Mot^{eff}(s_n) : \mathcal{SH}(k) \to DM^{eff}(k)$$

and a natural isomorphism

$$\varphi_n : EM_{\mathbb{A}^1}^{eff} \circ Mot^{eff}(s_n) \to s_n^{eff} \circ \Omega^\infty_{\mathbb{P}^1}$$

of functors from $\mathcal{SH}(k)$ to $\mathcal{SH}_{S^1}(k)$.

In other words, for $\mathcal{E} \in \mathcal{SH}(k)$, there is a canonical lifting of the slice $s_n^{eff}(\Omega^\infty_{\mathbb{P}^1}\mathcal{E})$ to a motive $Mot^{eff}(s_n)(\mathcal{E})$.

Proof. By Pelaez [18, theorem 3.3], there is a functor

$$Mot(s_n) : \mathcal{SH}(k) \to DM(k)$$

and a natural isomorphism

$$\Phi_n : EM_{\mathbb{A}^1} \circ Mot(s_n) \to s_n$$

i.e., the slice $s_n\mathcal{E}$ lifts canonically to a motive $Mot(s_n)(\mathcal{E})$. Now apply the 0-complex functor to define

$$Mot^{eff}(s_n) := \Omega^\infty_{\mathbb{P}^1, mot} \circ Mot(s_n).$$

We have canonical isomorphisms

$$\begin{aligned} EM_{\mathbb{A}^1}^{eff} \circ \Omega^\infty_{\mathbb{P}^1, mot} \circ Mot(s_n) &\cong \Omega^\infty_{\mathbb{P}^1} \circ EM_{\mathbb{A}^1} \circ Mot(s_n) \\ &\cong \Omega^\infty_{\mathbb{P}^1} \circ s_n \\ &\cong s_n^{eff} \circ \Omega^\infty_{\mathbb{P}^1}. \end{aligned}$$

Indeed, the first isomorphism is (1.1) and the second is Pelaez's isomorphism Φ_n. For the third, we have given in [11] an explicit model for s_n in terms of the functors s_m^{eff} as follows: given a $\mathbb{P}^1$-spectrum E, represented as a sequence of S^1-spectra $E_0, E_1, \ldots$ together with bonding maps $\Sigma_{\mathbb{P}^1} E_n \to E_{n+1}$, suppose that E is fibrant. In particular, the adjoints $E_n \to \Omega_{\mathbb{P}^1} E_{n+1}$ of the bonding maps are weak equivalences and $E_0 = \Omega^\infty_{\mathbb{P}^1} E$. It follows from [11, theorem 9.0.3] that $s_n E$ is represented by the sequence $(s_n^{eff} E_0, s_{n+1}^{eff} E_1, \ldots, s_{n+m}^{eff} E_m, \ldots)$, with certain bonding maps (defined in [11, §8.3]). In addition, by [11, theorem 4.1.1] this new sequence is termwise weakly equivalent to its fibrant model. This defines the natural isomorphism $\Omega^\infty_{\mathbb{P}^1} s_n E \cong s_n^{eff} E_0 \cong s_n^{eff} \Omega_{\mathbb{P}^1} E$. □

In other words, the slices of an infinite $\mathbb{P}^1$-loop spectrum are effective motives.

2. An example

We now show that the 0th slice of an S^1-spectrum is not always a motive. In fact, we will give an example of an Eilenberg-Maclane spectrum whose 0th slice does not admit transfers.
For this, note the following:

Lemma 2.1. *Let $p : Y \to X$ be a finite Galois cover in $\mathbf{Sm}/k$, with Galois group G. Let $\mathcal{F}$ be a presheaf with transfers on $\mathbf{Sm}/k$. Then the composition*

$$p^* \circ p_* : \mathcal{F}(Y) \to \mathcal{F}(Y)$$

is given by

$$p^* \circ p_*(x) = \sum_{g \in G} g^*(x)$$

Proof. Letting $\Gamma_p \subset Y \times X$ be the graph of p, and $\Gamma_g \subset Y \times Y$ the graph of $g : Y \to Y$ for $g \in G$, one computes that

$$\Gamma_p^t \circ \Gamma_p = \sum_{g \in G} \Gamma_g,$$

whence the result. □

Now let C be a smooth projective curve over k, having no k-rational points. We assume that C has genus $g > 0$, so every map $\mathbb{A}^1_F \to C_F$ over a field $F \supset k$ is constant (C is $\mathbb{A}^1$-*rigid*).
Let $\mathbb{Z}_C$ be the representable presheaf:

$$\mathbb{Z}_C(Y) := \mathbb{Z}[\mathrm{Hom}_{\mathbf{Sm}/k}(Y, C)].$$

$\mathbb{Z}_C$ is automatically a Nisnevich sheaf; since C is $\mathbb{A}^1$-rigid, $\mathbb{Z}_C$ is also homotopy invariant. Furthermore $\mathbb{Z}_C$ is a *birational* sheaf, that is, for each dense open immersion $U \to Y$ in $\mathbf{Sm}/k$, the restriction map $\mathbb{Z}_C(Y) \to \mathbb{Z}_C(U)$ is an isomorphism. To see this, it suffices to show that $\mathrm{Hom}_{\mathbf{Sm}/k}(Y, C) \to \mathrm{Hom}_{\mathbf{Sm}/k}(U, C)$ is an isomorphism, and for this, take a morphism $f : U \to C$. Then the projection to Y of the closure $\bar{\Gamma}$ of the graph of f in $Y \times C$ is proper and birational. But since Y is regular, each fiber of $\bar{\Gamma} \to Y$ is rationally connected, hence maps to a point of C, and thus $\bar{\Gamma} \to Y$ is birational and 1-1. By Zariski's main theorem, $\bar{\Gamma} \to Y$ is an isomorphism, hence f extends to $\bar{f} : Y \to C$, as claimed.
Next, $\mathbb{Z}_C$ satisfies Nisnevich excision. This is just a general property of birational sheaves. In fact, let

$$\begin{tikzcd} V \arrow[r, "j_V"] \arrow[d, "f_{|V}"'] & Y \arrow[d, "f"] \\ U \arrow[r, "j_U"'] & X \end{tikzcd}$$

be an elementary Nisnevich square, i.e., the square is cartesian, f is étale, j_U and j_V are open immersions, and f induces an isomorphism $Y \setminus V \to X \setminus U$.

We may assume that U and V are dense in X and Y. Let $\mathcal{F}$ be a birational sheaf on $\mathbf{Sm}/k$, and apply $\mathcal{F}$ to this diagram. This gives us the square

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "j_U^*"] \arrow[d, "f^*"'] & \mathcal{F}(U) \arrow[d, "f_{|V}^*"] \\
\mathcal{F}(Y) \arrow[r, "j_V^*"'] & \mathcal{F}(V)
\end{tikzcd}$$

As the horizontal arrows are isomorphisms, we have the exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(Y) \to \mathcal{F}(V) \to 0.$$

Thus, $\mathcal{F}$ transforms elementary Nisnevich squares to distinguished triangles in $D(\mathbf{Ab})$; by definition, $\mathcal{F}$ therefore satisfies Nisnevich excision.

Let $EM_s(\mathbb{Z}_C)$ denote presheaf of Eilenberg-Maclane spectra on $\mathbf{Sm}/k$ associated to $\mathbb{Z}_C$, that is, for $U \in \mathbf{Sm}/k$, $EM_s(\mathbb{Z}_C)(U) \in \mathbf{Spt}$ is the Eilenberg-Maclane spectrum associated to the abelian group $\mathbb{Z}_C(U)$. Since $\mathbb{Z}_C$ is homotopy invariant and satisfies Nisnevich excision, $EM_s(\mathbb{Z}_C)$ is weakly equivalent as a presheaf on $\mathbf{Sm}/k$ to its fibrant model in $\mathcal{SH}_{S^1}(k)$ ($EM_s(\mathbb{Z}_C)$ is *quasi-fibrant*)[2]. In addition, the canonical map

$$EM_s(\mathbb{Z}_C) \to s_0(EM_s(\mathbb{Z}_C))$$

is an isomorphism in $\mathcal{SH}_{S^1}(k)$. Indeed, since $EM_s(\mathbb{Z}_C)$ is quasi-fibrant, a quasi-fibrant model for $s_0(EM_s(\mathbb{Z}_C))$ may be computed by the method of [11, §5] as follows: Take $Y \in \mathbf{Sm}/k$ and let $F = k(Y)$. Let $\Delta^n_{F,0}$ be the *semi-local* algebraic n-simplex over F, that is,

$$\Delta^n_{F,0} = \mathrm{Spec}(\mathcal{O}_{\Delta^n_F,v});\ v = \{v_0, \ldots, v_n\}.$$

The assignment $n \mapsto \Delta^n_{F,0}$ forms a cosimplicial subscheme of $n \mapsto \Delta^n_F$ and for a quasi-fibrant S^1-spectrum E, there is a natural isomorphism in $\mathcal{SH}$

$$s_0(E)(Y) \cong E(\Delta^*_{F,0}),$$

where $E(\Delta^*_{F,0})$ denotes the total spectrum of the simplicial spectrum $n \mapsto E(\Delta^n_{F,0})$. If now E happens to be a birational S^1-spectrum, meaning that $j^* : E(Y) \to E(U)$ is a weak equivalence for each dense open immersion $j : U \to Y$ in $\mathbf{Sm}/k$, then the restriction map

$$j^* : E(\Delta^*_Y) \to E(\Delta^*_{F,0}) \cong s_0(E)(Y)$$

is a weak equivalence. Thus, as E is quasi-fibrant and hence homotopy invariant, we have the sequence of isomorphisms in $\mathcal{SH}$

$$E(Y) \to E(\Delta^*_Y) \to E(\Delta^*_{F,0}) \cong s_0(E)(Y),$$

and hence $E \to s_0(E)$ is an isomorphism in $\mathcal{SH}_{S^1}(k)$. Taking $E = EM_s(\mathbb{Z}_C)$ verifies our claim.

[2]The referee has pointed out that, using the standard model $(\mathbb{Z}_C, B\mathbb{Z}_C, \ldots, B^n\mathbb{Z}_C, \ldots)$ for $EM_s(\mathbb{Z}_C)$, $EM_s(\mathbb{Z}_C)$ is actually fibrant in the projective model structure.

Finally, $\mathbb{Z}_C$ does not admit transfers. Indeed, suppose $\mathbb{Z}_C$ has transfers. Let $k \to L$ be a Galois extension such that $C(L) \neq \emptyset$; let G be the Galois group. Since $\mathbb{Z}_C(k) = \{0\}$ (as we have assumed that $C(k) = \emptyset$), the push-forward map

$$p_* : \mathbb{Z}_C(L) \to \mathbb{Z}_C(k)$$

is the zero map, hence $p^* \circ p_* = 0$. But for each L-point x of C, lemma 2.1 tells us that

$$p^* \circ p_*(x) = \sum_{g \in G} x^g \neq 0,$$

a contradiction.
Thus the homotopy sheaf

$$\pi_0(s_0 EM_s(\mathbb{Z}_C)) = \pi_0(EM_s(\mathbb{Z}_C)) = \mathbb{Z}_C$$

does not admit transfers, giving us the example we were seeking.
Even if we ask for transfers in a weaker sense, namely, that there is a functorial separated filtration $F^*\mathbb{Z}_C$ admitting transfers on the associated graded $gr^*_F\mathbb{Z}_C$, a slight extension of the above argument shows that this is not possible as long as the filtration on $\mathbb{Z}_C(L)$ is finite. Indeed, p^*p_* would send $F^n\mathbb{Z}_C(L)$ to $F^{n+1}\mathbb{Z}_C(L)$, so $(p^*p_*)^N = 0$ for some $N \geq 1$, and hence $N \cdot \sum_{g \in G} x^g = 0$, a contradiction.

3. Co-transfer

In this section, k will be an arbitrary perfect field. We recall how one uses the deformation to the normal bundle to define the "co-transfer"

$$(\mathbb{P}^1_F, 1_F) \to (\mathbb{P}^1_F(x), 1_F)$$

for a closed point $x \in \mathbb{A}^1_F \subset \mathbb{P}^1_F$, with chosen generator $f \in m_x/m_x^2$. For later use, we work in a somewhat more general setting: Let S be a smooth finite type k-scheme and x a regular closed subscheme of $\mathbb{P}^1_S \setminus \{1\} \subset \mathbb{P}^1_S$, such that the projection $x \to S$ is finite. Let $m_x \subset \mathcal{O}_{\mathbb{P}^1_S}$ be the ideal sheaf of x. We assume that the invertible sheaf m_x/m_x^2 on x is isomorphic to the trivial invertible sheaf $\mathcal{O}_x$, and we choose a generator $f \in \Gamma(x, m_x/m_x^2)$ over $\mathcal{O}_x$.
We will eventually replace S with a semi-local affine scheme, $S = \operatorname{Spec} R$, for R a smooth semi-local k-algebra, essentially of finite type over k, for instance, $R = F$ a finitely generated separable field extension of k. Although this will take us out of the category $\mathcal{H}(k)$, this will not be a problem: when we work with a smooth scheme Y which is essentially of finite type over k, we will consider Y as a pro-object in $\mathcal{H}(k)$, and we will be interested in functors on $\mathcal{H}(k)$ of the form $\operatorname{Hom}_{\mathcal{H}(k)}(Y, -)$, which will then be a well-defined filtered colimit of co-representable functors.
Let $(X_0 : X_1)$ be the standard homogeneous coordinates on $\mathbb{P}^1$. We let $s := X_1/X_0$ be the standard parameter on $\mathbb{P}^1$, and as usual, write $0 = (1:0)$, $\infty = (0:1)$, $1 = (1:1)$. We often write $0, 1, \infty$ for the subschemes $0_X, 1_X, \infty_X$ of $\mathbb{P}^1_X$.

Let $\mu : W_x \to \mathbb{P}^1_S \times \mathbb{A}^1$ be the blow-up of $\mathbb{P}^1_S \times \mathbb{A}^1$ along $(x, 0)$ with exceptional divisor E. Let s_x, C_0 be the proper transforms $s_x = \mu^{-1}[x \times \mathbb{A}^1]$, $C_0 = \mu^{-1}[\mathbb{P}^1 \times 0]$. Let t be the standard parameter on $\mathbb{A}^1$ and let $\tilde{f}$ be a local lifting of f to a section of m_x; the rational function $\tilde{f}/t$ restricts to a well-defined rational parameter on E, independent of the choice of lifting, and thus defines a globally defined isomorphism

$$f/t : E \to \mathbb{P}^1_x.$$

We identify E with $\mathbb{P}^1_x$ by sending $s_x \cap E$ to 0, $C_0 \cap E$ to 1 and the section on E defined by $f/t = 1$ to ∞. Denote this isomorphism by

$$\varphi_f : \mathbb{P}^1_x \to E;$$

we write $(0, 1, \infty)$ for $(s_x \cap E, C_0 \cap E, f/t = 1)$, when the context makes it clear we are referring to subschemes of E.

We let $W_x^{(s_x)}$, $E^{(0)}$, $(\mathbb{P}^1_F)^{(0)}$ be following homotopy push-outs

$$\begin{aligned} W_x^{(s_x)} &:= (W_x, W_x \setminus s_x), \\ E^{(0)} &:= (E, E \setminus 0), \\ (\mathbb{P}^1_S)^{(0)} &:= (\mathbb{P}^1_S, \mathbb{P}^1_S \setminus 0). \end{aligned}$$

Since $(\mathbb{A}^1_x, 0) \cong *$ in $\mathcal{H}_\bullet(k)$, the respective identity maps induce isomorphisms

$$\begin{aligned} &(E, 1) \to E^{(0)}, \\ &(\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_S)^{(0)}. \end{aligned}$$

Composing with the isomorphism $\varphi_f : (\mathbb{P}^1_x, 1) \to (E, 1)$, the inclusion $E \to W_x$ induces the map

$$i_{0,f} : (\mathbb{P}^1_x, 1) \to W_x^{(s_x)}.$$

The proof of the homotopy purity theorem of Morel-Voevodsky [15, theorem 2.23] yields as a special case that $i_{0,f}$ is an isomorphism in $\mathcal{H}_\bullet(k)$. This enables us to define the "co-transfer map" as follows:

Definition 3.1. Let $x \subset \mathbb{P}^1_S \setminus 1_S$ be a closed subscheme, smooth over k and finite over S, and suppose that m_x/m_x^2 is a free $\mathcal{O}_x$-module with generator f. The map

$$\textit{co-tr}_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1)$$

in $\mathcal{H}_\bullet(k)$ is defined to be the composition

$$(\mathbb{P}^1_S, 1) \xrightarrow{i_1} W_x^{(s_x)} \xrightarrow{i_{0,f}^{-1}} (\mathbb{P}^1_x, 1).$$

Let $\mathcal{X} \in \mathcal{H}_\bullet(k)$ be a $\mathbb{P}^1$-loop space, i.e., $\mathcal{X} \cong \Omega^1_{\mathbb{P}}\mathcal{Y} := Maps_\bullet(\mathbb{P}^1, \mathcal{Y})$ for some $\mathcal{Y} \in \mathcal{H}_\bullet(k)$. For $x \subset \mathbb{P}^1_S$ and f as above, one has the transfer map

$$\mathcal{X}(x) \to \mathcal{X}(S)$$

in $\mathcal{H}_\bullet$ defined by pre-composing with the co-transfer map

$$\textit{co-tr}_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1).$$

We will find modification of this construction useful in the sequel, namely, in the proof of lemma 5.9 and lemma 5.11. Let $s_1 := 1_S \times \mathbb{A}^1 \subset \mathbb{P}^1_S \times \mathbb{A}^1$; as $W_x \to \mathbb{P}^1_S \times \mathbb{A}^1$ is an isomorphism over a neighborhood of s_1, we view s_1 as a closed subscheme of W_x. We write W for W_x, etc., when the context makes the meaning clear.

Lemma 3.2. *Let $x \subset \mathbb{P}^1_S \setminus 1_S$ be a closed subscheme, smooth over k. Suppose that $x \to S$ is finite and étale. Then the identity on W induces an isomorphism*

$$(W, C_0 \cup s_1) \to W^{(s_x)}$$

in $\mathcal{H}_\bullet(k)$.

Proof. As $s_1 \cong \mathbb{A}^1_S$, with $C_0 \cap s_1 = 0_S$, the inclusion $C_0 \to C_0 \cup s_1$ is an isomorphism in $\mathcal{H}(k)$. Thus, we need to show that $(W, C_0) \to W^{(s_x)}$ is an isomorphism in $\mathcal{H}_\bullet(k)$. As $W^{(s_x)} = (W, W \setminus s_x)$, we need to show that $C_0 \to W \setminus s_x$ is an isomorphism in $\mathcal{H}(k)$. To aid in the proof, we will prove a more general result, namely, let $U \subset \mathbb{P}^1_S$ be a open subscheme containing x. We consider W as a scheme over $\mathbb{P}^1_S$ via the composition

$$W \xrightarrow{\mu} \mathbb{P}^1_S \times \mathbb{A}^1_S \xrightarrow{p_1} \mathbb{P}^1_S$$

and for a subscheme Z of W, let Z_U denote the pull-back $Z \times_{\mathbb{P}^1_S} U$. Then we will show that

$$C_{0U} \to W_U \setminus s_x$$

is an isomorphism in $\mathcal{H}(k)$.

We first reduce to the case in which $x \to S$ is an isomorphism (in $\mathbf{Sm}/k$). For this, we have the étale map $q : \mathbb{P}^1_x \to \mathbb{P}^1_S$ and the canonical x-point of $\mathbb{P}^1_x$, which we write as $\tilde{x}$. Let $U(x) \subset \mathbb{P}^1_x$ be a Zariski open neighborhood of $\tilde{x}$ such that $q^{-1}(x) \cap U(x) = \{\tilde{x}\}$. This gives us the elementary Nisnevich square

$$\mathcal{U}(x) := \begin{array}{ccc} U(x) \setminus \tilde{x} & \longrightarrow & U(x) \\ \downarrow & & \downarrow \\ \mathbb{P}^1_S \setminus x & \longrightarrow & \mathbb{P}^1_S; \end{array}$$

for each $\mathbb{P}^1_S$-scheme $Z \to \mathbb{P}^1_S$ we thus have the elementary Nisnevich square $\mathcal{U}(x) \times_{\mathbb{P}^1_S} Z$, giving a Nisnevich cover of Z.

Let $V \subset \mathbb{P}^1_S$ be an open subscheme with $x \cap V = \emptyset$. Then $W_V \to V \times_S \mathbb{A}^1_S$ is an isomorphism and $W_V \cap s_x = \emptyset$. Similarly $C_{0V} \to V$ is an isomorphism, and thus $C_{0V} \to W_V$ is an isomorphism in $\mathcal{H}(k)$. Replacing S with x, and considering the map of elementary Nisnevich squares

$$\mathcal{U}(x) \times_{\mathbb{P}^1_S} C_{0U} \to \mathcal{U}(x) \times_{\mathbb{P}^1_S} (W_U \setminus s_x)$$

induced by $C_0 \to W \setminus s_x$, we achieve the desired reduction. A similar Mayer-Vietoris argument allows us to replace S with a Zariski open cover of S, so, changing notation, we may assume that x is the point $0 := (1:0)$ of $\mathbb{P}^1_S$.

