# Hyperbolicity of Orthogonal Involutions

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with an Appendix by Jean-Pierre Tignol

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Abstract. We show that a non-hyperbolic orthogonal involution on a central simple algebra over a field of characteristic $\neq 2$ remains non-hyperbolic over some splitting field of the algebra.

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## 1. Introduction

Throughout this note (besides of §3 and §4) $F$ is a field of characteristic $\neq 2$. The basic reference for the material related to involutions on central simple algebras is [13]. The *degree* $\deg A$ of a (finite-dimensional) central simple $F$-algebra $A$ is the integer $\sqrt{\dim_F A}$; the *index* $\operatorname{ind} A$ of $A$ is the degree of a central division algebra Brauer-equivalent to $A$. An orthogonal involution $\sigma$ on $A$ is *hyperbolic*, if the hermitian form $A \times A \to A$, $(a,b) \mapsto \sigma(a) \cdot b$ on the right $A$-module $A$ is so. This means that the variety $X\big((\deg A)/2; (A,\sigma)\big)$ of §2 has a rational point.
The main result of this paper is as follows (the proof is given in §7):

Theorem 1.1 (Main theorem). *A non-hyperbolic orthogonal involution $\sigma$ on a central simple $F$-algebra $A$ remains non-hyperbolic over the function field of the Severi-Brauer variety of $A$.*

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To explain the statement of Abstract, let us note that the function field $L$ of the Severi-Brauer variety of a central simple algebra $A$ is a *splitting field* of $A$, that is, the $L$-algebra $A_L$ is Brauer-trivial.
A stronger version of Theorem 1.1, where the word "non-hyperbolic" (in each of two appearances) is replaced by "anisotropic", is, in general, an open conjecture, cf. [11, Conjecture 5.2].
Let us recall that the index of a central simple algebra possessing an orthogonal involution is a power of 2. Here is the complete list of indices $\operatorname{ind} A$ and coindices $\operatorname{coind} A = \deg A/\operatorname{ind} A$ of $A$ for which Theorem 1.1 is known (over arbitrary fields of characteristic $\neq 2$), given in the chronological order:

- $\operatorname{ind} A = 1$ — trivial;
- $\operatorname{coind} A = 1$ (the stronger version) — [11, Theorem 5.3];
- $\operatorname{ind} A = 2$ — [5] and independently (the stronger version) [16, Corollary 3.4];
- $\operatorname{coind} A$ odd — [7, appendix by Zainoulline] and independently [12, Theorem 3.3];
- $\operatorname{ind} A = 4$ and $\operatorname{coind} A = 2$ — [19, Proposition 3];
- $\operatorname{ind} A = 4$ — [8, Theorem 1.2].

Let us note that Theorem 1.1 for any given $(A, \sigma)$ with $\operatorname{coind} A = 2$ implies the stronger version of Theorem 1.1 for this $(A, \sigma)$: indeed, by [12, Theorem 3.3], if $\operatorname{coind} A = 2$ and $\sigma$ becomes isotropic over the function field of the Severi-Brauer variety, then $\sigma$ becomes hyperbolic over this function field and the weaker version applies. Therefore we get

Theorem 1.2. *An anisotropic orthogonal involution on a central simple $F$-algebra of coindex 2 remains anisotropic over the function field of the Severi-Brauer variety of the algebra.* □

Sivatski's proof of the case with $\deg A = 8$ and $\operatorname{ind} A = 4$, mentioned above, is based on the following theorem, due to Laghribi:

Theorem 1.3 ([14, Théorème 4]). *Let $\varphi$ be an anisotropic quadratic form of dimension 8 and of trivial discriminant. Assume that the index of the Clifford algebra $C$ of $\varphi$ is 4. Then $\varphi$ remains anisotropic over the function field $F(X_1)$ of the Severi-Brauer variety $X_1$ of $C$.*

The following alternate proof of Theorem 1.3, given by Vishik, is a prototype of our proof of Main theorem (Theorem 1.1). Let $Y$ be the projective quadric of $\varphi$ and let $X_2$ be the Albert quadric of a biquaternion division algebra Brauer-equivalent to $C$. Assume that $\varphi_{F(X_1)}$ is isotropic. Then for any field extension $E/F$, the Witt index of $\varphi_E$ is at least 2 if and only if $X_2(E) \neq \emptyset$. By [21, Theorem 4.15] and since the Chow motive $M(X_2)$ of $X_2$ is indecomposable, it follows that the motive $M(X_2)(1)$ is a summand of the motive of $Y$. The complement summand of $M(Y)$ is then given by a *Rost projector* on $Y$ in the sense of Definition 5.1. Since $\dim Y + 1$ is not a power of 2, it follows that $Y$ is isotropic (cf. [6, Corollary 80.11]).

After introducing some notation in §2 and discussing some important general principles concerning Chow motives in §3, we produce in §4 a replacement of [21, Theorem 4.15] (used right above to split off the summand $M(X_2)(1)$ from the motive of $Y$) valid for more general (as projective quadrics) algebraic varieties (see Proposition 4.6). In §5 we reproduce some recent results due to Rost concerning the modulo 2 Rost correspondences and Rost projectors on more general (as projective quadrics) varieties. In §6 we apply some standard motivic decompositions of projective homogeneous varieties to certain varieties related to a central simple algebra with an isotropic orthogonal involution. We also reproduce (see Theorem 6.1) some results of [9] which contain the needed generalization of indecomposability of the motive of an Albert quadric used in the previous paragraph. Finally, in §7 we prove Main theorem (Theorem 1.1) following the strategy of [8] and using results of [9] which were not available at the time of [8].

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## 2. Notation

We understand under a *variety* a separated scheme of finite type over a field.
Let $D$ be a central simple $F$-algebra. The $F$-dimension of any right ideal in $D$ is divisible by $\deg D$; the quotient is the *reduced dimension* of the ideal. For any integer $i$, we write $X(i;D)$ for the generalized Severi-Brauer variety of the right ideals in $D$ of reduced dimension $i$. In particular, $X(0;D) = \operatorname{Spec} F = X(\deg D; D)$ and $X(i,D) = \emptyset$ for $i < 0$ and for $i > \deg D$.
More generally, let $V$ be a right $D$-module. The $F$-dimension of $V$ is then divisible by $\deg D$ and the quotient $\operatorname{rdim} V = \dim_F V/\deg D$ is called the *reduced dimension* of $V$. For any integer $i$, we write $X(i;V)$ for the projective homogeneous variety of the $D$-submodules in $V$ of reduced dimension $i$ (non-empty iff $0 \le i \le \operatorname{rdim} V$). For a finite sequence of integers $i_1, \dots, i_r$, we write $X(i_1 \subset \dots \subset i_r; V)$ for the projective homogeneous variety of flags of the $D$-submodules in $V$ of reduced dimensions $i_1, \dots, i_r$ (non-empty iff $0 \le i_1 \le \dots \le i_r \le \operatorname{rdim} V$).
Now we additionally assume that $D$ is endowed with an orthogonal involution $\tau$. Then we write $X(i;(D,\tau))$ for the variety of the totally isotropic right ideals in $D$ of reduced dimension $i$ (non-empty iff $0 \le i \le \deg D/2$).
If moreover $V$ is endowed with a hermitian (with respect to $\tau$) form $h$, we write $X(i;(V,h))$ for the variety of the totally isotropic $D$-submodules in $V$ of reduced dimension $i$.
We refer to [10] for a detailed construction and basic properties of the above varieties. We only mention here that for the central simple algebra $A := \operatorname{End}_D V$ with the involution $\sigma$ adjoint to the hermitian form $h$, the varieties $X(i;(A,\sigma))$

and $X(i;(V,h))$ (for any $i \in \mathbb{Z}$) are canonically isomorphic. Besides, $\deg A = \operatorname{rdim} V$, and the following four conditions are equivalent:

(1) $\sigma$ is hyperbolic;
(2) $X((\deg A)/2;(A,\sigma))(F) \neq \emptyset$;
(3) $X((\operatorname{rdim} V)/2;(V,h))(F) \neq \emptyset$;
(4) $h$ is hyperbolic.

## 3. Krull-Schmidt principle

The characteristic of the base field $F$ is arbitrary in this section.
Our basic reference for Chow groups and Chow motives (including notation) is [6]. We fix an associative unital commutative ring $\Lambda$ (we shall take $\Lambda = \mathbb{F}_2$ in the application) and for a variety $X$ we write $\mathrm{CH}(X;\Lambda)$ for its Chow group with coefficients in $\Lambda$. Our category of motives is the category $\mathrm{CM}(F,\Lambda)$ of *graded Chow motives with coefficients in* $\Lambda$, [6, definition of §64]. By a *sum* of motives we always mean the *direct* sum.
We shall often assume that our coefficient ring $\Lambda$ is finite. This simplifies significantly the situation (and is sufficient for our application). For instance, for a finite $\Lambda$, the endomorphism rings of finite sums of Tate motives are also finite and the following easy statement applies:

Lemma 3.1. *An appropriate power of any element of any* finite *associative (not necessarily commutative) ring is idempotent.*

*Proof.* Since the ring is finite, any its element $x$ satisfies $x^a = x^{a+b}$ for some $a \geq 1$ and $b \geq 1$. It follows that $x^{ab}$ is an idempotent. $\square$

Let $X$ be a smooth complete variety over $F$. We call $X$ *split*, if its *integral* motive $M(X) \in \mathrm{CM}(F,\mathbb{Z})$ (and therefore its motive with any coefficients) is a finite sum of Tate motives. We call $X$ *geometrically split*, if it splits over a field extension of $F$. We say that $X$ satisfies the *nilpotence principle*, if for any field extension $E/F$ and any coefficient ring $\Lambda$, the kernel of the change of field homomorphism $\mathrm{End}(M(X)) \to \mathrm{End}(M(X)_E)$ consists of nilpotents. Any projective homogeneous variety is geometrically split and satisfies the nilpotence principle, [3, Theorem 8.2].

