Dimensions of Anisotropic Indefinite Quadratic Forms II

To Andrei Suslin on the occasion of his 60th birthday

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Abstract. The u-invariant and the Hasse number $\widetilde{u}$ of a field F of characteristic not 2 are classical and important field invariants pertaining to quadratic forms. They measure the suprema of dimensions of anisotropic forms over F that satisfy certain additional properties. We prove new relations between these invariants and a new characterization of fields with finite Hasse number (resp. finite u-invariant for nonreal fields), the first one of its kind that uses intrinsic properties of quadratic forms and which, conjecturally, allows an 'algebro-geometric' characterization of fields with finite Hasse number.

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1. Introduction

Throughout this paper, fields are assumed to be of characteristic different from 2 and quadratic forms over a field are always assumed to be finite-dimensional and nondegenerate. The u-invariant of a field F is one of the most important field invariants pertaining to quadratic forms. The definition as introduced by Elman and Lam [EL1] is as follows:

$$u(F) := \sup\{\dim\varphi \mid \varphi \text{ is an anisotropic torsion form over } F\}\ ,$$

where 'torsion' means torsion when considered as an element in the Witt ring WF. Note that over a formally real field (or real field for short) torsion forms are exactly the forms of total signature zero, whereas over a nonreal field, all forms are torsion.

If F is a real field, for a form φ over F to be isotropic, it is clearly necessary for φ to be indefinite at each ordering of F, i.e., for φ to be *totally indefinite* or *t.i.* for short. This leads to another field invariant, the *Hasse number* $\widetilde{u}$ defined as

$$\widetilde{u}(F) := \sup\{\dim\varphi \mid \varphi \text{ is an anisotropic t.i. form}\} .$$

One puts $\widetilde{u}(F) = 0$ if there are no anisotropic t.i. forms over F. Clearly, $u(F) \leq \widetilde{u}(F)$, with equality in the case of nonreal fields since being totally indefinite is then an empty condition.

In the present paper, we focus on finiteness criteria for u and $\widetilde{u}$ and on upper bounds on $\widetilde{u}$ in terms of u for fields with finite $\widetilde{u}$. To formulate these results, we need to introduce further properties. We refer to [L3] for all undefined terminology and basic facts about quadratic forms.

Recall that a quadratic form of type $\langle 1, -a_1\rangle \otimes \ldots \otimes \langle 1, -a_n\rangle$ ($a_i \in F^*$) is called an n-fold Pfister form, and we write $\langle\langle a_1, \ldots, a_n\rangle\rangle$ for short. P_nF (resp. GP_nF) denotes the set of all isometry classes of n-fold Pfister forms (resp. of forms similar to n-fold Pfister forms). A form φ is a Pfister neighbor if there exists a Pfister form π and $a \in F^*$ such that $\varphi \subset a\pi$ and $\dim\varphi > \frac{1}{2}\dim\pi$. Pfister forms are either hyperbolic or anisotropic, and if φ is a Pfister neighbor of a Pfister form π then φ is anisotropic iff π is anisotropic. Recall that the n-fold Pfister forms generate additively I^nF, the n-th power of the fundamental ideal IF of classes of even-dimensional forms in the Witt ring WF. The Arason-Pfister Hauptsatz [AP], APH for short, states that if $\varphi \in I^nF$, then $\dim\varphi < 2^n$ implies that φ is hyperbolic, and $\dim\varphi = 2^n$ implies $\varphi \in GP_nF$.

Let F be a real field and let X_F denote its space of orderings. X_F is a compact totally disconnected Hausdorff space with a subbasis of the topology given by the clopen sets $H(a) = \{P \in X_F \mid a >_P 0\}$, $a \in F^*$. φ is called positive (resp. negative) definite at $P \in X_F$ if $\operatorname{sgn}_P(\varphi) = \dim\varphi$ (resp. $\operatorname{sgn}_P(\varphi) = -\dim\varphi$), and indefinite at P if it is not definite at P. A totally positive definite (t.p.d.) form is a form that is positive definite at each $P \in X_F$.