Using the open cover of $\mathbb{P}^1_S$ by the affine open subsets $U_0 := \mathbb{P}^1_S \setminus 1$, $U_1 := \mathbb{P}^1_S \setminus 0$ and arguing as above, we may assume that U is a subset of U_0, which we identify with $\mathbb{A}^1_S$ by sending $(0, \infty)$ to $(0, 1)$. We may also assume that $0_S \subset U$. Using coordinates (t_1, t_2) for $\mathbb{A}^2$, (t_1, t_2, t_3) for $\mathbb{A}^3$, the scheme $W_{U_0} \setminus s_0$ is isomorphic to the closed subscheme of $\mathbb{A}^3_S$ defined by $t_2 = t_1t_3$, with μ being the projection $(t_1, t_2, t_3) \mapsto (t_1, t_2)$. C_{0U_0} is the subscheme of $W_{U_0} \setminus s_0$ defined by $t_3 = 0$. The projection $p_{13} : W_{U_0} \setminus s_0 \to \mathbb{A}^2_S$ is thus an isomorphism, sending C_0 to $\mathbb{A}^1_S \times 0$.
Let $y = U \setminus U_0$, so y is a closed subset disjoint from 0_S. Then

$$p_{13}(\mu^{-1}(y \times \mathbb{A}^1)) = y \times \mathbb{A}^1 \subset \mathbb{A}^2_S,$$

hence $p_{13} : W_U \setminus s_0 \to \mathbb{A}^2_S$ identifies $W_U \setminus s_0$ with $U \times \mathbb{A}^1$ and identifies C_{0U} with $U \times 0$. Thus $C_{0U} \to W_U \setminus s_0$ is an isomorphism in $\mathcal{H}(k)$, completing the proof. □

Lemma 3.3. *With hypotheses as in lemma 3.2, the inclusion $E \to W$ and isomorphism $\varphi_f : \mathbb{P}^1 \to E$ induces an isomorphism*

$$\tilde{i}_{0,f} : (\mathbb{P}^1_x, 1) \to (W_x, C_0 \cup s_1)$$

in $\mathcal{H}_\bullet(k)$.

Proof. We have the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1_x, 1) \arrow[r, "\tilde{i}_{0,f}"] \arrow[dr, "i_{0,f}"'] & (W, C_0 \cup s_1) \arrow[d] \\
& W^{(s_x)}.
\end{tikzcd}$$

The diagonal arrow is an isomorphism in $\mathcal{H}_\bullet(k)$ by Morel-Voevodsky; the vertical arrow is an isomorphism by lemma 3.2. □

Definition 3.4. Let $x \subset \mathbb{P}^1_S \setminus 1_S$ be a closed subscheme, smooth over k and finite and étale over S. Suppose that m_x/m_x^2 is a free $\mathcal{O}_x$-module with generator f. The map

$$\tilde{co\text{-}tr}_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1)$$

in $\mathcal{H}_\bullet(k)$ is defined to be the composition

$$(\mathbb{P}^1_S, 1) \xrightarrow{i_1} (W, C_0 \cup s_1) \xrightarrow{\tilde{i}_{0,f}^{-1}} (\mathbb{P}^1_x, 1).$$

Remark 3.5. Given S, x, f as in definition 3.4, we have

$$\tilde{co\text{-}tr}_{x,f} = co\text{-}tr_{x,f}.$$

This follows directly from the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1_S, 1) \arrow[r, "i_1"] \arrow[d, equal] & (W, C_0 \cup s_1) \arrow[d, "\mathrm{id}"] & (\mathbb{P}^1_x, 1) \arrow[l, "\tilde{i}_{0,f}"'] \arrow[d, equal] \\
(\mathbb{P}^1_S, 1) \arrow[r, "i_1"'] & W^{(s_x)} & (\mathbb{P}^1_x, 1). \arrow[l, "i_{0,f}"]
\end{tikzcd}$$

We examine some properties of $co\text{-}tr_{x,f}$. For any ordering (a, b, c) of $\{0, 1, \infty\}$, we let $\tau^a_{b,c}$ denote the automorphism of $\mathbb{P}^1$ that fixes a and exchanges b and c. For $u \in k^\times$, we let $\mu(u)$ be the automorphism of $\mathbb{P}^1$ that fixes 0 and ∞ and sends 1 to u. We first prove the following elementary result

Lemma 3.6. *The automorphism* $\rho := \tau^0_{1,\infty} \circ \mu(-1) \circ \tau^0_{1,\infty} \circ \tau^1_{0,\infty}$ *of* $(\mathbb{P}^1, 1)$ *is the identity in* $\mathcal{H}_\bullet(k)$.

Proof. $\tau^0_{1,\infty}\rho\tau^0_{1,\infty}$ is the automorphism of $(\mathbb{P}^1, \infty)$ given by the matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \mathrm{GL}_2(k).$$

Noting that elementary matrices of the form

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

all fix ∞ and thus define automorphisms of $(\mathbb{P}^1, \infty)$ that are $\mathbb{A}^1$-homotopic to the identity, we see that $\tau^0_{1,\infty}\rho\tau^0_{1,\infty} = \mathrm{id}$ on $(\mathbb{P}^1, \infty)$ in $\mathcal{H}_\bullet(k)$, and thus $\rho = \mathrm{id}$ on $(\mathbb{P}^1, 1)$ in $\mathcal{H}_\bullet(k)$. □

Lemma 3.7. *1. The map* $co\text{-}tr_{0,-s} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1)$ *is the identity.*

2. The map $co\text{-}tr_{\infty,-s^{-1}} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1)$ *is the map in* $\mathcal{H}_\bullet(k)$ *induced by the automorphism* $\tau^1_{0,\infty}$.

3. The map $co\text{-}tr_{\infty,s^{-1}} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1)$ *is the identity.*

Proof. Since $\tau^{1*}_{0,\infty}(-s) = -s^{-1}$, (2) follows from (1) by applying $\tau^1_{0,\infty}$. Next, we show that (2) implies (3). It follows directly from the definition of $co\text{-}tr_{*,*}$ that

$$co\text{-}tr_{\infty,s^{-1}} = \tau^0_{1,\infty} \circ \mu(-1) \circ \tau^0_{1,\infty} \circ co\text{-}tr_{\infty,-s^{-1}}.$$

Thus, assuming (2), we have

$$co\text{-}tr_{\infty,s^{-1}} = \tau^0_{1,\infty} \circ \mu(-1) \circ \tau^0_{1,\infty} \circ \tau^1_{0,\infty}$$

in $\mathcal{H}_\bullet(k)$; (3) then follows from lemma 3.6.
We now prove (1). Identify $\mathbb{A}^1$ with $\mathbb{P}^1 \setminus \{1\}$ sending 0 to 0 and 1 to ∞. The blow-up $W := W_0$ is thus identified with an open subscheme of the blow-up $\bar{\mu} : \bar{W} \to \mathbb{P}^1 \times \mathbb{P}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$ at $(0, 0)$.

The curve C_0 on $\bar{W}$ has self-intersection -1, and can thus be blown down via a morphism

$$\bar{\rho} : \bar{W} \to \bar{W}'.$$

Letting $q : \bar{W} \to \mathbb{P}^1$ be the composition

$$\bar{W} \xrightarrow{\bar{\mu}} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p_2} \mathbb{P}^1,$$

the fact that $q(C_0) = 0$ implies that q descends to a morphism

$$\bar{q}' : \bar{W}' \to \mathbb{P}^1.$$

As the complement $\bar{W}_\infty := \bar{W} \setminus W$ is disjoint from C_0 and $\bar{\rho}$ is proper, we have the open subscheme $W' := \bar{W}' \setminus \bar{\rho}(\bar{W}_\infty)$ of $\bar{W}'$ and the proper birational morphism

$$\rho : W \to W',$$

with $\rho(C_0) \cong \operatorname{Spec} k$ and with the restriction $W \setminus C_0 \to W' \setminus \rho(C_0)$ an isomorphism. In addition, $\bar{q}'$ restricts to the proper morphism

$$q' : W' \to \mathbb{P}^1 \setminus 1.$$

In addition, q' is a smooth and projective morphism with geometric fibers isomorphic to $\mathbb{P}^1$. Finally, we have

$$q'^{-1}(0) = \rho(E).$$

Let $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{1\}$ be the restriction of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, giving us the proper transform $\mu^{-1}[\Delta]$ on W and the image $\Delta' = \rho(\mu^{-1}[\Delta])$ on W'. Similarly, let $s'_0 = \rho(s_0)$, $s'_1 = \rho(s_1)$; note that $\rho(C_0) \subset s'_1$. It is easy to check that s'_0, Δ' and s'_1 give disjoint sections of $q' : W' \to \mathbb{P}^1 \setminus 1$, hence there is a unique isomorphism (over $\mathbb{P}^1 \setminus 1$) of W' with $\mathbb{P}^1 \times \mathbb{P}^1 \setminus 1$ sending (s'_0, s'_1, Δ') to $(0, 1, \infty) \times \mathbb{P}^1 \setminus 1$. We have in addition the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1, 1) \arrow[r, "i_{0,-s}"] \arrow[dr, "i'_0"'] & (W, W \setminus s_0) \arrow[d, "\rho"] & (\mathbb{P}^1, 1) \arrow[l, "i_1"'] \arrow[dl, "i'_1"] \\
& (W', W' \setminus s'_0) &
\end{tikzcd} \tag{3.1}$$

where i'_0 is the canonical identification of $\mathbb{P}^1$ with the fiber of W' over 0, sending $(0, 1, \infty)$ into (s'_0, s'_1, Δ'), and i'_1 is defined similarly.

We claim that the isomorphism $\rho : E \to q'^{-1}(0)$ is a pointed isomorphism

$$\rho : (E, 0, 1, \infty) \to (q'^{-1}(0), q'^{-1}(0) \cap s'_0, q'^{-1}(0) \cap s'_1, q'^{-1}(0) \cap \Delta').$$

Indeed, by definition $0 = E \cap s_0$ and $1 = E \cap C_0$. Since $\rho(s_0) = s'_0$ and $\rho(C_0) \subset s'_1$, we need only show that $\rho(\infty) \subset \Delta'$. To distinguish the two factors of $\mathbb{P}^1$, we write

$$x_1 = p_1^*(s), x_2 = p_2^*(s)$$

where s is the standard parameter on $\mathbb{P}^1$. Using this notation, ∞ is the subscheme of E defined by the equation $-x_1/t = 1$, where t is the standard parameter on $\mathbb{A}^1_S = \mathbb{P}^1_S \setminus 1_S$. As our identification of $\mathbb{P}^1 \setminus 1_S$ with $\mathbb{A}^1$ sends $0 \in \mathbb{A}^1$ to $0 \in \mathbb{P}^1$, $1 \in \mathbb{A}^1$ to ∞ in $\mathbb{P}^1$, the standard parameter t goes over to

the rational function $x_2/(x_2-1)$ on $\mathbb{P}^1$. As the image of $x_2/(x_2-1)$ in m_0/m_0^2 is the same as the image of $-x_2$ in m_0/m_0^2, ∞ is defined by $x_1/x_2 = 1$ on E, which is clearly the subscheme defined by $E \cap \Delta$. Via the isomorphism ρ, this goes over to $q'^{-1}(0) \cap \Delta'$, as desired.
It follows from the proof of [15, theorem 2.2.3] that all the morphisms in the diagram (3.1) are isomorphisms in $\mathcal{H}_\bullet(k)$; as $i_0'^{-1} \circ i_1'$ is clearly the identity, the lemma is proved. $\square$

The proof of the next result is easy and is left to the reader.

Lemma 3.8. *Let $S' \to S$ be a morphism of smooth finite type k-schemes. Let x be a closed subscheme of $\mathbb{P}^1_S \setminus \{1\}$, finite over S. Let $x' = x \times_S S' \subset \mathbb{P}^1_{S'}$. We suppose we have a generator f for m_x/m_x^2, and let f' be the extension to $m_{x'}/m_{x'}^2$. If either $S' \to S$ is smooth, or $S \to S'$ is flat and $x \to S$ is étale, then the diagram*

$$\begin{tikzcd}
(\mathbb{P}^1_{S'}, 1) \arrow[r, "co\text{-}tr_{x',f'}"] \arrow[d] & (\mathbb{P}^1_{x'}, 1) \arrow[d] \\
(\mathbb{P}^1_S, 1) \arrow[r, "co\text{-}tr_{x,f}"'] & (\mathbb{P}^1_x, 1)
\end{tikzcd}$$

is defined and commutes.

4. Co-group structure on $\mathbb{P}^1$

In this section, k will be an arbitrary perfect field. Let $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, which we consider as a pointed scheme with base-point 1. We recall the Mayer-Vietoris square for the standard cover of $\mathbb{P}^1$:

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "t_\infty"] \arrow[d, "t_0"'] & \mathbb{A}^1 \arrow[d, "j_\infty"] \\
\mathbb{A}^1 \arrow[r, "j_0"'] & \mathbb{P}^1.
\end{tikzcd}$$

Here j_0, j_∞, t_0, t_∞ are defined by $j_0(t) = (1:t)$, $j_\infty(t) = (t:1)$, $t_0(t) = t$ and $t_\infty(t) = t^{-1}$. This gives us the isomorphism in $\mathcal{H}_\bullet(k)$ of $\mathbb{P}^1$ with the homotopy push-out in the diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "t_\infty"] \arrow[d, "t_0"'] & \mathbb{A}^1 \\
\mathbb{A}^1; &
\end{tikzcd} \tag{4.1}$$

the contractibility of $\mathbb{A}^1$ gives us the canonical isomorphism

$$\alpha : S^1 \wedge \mathbb{G}_m \xrightarrow{\sim} (\mathbb{P}^1, 1). \tag{4.2}$$

This, together with the standard co-group structure on S^1, $\sigma : S^1 \to S^1 \vee S^1$, makes $(\mathbb{P}^1, 1)$ a co-group object in $\mathcal{H}_\bullet(k)$; let

$$\sigma_{\mathbb{P}^1} := \sigma \wedge \mathrm{id}_{\mathbb{G}_m} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1)$$

be the co-multiplication. In this section, we discuss a more algebraic description of this structure.

Let $f := (f_0, f_\infty)$ be a pair of generators for m_0/m_0^2, m_∞/m_∞^2. giving us the collapse map

$$\textit{co-tr}_{\{0,\infty\},f} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1).$$

Lemma 4.1. *Let s be the standard parameter X_1/X_0 on $\mathbb{P}^1$. For $f = (-s, s^{-1})$, we have $\sigma_{\mathbb{P}^1} = \textit{co-tr}_{\{0,\infty\},f}$ in $\mathcal{H}_\bullet(k)$.*

Proof. We first unwind the definition of $\sigma_{\mathbb{P}^1}$ in some detail. As above, we identify $\mathbb{P}^1$ with the push-out in the diagram (4.1) and thus $(\mathbb{P}^1, 1)$ is isomorphic to the push-out in the diagram

$$\begin{tikzcd}
(\mathbb{G}_m, 1) \vee (\mathbb{G}_m, 1) \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "(t_0 \vee t_\infty)"'] & (\mathbb{G}_m, 1) \\
(\mathbb{A}^1, 1) \vee (\mathbb{A}^1, 1). &
\end{tikzcd}$$

Let I denote a simplicial model of the interval admitting a "mid-point" $1/2$, for example, we can take $I = \Delta^1{}_1\vee_0 \Delta^1$. The isomorphism $\alpha : S^1 \wedge \mathbb{G}_m \to (\mathbb{P}^1, 1)$ in $\mathcal{H}_\bullet(k)$ arises via a sequence of comparison maps between push-outs in the following diagrams (we point $\mathbb{P}^1$, $\mathbb{A}^1$ and $\mathbb{G}_m$ with 1):

$$\begin{tikzcd}
\mathbb{G}_m \vee \mathbb{G}_m \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "(t_0\vee t_\infty)"'] & \mathbb{G}_m & \leftarrow & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m \arrow[r, "(\iota_0,\iota_1)"] \arrow[d, "t_0\vee t_\infty"'] & I_+ \wedge \mathbb{G}_m \\
\mathbb{A}^1 \vee \mathbb{A}^1 & & & 0_+ \wedge \mathbb{A}^1 \vee 1_+ \wedge \mathbb{A}^1 \arrow[d] & \\
 & & & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m \arrow[r, "(\iota_0,\iota_1)"] \arrow[d] & I_+ \wedge \mathbb{G}_m \\
 & & & * &
\end{tikzcd} \tag{4.3}$$

the first map is induced by the evident projections and the second by contracting $\mathbb{A}^1$ to $*$. Thus, the open immersion $\mathbb{G}_m \to \mathbb{P}^1$, $t \mapsto (1 : t)$, goes over to the map

$$\{1/2\}_+ \wedge \mathbb{G}_m \to I_+ \wedge \mathbb{G}_m \to S^1 \wedge \mathbb{G}_m,$$

the second map given by the bottom diagram in (4.3). This gives us the isomorphism

$$\rho : (\mathbb{P}^1, \mathbb{G}_m) \to S^1 \wedge \mathbb{G}_m \vee S^1 \wedge \mathbb{G}_m$$

in $\mathcal{H}_\bullet(k)$, yielding the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1,1) \arrow[r, "\alpha", "\sim"'] \arrow[d, "\pi"'] & S^1\wedge\mathbb{G}_m \arrow[d, "\sigma\wedge\mathrm{id}"] \\
(\mathbb{P}^1,\mathbb{G}_m) \arrow[r, "\sim", "\rho"'] & S^1\wedge\mathbb{G}_m\vee S^1\wedge\mathbb{G}_m,
\end{tikzcd}$$

where π is the canonical quotient map and α is the isomorphism (4.2).
If we consider the middle diagram in (4.3), we find a similarly defined isomorphism (in $\mathcal{H}_\bullet(k)$)

$$\epsilon : (\mathbb{P}^1,\mathbb{G}_m) \to (\mathbb{A}^1,\mathbb{G}_m)_{t_0}\vee(\mathbb{A}^1,\mathbb{G}_m)_{t_\infty},$$

where the subscripts t_0, t_∞ refer to the morphism $\mathbb{G}_m\to\mathbb{A}^1$ used.
The map from the middle diagram to the last diagram in (4.3) furnishes the commutative diagram of isomorphisms in $\mathcal{H}_\bullet(k)$:

$$\begin{tikzcd}
(\mathbb{A}^1,\mathbb{G}_m)_{t_0}\vee(\mathbb{A}^1,\mathbb{G}_m)_{t_\infty} \arrow[r, "\beta"] \arrow[dr, "\vartheta"'] & S^1\wedge\mathbb{G}_m\vee S^1\wedge\mathbb{G}_m \arrow[d, "\alpha\vee\alpha"] \\
& (\mathbb{P}^1,1)\vee(\mathbb{P}^1,1).
\end{tikzcd}$$

Putting this all together gives us the commutative diagram in $\mathcal{H}_\bullet(k)$:

$$\begin{tikzcd}
(\mathbb{P}^1,1) \arrow[r, "\sim", "\alpha"'] \arrow[d, "\gamma"'] & S^1\wedge\mathbb{G}_m \arrow[d, "\sigma\wedge\mathrm{id}"] \\
(\mathbb{P}^1,\mathbb{G}_m) \arrow[r, "\sim", "\rho"'] \arrow[d, "\epsilon"', "\sim"] & S^1\wedge\mathbb{G}_m\vee S^1\wedge\mathbb{G}_m \arrow[d, "\alpha\vee\alpha"] \\
(\mathbb{A}^1,\mathbb{G}_m)_{t_0}\vee(\mathbb{A}^1,\mathbb{G}_m)_{t_\infty} \arrow[r, "\sim", "\vartheta"'] \arrow[ur, "\beta", "\sim"'] & (\mathbb{P}^1,1)\vee(\mathbb{P}^1,1).
\end{tikzcd} \tag{4.4}$$

Letting $\delta := \vartheta\circ\epsilon$, we thus need to show that the map $\delta\circ\gamma : (\mathbb{P}^1,1)\to(\mathbb{P}^1,1)\vee(\mathbb{P}^1,1)$ is given by $\textit{co-tr}_{\{0,1\},f}$.
Write $(\mathbb{A}^1,\mathbb{G}_m) := (\mathbb{A}^1,\mathbb{G}_m)_{t_0}$. Letting η be the inverse on $\mathbb{G}_m$, $\eta(t)=t^{-1}$, we identify $(\mathbb{A}^1,\mathbb{G}_m)$ with $(\mathbb{A}^1,\mathbb{G}_m)_{t_\infty}$ via the isomorphism (id,η). The maps $j_0:\mathbb{A}^1\to\mathbb{P}^1$, $j_\infty:\mathbb{A}^1\to\mathbb{P}^1$ induce the isomorphisms in $\mathcal{H}_\bullet(k)$

$$\bar{j}_0 : (\mathbb{A}^1,\mathbb{G}_m)\to(\mathbb{P}^1, j_\infty(\mathbb{A}^1))$$
$$\bar{j}_\infty : (\mathbb{A}^1,\mathbb{G}_m)\to(\mathbb{P}^1, j_0(\mathbb{A}^1))$$

giving together the isomorphism $\tau : (\mathbb{P}^1,1)\vee(\mathbb{P}^1,1)\to(\mathbb{A}^1,\mathbb{G}_m)\vee(\mathbb{A}^1,\mathbb{G}_m)$, defined as the composition:

$$(\mathbb{P}^1,1)\vee(\mathbb{P}^1,1)\xrightarrow{\mathrm{id}\vee\mathrm{id}}(\mathbb{P}^1,j_\infty(\mathbb{A}^1))\vee(\mathbb{P}^1,j_0(\mathbb{A}^1))\xrightarrow{\bar{j}_0^{-1}\vee\bar{j}_\infty^{-1}}(\mathbb{A}^1,\mathbb{G}_m)\vee(\mathbb{A}^1,\mathbb{G}_m).$$