Corollary 3.2 ([9, Corollary 2.2]). *Assume that the coefficient ring $\Lambda$ is finite. Let $X$ be a geometrically split variety satisfying the nilpotence principle. Then an appropriate power of any endomorphism of the motive of $X$ is a projector.*

We say that the *Krull-Schmidt principle* holds for a given pseudo-abelian category, if every object of the category has one and unique decomposition in a finite direct sum of indecomposable objects. In the sequel, we are constantly using the following statement:

Corollary 3.3 ([4, Corollary 35], see also [9, Corollary 2.6]). *Assume that the coefficient ring $\Lambda$ is finite. The Krull-Schmidt principle holds for the pseudo-abelian Tate subcategory in $\mathrm{CM}(F,\Lambda)$ generated by the motives of the geometrically split $F$-varieties satisfying the nilpotence principle.* $\square$

Remark 3.4. Replacing the Chow groups $\mathrm{CH}(-;\Lambda)$ by the *reduced* Chow groups $\overline{\mathrm{CH}}(-;\Lambda)$ (cf. [6, §72]) in the definition of the category $\mathrm{CM}(F,\Lambda)$, we get a "simplified" motivic category $\overline{\mathrm{CM}}(F,\Lambda)$ (which is still sufficient for the main purpose of this paper). Working within this category, we do not need the nilpotence principle any more. In particular, the Krull-Schmidt principle holds (with a simpler proof) for the pseudo-abelian Tate subcategory in $\overline{\mathrm{CM}}(F,\Lambda)$ generated by the motives of the geometrically split $F$-varieties.

## 4. Splitting off a motivic summand

The characteristic of the base field $F$ is still arbitrary in this section.
In this section we assume that the coefficient ring $\Lambda$ is connected. We shall often assume that $\Lambda$ is finite.
Before climbing to the main result of this section (which is Proposition 4.6), let us do some warm up.
The following definition of [9] extends some terminology of [20]:

Definition 4.1. Let $M \in \mathrm{CM}(F,\Lambda)$ be a summand of the motive of a smooth complete irreducible variety of dimension $d$. The summand $M$ is called *upper*, if $\mathrm{CH}^0(M;\Lambda) \neq 0$. The summand $M$ is called *lower*, if $\mathrm{CH}_d(M;\Lambda) \neq 0$. The summand $M$ is called *outer*, if it is simultaneously upper and lower.

For instance, the whole motive of a smooth complete irreducible variety is an outer summand of itself. Another example of an outer summand is the motive given by a *Rost projector* (see Definition 5.1).
Given a correspondence $\alpha \in \mathrm{CH}_{\dim X}(X\times Y;\Lambda)$ between some smooth complete irreducible varieties $X$ and $Y$, we write $\operatorname{mult}\alpha \in \Lambda$ for the *multiplicity* of $\alpha$, [6, definition of §75]. Multiplicity of a composition of two correspondences is the product of multiplicities of the composed correspondences (cf. [11, Corollary 1.7]). In particular, multiplicity of a projector is idempotent and therefore $\in \{0,1\}$ because the coefficient ring $\Lambda$ is connected.
Characterizations of outer summands given in the two following Lemmas are easily obtained:

Lemma 4.2 (cf. [9, Lemmas 2.8 and 2.9]). *Let $X$ be a smooth complete irreducible variety. The motive $(X,p)$ given by a projector $p \in \mathrm{CH}_{\dim X}(X\times X;\Lambda)$ is upper if and only if* $\operatorname{mult} p = 1$. *The motive $(X,p)$ is lower if and only if* $\operatorname{mult} p^t = 1$, *where $p^t$ is the transpose of $p$.*

Lemma 4.3 (cf. [9, Lemma 2.12]). *Assume that a summand $M$ of the motive of a smooth complete irreducible variety of dimension $d$ decomposes into a sum of Tate motives. Then $M$ is upper if and only if the Tate motive $\Lambda$ is present in the decomposition; it is lower if and only if the Tate motive $\Lambda(d)$ is present in the decomposition.*

The following statement generalizes (the finite coefficient version of) [21, Corollary 3.9]:

Lemma 4.4. *Assume that the coefficient ring* $\Lambda$ *is finite. Let* $X$ *and* $Y$ *be smooth complete irreducible varieties such that there exist multiplicity* 1 *correspondences*

$$\alpha \in \mathrm{CH}_{\dim X}(X \times Y; \Lambda) \quad \text{and} \quad \beta \in \mathrm{CH}_{\dim Y}(Y \times X; \Lambda).$$

*Assume that* $X$ *is geometrically split and satisfies the nilpotence principle. Then there is an upper summand of* $M(X)$ *isomorphic to an upper summand of* $M(Y)$*. Moreover, for any upper summand* $M_X$ *of* $M(X)$ *and any upper summand* $M_Y$ *of* $M(Y)$*, there is an upper summand of* $M_X$ *isomorphic to an upper summand of* $M_Y$*.*

*Proof.* By Corollary 3.2, the composition $p := (\beta \circ \alpha)^{\circ n}$ for some $n \geq 1$ is a projector. Therefore $q := (\alpha \circ \beta)^{\circ 2n}$ is also a projector and the summand $(X, p)$ of $M(X)$ is isomorphic to the summand $(Y, q)$ of $M(Y)$. Indeed, the morphisms $\alpha : M(X) \to M(Y)$ and $\beta' := \beta \circ (\alpha \circ \beta)^{\circ(2n-1)} : M(Y) \to M(X)$ satisfy the relations $\beta' \circ \alpha = p$ and $\alpha \circ \beta' = q$.
Since $\operatorname{mult} p = (\operatorname{mult} \beta \cdot \operatorname{mult} \alpha)^n = 1$ and similarly $\operatorname{mult} q = 1$, the summand $(X, p)$ of $M(X)$ and the summand $(Y, q)$ of $M(Y)$ are upper by Lemma 4.2.
We have proved the first statement of Lemma 4.4. As to the second statement, let

$$p' \in \mathrm{CH}_{\dim X}(X \times X; \Lambda) \text{ and } q' \in \mathrm{CH}_{\dim Y}(Y \times Y; \Lambda)$$

be projectors such that $M_X = (X, p')$ and $M_Y = (Y, q')$. Replacing $\alpha$ and $\beta$ by $q' \circ \alpha \circ p'$ and $p' \circ \beta \circ q'$, we get isomorphic upper motives $(X, p)$ and $(Y, q)$ which are summands of $M_X$ and $M_Y$. $\square$

Remark 4.5. Assume that the coefficient ring $\Lambda$ is finite. Let $X$ be a geometrically split irreducible smooth complete variety satisfying the nilpotence principle. Then the complete motivic decomposition of $X$ contains precisely one upper summand and it follows by Corollary 3.3 (or by Lemma 4.4) that an upper indecomposable summands of $M(X)$ is unique up to an isomorphism. (Of course, the same is true for the lower summands.)

Here comes the needed replacement of [21, Theorem 4.15]:

Proposition 4.6. *Assume that the coefficient ring* $\Lambda$ *is finite. Let* $X$ *be a geometrically split, geometrically irreducible variety satisfying the nilpotence principle and let* $M$ *be a motive. Assume that there exists a field extension* $E/F$ *such that*

(1) *the field extension* $E(X)/F(X)$ *is purely transcendental;*
(2) *the upper indecomposable summand of* $M(X)_E$ *is also lower and is a summand of* $M_E$*.*