If φ is a form over F, we denote by $D_F(\varphi)$ those elements in F^* represented by φ, by $D_F(n)$ ($n \in \mathbb{N}$) those elements in F^* that can be written as a sum of n squares, and we write $D_F(\infty) = \bigcup_{n\in\mathbb{N}} D_F(n)$ for the nonzero sums of squares in F. If F is nonreal then $F^* = D_F(\infty)$, and if F is real then $D_F(\infty)$ is the set of all totally positive elements in F.

The *Pythagoras number* $p(F)$ of a field F is the smallest n such that $D_F(n) = D_F(\infty)$ if such an n exists, otherwise $p(F) = \infty$.

If F is real, then $x \in D_F(\varphi)$ clearly implies that $x >_P 0$ (resp. $x <_P 0$) if φ is positive (resp. negative) definite at P. If the converse also holds, i.e. if

$$\begin{aligned} D_F(\varphi) \quad = \quad \{x \in F^* \mid \quad & x >_P 0 \text{ (resp. } x <_P 0\text{) if } \varphi \text{ is} \\ & \text{positive (resp. negative) definite at } P\} \end{aligned}$$

then φ is called *signature-universal* (*sgn-universal* for short). Over a real field, a form is universal (in the usual sense) if and only if it is t.i. and sgn-universal.

One readily sees that if $\widetilde{u}(F) < \infty$ then any form φ with $\dim\varphi \geq \widetilde{u}(F)$ is sgn-universal.
The following properties of fields will be used repeatedly.

Definition 1.1. (i) F is said to satisfy the *strong approximation property* SAP if given any disjoint closed subsets U, V of X_F there exists $a \in F^*$ such that $U \subset H(a)$ and $V \subset H(-a)$.

(ii) A form φ over a real field F is said to have *effective diagonalization* ED if it has a diagonalization $\langle a_1, \ldots, a_n\rangle$ such that $H(a_i) \subset H(a_{i+1})$ for $1 \leq i \leq n-1$. F is said to be ED if each form over F has ED.

(iii) F is said to have property S_1 if for every binary torsion form β over F one has $D_F(\beta) \cap D_F(\infty) \neq \emptyset$.

(iv) F is said to have property $PN(n)$ for some $n \in \mathbb{N}$ if each form of dimension $2^n + 1$ over F is a Pfister neighbor.

Note that if F is a nonreal field, i.e., F has no orderings, then $F^* = D_F(\infty)$ and all forms over F are torsion, so F is SAP, ED and S_1.
The paper is structured as follows. In §2 we give a new proof of the fact that ED is equivalent to SAP plus S_1, a result originally due to Prestel-Ware [PW]. In §3 we prove that for a field, having finite Hasse number is equivalent to having finite u-invariant plus having property ED. This result is originally due to Elman-Prestel [EP], but we give a proof that also allows us to derive various estimates for $\widetilde{u}$ in terms of u that are better than any previously known such estimates. In §4, we prove that having finite Hasse number is equivalent to having property $PN(n)$ for some $n \geq 2$, in which case we give estimates on $\widetilde{u}$ in terms of n. Since property $PN(2)$ is equivalent to F being linked (see Lemma 4.3), we will thus also recover as corollary a famous result on the u-invariant and the Hasse number of linked fields due to Elman-Lam [EL2], [E] (Corollary 4.12). We also explain how our results, conjecturally, provide an 'algebro-geometric' criterion for the finiteness of $\widetilde{u}$ (resp. u in case of nonreal fields).

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2. ED equals SAP plus S_1

The following theorem is due to Prestel-Ware [PW]. We give a new proof based mainly on the study of binary forms.

Theorem 2.1. *F has ED if and only if F has SAP and S_1.*

To prove this, we use alternative descriptions of the properties involved.