By comparing with the push-out diagrams in (4.3), we see that τ is the inverse to ϑ. As $\tau^1_{0,\infty}$ exchanges j_0 and j_∞, this gives the identity

$$\vartheta = \vartheta_0 \vee \tau^1_{0,\infty} \circ \vartheta_0, \tag{4.5}$$

where ϑ_0 is the composition

$$(\mathbb{A}^1, \mathbb{G}_m) \xrightarrow{j_0} (\mathbb{P}^1, j_\infty(\mathbb{A}^1)) \xleftarrow[\sim]{\mathrm{id}} (\mathbb{P}^1, 1).$$

Let $W \to \mathbb{P}^1 \times \mathbb{A}^1$ be the blow-up at $(\{0,\infty\}, 0)$ with exceptional divisor E. Let g be the trivialization $g := (-s, -s^{-1})$ of $m_0/m_0^2 \times m_\infty/m_\infty^2$. We have the composition of isomorphisms in $\mathcal{H}_\bullet(k)$

$$(\mathbb{P}^1, \mathbb{G}_m) \xrightarrow{i_1} (W, W \setminus s_{\{0,\infty\}}) \xleftarrow{i_0} (E, C_0 \cap E) \xleftarrow{\varphi_g} (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1). \tag{4.6}$$

The open cover $(j_0, j_\infty) : \mathbb{A}^1 \amalg \mathbb{A}^1 \to \mathbb{P}^1$ of $\mathbb{P}^1$ gives rise to an open cover of W: Let $\mu' : W' \to \mathbb{A}^1 \times \mathbb{A}^1$ be the blow-up at $(0,0)$, then we have the lifting of (j_0, j_∞) to the open cover

$$(j_0', j_\infty') : W' \amalg W' \to W.$$

The cover (j_0, j_∞) induces the excision isomorphism in $\mathcal{H}_\bullet(k)$

$$(\hat{j}_0, \hat{j}_\infty) : (\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1, \mathbb{G}_m);$$

it is easy to see that $(\hat{j}_0, \hat{j}_\infty)$ is inverse to the isomorphism ϵ in diagram (4.4). Similarly, letting $s' \subset W'$ be the proper transform of $0 \times \mathbb{A}^1$ to W', the cover (j_0', j_∞') induces the excision isomorphism in $\mathcal{H}_\bullet(k)$

$$(\tilde{j}_0', \tilde{j}_\infty') : (W', W' \setminus s') \vee (W', W' \setminus s') \to (W, W \setminus s_{\{0,\infty\}}).$$

This extends to a commutative diagram of isomorphisms in $\mathcal{H}_\bullet(k)$

$$\begin{tikzcd}
(\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m) \arrow[r, "(\hat{j}_0, \hat{j}_\infty)"] \arrow[d, "i_1 \vee i_1"'] & (\mathbb{P}^1, \mathbb{G}_m) \arrow[d, "i_1"] \\
(W', W' \setminus s') \vee (W', W' \setminus s') \arrow[r, "(\tilde{j}_0', \tilde{j}_\infty')"] & (W, W \setminus s_{\{0,\infty\}}) \\
(E', E' \cap C_0') \vee (E', E' \cap C_0') \arrow[r, "(\tilde{j}_{E0}', \tilde{j}_{E\infty}')"] \arrow[u, "i_0 \vee i_0"] & (E, E \cap C_0) \arrow[u, "i_0"'] \\
(\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1) \arrow[r, equal] \arrow[u, "\varphi_{-s} \vee \varphi_{-s}"] & (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1). \arrow[u, "\varphi_g"']
\end{tikzcd} \tag{4.7}$$

Indeed, the commutativity is obvious, except on the bottom square. On the first summand $(\mathbb{P}^1, 1)$, the commutativity is also obvious, since both φ_{-s} and φ_g are defined on this factor using the generator $-s$ for m_0/m_0^2, and on the second factor, the map $\tilde{j}_\infty$ sends $-s$ to $-s^{-1}$, which gives the desired commutativity. Examining the push-out diagram (4.3), we see that the map

$$(\hat{j}_0, \hat{j}_\infty) : (\mathbb{A}^1, \mathbb{G}_m) \vee (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1, \mathbb{G}_m)$$

is inverse to the map ϵ in diagram (4.4).
Let $W_0 \to \mathbb{P}^1 \times \mathbb{A}^1$ be the blow-up along $(0,0)$, E^0 the exceptional divisor, C_0^0 the proper transform of $\mathbb{P}^1 \times 0$. The inclusion j_0 induces the excision isomorphism in $\mathcal{H}_\bullet(k)$

$$j : (\mathbb{A}^1, \mathbb{G}_m) \to (\mathbb{P}^1, j_\infty(\mathbb{A}^1))$$

and gives us the commutative diagram

$$\begin{tikzcd}
 & (\mathbb{P}^1,1) \vee (\mathbb{P}^1,1) \arrow[d] \\
(\mathbb{A}^1,\mathbb{G}_m) \vee (\mathbb{A}^1,\mathbb{G}_m) \arrow[ur, "\vartheta_0\vee\vartheta_0"] \arrow[r, "(j\vee j)"] \arrow[d, "i_1\vee i_1"'] & (\mathbb{P}^1, j_\infty(\mathbb{A}^1)) \vee (\mathbb{P}^1, j_\infty(\mathbb{A}^1)) \arrow[d, "i_1\vee i_1"'] \\
(W', W'\setminus s') \vee (W', W'\setminus s') \arrow[r, "(\tilde{j}\vee\tilde{j})"] & (W_0, W_0\setminus s_0) \vee (W_0, W_0\setminus s_0) \\
(E', E'\cap C_0') \vee (E', E'\cap C_0') \arrow[u, "i_0\vee i_0"'] \arrow[r, "(\tilde{j}_{E'}\vee\tilde{j}_{E'})"] & (E^0, E^0\cap C_0^0) \vee (E^0, E^0\cap C_0^0) \arrow[u, "i_0\vee i_0"'] \\
(\mathbb{P}^1,1) \vee (\mathbb{P}^1,1) \arrow[u, "\varphi_{-s}\vee\varphi_{-s}"] \arrow[r, equal] & (\mathbb{P}^1,1) \vee (\mathbb{P}^1,1). \arrow[u, "\varphi_{-s}\vee\varphi_{-s}"]
\end{tikzcd}$$

By lemma 3.7 the composition along the right-hand side of this diagram is the identity on $(\mathbb{P}^1,1)\vee(\mathbb{P}^1,1)$, and thus the composition along the left-hand side is $\vartheta_0 \vee \vartheta_0 : (\mathbb{A}^1,\mathbb{G}_m)\vee(\mathbb{A}^1,\mathbb{G}_m) \to (\mathbb{P}^1,1)\vee(\mathbb{P}^1,1)$. Referring to diagram (4.4), as $\epsilon = (\hat{j}_0, \hat{j}_\infty)^{-1}$, it follows from (4.5) that the composition along the right-hand side of (4.7) is the map $(\mathrm{id} \vee \tau^1_{0,\infty}) \circ \delta$. As the right-hand side of (4.7) is the deformation diagram used to define $co\text{-}tr_{\{0,\infty\},g}$, we see that

$$co\text{-}tr_{\{0,\infty\},g} = (\mathrm{id} \vee \tau^1_{0,\infty}) \circ \sigma_{\mathbb{P}^1}.$$

Noting that f and g differ only by the trivialization at ∞, changing s^{-1} to $-s^{-1}$, we thus have

$$co\text{-}tr_{\{0,\infty\},f} = (\mathrm{id} \vee \tau^0_{1,\infty} \circ \mu(-1) \circ \tau^0_{1,\infty}) \circ co\text{-}tr_{\{0,\infty\},g}.$$

By lemma 3.6, we have

$$co\text{-}tr_{\{0,\infty\},f} = (\mathrm{id} \vee \tau^1_{0,\infty}) \circ co\text{-}tr_{\{0,\infty\},g} = \sigma_{\mathbb{P}^1}.$$

□

5. Slice localizations and co-transfer

In general, the co-transfer maps do not have the properties necessary to give a loop-spectrum $\Omega_{\mathbb{P}^1}E$ an action by correspondences. However, if we pass to a certain localization of $\mathcal{SH}_{S^1}(k)$ defined by the slice filtration, the co-transfer maps both extend to arbitrary correspondences and respect the composition

of correspondences. This will lead to the action of correspondences on $s_0\Omega_{\mathbb{P}^1}E$ we wish to construct. In this section, k will be an arbitrary perfect field.
We have the localizing subcategory $\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$, generated (as a localizing subcategory) by objects of the form $\Sigma^n_{\mathbb{P}^1}E$, for $E\in\mathcal{SH}_{S^1}(k)$. We let $\mathcal{SH}_{S^1}(k)/f_n$ denote the localization of $\mathcal{SH}_{S^1}(k)$ with respect to $\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$:

$$\mathcal{SH}_{S^1}(k)/f_n=\mathcal{SH}_{S^1}(k)/\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k).$$

Remark 5.1. Pelaez [17, corollary 3.2.40] has shown that there is a model structure on $\mathbf{Spt}_{S^1}(k)$ with homotopy category equivalent to $\mathcal{SH}_{S^1}(k)/f_n$; in particular, this localization of $\mathcal{SH}_{S^1}(k)$ does exist.

Remark 5.2. In the proofs of some of the next few results we will use the following fact, which relies on our ground field k being perfect: Let $V\subset U$ be a Zariski open subset of some $U\in\mathbf{Sm}/k$. Then we can filter U by open subschemes

$$V=U^{N+1}\subset U^N\subset\ldots\subset U^0=U$$

such that $U^{i+1}=U^i\setminus C_i$, with $C_i\subset U^i$ smooth and having trivial normal bundle in U^i for $i=0,\ldots,N$. Indeed, let $C=U\setminus V$, with reduced scheme structure. As k is perfect, there is a dense open subscheme C_{sm} of C which is smooth over k, and there is a non-empty open subscheme $C_1\subset C_{sm}$ such that the restriction of $\mathcal{I}_C/\mathcal{I}_C^2$ to C_1 is a free sheaf of rank equal to the codimension of C_1 in U. We let $U^1=U\setminus C_1$, and then proceed by noetherian induction.

Lemma 5.3. *Let $V\to U$ be a dense open immersion in $\mathbf{Sm}/k$, $n\ge 1$ an integer. Then the induced map*

$$\Sigma^n_{\mathbb{P}^1}V_+\to\Sigma^n_{\mathbb{P}^1}U_+$$

is an isomorphism in $\mathcal{SH}_{S^1}(k)/f_{n+1}$.

Proof. Filter U by open subschemes

$$V=U^{N+1}\subset U^N\subset\ldots\subset U^0=U$$

as in remark 5.2. Write $U^{i+1}=U^i\setminus C_i$, with C_i having trivial normal bundle in U_i, of rank say r_i, for $i=0,\ldots,N$.
By the Morel-Voevodsky purity theorem [15, theorem 2.23], the cofiber of $U^{i+1}\to U^i$ is isomorphic in $\mathcal{H}_\bullet(k)$ to $\Sigma^{r_i}_{\mathbb{P}^1}C_{i+}$, and thus the cofiber of $\Sigma^n_{\mathbb{P}^1}U^{i+1}_+\to\Sigma^n_{\mathbb{P}^1}U^i_+$ is isomorphic to $\Sigma^{r_i+n}_{\mathbb{P}^1}C_{i+}$. Since V is dense in U, we have $r_i\ge 1$ for all i, proving the lemma. □

Take $W\in\mathbf{Sm}/k$. By excision and homotopy invariance, we have a canonical isomorphism

$$\psi_{W,r}:\mathbb{A}^r_W/\mathbb{A}^r_W\setminus 0_W\to\Sigma^r_{\mathbb{P}^1}W_+$$

in $\mathcal{H}_\bullet(k)$. The action of the group-scheme GL_r/k on $\mathbb{A}^r$ gives an action of the group of sections $\mathrm{GL}_r(W)$ on $\mathbb{A}^r_W/\mathbb{A}^r_W\setminus 0_W$, giving us for each $g\in\mathrm{GL}_r(W)$ the isomorphism

$$\psi^g_{W,r}:=:\psi_{W,r}\circ g:\mathbb{A}^r_W/\mathbb{A}^r_W\setminus 0_W\to\Sigma^r_{\mathbb{P}^1}W_+.$$

Lemma 5.4. *For each $g\in\mathrm{GL}_r(W)$, we have $\psi^g_{W,r}=\psi_{W,r}$ in $\mathcal{SH}_{S^1}/f_{r+1}$.*

Proof. The action $\mathrm{GL}_r \times \mathbb{A}^r \to \mathbb{A}^r$ composed with $\psi_{W,r}$ gives us the morphism in $\mathcal{H}_\bullet(k)$

$$\Psi_W : (W \times \mathrm{GL}_r)_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to \Sigma^r_{\mathbb{P}^1} W_+;$$

for each section $g \in \mathrm{GL}_r(W)$, composing with the corresponding section $s_g : W \to W \times \mathrm{GL}_r$ gives the map

$$\Psi_W \circ s_g : W_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to \Sigma^r_{\mathbb{P}^1} W_+$$

which is clearly equal to $\psi^g_{W,r}$.

The open immersion $j : W \times \mathrm{GL}_r \to W \times \mathbb{A}^{r^2}$ is by lemma 5.3 an isomorphism in $\mathcal{SH}_{S^1}(k)/f_1$; as $(\mathbb{A}^r/\mathbb{A}^r \setminus 0) \cong \Sigma^r_{\mathbb{P}^1} \operatorname{Spec} k_+$, we see that the induced map

$$j \wedge \mathrm{id} : (W \times \mathrm{GL}_r)_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to (W \times \mathbb{A}^{r^2})_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0)$$

is an isomorphism in $\mathcal{SH}_{S^1}(k)/f_{r+1}$, and thus the projection

$$(W \times \mathrm{GL}_r)_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to W_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0)$$

is also an isomorphism in $\mathcal{SH}_{S^1}(k)/f_{r+1}$. From this it follows that the maps

$$s_g \wedge \mathrm{id}, s_{\mathrm{id}} \wedge \mathrm{id} : W_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0) \to (W \times \mathrm{GL}_r)_+ \wedge (\mathbb{A}^r/\mathbb{A}^r \setminus 0)$$

are equal in $\mathcal{SH}_{S^1}(k)/f_{r+1}$, hence $\psi^g_{W,r} = \psi_{W,r}$ in $\mathcal{SH}_{S^1}/f_{r+1}$. □

As application we have the following result

Proposition 5.5. *1. Let S be in $\mathbf{Sm}/k$. Let $x \subset \mathbb{P}^1_S \setminus 1_S$ be a closed subscheme, smooth over k and finite over S, such that the co-normal bundle m_x/m_x^2 is trivial. Then the maps*

$$\text{co-}tr_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1)$$

in $\mathcal{SH}_{S^1}(k)/f_2$ are independent of the choice of generator f for m_x/m_x^2. If $S = \operatorname{Spec} \mathcal{O}_{X,x}$ for x a finite set of points on some $X \in \mathbf{Sm}/k$, the analogous independence holds, this time as morphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$.

2. Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be a k-automorphism, with $g(1) = 1$. Then $g : (\mathbb{P}^1, 1) \to (\mathbb{P}^1, 1)$ is the identity in $\mathcal{SH}_{S^1}(k)/f_2$.

3. Take $a, b \in \mathbb{P}^1(k)$, with $a \neq b$ and $a, b \neq 1$. The canonical isomorphism $a \amalg b \to \operatorname{Spec} k \amalg \operatorname{Spec} k$ gives the canonical identification $(\mathbb{P}^1_{a,b}, 1) \cong (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1)$. Then for each choice of generator f for $m_{a,b}/m^2_{a,b}$, the map

$$\text{co-}tr_{a,b,f} : (\mathbb{P}^1, 1) \to (\mathbb{P}^1_{a,b}, 1) \cong (\mathbb{P}^1, 1) \vee (\mathbb{P}^1, 1)$$

is equal in $\mathcal{SH}_{S^1}(k)/f_2$ to the co-multiplication $\sigma_{\mathbb{P}^1}$.

Proof. (1) Suppose that we have generators f, f' for m_x/m_x^2. There is thus a unit $a \in \mathcal{O}^*_x$ with $f' = af$. Note that $\text{co-}tr_{x,f'} = g \circ \text{co-}tr_{x,f}$, where $g : \mathbb{P}^1_x \to \mathbb{P}^1_x$ is the automorphism $\tau^0_{1,\infty}\mu(a)\tau^1_{1,\infty}$. By lemma 5.4, the map

$$\mu(a) = \psi^a_{\operatorname{Spec} k,1} \circ \psi^{-1}_{\operatorname{Spec} k,1} : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$$

is the identity in $\mathcal{SH}_{S^1}(k)/f_2$, whence (1).

For (2), we may replace 1 with ∞. The affine group of isomorphisms $g : \mathbb{P}^1 \to \mathbb{P}^1$ with $g(\infty) = \infty$ is generated by the matrices of the form

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix},$$

with $u \in k^\times$ and $\lambda \in k$. Clearly the automorphisms of the second type act as the identity on $(\mathbb{P}^1, \infty)$ in $\mathcal{H}_\bullet(k)$; the automorphisms of the first type act by the identity on $(\mathbb{P}^1, \infty)$ in $\mathcal{SH}_{S^1}(k)/f_2$ by lemma 5.4.
(3). Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be the automorphism sending $(0, 1, \infty)$ to $(a, 1, b)$. Choose a generator f for $m_{a,b}/m_{a,b}^2$, then g^*f gives a generator for $m_{0,\infty}/m_{0,\infty}^2$. The automorphism g extends to an isomorphism $\tilde{g} : W_{0,\infty} \to W_{a,b}$, giving us a commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1, 1) \arrow[r, "i_1"] \arrow[d, "g"'] & (W_{0,\infty}, W_{0,\infty} \setminus s_{0,\infty}) \arrow[d, "\tilde{g}"'] & (\mathbb{P}^1_{0,\infty}, 1) \arrow[l, "i_{0,g^*f}"'] \arrow[d, "\beta"] \\
(\mathbb{P}^1, 1) \arrow[r, "i_1"'] & (W_{a,b}, W_{a,b} \setminus s_{a,b}) & (\mathbb{P}^1_{a,b}, 1) \arrow[l, "i_{0,f}"]
\end{tikzcd}$$

where $\beta : \mathbb{P}^1_{0,\infty} \to \mathbb{P}^1_{a,b}$ is canonical isomorphism over $(0, \infty) \to (a, b)$. This gives us the identity in $\mathcal{H}_\bullet(k)$:

$$\textit{co-tr}_{a,b,f} \circ g = \beta \circ \textit{co-tr}_{0,\infty,g^*f}.$$

By (1), the maps $\textit{co-tr}_{a,b,g^*f}$ and $\textit{co-tr}_{0,\infty,f}$ are independent (in $\mathcal{SH}_{S^1}(k)/f_2$) of the choice of f and by (2), g is the identity in $\mathcal{SH}_{S^1}(k)/f_2$. For suitable f, lemma 4.1 tells us $\textit{co-tr}_{0,\infty,f} = \sigma_{\mathbb{P}^1}$, completing the proof of (3). $\square$

As the map

$$\textit{co-tr}_{x,f} : (\mathbb{P}^1_S, 1) \to (\mathbb{P}^1_x, 1)$$

in $\mathcal{SH}_{S^1}(k)/f_2$ is independent of the choice of generator $f \in m_x/m_x^2$; we denote this map by $\textit{co-tr}_x$.
We have one additional application of lemma 5.4.

Lemma 5.6. *Let $W \subset U$ be a codimension $\geq r$ closed subscheme of $U \in \mathbf{Sm}/k$, let $w_1, \ldots, w_m$ be the generic points of W of codimension $= r$ in U. Then there is a canonical isomorphism of pro-objects in in $\mathcal{SH}_{S^1}/f_{r+1}$*

$$(U, U \setminus W) \cong \oplus_{i=1}^m \Sigma^r_{\mathbb{P}^1} w_{i+}.$$

Specifically, letting $m_i \subset \mathcal{O}_{U,w_i}$ be the maximal ideal, this isomorphism is independent of any choice of isomorphism $m_i/m_i^2 \cong k(w_i)^r$.

Proof. Let $w = \{w_1, \ldots .w_m\}$ and let $\mathcal{O}_{U,w}$ denote the semi-local ring of w in U. Consider the projective system $\mathcal{V} := \{V_\alpha\}$ consisting of open subschemes of U of the form $V_\alpha = U \setminus C_\alpha$, where C_α is a closed subset of W containing no generic point w_i of W.