*Then the upper indecomposable summand of* $M(X)$ *is a summand of* $M$*.*

*Proof.* We may assume that $M = (Y, p, n)$ for some irreducible smooth complete $F$-variety $Y$, a projector $p \in \mathrm{CH}_{\dim Y}(Y \times Y; \Lambda)$, and an integer $n$.
By the assumption (2), we have morphisms of motives $f : M(X)_E \to M_E$ and $g : M_E \to M(X)_E$ with $\operatorname{mult}(g \circ f) = 1$. By [9, Lemma 2.14], in order to

prove Proposition 4.6, it suffices to construct morphisms $f' : M(X) \to M$ and $g' : M \to M(X)$ (over $F$) with $\mathrm{mult}(g' \circ f') = 1$.
Let $\xi : \operatorname{Spec} F(X) \to X$ be the generic point of the (irreducible) variety $X$. For any $F$-scheme $Z$, we write $\xi_Z$ for the morphism $\xi_Z = (\xi \times \mathrm{id}_Z) : Z_{F(X)} = \operatorname{Spec}_{F(X)} \times Z \to X \times Z$. Note that for any $\alpha \in \mathrm{CH}(X \times Z)$, the image $\xi_Z^*(\alpha) \in \mathrm{CH}(Z_{F(X)})$ of $\alpha$ under the pull-back homomorphism $\xi_Z^* : \mathrm{CH}(X \times Z, \Lambda) \to \mathrm{CH}(Z_{F(X)}, \Lambda)$ coincides with the composition of correspondences $\alpha \circ [\xi]$, [6, Proposition 62.4(2)], where $[\xi] \in \mathrm{CH}_0(X_{F(X)}, \Lambda)$ is the class of the point $\xi$:

$$(*) \qquad \xi_Z^*(\alpha) = \alpha \circ [\xi].$$

In the commutative square

$$\begin{array}{ccc} \mathrm{CH}(X_E \times Y_E; \Lambda) & \xrightarrow{\xi^*_{Y_E}} & \mathrm{CH}(Y_{E(X)}; \Lambda) \\ {\scriptstyle \mathrm{res}_{E/F}} \uparrow & & {\scriptstyle \mathrm{res}_{E(X)/F(X)}} \uparrow \\ \mathrm{CH}(X \times Y; \Lambda) & \xrightarrow{\xi^*_{Y}} & \mathrm{CH}(Y_{F(X)}; \Lambda) \end{array}$$

the change of field homomorphism $\mathrm{res}_{E(X)/F(X)}$ is surjective[1] because of the assumption (1) by the homotopy invariance of Chow groups [6, Theorem 57.13] and by the localization property of Chow groups [6, Proposition 57.11]. Moreover, the pull-back homomorphism $\xi_Y^*$ is surjective by [6, Proposition 57.11]. It follows that there exists an element $f' \in \mathrm{CH}(X \times Y; \Lambda)$ such that $\xi^*_{Y_E}(f'_E) = \xi^*_{Y_E}(f)$.
Recall that $\mathrm{mult}(g \circ f) = 1$. On the other hand, $\mathrm{mult}(g \circ f'_E) = \mathrm{mult}(g \circ f)$. Indeed, $\mathrm{mult}(g \circ f) = \deg \xi^*_{X_E}(g \circ f)$ by [6, Lemma 75.1], where $\deg : \mathrm{CH}(X_{E(X)}) \to \Lambda$ is the degree homomorphism. Furthermore, $\xi^*_{X_E}(g \circ f) = (g \circ f) \circ [\xi_E]$ by $(*)$. Finally, $(g \circ f) \circ [\xi_E] = g \circ (f \circ [\xi_E])$ and $f \circ [\xi_E] = \xi^*_{Y_E}(f) = \xi^*_{Y_E}(f'_E)$ by the construction of $f'$.
Replacing $f'$ be the composition $p \circ f'$, we get a morphism $f' : M(X) \to M$. Since the composition $g \circ f'_E$ is not changed, we still have $\mathrm{mult}(g \circ f'_E) = 1$.
Since $\mathrm{mult}(g \circ f'_E) = 1$ and the indecomposable upper summand of $M(X)_E$ is lower, we have $\mathrm{mult}((f'_E)^t \circ g^t) = 1$. Therefore we may apply the above procedure to the dual morphisms

$$g^t : M(X)_E \to (Y, p, \dim X - \dim Y - n)_E$$
$$\text{and} \quad (f'_E)^t : (Y, p, \dim X - \dim Y - n)_E \to M(X)_E.$$

This way we get a morphism $g' : M \to M(X)$ such that $\mathrm{mult}((f')^t \circ (g')^t) = 1$. It follows that $\mathrm{mult}(g' \circ f') = 1$. □

REMARK 4.7. Replacing $\mathrm{CM}(F, \Lambda)$ by $\overline{\mathrm{CM}}(F, \Lambda)$ in Proposition 4.6, we get a weaker version of Proposition 4.6 which is still sufficient for our application. The nilpotence principle is no more needed in the proof of the weaker version. Because of that, there is no more need to assume that $X$ satisfies the nilpotence principle.

[1] In fact, $\mathrm{res}_{E(X)/F(X)}$ is even an isomorphism, but we do not need its injectivity (which can be obtained with a help of a specialization).

## 5. Rost correspondences

In this section, $X$ stands for a smooth complete geometrically irreducible variety of a positive dimension $d$.
The coefficient ring $\Lambda$ of the motivic category is $\mathbb{F}_2$ in this section. We write $\mathrm{Ch}(-)$ for the Chow group $\mathrm{CH}(-;\mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$. We write $\deg_{X/F}$ for the degree homomorphism $\mathrm{Ch}_0(X) \to \mathbb{F}_2$.

Definition 5.1. An element $\rho \in \mathrm{Ch}_d(X \times X)$ is called a *Rost correspondence* (on $X$), if $\rho_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_{F(X)})$ of degree 1. A *Rost projector* is a Rost correspondence which is a projector.

Remark 5.2. Our definition of a Rost correspondence differs from the definition of a *special correspondence* in [17]. Our definition is weaker in the sense that a special correspondence on $X$ (which is an element of the *integral* Chow group $\mathrm{CH}_d(X \times X)$) considered modulo 2 is a Rost correspondence but not any Rost correspondence is obtained this way. This difference gives a reason to reproduce below some results of [17]. Actually, some of the results below are formally more general than the corresponding results of [17]; their proofs, however, are essentially the same.

Remark 5.3. Clearly, the set of all Rost correspondences on $X$ is stable under transposition and composition. In particular, if $\rho$ is a Rost correspondence, then its both symmetrizations $\rho^t \circ \rho$ and $\rho \circ \rho^t$ are (symmetric) Rost correspondences. Writing $\rho_{F(X)}$ as in Definition 5.1, we have $(\rho^t \circ \rho)_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_1$ (and $(\rho \circ \rho^t)_{F(X)} = \chi_2 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$).

Lemma 5.4. *Assume that the variety $X$ is projective homogeneous. Let $\rho \in \mathrm{Ch}_d(X \times X)$ be a projector. If there exists a field extension $E/F$ such that $\rho_E = \chi_1 \times [X_E] + [X_E] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_E)$ of degree 1, then $\rho$ is a Rost projector.*

*Proof.* According to [3, Theorem 7.5], there exist some integer $n \geq 0$ and for $i = 1, \dots, n$ some integers $r_i > 0$ and some projective homogeneous varieties $X_i$ satisfying $\dim X_i + r_i < d$ such that for $M = \bigoplus_{i=1}^n M(X_i)(r_i)$ the motive $M(X)_{F(X)}$ decomposes as $\mathbb{F}_2 \oplus M \oplus \mathbb{F}_2(d)$. Since there is no non-zero morphism between different summands of this three terms decomposition, the ring $\operatorname{End} M(X)$ decomposes in the product of rings

$$\operatorname{End} \mathbb{F}_2 \times \operatorname{End} M \times \operatorname{End} \mathbb{F}_2(d) = \mathbb{F}_2 \times \operatorname{End} M \times \mathbb{F}_2.$$

Let $\chi \in \mathrm{Ch}_0(X_{F(X)})$ be a 0-cycle class of degree 1. We set

$$\begin{aligned}\rho' = \chi \times [X_{F(X)}] + [X_{F(X)}] \times \chi &\in \mathbb{F}_2 \times \mathbb{F}_2 \\ &\subset \mathbb{F}_2 \times \operatorname{End} M \times \mathbb{F}_2 = \operatorname{End} M(X)_{F(X)} = \mathrm{Ch}_d(X_{F(X)} \times X_{F(X)})\end{aligned}$$

and we show that $\rho_{F(X)} = \rho'$. The difference $\varepsilon = \rho_{F(X)} - \rho'$ vanishes over $E(X)$. Therefore $\varepsilon$ is a nilpotent element of $\operatorname{End} M$. Choosing a positive integer $m$

with $\varepsilon^m = 0$, we get

$$\rho_{F(X)} = \rho^m_{F(X)} = (\rho' + \varepsilon)^m = (\rho')^m + \varepsilon^m = (\rho')^m = \rho'. \qquad \square$$

LEMMA 5.5. *Let $\rho \in \mathrm{Ch}_d(X \times X)$ be a projector. The motive $(X, \rho)$ is isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2(d)$ iff $\rho = \chi_1 \times [X] + [X] \times \chi_2$ for some some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X)$ of degree 1.*

*Proof.* A morphism $\mathbb{F}_2 \oplus \mathbb{F}_2(d) \to (X, \rho)$ is given by some

$$f \in \mathrm{Hom}\big(\mathbb{F}_2, M(X)\big) = \mathrm{Ch}_0(X) \ \text{ and } \ f' \in \mathrm{Hom}\big(\mathbb{F}_2(d), M(X)\big) = \mathrm{Ch}_d(X).$$

A morphism in the inverse direction is given by some

$$g \in \mathrm{Hom}(M(X), \mathbb{F}_2) = \mathrm{Ch}^0(X) \ \text{ and } \ g' \in \mathrm{Hom}(M(X), \mathbb{F}_2(d)) = \mathrm{Ch}^d(X).$$

The two morphisms $\mathbb{F}_2 \oplus \mathbb{F}_2(d) \leftrightarrow (X, \rho)$ are mutually inverse isomorphisms iff $\rho = f \times g + f' \times g'$ and $\deg_{X/F}(fg) = 1 = \deg_{X/F}(f'g')$. The degree condition means that $f' = [X] = g$ and $\deg_{X/F}(f) = 1 = \deg_{X/F}(g')$. $\square$

COROLLARY 5.6. *If $X$ is projective homogeneous and $\rho$ is a projector on $X$ such that*

$$(X, \rho)_E \simeq \mathbb{F}_2 \oplus \mathbb{F}_2(d)$$

*for some field extension $E/F$, then $\rho$ is a Rost projector.* $\square$

A smooth complete variety is called *anisotropic*, if the degree of its any closed point is even.