Lemma 2.2. *Let F be a real field.*

(i) *F is SAP if and only if for all $a,b\in F^*$ there exists $c\in F^*$ such that $H(c)=H(a)\cap H(b)$ (or, equivalently, there exists $d\in F^*$ such that $H(d)=H(a)\cup H(b)$).*
(ii) *F is ED if and only if for all $a,b\in F^*$, there exist $c,d\in F^*$ such that $\langle a,b\rangle\cong\langle c,d\rangle$ and $H(c)=H(a)\cap H(b)$ (or, equivalently, $H(d)=H(a)\cup H(b)$).*
(iii) *F has property S_1 if and only if, for all $a\in F^*$, $s\in D_F(\infty)$, and $x\in D_F(\langle 1,as\rangle)$, there exists $t\in D_F(\infty)$ such that $tx\in D_F(\langle 1,a\rangle)$.*

Proof. (i) This is well known, see, e.g., [L1, Prop. 17.2].
(ii) The 'only if' is nothing else but ED for binary forms. As for the converse, we use induction on the dimension n of forms. Forms of dimension ≤ 2 have ED by assumption. So let φ be a form of dimension $n\geq 3$. Then we can write $\varphi=\langle a_1,\ldots,a_n\rangle$ and we may assume that $\langle a_2,\ldots,a_n\rangle$ is already an ED. Write $\langle a_1,a_2\rangle\cong\langle b_1,b_2\rangle$ with $H(b_1)=H(a_1)\cap H(a_2)$ (so $\langle b_1,b_2\rangle$ is an ED of $\langle a_1,a_2\rangle$). Then $\varphi\cong\langle b_1,b_2,a_3,\ldots,a_n\rangle$. Now let $\langle c_2,\ldots,c_n\rangle$ be an ED of $\langle b_2,a_3,\ldots,a_n\rangle$. Then one readily checks that $\langle b_1,c_2,\ldots,c_n\rangle$ is an ED of φ.
(iii) 'if': Let $\langle u,v\rangle\cong u\langle 1,uv\rangle$ be torsion. Then $uv=-s$ with $s\in D_F(\infty)$. Put $a=-s$. Then $\langle 1,-1\rangle\cong\langle 1,as\rangle$ which is hyperbolic and hence represents u. But then, by assumption, there exists $t\in D_F(\infty)$ such that tu is represented by $\langle 1,a\rangle\cong\langle 1,-s\rangle$ and hence t is represented by $u\langle 1,-s\rangle\cong\langle u,v\rangle$.
'only if': $x\in D_F(\langle 1,sa\rangle)$ implies that there exists $y\in F^*$ such that $\langle 1,sa\rangle\cong\langle x,y\rangle$. Now the torsion form $xa\langle s,-1\rangle$ represents some $u\in D_F(\infty)$ by S_1. Hence $\langle sa,-a\rangle\cong\langle xu,-xus\rangle$ and hence

$$\langle 1,sa,-a\rangle\cong\langle 1,xu,-xus\rangle\cong\langle -a,x,y\rangle$$

Thus, $\langle 1,a\rangle=\langle x,xus,-xu,y\rangle$ in WF, so $x\langle 1,us,-u,xy\rangle$ is isotropic and there exists $v\in D_F(\langle 1,us\rangle)\cap D_F(\langle u,-xy\rangle)$. Note that $us\in D_F(\infty)$, so $v\in D_F(\infty)$. Hence, $\langle 1,us\rangle\cong\langle v,vus\rangle$ and $\langle -u,xy\rangle\cong\langle -v,vuxy\rangle$, and we get $\langle 1,a\rangle\cong x\langle vus,vuxy\rangle\cong\langle xvus,vuy\rangle$, thus $xt\in D_F(\langle 1,a\rangle)$ with $t:=vus\in D_F(\infty)$. $\square$