Take $V_\alpha \in \mathcal{V}$. By applying remark 5.2, and noting that $U \setminus V_\alpha$ has codimension $\geq r+1$ in U, the argument used in the proof of lemma 5.3 shows that the cofiber of

$$(V_\alpha, V_\alpha \setminus W) \to (U, U \setminus W)$$

is in $\Sigma^{r+1}_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$. On the other hand, the collection of $V_\alpha \in \mathcal{V}$ such that $V_\alpha \cap W$ is smooth and has on each connected component a trivial normal bundle in V_α forms a cofinal subsystem $\mathcal{V}'$ in $\mathcal{V}$. For each $V_\alpha \in \mathcal{V}'$, we have $V_\alpha \cap W = \amalg_{i=1}^m W_i^\alpha$, with w_i the unique generic point of W_i^α, and we have the isomorphism

$$(V_\alpha, V_\alpha \setminus W) \cong \bigvee_{i=1}^m \Sigma^r_{\mathbb{P}^1} W^\alpha_{i+}$$

in $\mathcal{H}_\bullet(k)$. Since w_i is equal to the projective limit of the W_i^α, we have the desired isomorphism of pro-objects in $\mathcal{SH}_{S^1}(k)/f_{r+1}$.

We need only verify that the resulting isomorphism $(U, U \setminus W) \cong \oplus_{r=1}^m \Sigma^r_{\mathbb{P}^1} w_{i+}$ is independent of any choices. Let $V = \operatorname{Spec} \mathcal{O}_{U,W}$, and let $\mathcal{O}$ denote the henselization of w in V. We have the canonical excision isomorphism (of pro-objects in $\mathcal{H}_\bullet(k)$)

$$(V, V \setminus V \cap W) \cong (\operatorname{Spec} \mathcal{O}, \operatorname{Spec} \mathcal{O} \setminus w).$$

A choice of isomorphism $m_w/m_w^2 \cong k(w)^r$ gives the isomorphism in pro-$\mathcal{H}_\bullet(k)$

$$\Sigma^r_{\mathbb{P}^1} w_+ \cong (\operatorname{Spec} \mathcal{O}, \operatorname{Spec} \mathcal{O} \setminus w);$$

this choice of isomorphism is thus the only choice involved in constructing our isomorphism $(U, U \setminus W) \cong \oplus_{i=1}^m \Sigma^r_{\mathbb{P}^1} w_{i+}$. Explicitly, the choice of isomorphism $m_w/m_w^2 \cong k(w)^r$ is reflected in the isomorphism $(\operatorname{Spec} \mathcal{O}, \operatorname{Spec} \mathcal{O} \setminus w) \cong \Sigma^r_{\mathbb{P}^1} w_+$ through the identification of the exceptional divisor of the blow-up of $V \times \mathbb{A}^1$ along $w \times 0$ with $\mathbb{P}^r_w$. The desired independence now follows from lemma 5.4. □

The computation which is crucial for enabling us to introduce transfers on the higher slices of S^1-spectra is the following:

Lemma 5.7. *Let $\mu_n : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$ be the map $\mu_n(t_0 : t_1) = (t_0^n : t_1^n)$. Assume the characteristic of k is prime to $n!$. Then in $\mathcal{SH}_{S^1}(k)/f_2$, μ_n is multiplication by n.*

Proof. The proof goes by induction on n, starting with $n = 1, 2$. The case $n = 1$ is trivial. For $n = 2$, lemma 5.6 gives us the canonical isomorphisms in $\mathcal{SH}_{S^1}(k)/f_2$

$$(\mathbb{P}^1, \mathbb{P}^1 \setminus \{\pm 1\}) \xrightarrow{\alpha_{\pm 1}} (\mathbb{P}^1, \infty) \vee (\mathbb{P}^1, \infty); \quad (\mathbb{P}^1, \mathbb{P}^1 \setminus \{1\}) \xrightarrow{\alpha_1} (\mathbb{P}^1, \infty).$$

In addition, we have the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1, \infty) \arrow[r] \arrow[d, "\mu_2"'] & (\mathbb{P}^1, \mathbb{P}^1 \setminus \{\pm 1\}) \arrow[d, "\mu_2"] \\
(\mathbb{P}^1, \infty) \arrow[r] & (\mathbb{P}^1, \mathbb{P}^1 \setminus \{1\})
\end{tikzcd}$$

The bottom horizontal arrow is an isomorphism in $\mathcal{H}_\bullet(k)$. We claim the diagram

$$\begin{tikzcd}
(\mathbb{P}^1, \mathbb{P}^1 \setminus \{\pm 1\}) \arrow[r, "\alpha_{\pm 1}"] \arrow[d, "\mu_2"] & (\mathbb{P}^1, \infty) \vee (\mathbb{P}^1, \infty) \arrow[d, "(\mathrm{id},\mathrm{id})"] \\
(\mathbb{P}^1, \mathbb{P}^1 \setminus \{1\}) \arrow[r, "\alpha_1"'] & (\mathbb{P}^1, \infty)
\end{tikzcd}$$

commutes in $\mathcal{SH}_{S^1}(k)/f_2$. Indeed, the isomorphism $\alpha_{\pm 1}$ arises from the Morel-Voevodsky homotopy purity isomorphism identifying $(\mathbb{P}^1, \mathbb{P}^1 \setminus \{\pm 1\})$ canonically with the Thom space of the tangent space $T(\mathbb{P}^1)_{\pm 1}$ of $\mathbb{P}^1$ at ± 1, followed by the isomorphism

$$Th(T(\mathbb{P}^1)_{\pm 1}) \cong \Sigma_{\mathbb{P}^1}(\pm 1_+) = (\mathbb{P}^1, \infty) \vee (\mathbb{P}^1, \infty)$$

induced by a choice of basis for $T(\mathbb{P}^1)_{\pm 1}$ (which plays no role in $\mathcal{SH}_{S^1}(k)/f_2$). Similarly the map α_1 arises from a canonical isomorphism of $(\mathbb{P}^1, \mathbb{P}^1 \setminus \{1\})$ with $Th(T(\mathbb{P}^1)_1)$ followed by the isomorphism

$$Th(T(\mathbb{P}^1)_1) \to (\mathbb{P}^1, \infty)$$

induced by a choice of basis. As the map μ_2 is étale over 1, the differential

$$d\mu_2 : T(\mathbb{P}^1)_{\pm 1} \to T(\mathbb{P}^1)_1$$

is isomorphic to the sum map

$$\mathbb{A}^1 \oplus \mathbb{A}^1 \to \mathbb{A}^1.$$

As this sum map induces $(\mathrm{id}, \mathrm{id})$ on the Thom spaces, we have verified our claim.

Using proposition 5.5 and we see that this diagram together with the isomorphisms $\alpha_{\pm 1}$ and α_1 gives us the factorization of μ_2 (in $\mathcal{SH}_{S^1}(k)/f_2$) as

$$(\mathbb{P}^1, \infty) \xrightarrow{\sigma} (\mathbb{P}^1, \infty) \vee (\mathbb{P}^1, \infty) \xrightarrow{(\mathrm{id},\mathrm{id})} (\mathbb{P}^1, \infty).$$

Here σ is the co-multiplication (using ∞ instead of 1 as base-point). Since $(\mathrm{id}, \mathrm{id}) \circ \sigma$ is multiplication by 2, this takes care of the case $n = 2$.

In general, we consider the map $\rho_n : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$ sending $(t_0 : t_1)$ to $(w_0 : w_1) := (t_0^n : t_1^n - t_0 t_1^{n-1} + t_0^n)$. We may form the family of morphisms

$$\rho_n(s) : (\mathbb{P}^1 \times \mathbb{A}^1, \infty \times \mathbb{A}^1) \to (\mathbb{P}^1 \times \mathbb{A}^1, \infty \times \mathbb{A}^1)$$

sending $(t_0 : t_1, s)$ to $(t_0^n : t_1^n - s t_0 t_1^{n-1} + s t_0^n)$. By homotopy invariance, we have $\rho_n(0) = \rho_n(1)$, and thus $\rho_n = \mu_n$ in $\mathcal{H}_\bullet(k)$.

As above, we localize around $w := w_1/w_0 = 1$. Note that $\rho_n^{-1}(1) = \{0, 1\}$. We replace the target $\mathbb{P}^1$ with the henselization $\mathcal{O}$ at $w = 1$, and see that $\mathbb{P}^1 \times_{\rho_n} \mathcal{O}$ breaks up into two components via the factorization $w - 1 = t(t^{n-1} - 1)$, $t = t_1/t_0$. On the component containing 1, the map ρ_n is isomorphic to a hensel local version of μ_{n-1}, and on the component containing 0, the map ρ_n is isomorphic to the identity.

Using Nisnevich excision and proposition 5.5(1), we thus have the following commutative diagram (in $\mathcal{SH}_{S^1}(k)/f_2$)

$$\begin{tikzcd}
(\mathbb{P}^1,\infty) \arrow[r] \arrow[d, "\rho_n"'] & (\mathbb{P}^1,\mathbb{P}^1\setminus\{0,1\}) \arrow[r, "\sim"] \arrow[d, "\rho_n"'] & (\mathbb{P}^1,\infty)\vee(\mathbb{P}^1,\infty) \arrow[d, "\mu_{n-1}\vee\mathrm{id}"] \\
(\mathbb{P}^1,\infty) \arrow[r] & (\mathbb{P}^1,\mathbb{P}^1\setminus\{1\}) \arrow[r, "\sim"'] & (\mathbb{P}^1,\infty)
\end{tikzcd}$$

By proposition 5.5(2), the upper row is the co-multiplication (in $\mathcal{SH}_{S^1}(k)/f_2$), and thus

$$\rho_n = \mu_{n-1} + \mathrm{id}$$

in $\mathcal{SH}_{S^1}(k)/f_2$. As $\rho_n = \mu_n$ in $\mathcal{H}_\bullet(k)$, our induction hypothesis gives $\mu_n = n\cdot\mathrm{id}$, and the induction goes through. □

While we are on the subject, we might as well note that

Remark 5.8. The co-group $((\mathbb{P}^1,1),\sigma_{\mathbb{P}^1})$ in $\mathcal{SH}_{S^1}(k)/f_2$ is co-commutative.

As pointed out by the referee, every object in $\mathcal{SH}_{S^1}(k)/f_2$ is a co-commutative co-group, since $\mathcal{SH}_{S^1}(k)/f_2$ is a triangulated category and hence each object is a double suspension. In addition, the co-group structure $((\mathbb{P}^1,1),\sigma_{\mathbb{P}^1})$ is isomorphic in $\mathcal{H}_\bullet(k)$ to the co-group structure on $S^1\wedge\mathbb{G}_m$ induced by the co-group structure on S^1, so the "triangulated" co-group structure on $\mathbb{P}^1$ agrees with the one we have given.
One should, however, be able to reproduce our entire theory "modulo $\Sigma^2_{\mathbb{P}^1}$" in the unstable category. We have not done this here, as we do not at present have available a theory of the motivic Postnikov tower in the $\mathcal{H}_\bullet(k)$. We expect that, given such a theory, the results of this section would hold in the unstable setting and in particular, that the co-group $((\mathbb{P}^1,1),\sigma_{\mathbb{P}^1})$ would be co-commutative "modulo $\Sigma^2_{\mathbb{P}^1}$".

We now return to our study of properties of the co-transfer map in $\mathcal{SH}_{S^1}(k)/f_2$. We will find it convenient to work in the setting of smooth schemes essentially of finite type over k; as mentioned at the beginning of §3, we consider schemes Y essentially of finite type over k as pro-objects in $\mathcal{H}(k)$, $\mathcal{SH}_{S^1}(k)$, etc. In the end, we use scheme essentially of finite type over k only as a tool to construct maps in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$ between objects of $\mathcal{SH}_{S^1}(k)/f_{n+1}$; this will in the end give us morphisms in $\mathcal{SH}_{S^1}(k)/f_{n+1}$, as the functor $\mathcal{SH}_{S^1}(k)/f_{n+1}\to$ pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$ is fully faithful,
Suppose we have a semi-local smooth k-algebra A, essentially of finite type, and a finite extension $A\to B$, with B smooth over k. Suppose further that B is generated as an A-algebra by a single element $x\in B$:

$$B = A[x].$$

We say in this case that B is a *simply generated* A-algebra.
Let $\tilde f\in A[T]$ be the monic minimal polynomial of x, giving us the point x' of $\mathbb{A}^1_A = \operatorname{Spec} A[T]$ with ideal $(\tilde f)$. We identify $\mathbb{A}^1_A$ with $\mathbb{P}^1_A\setminus\{1\}$ as usual, giving

us the subscheme x of $\mathbb{P}^1_A \setminus \{1\}$, smooth over k and finite over $\operatorname{Spec} A$, in fact, canonically isomorphic to $\operatorname{Spec} B$ over $\operatorname{Spec} A$ via the choice of generator x. Let

$$\varphi_x : x \to \operatorname{Spec} B$$

be this isomorphism. We let f be the generator of m_x/m_x^2 determined by $\tilde{f}$. Via the composition

$$(\mathbb{P}^1_A, 1) \xrightarrow{co\text{-}tr_{x,f}} (\mathbb{P}^1_x, 1) \xrightarrow{\varphi_x \times \mathrm{id}} (\mathbb{P}^1_B, 1)$$

we have the morphism

$$co\text{-}tr_x : (\mathbb{P}^1_A, 1) \to (\mathbb{P}^1_B, 1)$$

in pro-$\mathcal{H}_\bullet(k)$.

Lemma 5.9. *Suppose that* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is étale over each generic point of* $\operatorname{Spec} A$*. Then the map* $co\text{-}tr_x : (\mathbb{P}^1_A, 1) \to (\mathbb{P}^1_B, 1)$ *in pro-*$\mathcal{SH}_{S^1}(k)/f_2$ *is independent of the choice of generator* x *for* B *over* A*.*

Via this result, we may write $co\text{-}tr_{B/A}$ for $co\text{-}tr_x$.

Proof. We use a deformation argument; we first localize to reduce to the case of an étale extension $A \to B$. For this, let $a \in A$ be a non-zero divisor, and let x be a generator for B as an A-algebra. Then x is a generator for $B[a^{-1}]$ as an $A[a^{-1}]$-algebra and by lemma 3.8 we have the commutative diagram

$$\begin{tikzcd} \mathbb{P}^1_{A[a^{-1}]} \arrow[r] \arrow[d, "co\text{-}tr_x"'] & \mathbb{P}^1_A \arrow[d, "co\text{-}tr_x"] \\ \mathbb{P}^1_{B[a^{-1}]} \arrow[r] & \mathbb{P}^1_B, \end{tikzcd}$$

with horizontal arrows isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$. Thus, we may assume that $A \to B$ is étale.

Suppose we have generators $x \neq x'$ for B over A; let $d = [B : A]$. Let s be an indeterminate, let $x(s) = sx + (1-s)x' \in B[s]$, and consider the extension $\tilde{B}_s := A[s][x(s)]$ of $A[s]$, considered as a subalgebra of $B[s]$. Clearly $\tilde{B}_s$ is finite over $A[s]$.

Let $m_A \subset A$ be the Jacobson radical, and let $A(s)$ be the localization of $A[s]$ at the ideal $(m_A A[s] + s(s-1))$. In other words, $A(s)$ is the semi-local ring of the set of closed points $\{(0,a),(1,a)\}$ in $\mathbb{A}^1 \times \operatorname{Spec} A$, as a runs over the closed points of $\operatorname{Spec} A$. Define $B(s) := B \otimes_A A(s)$ and $B_s := \tilde{B}_s \otimes_A A(s) \subset B(s)$. Let $y = (1,a)$ be a closed point of $A(s)$, with maximal ideal m_y, and let x_y be the image of x in $B(s)/m_y B(s)$. Clearly x_y is in the image of $B_s \to B(s)/m_y B(s)$, hence $B_s \to B(s)/m_y B(s)$ is surjective. Similarly, $B_s \to B(s)/m_y B(s)$ is surjective for all y of the form $(0,a)$; by Nakayama's lemma $B_s = B(s)$. Also, $B(s)$ and $A(s)$ are regular and $B(s)$ is finite over $A(s)$, hence $B(s)$ is flat over $A(s)$ and thus $B(s)$ is a free $A(s)$-module of rank d. Finally, $B(s)$ is clearly unramified over $A(s)$, hence $A(s) \to B(s)$ is étale.

Using Nakayama's lemma again, we see that $B(s)$ is generated as an $A(s)$ module by $1, x(s), x(s)^2, \ldots, x(s)^{d-1}$. It follows that $x(s)$ satisfies a monic polynomial equation of degree d over $A(s)$, thus $x(s)$ admits a monic minimal polynomial f_s of degree d over $A(s)$. Sending T to $x(s)$ defines an isomorphism

$$\varphi_s : A(s)[T]/(f_s) \to B(s).$$

We let $x_s \subset \mathbb{A}^1_{A(s)} = \mathbb{P}^1_{A(s)} \setminus \{1\}$ be the closed subscheme of $\mathbb{P}^1_{A(s)}$ corresponding to f_s; the isomorphism φ_s gives us the isomorphism

$$\varphi_s : x_s \to \operatorname{Spec} B(s).$$

Thus, we may define the map

$$\textit{co-tr}_{x(s)} : (\mathbb{P}^1_{A(s)}, 1) \to (\mathbb{P}^1_{B(s)}, 1)$$

giving us the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^1_A, 1) \arrow[r, "i_0"] \arrow[d, "\textit{co-tr}_{x'}"'] & (\mathbb{P}^1_{A(s)}, 1) \arrow[d, "\textit{co-tr}_{x(s)}"'] & (\mathbb{P}^1_A, 1) \arrow[l, "i_1"'] \arrow[d, "\textit{co-tr}_x"] \\
(\mathbb{P}^1_B, 1) \arrow[r, "i_0"'] & (\mathbb{P}^1_{B(s)}, 1) & (\mathbb{P}^1_B, 1) \arrow[l, "i_1"]
\end{tikzcd}$$

By lemma 5.3, the map $(\mathbb{P}^1_{A(s)}, 1) \to (\mathbb{P}^1_{A[s]}, 1)$ is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_2$. By homotopy invariance, it follows that the maps i_0, i_1 are isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$, inverse to the map $(\mathbb{P}^1_{A(s)}, 1) \to (\mathbb{P}^1_A, 1)$ induced by the projection $\operatorname{Spec} A(s) \to \operatorname{Spec} A$. Therefore $\textit{co-tr}_{x'} = \textit{co-tr}_x$, as desired. $\square$

Lemma 5.10. $\textit{co-tr}_{A/A} = \mathrm{id}_{(\mathbb{P}^1_A, 1)}$.