LEMMA 5.7 ([17, Lemma 9.2], cf. [18, proof of Lemma 6.2]). *Assume that $X$ is anisotropic and possesses a Rost correspondence $\rho$. Then for any integer $i \neq d$ and any elements $\alpha \in \mathrm{Ch}_i(X)$ and $\beta \in \mathrm{Ch}^i(X_{F(X)})$, the image of the product $\alpha_{F(X)} \cdot \beta \in \mathrm{Ch}_0(X_{F(X)})$ under the degree homomorphism $\deg_{X_{F(X)}/F(X)} : \mathrm{Ch}_0(X_{F(X)}) \to \mathbb{F}_2$ is 0.*

*Proof.* Let $\gamma \in \mathrm{Ch}^i(X \times X)$ be a preimage of $\beta$ under the surjection

$$\xi_X^* : \mathrm{Ch}^i(X \times X) \to \mathrm{Ch}^i(\mathrm{Spec}\, F(X) \times X)$$

(where $\xi_X^*$ is as defined in the proof of Proposition 4.6). We consider the 0-cycle class

$$\delta = \rho \cdot ([X] \times \alpha) \cdot \gamma \in \mathrm{Ch}_0(X \times X).$$

Since $X$ is anisotropic, so is $X \times X$, and it follows that $\deg_{(X\times X)/F} \delta = 0$. Therefore it suffices to show that $\deg_{(X\times X)/F} \delta = \deg_{X_{F(X)}/F(X)}(\alpha_{F(X)} \cdot \beta)$. We have $\deg_{(X\times X)/F} \delta = \deg_{(X\times X)_{F(X)}/F(X)}(\delta_{F(X)})$ and

$$\delta_{F(X)} = (\chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2) \cdot ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)} = \\ (\chi_1 \times [X_{F(X)}]) \cdot ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}$$

(because $i \neq d$) where $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_{F(X)})$ are as in Definition 5.1. For the first projection $pr_1 : X_{F(X)} \times X_{F(X)} \to X_{F(X)}$ we have

$$\deg_{(X\times X)_{F(X)}/F(X)} \delta_{F(X)} = \deg_{X_{F(X)}/F(X)}(pr_1)_*(\delta_{F(X)})$$

and by the projection formula

$$(pr_1)_*(\delta_{F(X)}) = \chi_1 \cdot (pr_1)_*\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big).$$

Finally,

$$(pr_1)_*\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) = \operatorname{mult}\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) \cdot [X_{F(X)}]$$

and

$$\operatorname{mult}\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) = \operatorname{mult}\big(([X] \times \alpha) \cdot \gamma\big).$$

Since $\operatorname{mult}\chi = \deg_{X_{F(X)}/F(X)} \xi_X^*(\chi)$ for any element $\chi \in \operatorname{Ch}_d(X \times X)$ by [6, Lemma 75.1], it follows that

$$\operatorname{mult}\big(([X] \times \alpha) \cdot \gamma\big) = \deg(\alpha_{F(X)} \cdot \beta). \qquad \square$$

For anisotropic $X$, we consider the homomorphism $\deg/2 : \operatorname{Ch}_0(X) \to \mathbb{F}_2$ induced by the homomorphism $\operatorname{CH}_0(X) \to \mathbb{Z}$, $\alpha \mapsto \deg(\alpha)/2$.

Corollary 5.8. *Assume that $X$ is anisotropic and possesses a Rost correspondence. Then for any integer $i \neq d$ and any elements $\alpha \in \operatorname{Ch}_i(X)$ and $\beta \in \operatorname{Ch}^i(X)$ with $\beta_{F(X)} = 0$ one has* $(\deg/2)(\alpha \cdot \beta) = 0$.

*Proof.* Let $\beta' \in \operatorname{CH}^i(X)$ be an integral representative of $\beta$. Since $\beta_{F(X)} = 0$, we have $\beta'_{F(X)} = 2\beta''$ for some $\beta'' \in \operatorname{CH}^i(X_{F(X)})$. Therefore

$$(\deg/2)(\alpha \cdot \beta) = \deg_{X_{F(X)}/F(X)}\big(\alpha_{F(X)} \cdot (\beta'' \mod 2)\big) = 0$$

by Lemma 5.7. $\square$

Corollary 5.9. *Assume that $X$ is anisotropic and possesses a Rost correspondence $\rho$. For any integer $i \notin \{0, d\}$ and any $\alpha \in \operatorname{Ch}_i(X)$ and $\beta \in \operatorname{Ch}^i(X)$ one has*

$$(\deg/2)\big((\alpha \times \beta) \cdot \rho\big) = 0.$$

*Proof.* Let $\alpha' \in \operatorname{CH}_i(X)$ and $\beta' \in \operatorname{CH}^i(X)$ be integral representatives of $\alpha$ and $\beta$. Let $\rho' \in \operatorname{CH}_d(X \times X)$ be an integral representative of $\rho$. It suffices to show that the degree of the 0-cycle class $(\alpha' \times \beta') \cdot \rho' \in \operatorname{CH}_0(X \times X)$ is divisible by 4.

Let $\chi_1$ and $\chi_2$ be as in Definition 5.1. Let $\chi'_1, \chi'_2 \in \operatorname{CH}_0(X_{F(X)})$ be integral representatives of $\chi_1$ and $\chi_2$. Then $\rho'_{F(X)} = \chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2 + 2\gamma$ for some $\gamma \in \operatorname{CH}_d(X_{F(X)} \times X_{F(X)})$. Therefore (since $i \notin \{0, d\}$)

$$(\alpha'_{F(X)} \times \beta'_{F(X)}) \cdot \rho'_{F(X)} = 2(\alpha'_{F(X)} \times \beta'_{F(X)}) \cdot \gamma.$$

Applying the projection $pr_1$ onto the first factor and the projection formula, we get twice the element $\alpha'_{F(X)} \cdot (pr_1)_*\big(([X_{F(X)}] \times \beta'_{F(X)}) \cdot \gamma\big)$ whose degree is even by Lemma 5.7 (here we use once again the condition that $i \neq d$). $\square$

Lemma 5.10. *Assume that $X$ is anisotropic and possesses a Rost correspondence $\rho$. Then* $(\deg/2)(\rho^2) = 1$.

*Proof.* Let $\chi_1$ and $\chi_2$ be as in Definition 5.1. Let $\chi'_1, \chi'_2 \in \mathrm{CH}_0(X_E)$ be integral representatives of $\chi_1$ and $\chi_2$. The degrees of $\chi'_1$ and $\chi'_2$ are odd. Therefore, the degree of the cycle class

$$(\chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2)^2 = 2(\chi'_1 \times \chi'_2) \in \mathrm{CH}_0(X_{F(X)} \times X_{F(X)})$$

is not divisible by 4.
Let $\rho' \in \mathrm{CH}_d(X \times X)$ be an integral representative of $\rho$. Since $\rho'_{F(X)}$ is $\chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2$ modulo 2, $(\rho'_{F(X)})^2$ is $(\chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2)^2$ modulo 4. Therefore $(\deg/2)(\rho^2) = 1$. □

THEOREM 5.11 ([17, Theorem 9.1], see also [18, proof of Lemma 6.2]). *Let $X$ be an anisotropic smooth complete geometrically irreducible variety of a positive dimension $d$ over a field $F$ of characteristic $\neq 2$ possessing a Rost correspondence. Then the degree of the highest Chern class $c_d(-T_X)$, where $T_X$ is the tangent bundle on $X$, is not divisible by 4.*

*Proof.* In this proof, we write $c_\bullet(-T_X)$ for the total Chern class $\in \mathrm{Ch}(X)$ in the Chow group with coefficient in $\mathbb{F}_2$. It suffices to show that $(\deg/2)(c_d(-T_X)) = 1$.
Let $\mathrm{Sq}^X_\bullet : \mathrm{Ch}(X) \to \mathrm{Ch}(X)$ be the modulo 2 homological Steenrod operation, [6, §59]. We have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Ch}_d(X \times X) \arrow[dl, "(pr_1)_*"'] \arrow[dd, "\mathrm{Sq}_d^{X\times X}"] & \\
\mathrm{Ch}_d(X) \arrow[dd, "\mathrm{Sq}_d^X"'] & & \\
 & \mathrm{Ch}_0(X \times X) \arrow[dl, "(pr_1)_*"'] \arrow[dr, "(pr_2)_*"] \arrow[dd, "\deg/2"] & \\
\mathrm{Ch}_0(X) \arrow[dr, "\deg/2"'] & & \mathrm{Ch}_0(X) \arrow[dl, "\deg/2"] \\
 & \mathbb{F}_2 &
\end{tikzcd}$$