Proof of Theorem 2.1. 'only if': Clearly, ED implies SAP. Now let $\langle a,b\rangle$ be any binary torsion form. Then $\mathrm{sgn}_P(\langle a,b\rangle)=0$, so $H(a)\cap H(b)=\emptyset$, and by ED, there exists $c\in -D_F(\infty)$ and $d\in D_F(\infty)$ such that $\langle a,b\rangle\cong\langle c,d\rangle$, in particular, d is a totally positive element represented by $\langle a,b\rangle$ and we have established S_1.
'if': Let F be SAP and S_1. We will verify the alternative description of ED from Lemma 2.2(ii). Let $\langle a,b\rangle$ be any binary form. By SAP, there exists $d'\in F^*$ such that $H(a)\cup H(b)=H(d')$. Then $\langle a,b,-d'\rangle$ is t.i., thus the form $\varphi\cong\langle a,b,-d',-d'ab\rangle\cong -d'\langle\langle ad',bd'\rangle\rangle$ has total signature zero and is therefore torsion. Hence, there exists some $n\in F$ such that for $\sigma_n\cong\langle\langle -1\rangle\rangle^{\otimes n}\cong\langle 1,1\rangle^{\otimes n}$, we have that $\sigma_n\otimes\langle a,b,-d',-d'ab\rangle\in GP_{n+2}F$ is hyperbolic. But then its Pfister neighbor $\sigma_n\otimes\langle a,b\rangle\perp\langle -d'\rangle$ is isotropic. It follows that there exist $u,v\in D_F(\sigma_n)\subset D_F(\infty)$ such that $d'\in D_F(\langle ua,vb\rangle)$, and hence $ad'u\in D_F(\langle 1,abuv\rangle)$. Now $uv\in D_F(\infty)$, and by Lemma 2.2(iii), there exists $w\in D_F(\infty)$ such that $ad'uw\in D_F(\langle 1,ab\rangle)$, i.e. $d:=d'uw\in D_F(\langle a,b\rangle)$.

In particular, there exists $c \in F^*$ such that $\langle a, b\rangle \cong \langle c, d\rangle$. Since $uw \in D_F(\infty)$, we have $H(d) = H(d') = H(a) \cup H(b)$ as required. $\square$

3. Relations between the Hasse number and the u-invariant

In this section, we will only consider real fields since for nonreal fields $u = \widetilde{u}$, and most of the statements below are trivially true. It is quite possible for a real field F that $u(F)$ is finite but $\widetilde{u}(F)$ is infinite. Elman-Prestel [EP, Th. 2.5] gave the following necessary and sufficient criterion for the finiteness of $\widetilde{u}(F)$:

Theorem 3.1. *$\widetilde{u}(F) < \infty$ if and only if $u(F) < \infty$ and F has ED.*

The main purpose of this section is to give a new and elementary proof of this statement that in the case of ED-fields will allow us at the same time to derive upper bounds for $\widetilde{u}$ in terms of u that considerably improve previous upper bounds obtained by Elman-Prestel [EP, Prop. 2.7] and Hornix [Hor1, Th. 3.9]. The following remark is well known and will be useful.

Remark 3.2. For any field F, if $p(F) > 2^n$ then $\widetilde{u}(F) \geq u(F) \geq 2^{n+1}$. In particular, $p(F) \leq u(F) \leq \widetilde{u}(F)$.

Proposition 3.3. *Suppose that F has ED and that there exists an n-dimensional t.p.d. sgn-universal form ρ. Then*

$$\widetilde{u}(F) \leq \frac{n}{2}(u(F) + 2)\ .$$

Proof. We may clearly assume that $u(F)$ (and hence $p(F)$) is finite. The form $p(F) \times \langle 1\rangle$ is t.p.d. and sgn-universal, so we may assume that $n \leq p(F)$. If $n = 1$ then F is obviously pythagorean and $u(F) = 0$. Since F has ED, any t.i. form φ over F contains a binary torsion form β as a subform. But then β is isotropic as $u(F) = 0$, hence φ is isotropic. It follows that $\widetilde{u}(F) = 0$ and the above inequality is clearly satisfied. So we may assume that $2 \leq n \leq p(F) = p$ and we have $\widetilde{u}(F) \geq u(F) \geq p \geq n$ by Remark 3.2.

It suffices to consider the case $\widetilde{u}(F) > u(F)$. Let φ_0 be any anisotropic t.i. form with $\dim \varphi_0 > u(F)$, and write $\dim \varphi_0 = m = rn + k + 1$ with $r \geq 1$ and $0 \leq k \leq n - 1$. Since F is ED and thus SAP, we may assume after scaling that $0 \leq \operatorname{sgn}_P \varphi_0 \leq \dim \varphi_0 - 2 = rn + k - 1$ for all orderings P on F.