Proof. We may choose 0 as the generator for A over A, which gives us the point $x = 0 \in \mathbb{P}^1_A$. The result now follows from lemma 3.7. $\square$

Lemma 5.11. *Let $A \to C$ be a finite simply generated extension and $A \subset B \subset C$ a sub-extension, with B also simply generated over A. We suppose that A, B and C are smooth over k, that $A \to B$ and $A \to C$ are étale over each generic point of* $\operatorname{Spec} A$, *and $B \to C$ is étale over each generic point of* $\operatorname{Spec} B$. *Then*

$$\textit{co-tr}_{C/A} = \textit{co-tr}_{C/B} \circ \textit{co-tr}_{B/A}.$$

Proof. This is another deformation argument. As in the proof of lemma 5.9, we may assume that $A \to B$, $B \to C$ and $A \to C$ are étale extensions; we retain the notation from the proof of lemma 5.9. Let y be a generator for C over A, x a generator for B over A. These generators give us corresponding closed subschemes $y, x \subset \mathbb{P}^1_A$ and $y_B \subset \mathbb{P}^1_B$. Let $y(s) = sy + (1-s)x$, giving $y(s) \subset \mathbb{P}^1_{A(s)}$. Note that $y(1) = y$, $y(0)_{\mathrm{red}} = x$
As in the proof of lemma 5.9, the element $y(s)$ of $C(s)$ is a generator over $A(s)$ after localizing at the points of $\operatorname{Spec} A(s)$ lying over $s = 1$. The subscheme $y(s)$ in a neighborhood of $s = 0$ is not in general regular, hence $y(s)$ is not a generator of $C(s)$ over $A(s)$. However, let $\mu : W := W_x \to \mathbb{P}^1 \times \mathbb{A}^1$ be the blow-up along

$\{(x,0)\}$, and let $\tilde{y} \subset W_{A(s)}$ be the proper transform $\mu^{-1}[y]$. An elementary local computation shows that this blow-up resolves the singularities of $y(s)$, and that $\tilde{y}$ is étale over $A(s)$; the argument used in the proof of lemma 5.9 goes through to show that $A(s)(\tilde{y}) \cong C(s)$. In addition, let C_0 be the proper transform to $W_{A(s)}$ of $\mathbb{P}^1 \times 0$ and E the exceptional divisor, then $\tilde{y}(0)$ is disjoint from C_0. Finally, after identifying E with $\mathbb{P}^1_{A[x]} = \mathbb{P}^1_B$ (using the monic minimal polynomial of x as a generator for m_x), we may consider $\tilde{y}(0)$ as a closed subscheme of $\mathbb{P}^1_B$; the isomorphism $A(s)(\tilde{y}) \cong C(s)$ leads us to conclude that $A(\tilde{y}(1)) = B(\tilde{y}(0)) = C$. By lemma 5.9, we may use $\tilde{y}(0)$ to define $\textit{co-tr}_{C/B}$. The map $\textit{co-tr}_{C/A}$ in pro-$\mathcal{SH}_{S^1}(k)/f_2$ is defined via the diagram

$$(\mathbb{P}^1_A, 1) \to (\mathbb{P}^1_A, \mathbb{P}^1_A \setminus y) \cong (\mathbb{P}^1_C, 1)$$

where the various choices involved lead to equal maps. By lemma 5.3, $W_{A(s)} \to W_{A[s]}$ is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_2$; by homotopy invariance, the projection $W_{A(s)} \to W$ is also an isomorphism pro-$\mathcal{SH}_{S^1}(k)/f_2$. The inclusions $i_1 : \mathbb{P}^1_A \to W_{A(s)}$, $i_0 : \mathbb{P}^1_{A[x]} \to W_{A(s)}$ induce isomorphisms (in pro-$\mathcal{SH}_{S^1}(k)/f_2$)

$$(\mathbb{P}^1_A, \mathbb{P}^1_A \setminus y) = (\mathbb{P}^1_A, \mathbb{P}^1_A \setminus y(1)) \cong (W_{A(s)}, W_{A(s)} \setminus \tilde{y}(s)) \cong (\mathbb{P}^1_{A[x]}, \mathbb{P}^1_{A[x]} \setminus \tilde{y}(0)).$$

As in the proof of lemma 5.9, we can use homotopy invariance to see that $\textit{co-tr}_{C/A}$ is also equal to the composition

$$\begin{aligned}(\mathbb{P}^1_A, 1) \to (\mathbb{P}^1_A, \mathbb{P}^1_A \setminus y) &\xrightarrow{i_1} (W_{A(s)}, W_{A(s)} \setminus \tilde{y}(s)) \\ &\xrightarrow{i_0^{-1}} (\mathbb{P}^1_{A[x]}, \mathbb{P}^1_{A[x]} \setminus \tilde{y}(0)) \cong (\mathbb{P}^1_C, 1).\end{aligned}$$

Now let $s_{1A(s)}$ be the transform to $W_{A(s)}$ of the 1-section. By lemma 3.3, the inclusion $i_0 : (\mathbb{P}^1_{A[x]}, 1) \to (W_{A(s)}, C_0 \cup s_{1A(s)})$ is an isomorphism in pro-$\mathcal{H}_\bullet(k)$. The above factorization of $\textit{co-tr}_{C/A}$ shows that $\textit{co-tr}_{C/A}$ is also equal to the composition

$$(\mathbb{P}^1_A, 1) \xrightarrow{i_1} (W_{A(s)}, C_0 \cup s_{1A(s)}) \xrightarrow{i_0^{-1}} (\mathbb{P}^1_{A[x]}, 1) \to (\mathbb{P}^1_{A[x]} \setminus \tilde{y}(0)) \cong (\mathbb{P}^1_C, 1).$$

Using remark 3.5, this latter composition is $\textit{co-tr}_{C/B} \circ (\textit{co-tr}_{B/A})$, as desired. □

Remark 5.12. 1. Suppose we have simply generated finite generically étale extensions $A_1 \to B_1$, $A_2 \to B_2$, with A_i smooth, semi-local and essentially of finite type over k. Then

$$\textit{co-tr}_{B_1 \times B_2/A_1 \times A_2} = \textit{co-tr}_{B_1/A_1} \vee \textit{co-tr}_{B_2/A_2}$$

where we make the evident identification $(\mathbb{P}^1_{B_1 \times B_2}, 1) = (\mathbb{P}^1_{B_1}, 1) \vee (\mathbb{P}^1_{B_2}, 1)$ and similarly for A_1, A_2.

2. Let B_1, B_2 be simply generated finite generically étale A algebras and let $B = B_1 \times B_2$. As a special case of lemma 5.11, we have

$$\textit{co-tr}_{B/A} = (\textit{co-tr}_{B_1/A} \vee \textit{co-tr}_{B_2/A}) \circ \sigma_{\mathbb{P}^1_A}$$

Indeed, we may factor the extension $A \to B$ as $A \xrightarrow{\delta} A \times A \to B_1 \times B_2 = B$. We then use (1) and note that $\sigma_{\mathbb{P}^1_A} = co\text{-}tr_{A\times A/A}$ by lemma 4.1.

Next, we make a local calculation. Let (A, m) be a local ring of essentially finite type and smooth over k. Let $s \in m$ be a parameter, let $B = A[T]/T^n - s$ and let $t \in B$ be the image of T. Set $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/(s)$, $W = \operatorname{Spec} B/(t)$; the extension $A \to B$ induces an isomorphism $\alpha : W \xrightarrow{\sim} Z$. We write $co\text{-}tr_{Y/X}$ for $co\text{-}tr_{B/A}$, etc. This gives us the diagram in pro-$\mathcal{SH}_{S^1}(k)/f_2$

$$\begin{tikzcd} \mathbb{P}^1_Z \arrow[r, "i_Z"] & \mathbb{P}^1_X \arrow[d, "co\text{-}tr_{Y/X}"] \\ \mathbb{P}^1_W \arrow[u, "\alpha"] \arrow[r, "i_W"'] & \mathbb{P}^1_Y. \end{tikzcd}$$

Lemma 5.13. *Suppose that $n!$ is prime to* $\operatorname{char} k$. *In pro-$\mathcal{SH}_{S^1}(k)/f_2$ we have*

$$co\text{-}tr_{Y/X} \circ i_Z \circ \alpha = n \times i_W.$$

Proof. First, suppose we have a Nisnevich neighborhood $f : X' \to X$ of Z in X, giving us the Nisnevich neighborhood $g : Y' := Y \times_X X' \to Y$ of W in Y. As

$$co\text{-}tr_{Y/X} \circ f = g \circ co\text{-}tr_{Y'/X'}$$

we may replace X with X', Y with Y'. Similarly, we reduce to the case of A a hensel DVR, i.e., the henselization of $0 \in \mathbb{A}^1_F$ for some field F, $Z = W = 0$, with s the image in A of the canonical coordinate on $\mathbb{A}^1_F$.

The map $co\text{-}tr_{Y/X}$ is defined by the closed immersion

$$Y \xrightarrow{i_Y} \mathbb{A}^1_X = \mathbb{P}^1_X \setminus 1_X \subset \mathbb{P}^1_X$$

where i_Y is the closed subscheme of $\mathbb{A}^1 = \operatorname{Spec} A[T]$ defined by $T^n - s$, together with the isomorphism

$$(\mathbb{P}^1_X, \mathbb{P}^1_X \setminus Y) \cong \mathbb{P}^1_Y$$

furnished by the blow-up $\mu : W_Y \to \mathbb{A}^1_X \times \mathbb{A}^1$ of $\mathbb{A}^1_X \times \mathbb{A}^1$ along $(Y, 0)$. The composition $co\text{-}tr_{Y/X} \circ i_Z \circ \alpha$ is given by the composition

$$\begin{aligned} (\mathbb{P}^1_W, 1) \cong (\mathbb{P}^1_W, \mathbb{P}^1_W \setminus 0_W) &\xrightarrow{\alpha} (\mathbb{P}^1_Z, \mathbb{P}^1_Z \setminus 0_Z) \\ &\xrightarrow{i_Z} (\mathbb{P}^1_X, \mathbb{P}^1_X \setminus 0_X) \xleftarrow{\mathrm{id}} (\mathbb{P}^1_X, 1) \to (\mathbb{P}^1_X, \mathbb{P}^1_X \setminus Y) \cong (\mathbb{P}^1_Y, 1). \end{aligned}$$

In both cases, the isomorphisms (in pro-$\mathcal{SH}_{S^1}(k)/f_2$) are independent of a choice of trivialization of the various normal bundles. Let $U \to \mathbb{P}^1_X$ be the hensel local neighborhood of 0_Z in $\mathbb{P}^1_X$, $\operatorname{Spec} \mathcal{O}^h_{\mathbb{P}^1_X, 0_Z}$. Let $p : U \to X$ be the map induced by the projection $p_X : \mathbb{P}^1_X \to X$ and let $U_Z = p^{-1}(Z)$, with inclusion $i_Z : U_Z \to U$. We may use excision to rewrite the above description of $co\text{-}tr_{Y/X} \circ i_Z \circ \alpha$ as a composition as

$$(\mathbb{P}^1_W, 1) \cong (\mathbb{P}^1_Z, 1) \cong (U_Z, U_Z \setminus 0_Z) \xrightarrow{i_Z} (U, U \setminus Y) \cong (\mathbb{P}^1_Y, 1).$$

Similarly, letting $i_0 : X \to X \times \mathbb{P}^1$ be the 0-section, the map i_W may be given by the composition

$$(\mathbb{P}^1_W, 1) \cong (X, X \setminus Z) \xrightarrow{i_0} (U, U \setminus Y) \cong (\mathbb{P}^1_Y, 1);$$

again, the isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$ are independent of choice of trivializations of the various normal bundles.

We write (s, t) for the parameters on U induced by the functions s, T on $\mathbb{A}^1_X$. We change coordinates in U by the isomorphism $(s, t) \mapsto (s - t^n, t)$. This transforms Y to the subscheme $s = 0$, is the identity on the 0-section, and transforms $s = 0$ to $t^n + s = 0$. Replacing s with $-s$, we have just switched the roles of Y and U_Z. Let

$$\varphi : U_Z \to U$$

be the map $\varphi(t) = (t^n, t)$. After making our change of coordinates, the map $co\text{-}tr_{Y/X} \circ i_Z \circ \alpha$ is identified with

$$(\mathbb{P}^1_W, 1) \cong (U_Z, U_Z \setminus 0_Z) \xrightarrow{\varphi} (U, U \setminus U_Z) \cong (\mathbb{P}^1_Y, 1)$$

while the description of i_W becomes

$$(\mathbb{P}^1_W, 1) \cong (X, X \setminus Z) \xrightarrow{i_0} (U, U \setminus U_Z) \cong (\mathbb{P}^1_Y, 1);$$

here we are using lemma 5.4 to conclude that the automorphism $(x_0 : x_1) \mapsto (-x_0 : x_1)$ of $\mathbb{P}^1_W$ induces the identity on $(\mathbb{P}^1_W, \infty)$ in pro-$\mathcal{SH}_{S^1}(k)/f_2$.

We now construct an $\mathbb{A}^1$-family of maps $(U_Z, U_Z \setminus 0_Z) \to (U, U \setminus U_Z)$. Let

$$\Phi : U_Z \times \mathbb{A}^1 \to U$$

be the map $\Phi(t, v) = (t^n, vt)$. Note that Φ defines a map of pairs

$$\Phi : (U_Z, U_Z \setminus 0_Z) \times \mathbb{A}^1 \to (U, U \setminus U_Z).$$

Clearly $\Phi(-, 1) = \varphi$ while $\Phi(-, 0)$ factors as

$$U_Z \xrightarrow{\mu_n} U_Z \xrightarrow{\beta} X \xrightarrow{i_0} U$$

where μ_n is the map $t \mapsto t^n$ and β is the isomorphism $\beta(t) = s$. Thus, we can rewrite $co\text{-}tr_{Y/X} \circ i_Z \circ \alpha$ as

$$(\mathbb{P}^1_W, 1) \cong (X, X \setminus Z) \xrightarrow{\mu_n} (X, X \setminus Z) \xrightarrow{i_0} (U, U \setminus U_Z) \cong (\mathbb{P}^1_Y, 1)$$

We identify X with the hensel neighborhood of 0_Z in $\mathbb{P}^1_Z$. Using excision again, we have the commutative diagram in pro-$\mathcal{H}_\bullet(k)$

$$\begin{tikzcd}
(X, X \setminus Z) \arrow[r, "\mu_n"] \arrow[d] & (X, X \setminus Z) \arrow[d] \\
(\mathbb{P}^1_Z, \mathbb{P}^1_Z \setminus 0_Z) \arrow[r, "\mu_n"] & (\mathbb{P}^1_Z, \mathbb{P}^1_Z \setminus 0_Z) \\
(\mathbb{P}^1_Z, \infty) \arrow[r, "\mu_n^Z"] \arrow[u] & (\mathbb{P}^1_Z, \infty) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are all isomorphisms. By lemma 5.7 the bottom map is multiplication by n, which completes the proof. □

Lemma 5.14. *Let $A \to B$ be a finite simple étale extension, A as above. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, let $i_x : x \to X$ be the closed point of X and $i_y : y \to Y$ the inclusion of $y := x \times_X Y$. Then*

$$co\text{-}tr_{Y/X} \circ i_x = i_y \circ co\text{-}tr_{y/x}.$$

Proof. Take an embedding of Y in $\mathbb{A}^1_X = \mathbb{P}^1_X \setminus 1_X \subset \mathbb{P}^1_X$; the fiber of $Y \to \mathbb{A}^1_X$ over $x \to X$ is thus an embedding $y \to \mathbb{A}^1_x = \mathbb{P}^1_x \setminus 1_x \subset \mathbb{P}^1_x$. The result follows easily from the commutativity of the diagram

$$\begin{tikzcd} \mathbb{P}^1_x \setminus y \arrow[r] \arrow[d] & \mathbb{P}^1_x \arrow[d] \\ \mathbb{P}^1_X \setminus Y \arrow[r] & \mathbb{P}^1_X \end{tikzcd}$$

□

Proposition 5.15. *Let $A \to B$ be a finite generically étale extension, with A a DVR and B a semi-local principal ideal ring. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, let $i_x : x \to X$ be the closed point of X and $i_y : y \to Y$ the inclusion of $y := x \times_X Y$. Write $y = \{y_1, \ldots, y_r\}$, with each y_i irreducible. Let n_i denote the ramification index of y_i; suppose that $n_i!$ is prime to* $\operatorname{char} k$ *for each i. Then*

$$co\text{-}tr_{Y/X} \circ i_x = \sum_{i=1}^{r} n_i \cdot i_{y_i} \circ co\text{-}tr_{y_i/x}.$$

Proof. We note that every such extension is simple. By passing to the henselization $A \to A^h$, we may assume A is hensel. By remark 5.12(2), we may assume that $r = 1$. Let $A \to B_0 \subset B$ be the maximal unramified subextension. As $co\text{-}tr_{B/A} = co\text{-}tr_{B/B_0} \circ co\text{-}tr_{B_0/B}$, we reduce to the two cases $A = B_0$, $B = B_0$. We note that a finite separable extension of hensel DVRs $A \to B$ with trivial residue field extension degree and ramification index prime to the characteristic is isomorphic to an extension of the form $t^n = s$ for some $s \in m_A \setminus m_A^2$. Thus, the first case is lemma 5.13, the second is lemma 5.14. □

Consider the functor

$$(\mathbb{P}^1_?, 1) : \mathbf{Sm}/k \to \mathcal{SH}_{S^1}(k)/f_2$$

sending X to $(\mathbb{P}^1_X, 1) \in \mathcal{SH}_{S^1}(k)/f_2$, which we consider as a $\mathcal{SH}_{S^1}(k)/f_2$-valued presheaf on $\mathbf{Sm}/k^{\mathrm{op}}$ (we could also write this functor as $X \mapsto \Sigma^\infty_{\mathbb{P}^1} X_+$). We proceed to extend $(\mathbb{P}^1_?, 1)$ to a presheaf on $SmCor(k)^{\mathrm{op}}$; we will assume that $\operatorname{char} k = 0$, so we do not need to worry about inseparability.
We first define the action on the generators of $\operatorname{Hom}_{SmCor}(X, Y)$, i.e., on irreducible $W \subset X \times Y$ such that $W \to X$ is finite and surjective over some component of X. As $\mathcal{SH}_{S^1}(k)/f_2$ is an additive category, it suffices to consider the case of irreducible X. Let $U \subset X$ be a dense open subscheme. Then the map

$(\mathbb{P}^1_U, 1) \to (\mathbb{P}^1_X, 1)$ induced by the inclusion is an isomorphism in $\mathcal{SH}_{S^1}(k)/f_2$. We may therefore define the morphism

$$(\mathbb{P}^1_?, 1)(W) : (\mathbb{P}^1_X, 1) \to (\mathbb{P}^1_Y, 1)$$

in $\mathcal{SH}_{S^1}(k)/f_2$ as the composition (in pro-$\mathcal{SH}_{S^1}(k)/f_2$)

$$(\mathbb{P}^1_X, 1) \cong (\mathbb{P}^1_{k(X)}, 1) \xrightarrow{co\text{-}tr_{k(W)/k(X)}} (\mathbb{P}^1_{k(W)}, 1) \xrightarrow{p_2} (\mathbb{P}^1_Y, 1).$$

We extend linearly to define $(\mathbb{P}^1_?, 1)$ on $\mathrm{Hom}_{SmCor}(X, Y)$.
Suppose that $\Gamma_f \subset X \times Y$ is the graph of a morphism $f : X \to Y$. It follows from lemma 5.10 that $co\text{-}tr_{k(\Gamma_f)/k(X)}$ is the inverse to the isomorphism $p_1 : (\mathbb{P}^1_{k(\Gamma_f)}, 1) \to (\mathbb{P}^1_{k(X)}, 1)$. Thus, the composition

$$(\mathbb{P}^1_{k(X)}, 1) \xrightarrow{co\text{-}tr_{k(\Gamma_f)/k(X)}} (\mathbb{P}^1_{k(\Gamma_f)}, 1) \xrightarrow{p_2} (\mathbb{P}^1_Y, 1)$$

is the map induced by the restriction of f to $\mathrm{Spec}\, k(X)$. Since $(\mathbb{P}^1_{k(X)}, 1) \to (\mathbb{P}^1_X, 1)$ is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_2$, it follows that $(\mathbb{P}^1_?, 1)(\Gamma_f) = f$, i.e., our definition of $(\mathbb{P}^1_?, 1)$ on $\mathrm{Hom}_{SmCor}(X, Y)$ really is an extension of its definition on $\mathrm{Hom}_{\mathbf{Sm}/k}(X, Y)$.
The main point is to check functoriality.

Lemma 5.16. *Suppose* $\mathrm{char}\, k = 0$. *For* $\alpha \in \mathrm{Hom}_{SmCor}(X, Y)$, $\beta \in \mathrm{Hom}_{SmCor}(Y, Z)$, *we have*

$$(\mathbb{P}^1_?, 1)(\beta \circ \alpha) = (\mathbb{P}^1_?, 1)(\beta) \circ (\mathbb{P}^1_?, 1)(\alpha)$$

Proof. It suffices to consider the case of irreducible finite correspondences $W \subset X \times Y$, $W' \subset Y \times Z$. If W is the graph of a flat morphism, the result follows from lemma 3.8.
As the action of correspondences is defined at the generic point, we may replace X with $\eta := \mathrm{Spec}\, k(X)$. Then W becomes a closed point of Y_η and the correspondence $W_\eta : \eta \to Y$ factors as $p_2 \circ i_{W_\eta} \circ p_1^t$, where $p_1 : W_\eta \to \eta$ and $p_2 : Y_\eta \to Y$ are the projections.
Let $W'_\eta \subset Y_\eta \times Z$ be the pull-back of W'. As we have already established naturality with respect to pull-back by flat maps, we reduce to showing

$$(\mathbb{P}^1_?, 1)(W'_\eta \circ i_{W_\eta}) = (\mathbb{P}^1_?, 1)(W'_\eta) \circ (\mathbb{P}^1_?, 1)(i_{W_\eta}).$$

Since Y is quasi-projective, we can find a sequence of closed subschemes of Y_η

$$W_\eta = W_0 \subset W_1 \subset \ldots \subset W_{d-1} \subset W_d = Y_\eta$$

such that W_i is smooth of codimension $d-i$ on Y_η. Using again the fact the *co-tr* is defined at the generic point, and that we have already proven functoriality with respect to composition of morphisms, we reduce to the case of $Y = \mathrm{Spec}\, \mathcal{O}$ for some DVR $\mathcal{O}$, and i_η the inclusion of the closed point η of Y.
Let $W'' \to W'$ be the normalization of W'. Using functoriality with respect to morphisms in $\mathbf{Sm}/k$ once more, we may replace Z with W'' and W' with the transpose of the graph of the projection $W'' \to Y$. Changing notation, we may assume that W' is the transpose of the graph of a finite morphism $Z \to Y$.

This reduces us to the case considered in proposition 5.15; this latter result completes the proof. □

We will collect the results of this section, generalized to higher loops, in theorem 6.1 of the next section.

6. Higher loops

The results of these last sections carry over immediately to statements about the n-fold smash product $(\mathbb{P}^1,1)^{\wedge n}$ for $n \geq 1$. For clarity and completeness, we list these explicitly in an omnibus theorem.