Since $(pr_1)_*(\rho) = [X]$ and $\mathrm{Sq}_d^X([X]) = c_d(-T_X)$ [6, formula (60.1)], it suffices to show that

$$(\deg/2)\big(\mathrm{Sq}_d^{X\times X}(\rho)\big) = 1.$$

We have $\mathrm{Sq}_\bullet^{X\times X} = c_\bullet(-T_{X\times X}) \cdot \mathrm{Sq}^\bullet_{X\times X}$, where $\mathrm{Sq}^\bullet$ is the cohomological Steenrod operation, [6, §61]. Therefore

$$\mathrm{Sq}_d^{X\times X}(\rho) = \sum_{i=0}^{d} c_{d-i}(-T_{X\times X}) \cdot \mathrm{Sq}^i_{X\times X}(\rho).$$

The summand with $i = d$ is $\mathrm{Sq}^d_{X\times X}(\rho) = \rho^2$ by [6, Theorem 61.13]. By Lemma 5.10, its image under deg/2 is 1.

Since $c_\bullet(-T_{X\times X}) = c_\bullet(-T_X)\times c_\bullet(-T_X)$ and $\mathrm{Sq}^0 = \mathrm{id}$, the summand with $i=0$ is

$$\left(\sum_{j=0}^{d} c_j(-T_X)\times c_{d-j}(-T_X)\right)\cdot\rho.$$

Its image under deg/2 is 0 because

$$(\deg/2)\Big(\big(c_0(-T_X)\times c_d(-T_X)\big)\cdot\rho\Big) = (\deg/2)(c_d(-T_X)) = (\deg/2)\Big(\big(c_d(-T_X)\times c_0(-T_X)\big)\cdot\rho\Big)$$

while for $j\notin\{0,d\}$, we have $(\deg/2)\Big(\big(c_j(-T_X)\times c_{d-j}(-T_X)\big)\cdot\rho\Big) = 0$ by Corollary 5.9.

Finally, for any $i$ with $0<i<d$ the $i$th summand is the sum

$$\sum_{j=0}^{d-i}\big(c_j(-T_X)\times c_{d-i-j}(-T_X)\big)\cdot\mathrm{Sq}^i_{X\times X}(\rho).$$

We shall show that for any $j$ the image of the $j$th summand under deg/2 is 0. Note that the image under deg/2 coincides with the image under the composition $(\deg/2)\circ(pr_1)_*$ and also under the composition $(\deg/2)\circ(pr_2)_*$ (look at the above commutative diagram). By the projection formula we have

$$(pr_1)_*\Big(\big(c_j(-T_X)\times c_{d-i-j}(-T_X)\big)\cdot\mathrm{Sq}^i_{X\times X}(\rho)\Big) = c_j(-T_X)\cdot(pr_1)_*\Big(\big([X]\times c_{d-i-j}(-T_X)\big)\cdot\mathrm{Sq}^i_{X\times X}(\rho)\Big)$$

and the image under deg/2 is 0 for positive $j$ by Corollary 5.8 applied to $\alpha = c_j(-T_X)$ and $\beta = (pr_1)_*\Big(\big([X]\times c_{d-i-j}(-T_X)\big)\cdot\mathrm{Sq}^i_{X\times X}(\rho)\Big)$. Corollary 5.8 can be indeed applied, because since $\rho_{F(X)} = \chi_1\times[X_{F(X)}]+[X_{F(X)}]\times\chi_2$ and $i>0$, we have $\mathrm{Sq}^i_{(X\times X)_{F(X)}}(\rho)_{F(X)} = 0$ and therefore $\beta_{F(X)} = 0$.

For $j=0$ we use the projection formula for $pr_2$ and Corollary 5.8 with $\alpha = c_{d-i}(-T_X)$ and $\beta = (pr_2)_*\big(\mathrm{Sq}^i_{X\times X}(\rho)\big)$. □

REMARK 5.12. The reason of the characteristic exclusion in Theorem 5.11 is that its proof makes use of Steenrod operations on Chow groups with coefficients in $\mathbb{F}_2$ which (the operations) -are not available in characteristic 2.

We would like to mention

LEMMA 5.13 ([17, Lemma 9.10]). *Let $X$ be an anisotropic smooth complete equidimensional variety over a field of arbitrary characteristic. If $\dim X+1$ is not a power of 2, then the degree of the integral 0-cycle class $c_{\dim X}(-T_X)\in\mathrm{CH}_0(X)$ is divisible by 4.*

COROLLARY 5.14 ([17, Corollary 9.12]). *In the situation of Theorem 5.11, the integer $\dim X+1$ is a power of 2.* □

## 6. Motivic decompositions of some isotropic varieties

The coefficient ring $\Lambda$ is $\mathbb{F}_2$ in this section. Throughout this section, $D$ is a central division $F$-algebra of degree $2^r$ with some positive integer $r$.

We say that motives $M$ and $N$ are *quasi-isomorphic* and write $M \approx N$, if there exist decompositions $M \simeq M_1 \oplus \cdots \oplus M_m$ and $N \simeq N_1 \oplus \cdots \oplus N_n$ such that

$$M_1(i_1) \oplus \cdots \oplus M_m(i_m) \simeq N_1(j_1) \oplus \cdots \oplus N_n(j_n)$$

for some (shift) integers $i_1, \dots, i_m$ and $j_1, \dots, j_n$.

We shall use the following

Theorem 6.1 ([9, Theorems 3.8 and 4.1]). *For any integer $l = 0, 1, \dots, r$, the upper indecomposable summand $M_l$ of the motive of the generalized Severi-Brauer variety $X(2^l; D)$ is lower. Besides of this, the motive of any finite direct product of any generalized Severi-Brauer varieties of $D$ is quasi-isomorphic to a finite sum of $M_l$ (with various $l$).*

For the rest of this section, we fix an orthogonal involution on the algebra $D$.

Lemma 6.2. *Let $n$ be an positive integer. Let $h$ be a hyperbolic hermitian form on the right $D$-module $D^{2n}$ and let $Y$ be the variety $X(n \deg D; (D^{2n}, h))$ (of the* maximal *totally isotropic submodules). Then the motive $M(Y)$ is isomorphic to a finite sum of several shifted copies of the motives $M_0, M_1, \dots, M_r$.*

*Proof.* By [10, §15] the motive of the variety $Y$ is quasi-isomorphic to the motive of the "total" variety

$$X(*; D^n) = \coprod_{i \in \mathbb{Z}} X(i; D^n) = \coprod_{i=0}^{2^r n} X(i; D^n)$$

of $D$-submodules in $D^n$ (the range limit $2^r n$ is the reduced dimension of the $D$-module $D^n$). (Note that in our specific situation we always have $i = j$ in the flag varieties $X(i \subset j; D^n)$ which appear in the general formula of [10, Следствие 15.14].) Furthermore, $M(X(*; D^n)) \approx M(X(*; D))^{\otimes n}$ by [10, Следствие 10.19]. Therefore the motive of $Y$ is a direct sum of the motives of products of generalized Severi-Brauer varieties of $D$. (One can also come to this conclusion by [2] computing the semisimple anisotropic kernel of the connected component of the algebraic group $\mathrm{Aut}(D^{2n}, h)$.) We finish by Theorem 6.1. □

As before, we write $\mathrm{Ch}(-)$ for the Chow group $\mathrm{CH}(-; \mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$. We recall that a smooth complete variety is called *anisotropic*, if the degree of its any closed point is even (the empty variety is anisotropic). The following statement is a particular case of [9, Lemma 2.21].