Let R be a semi-local k-algebra, smooth and essentially of finite type over k, and let $x \subset \mathbb{P}^1_R$ and f be as in section 3. For $n \geq 1$, define

$$co\text{-}tr^n_{x,f} : \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} R_+ \to \Sigma^n_{\mathbb{P}^1} x_+$$

to be the map $\Sigma^{n-1}_{\mathbb{P}^1}(co\text{-}tr_{x,f})$ (in pro-$\mathcal{H}_\bullet(k)$).

Similarly, let A be a semi-local k-algebra, smooth and essentially of finite type over k. Let $B = A[x]$ be a simply generated finite generically étale A-algebra. For $n \geq 1$, define

$$co\text{-}tr^n_x : \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} A_+ \to \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} B_+$$

to be the map $\Sigma^{n-1}_{\mathbb{P}^1}(co\text{-}tr_x)$ (in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$).

Theorem 6.1. *1.* $co\text{-}tr^n_{0,-s} = \mathrm{id}$.

2. Let $R \to R'$ be a flat extension of smooth semi-local k-algebras, essentially of finite type over k. Let x be a smooth closed subscheme of $\mathbb{P}^1_R \setminus \{1\}$, finite and generically étale over R. Let $x' = x \times_R R' \subset \mathbb{P}^1_{R'}$. Let f be a generator for m_x/m_x^2, and let f' be the extension to $m_{x'}/m^2_{x'}$. Suppose that either $R \to R'$ is smooth or that $x \to \operatorname{Spec} R$ is étale. Then the diagram

$$\begin{tikzcd}
\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} R'_+ \arrow[r, "co\text{-}tr^n_{x',f'}"] \arrow[d] & \Sigma^n_{\mathbb{P}^1} x'_+ \arrow[d] \\
\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} R_+ \arrow[r, "co\text{-}tr^n_{x,f}"'] & \Sigma^n_{\mathbb{P}^1} x_+
\end{tikzcd}$$

is well-defined and commutes.

3. The co-group structure $\Sigma^{n-1}_{\mathbb{P}^1}(\sigma_{\mathbb{P}^1})$ on $(\mathbb{P}^1,1)^{\wedge n}$ is given by the map

$$co\text{-}tr^n_{\{0,\infty\},(-s,s^{-1})} : (\mathbb{P}^1,1)^{\wedge n} \to (\mathbb{P}^1,1)^{\wedge n} \vee (\mathbb{P}^1,1)^{\wedge n}.$$

4. The co-group $((\mathbb{P}^1,1)^{\wedge n}, \Sigma^{n-1}_{\mathbb{P}^1}(\sigma_{\mathbb{P}^1}))$ in $\mathcal{SH}_{S^1}(k)/f_{n+1}$ is co-commutative.

5. For an extension $A \to B$ as above, the map $co\text{-}tr^n_x : \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} A_+ \to \Sigma^n_{\mathbb{P}^1} \operatorname{Spec} B_+$ is independent of the choice of x, and is denoted $co\text{-}tr^n_{B/A}$.

6. Suppose that $\operatorname{char} k = 0$*. The* $\mathcal{SH}_{S^1}(k)/f_{n+1}$*-valued presheaf on* $\mathbf{Sm}/k^{\mathrm{op}}$

$$\Sigma^n_{\mathbb{P}^1} ?_+ : \mathbf{Sm}/k \to \mathcal{SH}_{S^1}(k)/f_{n+1}$$

extends to an $\mathcal{SH}_{S^1}(k)/f_{n+1}$*-valued presheaf on* $SmCor(k)^{\mathrm{op}}$*, by sending a generator* $W \subset X \times Y$ *of* $\operatorname{Hom}_{SmCor}(X,Y)$ *to the morphism* $\Sigma^n_{\mathbb{P}^1} X_+ \to \Sigma^n_{\mathbb{P}^1} Y_+$ *in* $\mathcal{SH}_{S^1}(k)/f_{n+1}$ *determined by the diagram*

$$\begin{tikzcd}
\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} k(X)_+ \arrow[r, "\sim"] \arrow[d, "co\text{-}tr^n_{k(W)/k(X)}"'] & \Sigma^n_{\mathbb{P}^1} X_+ \\
\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} k(W)_+ \arrow[d, "p_2"'] & \\
\Sigma^n_{\mathbb{P}^1} Y_+ &
\end{tikzcd}$$

in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$*. The assertion that*

$$\Sigma^n_{\mathbb{P}^1} \operatorname{Spec} k(X)_+ \to \Sigma^n_{\mathbb{P}^1} X_+$$

is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$ *is part of the statement. We write the map in* $\mathcal{SH}_{S^1}(k)/f_{n+1}$ *associated to* $\alpha \in \operatorname{Hom}_{SmCor}(X,Y)$ *as*

$$co\text{-}tr^n(\alpha) : \Sigma^n_{\mathbb{P}^1} X_+ \to \Sigma^n_{\mathbb{P}^1} Y_+.$$

7. Supports and co-transfers

In this section, we assume that $\operatorname{char} k = 0$. We consider the following situation. Let $i : Y \to X$ be a codimension one closed immersion in $\mathbf{Sm}/k$, and let $Z \subset X$ be a pure codimension n closed subset of X such that $i^{-1}(Z) \subset Y$ also has pure codimension n. We let $T = i^{-1}(Z)$, $X^{(Z)} = (X, X \setminus Z)$, $Y^{(T)} = (Y, Y \setminus T)$, so that i induces the map of pointed spaces

$$i : Y^{(T)} \to X^{(Z)}.$$

Let z be the set of generic points of Z, $\mathcal{O}_{X,z}$ the semi-local ring of z in X, $X_z = \operatorname{Spec} \mathcal{O}_{X,z}$ and $X_z^{(z)} = (X_z, X_z \setminus z)$. We let t be the set of generic points of T, and let $\mathcal{O}_{X,t}$ be the semi-local ring of t in X, $X_t = \operatorname{Spec} \mathcal{O}_{X,t}$. Set $Y_t := X_t \times_X Y$ and let $Y_t^{(t)} = (Y_t, Y_t \setminus t)$.

Lemma 7.1. *There are canonical isomorphisms in pro-*$\mathcal{SH}_{S^1}(k)/f_{n+1}$

$$X^{(Z)} \cong X_z^{(z)} \cong \Sigma^n_{\mathbb{P}^1} z_+; \quad Y^{(T)} \cong Y_t^{(t)} \cong \Sigma^n_{\mathbb{P}^1} t_+.$$

Proof. This follows from lemma 5.6. □

Thus, the map $i : Y^{(T)} \to X^{(Z)}$ gives us the map in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$:

$$i : \Sigma^n_{\mathbb{P}^1} t_+ \to \Sigma^n_{\mathbb{P}^1} z_+. \tag{7.1}$$

On the other hand, we can define a map

$$i_{co\text{-}tr} : \Sigma^n_{\mathbb{P}^1} t_+ \to \Sigma^n_{\mathbb{P}^1} z_+ \tag{7.2}$$

as follows: Let $Z_t = Z \cap X_t \subset X_t$. Since Y has codimension one in X and intersects Z properly, t is a collection of codimension one points of Z, and thus Z_t is a semi-local reduced scheme of dimension one. Let $p : \tilde{Z}_t \to Z_t$ be the normalization, and let $\tilde{t} \subset \tilde{Z}_t$ be the set of points lying over $t \subset Z_t$. Write $\tilde{t}$ as a union of closed points, $\tilde{t} = \amalg_j \tilde{t}_j$. For each j, we let n_j denote the multiplicity at $\tilde{t}_j$ of the pull-back Cartier divisor $Y_t \times_{X_t} \tilde{Z}_t$, and let $t_j = p(\tilde{t}_j)$. This gives us the commutative diagram

$$\begin{tikzcd} \amalg_j \tilde{t}_j \arrow[r, equal] & \tilde{t} \arrow[r, "\tilde{i}"] \arrow[d, "p"] & \tilde{Z}_t \arrow[d, "p"] & z \arrow[l, "j"'] \arrow[dl] \\ & t \arrow[r, "i"'] & Z. & \end{tikzcd}$$

Note that j is an isomorphism in pro-$\mathcal{SH}_{S^1}(k)/f_1$. We define $i_{co\text{-}tr}$ to be the composition

$$\Sigma^n_{\mathbb{P}^1} t_+ \xrightarrow{\prod_j n_j co\text{-}tr^n_{\tilde{t}_j/t}} \oplus_j \Sigma^n_{\mathbb{P}^1} \tilde{t}_{j+} = \Sigma^n_{\mathbb{P}^1} \tilde{t}_+ \xrightarrow{\Sigma^n_{\mathbb{P}^1} \tilde{i}} \Sigma^n_{\mathbb{P}^1} \tilde{Z}_+ \xrightarrow{\Sigma^n_{\mathbb{P}^1} j^{-1}} \Sigma^n_{\mathbb{P}^1} z_+$$

in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$.

Lemma 7.2. *The morphisms* (7.1) *and* (7.2)*are equal in pro-*$\mathcal{SH}_{S^1}(k)/f_{n+1}$.

Proof. Using Nisnevich excision, we may replace X with the henselization of X along t; we may also assume that t is a single point. Via a limit argument, we may then replace X with a smooth affine scheme of dimension $n+1$ over $k(t)$; Z is thus a reduced closed subscheme of X of pure dimension one over $k(t)$. We may also assume that Y is the fiber over 0 of a morphism $X \to \mathbb{A}^1_{k(t)}$ for which the restriction to Z is finite.

As we are working in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$, we may replace (X, Z) with (X', Z') if there is a morphism $f : X \to X'$ over $\mathbb{A}^1_{k(t)}$ which makes (X, t) a hensel neighborhood of $(X', f(t))$ and such that the restriction of f to $f_Z : Z \to Z'$ is birational. Using Gabber's presentation lemma [6, lemma 3.1], we may assume that $X = \mathbb{A}^{n+1}_{k(t)}$, that t is the origin 0 and that Y is the coordinate hyperplane $X_{n+1} = 0$. We write F for $k(t)$ and write simply 0 for t.

After a suitable linear change of coordinates in $\mathbb{A}^{n+1}_F$, we may assume that each coordinate projection

$$q : \mathbb{A}^{n+1}_F \to \mathbb{A}^r_F$$
$$q(x_1, \ldots, x_{n+1}) = (x_{i_1}, \ldots, x_{i_r}),$$

$r = 1, \ldots, n$, restricts to a finite morphism on Z, and that $Z \to q(Z)$ is birational if $r \geq 2$.

We now reduce to the case in which Z is contained in the coordinate subspace $X' = \mathbb{A}^2_F$ defined by $X_1 = \ldots = X_{n-1} = 0$. For this, consider the map

$$m : \mathbb{A}^1 \times \mathbb{A}^{n+1}_F \to \mathbb{A}^1 \times \mathbb{A}^{n+1}_F$$
$$m(t, x_1, \ldots, x_{n+1}) = (t, tx_1, \ldots, tx_{n-1}, x_n, x_{n+1})$$

Let $\mathcal{Z} = m(\mathbb{A}^1 \times Z) \subset \mathbb{A}^1 \times \mathbb{A}^{n+1}_F$. By our finiteness assumptions, $\mathcal{Z}$ is a (reduced) closed subscheme of $\mathbb{A}^1 \times \mathbb{A}^{n+1}_F$, and each fiber $\mathcal{Z}_t \subset t \times \mathbb{A}^{n+1}_F$ is birationally isomorphic to $Z \times_F F(t)$. Consider the inclusion map

$$(\mathbb{A}^1 \times Y)^{(\mathbb{A}^1\times 0)} \to (\mathbb{A}^1 \times X)^{(\mathcal{Z})}$$

The maps

$$i_0, i_1 : Y^{(0)} \to (\mathbb{A}^1 \times Y)^{(\mathbb{A}^1\times 0)}$$

are clearly isomorphisms in pro-$\mathcal{H}_\bullet(k)$, and the maps

$$i_1 : X^{(Z)} \to (\mathbb{A}^1 \times X)^{(\mathcal{Z})}$$
$$i_0 : X^{(\mathcal{Z}_0)} \to (\mathbb{A}^1 \times X)^{(\mathcal{Z})}$$

are easily seen to be isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_{n+1}$. Combining this with the commutative diagram

$$\begin{tikzcd}
Y^{(0)} \arrow[r] \arrow[d, "i_1"'] & X^{(Z)} \arrow[d, "i_1"] \\
(\mathbb{A}^1 \times Y)^{(\mathbb{A}^1\times 0)} \arrow[r] & X^{(\mathcal{Z})} \\
Y^{(0)} \arrow[r] \arrow[u, "i_0"] & X^{(\mathcal{Z}_0)} \arrow[u, "i_0"']
\end{tikzcd}$$

shows that we can replace Z with $\mathcal{Z}_0 \subset X'$.
Having done this, we see that the map $Y^{(0)} \to X^{(Z)}$ is just the $n-1$-fold $\mathbb{P}^1$ suspension of the map

$$(Y \cap X')^{(0)} \to (X')^{(Z)}$$

This reduces us to the case $n = 1$.
Since $p_2 : Z \to \mathbb{A}^1_F$ is finite, we may take $X = \mathbb{P}^1 \times \mathbb{A}^1_F$ instead of $\mathbb{A}^1 \times \mathbb{A}^1_F$. Then the map $Y^{(0)} \to X^{(Z)}$ is isomorphic to $(\mathbb{P}^1 \times 0, \infty \times 0) \to X^{(Z)}$. We extend this to the isomorphic map

$$(\mathbb{P}^1 \times \mathbb{A}^1_F, \infty \times \mathbb{A}^1_F) \to X^{(Z)} = (\mathbb{P}^1 \times \mathbb{A}^1_F, \mathbb{P}^1 \times \mathbb{A}^1_F \setminus Z).$$

Let s be the generic point of $\mathbb{A}^1_F$, Z_s the fiber of p_2 over s. Then the inclusions

$$(\mathbb{P}^1 \times 0, \infty \times 0) \xrightarrow{j_0} (\mathbb{P}^1 \times \mathbb{A}^1_F, \infty \times \mathbb{A}^1_F) \xleftarrow{j_s} (\mathbb{P}^1 \times s, \infty \times s)$$
$$(\mathbb{P}^1 \times \mathbb{A}^1_F, \mathbb{P}^1 \times \mathbb{A}^1_F \setminus Z) \xleftarrow{j_s} (\mathbb{P}^1 \times s, \mathbb{P}^1_s \setminus Z_s)$$

are isomorphisms in pro-$\mathcal{SH}_{S^1}(k)/f_2$, and thus the map

$$i_0 : Y^{(0)} \cong (\mathbb{P}^1 \times 0, \infty \times 0) \to X^{(Z)} = (\mathbb{P}^1 \times \mathbb{A}^1_F, \mathbb{P}^1 \times \mathbb{A}^1_F \setminus Z)$$

is isomorphic in pro-$\mathcal{SH}_{S^1}(k)/f_2$ to the collapse map

$$(\mathbb{P}^1 \times s, \infty \times s) \to (\mathbb{P}^1 \times s, \mathbb{P}^1_s \setminus Z_s).$$

Therefore, the map

$$i : \Sigma_{\mathbb{P}^1} 0_+ \to \Sigma_{\mathbb{P}^1} z_+$$

we need to consider is equal to the co-transfer map

$$\textit{co-tr}_{Z_s/s} : \Sigma_{\mathbb{P}^1} s_+ \to \Sigma_{\mathbb{P}^1} z_{s+}$$

composed with the (canonical) isomorphisms

$$\Sigma_{\mathbb{P}^1} 0_+ \xrightarrow{i_0} \Sigma_{\mathbb{P}^1} \mathbb{A}^1_+ \cong \Sigma_{\mathbb{P}^1} s_+, \quad \Sigma_{\mathbb{P}^1} z_{s+} \cong \Sigma_{\mathbb{P}^1} z_+;$$

the latter isomorphism arising by noting that z_s is a generic point of Z over F. The result now follows directly from proposition 5.15. $\square$

Definition 7.3. 1. Take $X, X' \in \mathbf{Sm}/k$, and let $Z \subset X$, $Z' \subset X'$ be pure codimension n closed subsets. Take a generator $A \in \mathrm{Hom}_{SmCor}(X, X')$, $A \subset X \times X'$. Let $q : A^N \to A$ be the normalization of A. Let z be the set of generic points of Z, let a be the set of generic points of $A \cap X \times Z'$ and let $a' = q^{-1}(a)$. Suppose that

(1) $A^N \to X$ is étale on a neighborhood of a'
(2) $p_X(a)$ is contained in Z.

Let $\mathcal{O}_{A^N,a}$ be the semi-local ring of a' in A^N, and let $A^N_{a'} = \mathrm{Spec}\, \mathcal{O}_{A^N,a'}$; define X_z similarly. Define

$$\textit{co-tr}^n(W) : X^{(Z)} \to X'^{(Z')}$$

to be the map in $\mathcal{SH}_{S^1}(k)/f_{n+1}$ given by the following composition:

$$X^{(Z)} \cong X_z^{(z)} \cong \Sigma^n_{\mathbb{P}^1} z_+ \xrightarrow{\textit{co-tr}^n_{a'/z}} \Sigma^n_{\mathbb{P}^1} a'_+ \cong A^{N(a')}_{a'} \xrightarrow{p_{X'}} X'^{(Z')}.$$

2. Let $\mathrm{Hom}_{SmCor}(X, X')_{Z,Z'} \subset \mathrm{Hom}_{SmCor}(X, X')$ be the subgroup generated by A satisfying (a) and (b). We extend the definition of the morphism $\textit{co-tr}^n(A)$ to $\mathrm{Hom}_{SmCor}(X, X')_{Z,Z'}$ by linearity.

Note that we implicitly invoke lemma 7.1 to ensure that the isomorphisms used in the definition of $\textit{co-tr}^n(A)$ exist and are canonical; condition (1) implies in particular that A^N is smooth in a neighborhood of a', so we may use lemma 7.1 for the isomorphism $\Sigma^n_{\mathbb{P}^1} a'_+ \cong A^{N(a')}_{a'}$.

Lemma 7.4. *Take $X, X', X'' \in \mathbf{Sm}/k$, and let $Z \subset X$, $Z' \subset X'$ and $Z'' \subset X''$ be pure codimension n closed subsets. Take $\alpha \in \mathrm{Hom}_{SmCor}(X, X')_{Z,Z'}$, $\alpha' \in \mathrm{Hom}_{SmCor}(X', X'')_{Z',Z''}$. Then*

1. $\alpha' \circ \alpha$ is in $\mathrm{Hom}_{SmCor}(X, X'')_{Z,Z''}$

2. $\textit{co-tr}^n(\alpha') \circ \textit{co-tr}^n(\alpha) = \textit{co-tr}^n(\alpha' \circ \alpha)$.

Proof. For (1), we may assume that α and α' are generators A and A'. We may replace X, X' and X'' with the respective strict henselizations along z, z' and z''. Write $z = \{z_1, \ldots, z_r\}$, $z' = \{z'_1, \ldots, z'_s\}$, $z'' = \{z''_1, \ldots, z''_t\}$. Then the normalizations A^N and A'^N break up as a disjoint union of graphs of morphisms

$$f_{jk} : X_{z_k} \to X'_{z'_j}; \quad g_{ij} : X'_{z'_j} \to X''_{z''_i}$$

and $A' \circ A$ is thus the sum of the graphs of the compositions $g_{ij} \circ f_{jk}$. Therefore, each irreducible component of the normalization of the support of $A' \circ A$ is étale over X. This verifies condition (1) of definition 7.3; the condition (2) is easy and is left to the reader.
(2) follows directly from theorem 6.1(6). □

Proposition 7.5. *Let $i : \Delta_1 \to \Delta$ be a closed immersion of quasi-projective schemes in $\mathbf{Sm}/k$, take $X, X' \in \mathbf{Sm}/k$ and $\alpha \in \mathrm{Hom}_{SmCor}(X, X')$. Let $Z \subset X \times \Delta$, $Z' \subset X' \times \Delta$ be closed codimension n subsets. Suppose that*

1. *$Z_1 := Z \cap X \times \Delta_1$ and $Z'_1 := Z' \cap X' \times \Delta_1$ have codimension n in $X \times \Delta_1$, $X' \times \Delta_1$, respectively.*
2. *$\alpha \times \mathrm{id}_\Delta$ is in $\mathrm{Hom}_{SmCor}(X \times \Delta, X' \times \Delta)_{Z,Z'}$*
3. *$\alpha \times \mathrm{id}_{\Delta_1}$ is in $\mathrm{Hom}_{SmCor}(X \times \Delta_1, X' \times \Delta_1)_{Z_1,Z'_1}$*

Then the diagram in $\mathcal{SH}_{S^1}(k)/f_{n+1}$

$$\begin{array}{ccc} (X \times \Delta_1)^{(Z_1)} & \xrightarrow{co\text{-}tr^n(\alpha\times\mathrm{id})} & (X' \times \Delta_1)^{(Z'_1)} \\ \downarrow{\scriptstyle \mathrm{id}\times i} & & \downarrow{\scriptstyle \mathrm{id}\times i} \\ (X \times \Delta)^{(Z)} & \xrightarrow[co\text{-}tr^n(\alpha\times\mathrm{id})]{} & (X' \times \Delta)^{(Z')} \end{array}$$

commutes.

Proof. Since Δ is by assumption quasi-projective, we may factor $\Delta_1 \to \Delta$ as a sequence of closed codimension 1 immersions

$$\Delta_1 = \Delta^d \to \Delta^{d-1} \to \ldots \to \Delta^1 \to \Delta^0 = \Delta$$

such that each closed immersion $\Delta^i \to \Delta$ satisfies the conditions of the proposition. This reduces us to the case of a codimension one closed immersion.
We may replace $X \times \Delta$, $X' \times \Delta$, etc., with the respective semi-local schemes about the generic points of Z_1 and Z'_1. As Δ_1 has codimension one on Δ, it follows that the normalizations Z^N, Z'^N of Z and Z' are smooth over k. Let $\tilde{i} : \tilde{z} \to Z^N$, $\tilde{i}' : \tilde{z}' \to Z'^N$ be the points of Z^N, Z'^N lying over Z_1, Z'_1, respectively, which we write as a disjoint union of closed points

$$\tilde{z} = \amalg_j \tilde{z}_j; \quad \tilde{z}' = \amalg_j \tilde{z}'_j.$$

By lemma 7.1 and lemma 7.2, we may rewrite the diagram in the statement of the proposition as

$$
\begin{tikzcd}
\Sigma^n_{\mathbb{P}^1} Z_{1+} \arrow[r, "co\text{-}tr^n(\alpha\times\mathrm{id}_{Z_1^N})"] \arrow[d, "\sum_j m_j co\text{-}tr^n_{\tilde z_j/Z_1}"'] & \Sigma^n_{\mathbb{P}^1} Z'_{1+} \arrow[d, "\sum_j m'_j co\text{-}tr^n_{\tilde z'_j/Z'_1}"] \\
\Sigma^n_{\mathbb{P}^1} \tilde z_+ \arrow[d, "\tilde i"'] & \Sigma^n_{\mathbb{P}^1} \tilde z'_+ \arrow[d, "\tilde i'"] \\
\Sigma^n_{\mathbb{P}^1} Z^N \arrow[r, "co\text{-}tr^n(\alpha\times\mathrm{id}_{Z^N})"'] & \Sigma^n_{\mathbb{P}^1} Z'^N
\end{tikzcd}
$$

where $\alpha \times \mathrm{id}_{Z^N}$, $\alpha \times \mathrm{id}_{Z_1}$ denote the correspondences induced by $\alpha \times \mathrm{id}_\Delta$ and $\alpha \times \mathrm{id}_{\Delta_1}$, and the m_j, m'_j are the relevant intersection multiplicities. The commutativity of this diagram follows from theorem 6.1(6). $\square$

8. SLICES OF LOOP SPECTRA

Take $E \in \mathcal{SH}_{S^1}(k)$. Following Voevodsky's remarks in [22], Neeman's version of Brown representability [16] gives us the motivic Postnikov tower

$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to f_0E = E,$$

where $f_nE \to E$ is universal for morphisms from an object of $\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$ to E. The layer s_nE is the *nth slice* of E, and is characterized up to unique isomorphism by the distinguished triangle

$$f_{n+1}E \to f_nE \to s_nE \to \Sigma_s f_{n+1}E. \tag{8.1}$$

The fact that this distinguished triangle determines s_nE up to unique isomorphism rather than just up to isomorphism follows from

$$\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma^{n+1}_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k), s_nE) = 0 \tag{8.2}$$

To see this, just use the universal property of $f_{n+1}E \to E$ and the long exact sequence of Homs associated to the distinguished triangle (8.1). In particular, using the description of $\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(-,-)$ via right fractions we have

LEMMA 8.1. *For all $F, E \in \mathcal{SH}_{S^1}(k)$ and $n \geq 0$, the natural map*

$$\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(F, s_nE) \to \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(F, s_nE)$$

is an isomorphism.

See also [21, proposition 5-3]

We recall the *de-looping formula* [11, theorem 7.4.2]

$$s_n(\Omega_{\mathbb{P}^1}E) \cong \Omega_{\mathbb{P}^1}(s_{n+1}E)$$

for $n \geq 0$.

Take $F \in \mathbf{Spc}_\bullet(k)$. For $E \in \mathbf{Spt}_{S^1}(k)$, we have $\mathcal{H}om^{int}(F, E) \in \mathcal{SH}$, which for $F = X_+$ is just $E(X)$, and in general is formed as the homotopy limit associated to the description of F as a homotopy colimit of representable objects.

This gives us the "internal Hom" functor

$$\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(F,-):\mathcal{SH}_{S^1}(k)\to\mathcal{SH}_{S^1}(k)$$

and more generally

$$\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(F,-):\mathcal{SH}_{S^1}(k)/f_{n+1}\to\mathcal{SH}_{S^1}(k),$$

with natural transformation

$$\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(F,-)\to\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(F,-).$$

These have value on $E\in\mathbf{Spt}_{S^1}(k)$ defined by taking a fibrant model $\tilde{E}$ of E (in $\mathcal{SH}_{S^1}(k)$ or $\mathcal{SH}_{S^1}(k)/f_{n+1}$, as the case may be) and forming the presheaf on $\mathbf{Sm}/k$

$$X\mapsto\mathcal{H}om^{int}(F\wedge X_+,\tilde{E}).$$

Putting the de-looping formula together with lemma 8.1 gives us

Proposition 8.2. *For $E\in\mathcal{SH}_{S^1}(k)$ we have natural isomorphisms*

$$s_0(\Omega^n_{\mathbb{P}^1}E)\cong\Omega^n_{\mathbb{P}^1}s_nE\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}((\mathbb{P}^1,1)^{\wedge n},s_nE)$$

Proof. Indeed, the first isomorphism is just the de-looping isomorphism repeated n times. For the second, we have

$$\begin{aligned}\Omega^n_{\mathbb{P}^1}s_nE&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}((\mathbb{P}^1,1)^{\wedge n},s_nE)\\&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}((\mathbb{P}^1,1)^{\wedge n},s_nE)\end{aligned}$$

the second isomorphism following from lemma 8.1. □

Definition 8.3. Suppose that $\operatorname{char}k=0$. Take $E\in\mathcal{SH}_{S^1}(k)$, take $\alpha\in\mathrm{Hom}_{SmCor}(X,Y)$ and let $n\geq1$ be an integer. The *transfer*

$$\mathrm{Tr}_{Y/X}(\alpha):(\Omega^n_{\mathbb{P}^1}s_nE)(Y)\to(\Omega^n_{\mathbb{P}^1}s_nE)(X)$$

is the map in $\mathcal{SH}$ defined as follows:

$$\begin{aligned}(\Omega^n_{\mathbb{P}^1}s_nE)(Y)&=\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(Y_+,\Omega^n_{\mathbb{P}^1}s_nE)\\&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(\Sigma^n_{\mathbb{P}^1}Y_+,s_nE)\\&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(\Sigma^n_{\mathbb{P}^1}Y_+,s_nE)\\&\xrightarrow{co\text{-}tr^n(\alpha)^*}\mathcal{H}om_{\mathcal{SH}_{S^1}(k)/f_{n+1}}(\Sigma^n_{\mathbb{P}^1}X_+,s_nE)\\&\cong\mathcal{H}om_{\mathcal{SH}_{S^1}(k)}(X_+,\Omega^n_{\mathbb{P}^1}s_nE)\\&\cong(\Omega^n_{\mathbb{P}^1}s_nE)(X).\end{aligned}$$

Theorem 8.4. *Suppose that* $\operatorname{char}k=0$. *For $E\in\mathcal{SH}_{S^1}(k)$, the maps* $\mathrm{Tr}(\alpha)$ *extend the presheaf*

$$\Omega^n_{\mathbb{P}^1}s_nE:\mathbf{Sm}/k^{\mathrm{op}}\to\mathcal{SH}$$

to an $\mathcal{SH}$-valued presheaf with transfers

$$\Omega^n_{\mathbb{P}^1}s_nE:SmCor(k)^{\mathrm{op}}\to\mathcal{SH}$$

Proof. This follows from the definition of the maps $\mathrm{Tr}(\alpha)$ and theorem 6.1, the main point being that the maps $\mathrm{Tr}(\alpha)$ factor through the internal Hom in $\mathcal{SH}_{S^1}(k)/f_{n+1}$. $\square$

COROLLARY 8.5. *Suppose that* $\mathrm{char}\, k = 0$. *For* $E \in \mathcal{SH}_{S^1}(k)$, *there is an extension of the presheaf*

$$s_0\Omega_{\mathbb{P}^1}E : \mathbf{Sm}/k^{\mathrm{op}} \to \mathcal{SH}$$

to an $\mathcal{SH}$*-valued presheaf with transfers*

$$s_0\Omega_{\mathbb{P}^1}E : SmCor(k)^{\mathrm{op}} \to \mathcal{SH}.$$

Proof. This is just the case $n = 1$ of theorem 8.4, together with the de-looping isomorphism

$$s_0\Omega_{\mathbb{P}^1}E \cong \Omega_{\mathbb{P}^1}s_1E.$$

$\square$

Remark 8.6. The corollary is actually the main result, in that one can deduce theorem 8.4 from corollary 8.5 (applied to $\Omega^{n-1}_{\mathbb{P}^1}E$) and the de-looping formula

$$\Omega^n_{\mathbb{P}^1}s_nE \cong s_0\Omega^n_{\mathbb{P}^1}E = s_0\Omega_{\mathbb{P}^1}(\Omega^{n-1}_{\mathbb{P}^1}E).$$

As the maps $co\text{-}tr^n(\alpha)$ are defined by smashing $co\text{-}tr^1(\alpha)$ with an identity map, this procedure does indeed give back the maps

$$\mathrm{Tr}(\alpha) : \Omega^n_{\mathbb{P}^1}s_nE(Y) \to \Omega^n_{\mathbb{P}^1}s_nE(X)$$

as defined above.

proof of theorem 3. The weak transfers defined above give rise to homotopy invariant sheaves with transfers in the usual sense by taking the sheaves of homotopy groups of the motivic spectrum in question. $\square$

For instance, corollary 8.5 gives the sheaf $\pi_m(s_0\Omega_{\mathbb{P}^1}E)$ the structure of a homotopy invariant sheaf with transfers, in particular, an effective motive. In fact, these are *birational motives* in the sense of Kahn-Huber-Sujatha [7, 10], as s_0F is a birational S^1-spectrum for each S^1-spectrum F. The classical Postnikov tower thus gives us a spectral sequence

$$E^2_{p,q} := H^{-p}(X_{\mathrm{Nis}}, \pi_q(s_0\Omega_{\mathbb{P}^1}E)) \Longrightarrow \pi_{p+q}(s_0\Omega_{\mathbb{P}^1}E(X))$$

with E^2 term a "generalized motivic cohomology" of X. As the sheaves $\pi_q(s_0\Omega_{\mathbb{P}^1}E)$ are motives, we may replace Nisnevich cohomology with Zariski cohomology; as the sheaves $\pi_q(s_0\Omega_{\mathbb{P}^1}E)$ are birational, i.e., Zariski locally constant, the higher Zariski cohomology vanishes, giving us

$$\pi_n(s_0\Omega_{\mathbb{P}^1}E(X)) \cong H^0(X_{\mathrm{Zar}}, \pi_n(s_0\Omega_{\mathbb{P}^1}E)) = \pi_n(s_0\Omega_{\mathbb{P}^1}E(k(X)).$$

In short, we have shown that the 0th slice of a $\mathbb{P}^1$-loop spectrum has transfers in the weak sense. We have already seen in section 2 that this does not hold for an arbitrary object of $\mathcal{SH}_{S^1}(k)$; in the next section we will see that the higher slices of an arbitrary S^1-spectrum do have transfers, albeit in an even weaker sense than the one used above.

9. Transfers on the generalized cycle complex

We begin by recalling from [11, theorem 7.1.1] models for $f_nE(X)$ and $s_nE(X)$ that are reminiscent of Bloch's higher cycle complex [1]. To simplify the notation, we will always assume that we have taken a model $E \in \mathbf{Spt}_{S^1}(k)$ which is quasi-fibrant. For W a closed subset of some $Y \in \mathbf{Sm}/k$, $E^{(W)}(Y)$ will denote the homotopy fiber of the restriction map $E(Y) \to E(Y \setminus W)$.

We make use of the cosimplicial scheme $n \mapsto \Delta^n := \operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1$. A *face* F of Δ^n is a subscheme defined by $t_{i_1} = \ldots = t_{i_r} = 0$.

For a scheme X of finite type and locally equi-dimensional over k, let $\mathcal{S}_X^{(n)}(m)$ be the set of closed subsets W of $X \times \Delta^m$ of codimension $\geq n$, such that, for each face F of Δ^n, $W \cap X \times F$ has codimension $\geq n$ on $X \times F$ (or is empty). We order $\mathcal{S}_X^{(n)}(m)$ by inclusion.

For $X \in \mathbf{Sm}/k$, we let

$$E^{(n)}(X, m) := \varinjlim_{W \in \mathcal{S}_X^{(n)}(m)} E^{(W)}(X \times \Delta^m).$$

Similarly, for $0 \leq n \leq n'$, we define

$$E^{(n/n')}(X, m) := \varinjlim_{W \in \mathcal{S}_X^{(n)}(m), W' \in \mathcal{S}_X^{(n')}(m)} E^{(W \setminus W')}(X \times \Delta^m \setminus W')$$

The conditions on the intersections of W with $X \times F$ for faces F means that $m \mapsto \mathcal{S}_X^{(n)}(m)$ form a cosimplicial set, denoted $\mathcal{S}_X^{(n)}$, for each n and that $\mathcal{S}_X^{(n')}$ is a cosimplicial subset of $\mathcal{S}_X^{(n)}$ for $n \leq n'$. Thus the restriction maps for E make $m \mapsto E^{(n)}(X, m)$ and $m \mapsto E^{(n/n')}(X, m)$ simplicial spectra, denoted $E^{(n)}(X, -)$ and $E^{(n/n')}(X, -)$. We denote the associated total spectra by $|E^{(n)}(X, -)|$ and $|E^{(n/n')}(X, -)|$.

The inclusion $\mathcal{S}_X^{(n')}(m) \to \mathcal{S}_X^{(n)}(m)$ for $n \leq n'$ and the evident restriction maps give the sequence

$$|E^{(n')}(X, -)| \to |E^{(n)}(X, -)| \to |E^{(n/n')}(X, -)|$$

which is easily seen to be a weak homotopy fiber sequence.

We note that $|E^{(0)}(X, -)| = E(X \times \Delta^*)$; as E is homotopy invariant, the canonical map

$$E(X) \to |E^{(0)}(X, -)|$$

is thus a weak equivalence. We therefore have the tower in $\mathcal{SH}$

$$\ldots \to |E^{(n+1)}(X, -)| \to |E^{(n)}(X, -)| \to \ldots \to |E^{(0)}(X, -)| \cong E(X) \tag{9.1}$$

with nth layer isomorphic to $|E^{(n/n+1)}(X, -)|$. We call this tower the *homotopy coniveau tower* for $E(X)$. In this regard, one of the main results from [11] states

Theorem 9.1 ([11, theorem 7.1.1]). *There is a canonical isomorphism of the tower* (9.1) *with the motivic Postnikov tower evaluated at X:*

$$\ldots \to f_{n+1}E(X) \to f_nE(X) \to \ldots \to f_0E(X) = E(X),$$

giving a canonical isomorphism

$$s_nE(X) \cong |E^{(n/n+1)}(X, -)|.$$

We can further modify this description of $s_nE(X)$ as follows: Since s_n is an idempotent functor, we have

$$s_nE(X) \cong s_n(s_nE)(X) \cong |(s_nE)^{(n/n+1)}(X, -)|.$$

Note that $|(s_nE)^{(n/n+1)}(X, -)|$ fits into a weak homotopy fiber sequence

$$|(s_nE)^{(n+1)}(X, -)| \to |(s_nE)^{(n)}(X, -)| \to |(s_nE)^{(n/n+1)}(X, -)|.$$

Using theorem 9.1 in reverse, we have the isomorphism in $\mathcal{SH}$

$$|(s_nE)^{(n+1)}(X, -)| \cong f_{n+1}(s_nE)(X).$$

But as $f_{n+1} \circ f_n \cong f_{n+1}$, we see that $f_{n+1}(s_nE) \cong 0$ in $\mathcal{SH}_{S^1}(k)$ and thus

$$|(s_nE)^{(n)}(X, -)| \cong |(s_nE)^{(n/n+1)}(X, -)| \cong s_nE(X).$$

We may therefore use the simplicial model $|(s_nE)^{(n)}(X, -)|$ for $s_nE(X)$.
We will need a refinement of this construction, which takes into account the interaction of the support conditions with a given correspondence.

DEFINITION 9.2. Let $A \subset Y \times X$ be a generator in $\mathrm{Hom}_{SmCor}(Y, X)$; for each m, we let $A(m) \in \mathrm{Hom}_{SmCor}(Y \times \Delta^m, X \times \Delta^m)$ denote the correspondence $A \times \mathrm{id}_{\Delta^m}$. Let $\mathcal{S}^{(n)}_{X,A}(m)$ be the subset of $\mathcal{S}^{(n)}_X(m)$ consisting of those $W' \in \mathcal{S}^{(n)}_X(m)$ such that

(1) $W := p_{Y\times\Delta^m}(A \times \Delta^m \cap Y \times W')$ is in $\mathcal{S}^{(n)}_Y(m)$.
(2) $A(m)$ is in $\mathrm{Hom}_{SmCor}(Y \times \Delta^m, X \times \Delta^m)_{W,W'}$.

For an arbitrary $\alpha \in \mathrm{Hom}_{SmCor}(Y, X)$, write

$$\alpha = \sum_{i=1}^{r} n_iA_i$$

with the A_i generators and the n_i non-zero integers and define

$$\mathcal{S}^{(n)}_{X,\alpha}(m) := \cap_{i=1}^r \mathcal{S}^{(n)}_{X,A_i}(m).$$

If we have in addition to α a finite correspondence $\beta \in \mathrm{Hom}_{SmCor}(Z, Y)$, we let $\mathcal{S}^{(n)}_{X,\alpha,\beta}(m) \subset \mathcal{S}^{(n)}_{X,\alpha}(m)$ be the set of $W \subset X \times \Delta^m$ such that W is in $\mathcal{S}^{(n)}_{X,\alpha}(m)$ and $p^{Y\times X\times\Delta^m}_{Y\times\Delta^m}(Y \times W \cap |\alpha| \times \Delta^m)$ is in $\mathcal{S}^{(n)}_{Y,\beta}(m)$.

For $f : Y \to X$ a flat morphism, one has

$$\mathcal{S}^{(n)}_{X,\Gamma_f}(m) = \mathcal{S}^{(n)}_X(m)$$

and for $g : Z \to Y$ a flat morphism, and $\alpha \in \mathrm{Hom}_{SmCor}(Y, X)$, one has

$$\mathcal{S}^{(n)}_{X,\alpha,\Gamma_g}(m) = \mathcal{S}^{(n)}_{X,\alpha}(m)$$

Note that $m \mapsto \mathcal{S}^{(n)}_{X,\alpha}(m)$ and $m \mapsto \mathcal{S}^{(n)}_{X,\alpha,\beta}(m)$ define cosimplicial subsets of $m \mapsto \mathcal{S}^{(n)}_X(m)$. We define the simplicial spectra $E^{(n)}(X,-)_\alpha$ and $E^{(n)}(X,-)_{\alpha,\beta}$ using the support conditions $\mathcal{S}^{(n)}_{X,\alpha}(m)$ and $\mathcal{S}^{(n)}_{X,\alpha,\beta}(m)$ instead of $\mathcal{S}^{(n)}_X(m)$:

$$E^{(n)}(X,m)_\alpha := \varinjlim_{W\in\mathcal{S}^{(n)}_{X,\alpha}(m)} E^{(W)}(X\times\Delta^m)$$
$$E^{(n)}(X,m)_{\alpha,\beta} := \varinjlim_{W\in\mathcal{S}^{(n)}_{X,\alpha,\beta}(m)} E^{(W)}(X\times\Delta^m),$$

giving us the sequence of simplicial spectra

$$E^{(n)}(X,-)_{\alpha,\beta} \to E^{(n)}(X,-)_\alpha \to E^{(n)}(X,-).$$

The main "moving lemma" [12, theorem 2.6.2(2)] yields

Proposition 9.3. *For $X \in \mathbf{Sm}/k$ affine, and $E \in \mathbf{Spt}_{S^1}(k)$ quasi-fibrant, the maps*

$$|E^{(n)}(X,-)_{\alpha,\beta}| \to |E^{(n)}(X,-)_\alpha| \to |E^{(n)}(X,-)|$$

are weak equivalences.