Lemma 6.3. *Let $Z$ be an anisotropic $F$-variety with a projector $p \in \mathrm{Ch}_{\dim Z}(Z \times Z)$ such that the motive $(Z, p)_L \in \mathrm{CM}(L, \mathbb{F}_2)$ for a field extension $L/F$ is isomorphic to a finite sum of Tate motives. Then the number of the Tate summands is even. In particular, the motive in $\mathrm{CM}(F, \mathbb{F}_2)$ of any anisotropic $F$-variety does not contain a Tate summand.*

*Proof.* Mutually inverse isomorphisms between $(Z,p)_L$ and a sum of, say, $n$ Tate summands, are given by two sequences of homogeneous elements $a_1,\dots,a_n$ and $b_1,\dots,b_n$ in $\mathrm{Ch}(Z_L)$ with $p_L = a_1\times b_1+\dots+a_n\times b_n$ and such that for any $i,j=1,\dots,n$ the degree $\deg(a_ib_j)$ is 0 for $i\neq j$ and $1\in\mathbb{F}_2$ for $i=j$. The pull-back of $p$ via the diagonal morphism of $Z$ is therefore a 0-cycle class on $Z$ of degree $n$ (modulo 2). □

Lemma 6.4. *Let $n$ be an integer $\geq 0$. Let $h'$ be a hermitian form on the right $D$-module $D^n$ such that $h'_L$ is anisotropic for any finite odd degree field extension $L/F$. Let $h$ be the hermitian form on the right $D$-module $D^{n+2}$ which is the orthogonal sum of $h'$ and a hyperbolic $D$-plane. Let $Y'$ be the variety of totally isotropic submodules of $D^{n+2}$ of reduced dimension $2^r$ ($=\operatorname{ind} D$). Then the complete motivic decomposition of $M(Y')\in\mathrm{CM}(F,\mathbb{F}_2)$ (cf. Corollary 3.3) contains one summand $\mathbb{F}_2$, one summand $\mathbb{F}_2(\dim Y')$, and does not contain any other Tate motive.*

*Proof.* Since $Y'(F)\neq\emptyset$, $M(Y')$ contains an exemplar of the Tate motive $\mathbb{F}_2$ and an exemplar of the Tate motive $\mathbb{F}_2(\dim Y')$.
According to [10, Следствие 15.14] (see also [10, Следствие 15.9]), $M(Y')$ is quasi-isomorphic to the sum of the motives of the products

$$X(i\subset j;D)\times X(j-i;(D^n,h'))$$

where $i,j$ run over all integers (the product is non-empty only if $0\leq i\leq j\leq 2^r$). The choices $i=j=0$ and $i=j=2^r$ give two exemplars of the Tate motive $\mathbb{F}_2$ (up to a shift). The variety obtained by any other choice of $i,j$ but $i=0, j=2^r$ is anisotropic because the algebra $D$ is division. The variety with $i=0,j=2^r$ is anisotropic by the assumption involving the odd degree field extensions. Lemma 6.3 terminates the proof. □

## 7. Proof of Main theorem

We fix a central simple algebra $A$ of index $>1$ with a non-hyperbolic orthogonal involution $\sigma$. Since the involution is an isomorphism of $A$ with its dual, the exponent of $A$ is 2; therefore, the index of $A$ is a power of 2, say, $\operatorname{ind} A=2^r$ for a positive integer $r$. We assume that $\sigma$ becomes hyperbolic over the function field of the Severi-Brauer variety of $A$ and we are looking for a contradiction.
According to [12, Theorem 3.3], $\operatorname{coind} A=2n$ for some integer $n\geq 1$. We assume that Main theorem (Theorem 1.1) is already proven for all algebras (over all fields) of index $<2^r$ as well as for all algebras of index $2^r$ and coindex $<2n$.
Let $D$ be a central division algebra Brauer-equivalent to $A$. Let $X_0$ be the Severi-Brauer variety of $D$. Let us fix an (arbitrary) orthogonal involution $\tau$ on $D$ and an isomorphism of $F$-algebras $A\simeq\operatorname{End}_D(D^{2n})$. Let $h$ be a hermitian (with respect to $\tau$) form on the right $D$-module $D^{2n}$ such that $\sigma$ is adjoint to $h$. Then $h_{F(X_0)}$ is hyperbolic. Since the anisotropic kernel of $h$ also becomes hyperbolic over $F(X_0)$, our induction hypothesis ensures that $h$ is anisotropic. Moreover, $h_L$ is hyperbolic for any field extension $L/F$ such that $h_L$ is isotropic.

It follows by [1, Proposition 1.2] that $h_L$ is anisotropic for any finite odd degree field extension $L/F$.

Let $Y$ be the variety of totally isotropic submodules in $D^{2n}$ of reduced dimension $n \deg D$. (The variety $Y$ is a twisted form of the variety of maximal totally isotropic subspaces of a quadratic form studied in [6, Chapter XVI].) It is isomorphic to the variety of totally isotropic right ideals in $A$ of reduced dimension $(\deg A)/2$ $(=n2^r)$. Since $\sigma$ is hyperbolic over $F(X_0)$ and the field $F$ is algebraically closed in $F(X_0)$ (because the variety $X_0$ is geometrically integral), the discriminant of $\sigma$ is trivial. Therefore the variety $Y$ has two connected components $Y_+$ and $Y_-$ corresponding to the components $C_+$ and $C_-$ (cf. [6, Theorem 8.10]) of the Clifford algebra $C(A, \sigma)$. Note that the varieties $Y_+$ and $Y_-$ are projective homogeneous under the connected component of the algebraic group $\mathrm{Aut}(D^{2n}, h) = \mathrm{Aut}(A, \sigma)$.

The central simple algebras $C_+$ and $C_-$ are related with $A$ by the formula [13, (9.14)]:

$$[C_+] + [C_-] = [A] \in \mathrm{Br}(F).$$

Since $[C_+]_{F(X_0)} = [C_-]_{F(X_0)} = 0 \in \mathrm{Br}(F(X_0))$, we have $[C_+], [C_-] \in \{0, [A]\}$ and it follows that $[C_+] = 0$, $[C_-] = [A]$ up to exchange of the indices $+, -$.

By the index reduction formula for the varieties $Y_+$ and $Y_-$ of [15, page 594], we have: $\operatorname{ind} D_{F(Y_+)} = \operatorname{ind} D$, $\operatorname{ind} D_{F(Y_-)} = 1$.

Below we will work with the variety $Y_+$ and not with the variety $Y_-$. One reason of this choice is Lemma 7.1. Another reason of the choice is that we need $D_{F(Y_+)}$ to be a division algebra when applying Proposition 4.6 in the proof of Lemma 7.2.

Lemma 7.1. *For any field extension $L/F$ one has:*

a) $Y_-(L) \neq \emptyset \Leftrightarrow D_L$ *is Brauer-trivial* $\Leftrightarrow D_L$ *is Brauer-trivial and* $\sigma_L$ *is hyperbolic;*

b) $Y_+(L) \neq \emptyset \Leftrightarrow \sigma_L$ *is hyperbolic.*

*Proof.* Since $\sigma_{F(X_0)}$ is hyperbolic, $Y(F(X_0)) \neq \emptyset$. Since the varieties $Y_+$ and $Y_-$ become isomorphic over $F(X_0)$, each of them has an $F(X_0)$-point. Moreover, $X_0$ has an $F(Y_-)$-point. □

For the sake of notation simplicity, we write $Y$ for $Y_+$ (we will not meet the old $Y$ anymore).

The coefficient ring $\Lambda$ is $\mathbb{F}_2$ in this section. We use the $F$-motives $M_0, \ldots, M_r$ introduced in Theorem 6.1. Note that for any field extension $E/F$ such that $D_E$ is still a division algebra, we also have the $E$-motives $M_0, \ldots, M_r$.

Lemma 7.2. *The motive of $Y$ decomposes as $R_1 \oplus R_2$, where $R_1$ is quasi-isomorphic to a finite sum of several copies of the motives $M_0, \ldots, M_{r-1}$, and where $(R_2)_{F(Y)}$ is isomorphic to a finite sum of Tate motives including one exemplar of $\mathbb{F}_2$.*

*Proof.* According to Lemma 6.2, the motive $M(Y)_{F(Y)}$ is isomorphic to a sum of several shifted copies of the $F(Y)$-motives $M_0, \ldots, M_r$ (introduced in Theorem 6.1). Since $Y_{F(Y)} \neq \emptyset$, a (non-shifted) copy of the Tate motive $\mathbb{F}_2$ shows

up. If for some $l = 0, \dots, r-1$ there is at least one copy of $M_l$ (with a shift $j \in \mathbb{Z}$) in the decomposition, let us apply Proposition 4.6 taking as $X$ the variety $X_l = X(2^l; D)$, taking as $M$ the motive $M(Y)(-j)$, and taking as $E$ the function field $F(Y)$.
Since $D_E$ is a division algebra, condition (2) of Proposition 4.6 is fulfilled. Since $\operatorname{ind} D_{F(X)} < 2^r$, the hermitian form $h_{F(X)}$ is hyperbolic by the induction hypothesis; therefore the variety $Y_{F(X)}$ is rational (see Remark 7.1) and condition (1) of Proposition 4.6 is fulfilled as well.
It follows that the $F$-motive $M_l$ is a summand of $M(Y)(-j)$. Let now $M$ be the complement summand of $M(Y)(-j)$. By Corollary 3.3, the complete decomposition of $M_{F(Y)}$ is the complete decomposition of $M(Y)(-j)_{F(Y)}$ with one copy of $M_l$ erased. If $M_{F(Y)}$ contains one more copy of a shift of $M_l$ (for some $l = 0, \dots, r-1$), we once again apply Proposition 4.6 to the variety $X_l$ and an appropriate shift of $M$. Doing this until we can, we get the desired decomposition in the end. □