We proceed to the main construction of this section. Consider the simplicial model $|(s_nE)^{(n)}(X,-)|$ for $s_nE(X)$. For each m, we may consider the classical Postnikov tower (or rather, its dual version) for the spectrum $(s_nE)^{(n)}(X,m)$, which we write as

$$\ldots \to \tau_{\geq p+1}(s_nE)^{(n)}(X,m) \to \tau_{\geq p}(s_nE)^{(n)}(X,m) \to \ldots \to (s_nE)^{(n)}(X,m),$$

where

$$\tau_{\geq p+1}(s_nE)^{(n)}(X,m) \to (s_nE)^{(n)}(X,m)$$

is the p-connected cover of $(s_nE)^{(n)}(X,m)$. The pth layer in this tower is of course the pth suspension of the Eilenberg-Maclane spectrum on $\pi_p((s_nE)^{(n)}(X,m))$. Taking a functorial model for the p-connected cover, we have for each p the simplicial spectrum

$$m \mapsto \tau_{\geq p+1}(s_nE)^{(n)}(X,m)$$

giving us the tower of total spectra

(9.2)
$$\ldots \to |\tau_{\geq p+1}(s_nE)^{(n)}(X,-)| \to |\tau_{\geq p}(s_nE)^{(n)}(X,-)| \to \ldots \to |(s_nE)^{(n)}(X,-)|.$$

The pth layer in this tower are then (up to suspension) the Eilenberg-Maclane spectrum on the chain complex $\pi_p(s_nE)^{(n)}(X,*)$, with differential as usual the alternating sum of the face maps.

The chain complexes $\pi_p(s_nE)^{(n)}(X,*)$ are evidently functorial for smooth maps and inherit the homotopy invariance property from $(s_nE)^{(n)}(X,*)$ (see [12, theorem 3.3.5]). Somewhat more surprising is

Lemma 9.4. *The complexes $\pi_p(s_nE)^{(n)}(X,*)$ satisfy Nisnevich excision.*

Proof. Let $W \subset X \times \Delta^m$ be a closed subset in $\mathcal{S}_X^{(n)}(m)$, and let w be the set of generic points of W having codimension exactly n on $X \times \Delta^m$. Then

$$s_nE^{(W)}(X \times \Delta^m) \cong s_nE(\Sigma^n_{\mathbb{P}^1} w_+) \cong \Omega^n_{\mathbb{P}^1}(s_nE)(w) \cong s_0(\Omega^n_{\mathbb{P}^1}E)(w).$$

This gives us the following description of $\pi_p((s_nE)^{(n)}(X,m))$:

$$\pi_p((s_nE)^{(n)}(X,m)) \cong \oplus_w \pi_p(s_0(\Omega^n_{\mathbb{P}^1}E)(w))$$

where the direct sum is over the set $\mathcal{T}_X^{(n)}(m)$ of generic points of the irreducible $W \in \mathcal{S}_X^{(n)}(m)$ having codimension exactly n in $X \times \Delta^m$.
Now let $i : Z \to X$ be a closed subset with open complement $j : U \to X$. For each m, we thus have the exact sequence

$$\begin{aligned} 0 \to \oplus_{w \in Z \times \Delta^m \cap \mathcal{T}_X^{(n)}(m)} \pi_p(s_0(\Omega^n_{\mathbb{P}^1}E)(w)) \to \oplus_{w \in \mathcal{T}_X^{(n)}(m)} \pi_p(s_0(\Omega^n_{\mathbb{P}^1}E)(w)) \\ \to \oplus_{w \in \mathcal{T}_X^{(n)}(m) \cap U \times \Delta^m} \pi_p(s_0(\Omega^n_{\mathbb{P}^1}E)(w)) \to 0 \end{aligned}$$

Define the subcomplex $\pi_p(s_nE)^{(n)}(X,*)_Z$ of $\pi_p(s_nE)^{(n)}(X,*)$ and quotient complex $\pi_p(s_nE)^{(n)}(U_X,*)$ of $\pi_p(s_nE)^{(n)}(X,*)$ by taking supports in

$$\{W \in \mathcal{S}_X^{(n)}(m) \mid W \subset Z \times \Delta^m\}, \text{ resp. } \{W \cap U \times \Delta^m \mid W \in \mathcal{S}_X^{(n)}(m)\}.$$

We thus have the term-wise exact sequence of complexes

$$0 \to \pi_p(s_nE)^{(n)}(X,*)_Z \to \pi_p(s_nE)^{(n)}(X,*) \to \pi_p(s_nE)^{(n)}(U_X,*) \to 0$$

CLAIM. *The inclusion*

$$\pi_p(s_nE)^{(n)}(U_X,*) \xrightarrow{\iota} \pi_p(s_nE)^{(n)}(U,*)$$

is a quasi-isomorphism.

Proof of the claim. This follows using the localization technique [13, theorem 8.10] (for details, see [11, theorem 3.2.1]). In a few words, one takes an integer N and a $W \in \mathcal{S}_U^{(n)}(N)$. We assume that $(\mathrm{id}_U \times \Delta(\sigma))(W) = W$ for each permutation σ of the vertices of Δ^N, where $\Delta(\sigma) : \Delta^N \to \Delta^N$ is the affine-linear extension of σ. For $m \le N$, let $\mathcal{T}_X^{(n)}(m)_W \subset \mathcal{T}_X^{(n)}(m)$ be the set of points w such that $w \in (\mathrm{id}_U \times \Delta(g))^*(W)$ for some injective $g : [m] \to [N]$, and set

$$\pi_p(s_nE)^{(n)}(U,m)_W := \oplus_{w \in \mathcal{T}_X^{(n)}(m)_W} \pi_p(s_nE)^{(n)}(U,m) \subset \pi_p(s_nE)^{(n)}(U,m).$$

For $m > N$ set $\pi_p(s_nE)^{(n)}(U,m)_W = 0$. This gives us the subcomplex

$$\pi_p(s_nE)^{(n)}(U,*)_W \subset \pi_p(s_nE)^{(n)}(U,*);$$

clearly $\pi_p(s_nE)^{(n)}(U,*)$ is the colimit of the subcomplexes $\pi_p(s_nE)^{(n)}(U,*)_W$. Similarly, we have the subcomplex $\pi_p(s_nE)^{(n)}(U_X,*)_W$ of $\pi_p(s_nE)^{(n)}(U_X,*)$ and the inclusion

$$\iota_W : \pi_p(s_nE)^{(n)}(U_X,*)_W \to \pi_p(s_nE)^{(n)}(U,*)_W,$$

with $\pi_p(s_nE)^{(n)}(U_X,*)$ the colimit of the $\pi_p(s_nE)^{(n)}(U_X,*)_W$.

In [11, theorem 3.2.1], we have the formal sums of maps of simplices

$$\psi_W(m) = \sum_i n_i \psi_W(m)_i; \quad \psi_W(m)_i : \Delta^m \to \Delta^m$$
$$\Psi_W(m) = \sum_i m_i \Psi_W(m)_i; \quad \Psi_W(m)_i : \Delta^{m+1} \to \Delta^m$$

for $m = 0, \ldots, N$, such that the pull-back by the maps $\psi_W(m)$ define a map of complexes

$$\psi_W^* : \pi_p(s_nE)^{(n)}(U, *)_W \to \pi_p(s_nE)^{(n)}(U_X, *).$$

Additionally, the pull-back by the $\Psi_W(m)$ define homotopies of the map $\iota \circ \psi_W$ with the inclusion $\pi_p(s_nE)^{(n)}(U, *)_W \to \pi_p(s_nE)^{(n)}(U, *)$ and similarly of $\psi_W \circ \iota_W$ with the inclusion $\pi_p(s_nE)^{(n)}(U_X, *)_W \to \pi_p(s_nE)^{(n)}(U_X, *)$. The claim follows easily from this. $\square$

We therefore have the quasi-isomorphism
(9.3)

$$\pi_p(s_nE)^{(n)}(X, *)_Z \to \mathrm{cone}\left(\pi_p(s_nE)^{(n)}(X, *) \xrightarrow{j^*} \pi_p(s_nE)^{(n)}(U, *)\right)[-1].$$

Now let

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

be an elementary Nisnevich square, i.e., the square is cartesian, p is étale and induces an isomorphism $p : Z' := X' \setminus U' \to Z$. Clearly p induces an isomorphism

$$p^* : \pi_p(s_nE)^{(n)}(X, *)_Z \to \pi_p(s_nE)^{(n)}(X', *)_{Z'};$$

using the localization quasi-isomorphism (9.3), it follows that p^* induces a quasi-isomorphism

$$\mathrm{cone}\left(\pi_p(s_nE)^{(n)}(X, *) \xrightarrow{j^*} \pi_p(s_nE)^{(n)}(U, *)\right)[-1]$$
$$\xrightarrow{p^*} \mathrm{cone}\left(\pi_p(s_nE)^{(n)}(X', *) \xrightarrow{j^*} \pi_p(s_nE)^{(n)}(U', *)\right)[-1],$$

proving the lemma. $\square$

We will use the results of section 7 to give $X \mapsto \pi_p(s_nE)^{(n)}(X, *)$ the structure of a complex of homotopy invariant presheaves with transfer on $\mathbf{Sm}/k$, i.e. a motive.

For this, we consider the complexes $\pi_p(s_nE)^{(n)}(X, *)_\alpha$, $\pi_p(s_nE)^{(n)}(X, *)_{\alpha,\beta}$ constructed above. The refined support condition are constructed so that, for each $W \in \mathcal{S}^{(n)}_{X,\alpha}(m)$, $\alpha \times \mathrm{id}_{\Delta^m}$ is in $\mathrm{Hom}_{SmCor}(Y \times \Delta^m, X \times \Delta^m)_{W',W}$, where

$$W' = p_{Y \times \Delta^m}(Y \times \Delta^m \times W \cap |\alpha \times \mathrm{id}_{\Delta^m}|).$$

We may therefore use the morphism $co\text{-}tr^n(\alpha \times \mathrm{id}_{\Delta^m})$ to define the map

$$\mathrm{Tr}_{Y/X}(\alpha)(m) : \pi_p((s_nE)^{(n)}(X,m))_\alpha \to \pi_p(s_nE)^{(n)}(Y,m).$$

By proposition 7.5, the maps $\mathrm{Tr}_{Y/X}(m)$ define a map of complexes

$$\mathrm{Tr}_{Y/X}(\alpha) : \pi_p(s_nE)^{(n)}(X,*)_\alpha \to \pi_p(s_nE)^{(n)}(Y,*).$$

Similarly, given $\beta \in \mathrm{Hom}_{SmCor}(Z,Y)$, we have the map of complexes

$$\mathrm{Tr}_{Y/X}(\alpha)_\beta : \pi_p(s_nE)^{(n)}(X,*)_{\alpha,\beta} \to \pi_p(s_nE)^{(n)}(Y,*)_\beta.$$

Note that, due to possible cancellations occurring when one takes the composition $\alpha \circ \beta$, we have only an inclusion

$$\mathcal{S}^{(n)}_{X,\alpha,\beta}(m) \subset \mathcal{S}^{(n)}_{X,\alpha\circ\beta}(m)$$

giving us a natural comparison map

$$\iota_{\alpha,\beta} : \pi_p(s_nE)^{(n)}(X,*)_{\alpha,\beta} \to: \pi_p(s_nE)^{(n)}(X,*)_{\alpha\circ\beta}.$$

Using our moving lemma again, we see that $\iota_{\alpha,\beta}$ is a quasi-isomorphism in case X is affine.

Lemma 9.5. *Suppose* $\mathrm{char}\, k = 0$. *For*

$$\alpha \in \mathrm{Hom}_{SmCor}(Z,Y),\ \beta \in \mathrm{Hom}_{SmCor}(Z,Y),$$

we have

$$\mathrm{Tr}_{Z/Y}(\beta) \circ \mathrm{Tr}_{Y/X}(\alpha)_\beta = \mathrm{Tr}_{Z/X}(\alpha \circ \beta) \circ \iota_{\alpha,\beta}.$$

Proof. This follows from lemma 7.4. □

We have already noted that complexes $\pi_p(s_nE)^{(n)}(X,*)$ are functorial in X for flat morphisms in $\mathbf{Sm}/k$, in particular for smooth morphisms in $\mathbf{Sm}/k$. Let $\widetilde{\mathbf{Sm}}/k$ denote the subcategory of $\mathbf{Sm}/k$ with the same objects and with morphisms the smooth morphisms. The transfer maps we have defined on the refined complexes, together with the moving lemma 7.4 yield the following result:

Theorem 9.6. *Suppose* $\mathrm{char}\, k = 0$. *Consider the presheaf*

$$\pi_p(s_nE)^{(n)}(-,*)) : \widetilde{\mathbf{Sm}}/k^{\mathrm{op}} \to C^-(\mathbf{Ab})$$

on $\widetilde{\mathbf{Sm}}/k^{\mathrm{op}}$. *Let*

$$\iota : \widetilde{\mathbf{Sm}}/k \to SmCor(k)$$

be the evident inclusion and let

$$Q : C^-(\mathbf{Ab}) \to D^-(\mathbf{Ab})$$

be the evident additive functor. There is a complex of presheaves with transfers

$$\hat{\pi}_p((s_nE)^{(n)})^* : SmCor(k)^{\mathrm{op}} \to C^-(\mathbf{Ab})$$

and an isomorphism of functors from $\widetilde{\mathbf{Sm}}/k^{\mathrm{op}}$ *to* $D^-(\mathbf{Ab})$

$$Q \circ \pi_p(s_nE)^{(n)}(-,*)) \cong Q \circ \hat{\pi}_p((s_nE)^{(n)})^* \circ \iota.$$

Proof. We give a rough sketch of the construction here; for details we refer the reader to [9, proposition 2.2.3], which in turn is an elaboration of [12, theorem 7.4.1]. The construction of $\hat{\pi}_p((s_nE)^{(n)})^*$ is accomplished by first taking a homotopy limit over the complexes $\pi_p(s_nE)^{(n)}(X,*)_\alpha$. These are then functorial on $SmCor(k)^{\mathrm{op}}$, up to homotopy equivalences arising from the replacement of the index category for the homotopy limit with a certain cofinal subcategory. One then forms a regularizing homotopy colimit that is strictly functorial on $SmCor(k)^{\mathrm{op}}$, and finally, one replaces this presheaf with a fibrant model. The moving lemma for affine schemes (proposition 9.3) implies that the homotopy limit construction yields for each affine $X \in \mathbf{Sm}/k$ a complex canonically quasi-isomorphic to $\pi_p(s_nE)^{(n)}(X,*)$; this property is inherited by the regularizing homotopy colimit. As the complexes $\pi_p(s_nE)^{(n)}(X,*)$ satisfy Nisnevich excision (lemma 9.4) and are homotopy invariant for *all* X, this implies that $\hat{\pi}_p((s_nE)^{(n)})^*(X)$ is canonically isomorphic to $\pi_p(s_nE)^{(n)}(X,*)$ in $D^-(\mathbf{Ab})$ for all $X \in \mathbf{Sm}/k$. By construction, this isomorphism is natural with respect to smooth morphisms in $\mathbf{Sm}/k$. □

Corollary 9.7. *Suppose* $\mathrm{char}\, k = 0$. $\hat{\pi}_p((s_nE)^{(n)})^*$ *is a homotopy invariant complex of presheaves with transfer.*

Proof. By theorem 9.6, we have the isomorphism in $D^-(\mathbf{Ab})$

$$\hat{\pi}_p((s_nE)^{(n)})^*(X) \cong \pi_p(s_nE)^{(n)}(X,*).$$

for all $X \in \mathbf{Sm}/k$, natural with respect to smooth morphisms. As the presheaf $\pi_p(s_nE)^{(n)}(-,*)$ is homotopy invariant, so is $\hat{\pi}_p((s_nE)^{(n)})^*$. □

proof of theorem 2. As in the proof of theorem 9.6, the method of [12, theorem 7.4.1], shows that the tower (9.2) extends to a tower

$$\ldots \to \rho_{\geq p+1}s_nE \to \rho_{\geq p}s_nE \to \ldots \to s_nE \tag{9.4}$$

in $\mathcal{SH}_{S^1}(k)$ with value (9.2) at $X \in \mathbf{Sm}/k$, and with the cofiber of $\rho_{\geq p+1}s_nE \to \rho_{\geq p}s_nE$ naturally isomorphic to $EM^{eff}_{\mathbb{A}^1}(\hat{\pi}_p((s_nE)^{(n)})^*)$. By lemma 9.4 and corollary 9.7, the presheaves $\hat{\pi}_p((s_nE)^{(n)})^*$ define objects in $DM^{eff}_-(k)$. Thus, we have shown that the layers in the tower (9.4) have a motivic structure, proving theorem 2. □

10. The Friedlander-Suslin tower

As the reader has surely noticed, the lack of functoriality for the simplicial spectra $E^{(n)}(X,-)$ creates annoying technical problems when we wish to extend the construction of the homotopy coniveau tower to a tower in $\mathcal{SH}_{S^1}(k)$. In their work on the spectral sequence from motivic cohomology to K-theory, Friedlander and Suslin [4] have constructed a completely functorial version of the homotopy coniveau tower, using "quasi-finite supports". Unfortunately, the comparison between the Friedlander-Suslin version and $E^{(n)}(X,-)$ is proven in [4] only for K-theory and motivic cohomology. In this last section, we recall the

Friedlander-Suslin construction and form the conjecture that the Friedlander-Suslin tower is naturally isomorphic to the homotopy coniveau tower.

Let $\mathcal{Q}_X^{(n)}(m)$ be the set of closed subsets W of $\mathbb{A}^n \times X \times \Delta^m$ such that, for each irreducible component W' of W, the projection $W' \to X \times \Delta^m$ is quasi-finite. For $E \in \mathbf{Spt}_{S^1}(k)$, we let

$$E_{FS}^{(n)}(X,m) := \varinjlim_{W \in \mathcal{Q}_X^{(n)}(m)} E^{(W)}(\mathbb{A}^n \times X \times \Delta^m)$$

As the condition defining $\mathcal{Q}_X^{(n)}(m)$ are preserved under maps

$$\mathrm{id}_{\mathbb{A}^n} \times f \times g : \mathbb{A}^n \times X' \times \Delta^{m'} \to \mathbb{A}^n \times X \times \Delta^m,$$

where $f : X' \to X$ is an arbitrary map in $\mathbf{Sm}/k$, and $g : \Delta^{m'} \to \Delta^m$ is a structure map in Δ^*, the spectra $E_{FS}^{(n)}(X,m)$ define a simplicial spectrum $E_{FS}^{(n)}(X,-)$ and these simplicial spectra, for $X \in \mathbf{Sm}/k$, extend to a presheaf of simplicial spectra on $\mathbf{Sm}/k$:

$$E_{FS}^{(n)}(?,-) : \mathbf{Sm}/k^{\mathrm{op}} \to \Delta^{\mathrm{op}}\mathbf{Spt}.$$

Similarly, if we take the linear embedding $i_n : \mathbb{A}^n \to \mathbb{A}^{n+1} = \mathbb{A}^n \times \mathbb{A}^1$, $x \mapsto (x,0)$, the pull-back by $i_n \times \mathrm{id}$ preserves the support conditions, and thus gives a well-defined map of simplicial spectra

$$i_n^* : E_{FS}^{(n+1)}(X,-) \to E_{FS}^{(n)}(X,-),$$

forming the tower of presheaves on $\mathbf{Sm}/k$

$$\ldots \to E_{FS}^{(n+1)}(?,-) \to E_{FS}^{(n)}(?,-) \to \ldots \; . \tag{10.1}$$

We may compare $E_{FS}^{(n)}(X,-)$ and $E^{(n)}(X,-)$ using the method of [4] as follows: The simplicial spectra $E^{(n)}(X,-)$ are functorial for flat maps in $\mathbf{Sm}/k$, in the evident manner. They satisfy homotopy invariance, in that the pull-back map

$$p^* : E^{(n)}(X,-) \to E^{(n)}(\mathbb{A}^1 \times X,-)$$

induces a weak equivalence on the total spectra. We have the evident inclusion of simplicial sets

$$\mathcal{Q}_X^{(n)}(-) \hookrightarrow \mathcal{S}_{\mathbb{A}^n \times X}^{(n)}(-)$$

inducing the map

$$\varphi_{X,n} : E_{FS}^{(n)}(X,-) \to E^{(n)}(\mathbb{A}^n \times X,-).$$

Together with the weak equivalence $p^* : |E^{(n)}(X,-)| \to |E^{(n)}(\mathbb{A}^n \times X,-)|$, the maps $\varphi_{X,n}$ induce a map of towers of total spectra in $\mathcal{SH}$

$$\varphi_{X,*} : |E_{FS}^{(*)}(X,-)| \to |E^{(*)}(X,-)|. \tag{10.2}$$

Conjecture 10.1. *For each $X \in \mathbf{Sm}/k$ and each quasi-fibrant $E \in \mathbf{Spt}_{S^1}(k)$, the map* (10.2) *induces an isomorphism in $\mathcal{SH}$ of the towers of total spectra.*

Combined with the weak equivalence given by homotopy invariance and the results of [11], this would give us an isomorphism in $\mathcal{SH}_{S^1}(k)$:

$$f_n E \cong |E^{(n)}_{FS}(?, -)|.$$

As transfers in some form or other are used in the arguments relating the Friedlander-Suslin complex to the Bloch-type complexes in the known cases, a weaker form of the conjecture might be more reasonable:

Conjecture 10.2. *For each $X \in \mathbf{Sm}/k$ and each quasi-fibrant $E \in \mathbf{Spt}_{S^1}(k)$ with $s_0 E \cong 0$ in $\mathcal{SH}_{S^1}(k)$, the map* (10.2) *induces an isomorphism in $\mathcal{SH}$ of the towers of total spectra.*

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