Now let us consider a minimal right $D$-submodule $V \subset D^{2n}$ such that $V$ becomes isotropic over a finite odd degree field extension of $F(Y)$. We set $v = \dim_D V$. Clearly, $v \geq 2$ (because $D_{F(Y)}$ is a division algebra). For $v > 2$, let $Y'$ be the variety $X(2^r; (V, h|_V))$ of totally isotropic submodules in $V$ of reduced dimension $2^r$ (that is, of "$D$-dimension" 1). Writing $\tilde{F}$ for an odd degree field extension of $F(Y)$ with isotropic $V_{\tilde{F}}$, we have $Y'(\tilde{F}) \neq \emptyset$ (because $D_{\tilde{F}}$ is a division algebra). Therefore there exists a correspondence of odd multiplicity (that is, of multiplicity $1 \in \mathbb{F}_2$) $\alpha \in \mathrm{Ch}_{\dim Y}(Y \times Y')$.
If $v = 2$, then $h|_V$ becomes hyperbolic over (an odd degree extension of) $F(Y)$. Therefore $h|_V$ becomes hyperbolic over $F(X_0)$, and our induction hypothesis actually insures that $n = v = 2$. In this case we simply take $Y' := Y$ (our component).
The variety $Y'$ is projective homogeneous (in particular, irreducible) of dimension

$$\dim Y' = 2^{r-1}(2^r - 1) + 2^{2r}(v-2)$$

which is equal to a power of 2 minus 1 only if $r = 1$ and $v = 2$. Moreover, the variety $Y'$ is anisotropic (because the hermitian form $h$ is anisotropic and remains anisotropic over any finite odd degree field extension of the base field). Surprisingly, we can however prove the following

Lemma 7.3. *There is a Rost projector (Definition 5.1) on $Y'$.*

*Proof.* By the construction of $Y'$, there exists a correspondence of odd multiplicity (that is, of multiplicity $1 \in \mathbb{F}_2$) $\alpha \in \mathrm{Ch}_{\dim Y}(Y \times Y')$. On the other hand, since $h_{F(Y')}$ is isotropic, $h_{F(Y')}$ is hyperbolic and therefore there exist a rational map $Y' \dashrightarrow Y$ and a multiplicity 1 correspondence $\beta \in \mathrm{Ch}_{\dim Y'}(Y' \times Y)$ (e.g., the class of the closure of the graph of the rational map). Since the summand $R_2$ of $M(Y)$ given by Lemma 7.2 is upper (cf. Definition 4.1 and Lemma 4.3), by Lemma 4.4 there is an upper summand of $M(Y')$ isomorphic to a summand of $R_2$.

Let $\rho \in \mathrm{Ch}_{\dim Y'}(Y' \times Y')$ be the projector giving this summand. We claim that $\rho$ is a Rost projector. We prove the claim by showing that the motive $(Y', \rho)_{\tilde{F}}$ is isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y')$, cf. Corollary 5.6, where $\tilde{F}/F(Y)$ is a finite odd degree field extension such that $V$ becomes isotropic over $\tilde{F}$.
Since $(R_2)_{F(Y)}$ is a finite sum of Tate motives, the motive $(Y', \rho)_{\tilde{F}}$ is also a finite sum of Tate motives. Since $(Y', \rho)_{\tilde{F}}$ is upper, the Tate motive $\mathbb{F}_2$ is included (Lemma 4.3). Now, by the minimal choice of $V$, the hermitian form $(h|_V)_{\tilde{F}}$ satisfies the condition on $h$ in Lemma 6.4: $(h|_V)_{\tilde{F}}$ is an orthogonal sum of a hyperbolic $D_{\tilde{F}}$-plane and a hermitian form $h'$ such that $h'_L$ is anisotropic for any finite odd degree field extension $L/\tilde{F}$ of the base field $\tilde{F}$. Indeed, otherwise – if $h'_L$ is isotropic for some such $L$, the module $V_L$ contains a totally isotropic submodule $W$ of $D$-dimension 2; any $D$-hyperplane $V' \subset V$, considered over $L$, meets $W$ non-trivially; it follows that $V'_L$ is isotropic and this contradicts to the minimality of $V$. (This is a very standard argument in the theory of quadratic forms over field which we applied now to a hermitian form over a division algebra.)
Therefore, by Lemma 6.4, the complete motivic decomposition of $Y'_{\tilde{F}}$ has one copy of $\mathbb{F}_2$, one copy of $\mathbb{F}_2(\dim Y')$, and no other Tate summands. By Corollary 3.3 and anisotropy of the variety $Y'$ (see Lemma 6.3), it follows that

$$(Y', \rho)_{\tilde{F}} \simeq \mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y'). \qquad \square$$

If we are away from the case where $r = 1$ and $v = 2$, then Lemma 7.3 contradicts to Corollary 5.14 thus proving Main theorem (Theorem 1.1). Note that Corollary 5.14 is a formal consequence of Theorem 5.11 and Lemma 5.13. We can avoid the use of Lemma 5.13 by showing that $\deg c_{\dim Y'}(-T_{Y'})$ is divisible by 4 for our variety $Y'$. Indeed, if $v > 2$, then let $K$ be the field $F(t_1, \ldots, t_{v2^r})$ of rational functions over $F$ in $v2^r$ variables. Let us consider the (generic) diagonal quadratic form $\langle t_1, \ldots, t_{v2^r}\rangle$ on the $K$-vector space $K^{v2^r}$. Let $Y''$ be the variety of $2^r$-dimensional totally isotropic subspaces in $K^{v2^r}$. The degree of any closed point on $Y''$ is divisible by $2^{2^r}$. In particular, the integer $\deg c_{\dim Y''}(-T_{Y''})$ is divisible by $2^{2^r}$. Since over an algebraic closure $\bar{K}$ of $K$ the varieties $Y'$ and $Y''$ become isomorphic, we have

$$\deg c_{\dim Y'}(-T_{Y'}) = \deg c_{\dim Y''}(-T_{Y''}).$$

If $v = 2$ and $r > 1$, we can play the same game, taking as $Y''$ a component of the variety of $2^r$-dimensional totally isotropic subspaces of the (generic) diagonal quadratic form (of trivial discriminant) $\langle t_1, \ldots, t_{v2^r-1}, t_1 \ldots t_{v2^r-1}\rangle$, because the degree of any closed point on $Y''$ is divisible by $2^{2^r-1}$.
Finally, the remaining case where $r = 1$ and $v = 2$ needs a special argument (or reference). Indeed, in this case, the variety $Y'$ is a conic, and therefore Lemma 7.3 does not provide any information on $Y'$. Of course, a reference to [16] allows one to avoid consideration of the case of $r = 1$ (and any $v$) at all. Also, [13, §15.B] covers our special case of $r = 1$ and $v = 2$. Finally, to stay with the methods of this paper, we can do this special case as follows: if the anisotropic conic $Y'$ becomes isotropic over (an odd degree extension of) the

function field of the conic $X_0$, then $X_0$ becomes isotropic over the function field of $Y'$ and, therefore, of $Y$; but this is not the case because the algebra $D_{F(Y)}$ is not split by the very definition of $Y$ (we recall that $X_0$ is the Severi-Brauer variety of the quaternion algebra $D$).

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## Appendix A.

## Hyperbolicity of Symplectic and Unitary Involutions

by Jean-Pierre Tignol

The purpose of this note is to show how Karpenko's results in [4] and [6] can be used to prove the following analogues for symplectic and unitary involutions:

Theorem A.1. *Let $A$ be a central simple algebra of even degree over an arbitrary field $F$ of characteristic different from 2 and let $L$ be the function field over $F$ of the generalized Severi–Brauer variety $X_2(A)$ of right ideals of dimension $2\deg A$ (i.e., reduced dimension 2) in $A$ (see* [7, (1.16)]*). If a symplectic involution $\sigma$ on $A$ is not hyperbolic, then its scalar extension $\sigma_L = \sigma \otimes \mathrm{id}_L$ on $A_L = A \otimes_F L$ is not hyperbolic. Moreover, if $A$ is a division algebra then $\sigma_L$ is anisotropic.*

By a standard specialization argument, it suffices to find a field extension $L'/F$ such that $A_{L'}$ has index 2 and $\sigma_{L'}$ is not hyperbolic to prove the first part. If $A$ is a division algebra we need moreover $\sigma_{L'}$ anisotropic.

Theorem A.2. *Let $B$ be a central simple algebra of exponent 2 over an arbitrary field $K$ of characteristic different from 2, and let $\tau$ be a unitary involution on $B$. Let $F$ be the subfield of $K$ fixed under $\tau$ and let $M$ be the function field over $F$ of the Weil transfer $R_{K/F}(X(B))$ of the Severi–Brauer variety of $B$. If $\tau$ is not hyperbolic, then its scalar extension $\tau_M = \tau \otimes \mathrm{id}_M$ on $B_M = B \otimes_F M$ is not hyperbolic. Moreover, if $B$ is a division algebra, then $\tau_M$ is anisotropic.*

Again, by a standard specialization argument, it suffices to find a field extension $M'/F$ such that $B_{M'}$ is split and $\tau_{M'}$ is not hyperbolic ($\tau_{M'}$ anisotropic if $B$ is a division algebra).

A.1. SYMPLECTIC INVOLUTIONS. Consider the algebra of iterated twisted Laurent series in two indeterminates

$$\widehat{A} = A((\xi))((\eta; f))$$

where $f$ is the automorphism of $A((\xi))$ that maps $\xi$ to $-\xi$ and is the identity on $A$. Thus, $\xi$ and $\eta$ anticommute and centralize $A$. Let $x = \xi^2$ and $y = \eta^2$; the center of $\widehat{A}$ is the field of Laurent series $\widehat{F} = F((x))((y))$. Moreover, $\xi$ and $\eta$ generate over $\widehat{F}$ a quaternion algebra $(x, y)_{\widehat{F}}$, and we have $\widehat{A} = A \otimes_F (x, y)_{\widehat{F}}$. Let $\widehat{\sigma}$ be the involution on $\widehat{A}$ extending $\sigma$ and mapping $\xi$ to $-\xi$ and $\eta$ to $-\eta$. This involution is the tensor product of $\sigma$ and the canonical (conjugation) involution on $(x, y)_{\widehat{F}}$. Since $\sigma$ is symplectic, it follows that $\widehat{\sigma}$ is orthogonal.

PROPOSITION A.3. *If $\sigma$ is anisotropic (resp. hyperbolic), then $\widehat{\sigma}$ is anisotropic (resp. hyperbolic).*

*Proof.* If $\sigma$ is hyperbolic, then $A$ contains an idempotent $e$ such that $\sigma(e) = 1 - e$, see [7, (6.7)]. Since $(A, \sigma) \subset (\widehat{A}, \widehat{\sigma})$, this idempotent also lies in $\widehat{A}$ and satisfies $\widehat{\sigma}(e) = 1 - e$, hence $\widehat{\sigma}$ is hyperbolic. Now, suppose $\widehat{\sigma}$ is isotropic and let $a \in \widehat{A}$ be a nonzero element such that $\widehat{\sigma}(a)a = 0$. We may write

$$a = \sum_{i=z}^{\infty} a_i \eta^i$$

for some $a_i \in A((\xi))$ with $a_z \neq 0$. The coefficient of $\eta^{2z}$ in $\widehat{\sigma}(a)a$ is $(-1)^z f^z(\widehat{\sigma}(a_z)a_z)$, hence $\widehat{\sigma}(a_z)a_z = 0$. Now, let

$$a_z = \sum_{j=y}^{\infty} a_{jz} \xi^j$$

with $a_{jz} \in A$ and $a_{yz} \neq 0$. The coefficient of $\xi^{2y}$ in $\widehat{\sigma}(a_z)a_z$ is $(-1)^y \sigma(a_{yz})a_{yz}$, hence $\sigma(a_{yz})a_{yz} = 0$, which shows $\sigma$ is isotropic. □

*Proof of Theorem A.1.* Substituting for $(A, \sigma)$ its anisotropic kernel, we may assume $\sigma$ is anisotropic. Proposition A.3 then shows $(\widehat{A}, \widehat{\sigma})$ is anisotropic. Let $L'$ be the function field over $\widehat{F}$ of the Severi–Brauer variety of $\widehat{A}$. By Karpenko's theorem in [6], the algebra with involution $(\widehat{A}_{L'}, \widehat{\sigma}_{L'})$ is not hyperbolic. Therefore, it follows from Proposition A.3 that $(A_{L'}, \sigma_{L'})$ is not hyperbolic. In particular, $A_{L'}$ is not split since every symplectic involution on a split algebra is hyperbolic. On the other hand, $\widehat{A}_{L'}$ is split, hence $A_{L'}$ is Brauer-equivalent to $(x, y)_{L'}$. We have thus found a field $L'$ such that $A_{L'}$ has index 2 and $\sigma_{L'}$ is not hyperbolic, and the first part of Theorem A.1 follows. If $A$ is a division algebra, then $\widehat{A}$ also is division. Karpenko's theorem in [4] then shows that $\widehat{\sigma}_{L'}$ is anisotropic, hence $\sigma_{L'}$ is anisotropic since $(A_{L'}, \sigma_{L'}) \subset (\widehat{A}_{L'}, \widehat{\sigma}_{L'})$. □

Remark A.4. The last assertion in Theorem A.1 also holds if char $F = 2$, as a result of another theorem of Karpenko [5][2]: if $(A, \sigma)$ is a central division algebra with symplectic involution over a field $F$ of characteristic 2 and $Q = [x, y)_{F(x,y)}$ is a "generic" quaternion algebra where $x$ and $y$ are independent indeterminates over $F$, then $A \otimes_F Q$ is a central division algebra over $F(x, y)$ and we may consider on this algebra the quadratic pair $(\sigma \otimes \gamma, f_\otimes)$ where $\gamma$ is the conjugation involution on $Q$ and $f_\otimes$ is defined in [7, (5.20)]. By [5, Theorem 3.3], this quadratic pair remains anisotropic over the function field $L'$ of the Severi–Brauer variety of $A \otimes Q$, hence $\sigma_{L'}$ also is anisotropic, while $A_{L'}$ has index 2.

A.2. Unitary involutions. The proof of Theorem A.2 follows a line of argument similar to the proof of Theorem A.1. Since the exponent of $B$ is 2, the algebra $B$ carries an orthogonal involution $\nu$. Let $g = \nu \circ \tau$, which is an outer automorphism of $B$, and consider the algebra of twisted Laurent series

$$\widetilde{B} = B((\xi; g)).$$

It is readily checked that $\widetilde{B}$ carries an involution $\widetilde{\tau}$ extending $\tau$ such that $\widetilde{\tau}(\xi) = \xi$. To describe the center $\widetilde{F}$ of $\widetilde{B}$, pick an element $u \in \widetilde{B}$ such that

$$\nu(u) = \tau(u) = u \quad \text{and} \quad g^2(b) = ubu^{-1} \text{ for all } b \in B,$$

see [2, Lemma 3.1], and let $x = u^{-1}\xi^2$. Then $\widetilde{F} = F((x))$, and $\widetilde{B}$ is central simple over $\widetilde{F}$ by [1, Theorem 11.10]. By computing the dimension of the space of $\widetilde{\tau}$-symmetric elements as in [3, Proposition 1.9], we see that $\widetilde{\tau}$ is orthogonal. The algebra with involution $(\widetilde{B}, \widetilde{\tau})$ can be alternatively described as follows: let $\beta$ be the Brauer class of the central simple $F$-algebra $B_1 = B \oplus B\zeta$ where $\zeta^2 = u$ and $\zeta b = g(b)\zeta$ for all $b \in B$. Then $(\widetilde{B}, \widetilde{\tau})$ is the unique orthogonal quadratic extension of $(B, \tau)_{\widetilde{F}}$ with Brauer class $\beta_{\widetilde{F}} + (K, x)_{\widetilde{F}}$, see [3, Proposition 1.9].

Proposition A.5. *If $\tau$ is anisotropic (resp. hyperbolic), then $\widetilde{\tau}$ is anisotropic (resp. hyperbolic).*

*Proof.* If $\tau$ is hyperbolic, then $\widetilde{\tau}$ also is hyperbolic because $(B, \tau) \subset (\widetilde{B}, \widetilde{\tau})$. If $\widetilde{\tau}$ is isotropic, a leading term argument as in Proposition A.3 shows that $\tau$ is isotropic. □

*Proof of Theorem A.2.* Substituting for $(B, \tau)$ its anisotropic kernel, we may assume $\tau$ is anisotropic, hence $\widetilde{\tau}$ also is anisotropic. Let $M'$ be the function field over $\widetilde{F}$ of the Severi–Brauer variety of $\widetilde{B}$. By Karpenko's theorem in [6], the algebra with involution $(\widetilde{B}_{M'}, \widetilde{\tau}_{M'})$ is not hyperbolic, hence $(B_{M'}, \tau_{M'})$ is not hyperbolic. On the other hand, $\widetilde{B}_{M'}$ is split, and $B_{M'}$ is the centralizer of $K$ in $\widetilde{B}_{M'}$, hence $B_{M'}$ is split. We have thus found an extension $M'/F$ such that $B_{M'}$ is split and $\tau_{M'}$ is not hyperbolic, which proves the first part of Theorem A.2. If $B$ is a division algebra, then $\widetilde{B}$ also is a division algebra, and Karpenko's theorem in [4] shows that $\widetilde{\tau}_{M'}$ is anisotropic. Then $\tau_{M'}$ is anisotropic since $(B_{M'}, \tau_{M'}) \subset (\widetilde{B}_{M'}, \widetilde{\tau}_{M'})$. □

[2] I am grateful to N. Karpenko for calling my attention on this reference.

Remark A.6. As for symplectic involutions, the last assertion in Theorem A.2 also holds if $\operatorname{char} F = 2$, with almost the same proof: take $\ell \in K$ such that $\tau(\ell) = \ell + 1$, and consider the quadratic pair $(\widetilde{\tau}, f)$ on $\widetilde{B}$ where $f$ is defined by $f(s) = \operatorname{Trd}_{\widetilde{B}}(\ell s)$ for any $\widetilde{\tau}$-symmetric element $s \in \widetilde{B}$. If $B$ is a division algebra, then $\widetilde{B}$ is a division algebra, hence Karpenko's Theorem 3.3 in [5] shows that the quadratic pair $(\widetilde{\tau}, f)$ remains anisotropic after scalar extension to $M'$. Therefore, $\tau_{M'}$ is anisotropic.

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