Let $\varphi_1 = a_0(\varphi_0 \perp -\rho)_{\mathrm{an}}$, where a_0 is chosen such that $0 \leq \operatorname{sgn}_P \varphi_1$ for all orderings P.

If i_W denotes the Witt index, we have $i_W(\varphi_0 \perp -\rho) \leq n - 1$, for otherwise one could write $\varphi_0 \cong \rho \perp \tau$ for some form τ. Since φ_0 is t.i. and since F has ED, this implies that there exists $x \in D_F(\infty)$ such that $-x$ is represented by τ. But then the form φ_0 contains the subform $\rho \perp \langle -x\rangle$ which is isotropic as ρ is t.p.d. and sgn-universal, clearly a contradiction. This implies that

$$\dim \varphi_1 \geq \dim \varphi_0 + n - 2(n - 1) = (r - 1)n + (k + 1) + 2\ .$$

Note also that $\operatorname{sgn}_P(\varphi_0 \perp -\rho) = \operatorname{sgn}_P \varphi_0 - n$ for each ordering P. Hence, one obtains

$$\operatorname{sgn}_P \varphi_1 \leq \max\{(r - 1)n + k - 1, n\}$$

for each ordering P. Note that if $r \geq 2$, then φ_1 is again t.i. as $0 \leq \operatorname{sgn}_P \varphi_1 < \dim \varphi_1$ for all orderings P. Applying this procedure altogether $r-1$ times, we get a form φ_{r-1} which is anisotropic, t.i., and such that

$$\dim \varphi_{r-1} \geq n + (k+1) + 2(r-1) \ ,$$

$$0 \leq \operatorname{sgn}_P \varphi_{r-1} \leq \max\{n+k-1, n\} \text{ for all orderings } P.$$

We therefore have

$$\dim \varphi_{r-1} - \operatorname{sgn}_P \varphi_{r-1} \geq \min\{2r, k+2r-1\} \ .$$

Since $\dim \varphi_{r-1} - \operatorname{sgn}_P \varphi_{r-1}$ is even, this yields $\dim \varphi_{r-1} - \operatorname{sgn}_P \varphi_{r-1} \geq 2r$ for all orderings P. By ED, the anisotropic form φ_{r-1} contains a torsion subform φ_t of dimension $\geq 2r$. Hence $u(F) \geq 2r$ and thus $u(F)+2 \geq 2(r+1)$. On the other hand, by assumption $m = rn+k+1 \leq n(r+1)$. These two inequalities together imply $m \leq \frac{n}{2}(u(F)+2)$. It follows readily that $\widetilde{u}(F) \leq \frac{n}{2}(u(F)+2)$. □

Proof of Theorem 3.1. The 'only if' part is easy and left to the reader. As for the 'if' part, we have $\infty > u(F) \geq p(F)$ by Lemma 3.2, and if we put $\rho = p(F) \times \langle 1 \rangle$, then Proposition 3.3 immediately yields $\widetilde{u}(F) \leq \frac{p(F)}{2}(u(F)+2) < \infty$. □

For a real field F, let $\widetilde{m}(F)$ be the smallest integer $n \geq 1$ such that there exists an n-dimensional t.p.d. sgn-universal form, and $\widetilde{m}(F) = \infty$ if there are no t.p.d. sgn-universal forms (cf. [GV] where an analogous invariant $m(F)$ for anisotropic universal forms was introduced). If $p(F) < \infty$, we have that $p(F) \times \langle 1 \rangle$ is sgn-universal. Hence $\widetilde{m}(F) \leq p(F)$. With this new invariant, Proposition 3.3 immediately implies

Corollary 3.4. *Suppose that $\widetilde{u}(F) < \infty$. Then*

$$\widetilde{u}(F) \leq \frac{\widetilde{m}(F)}{2}(u(F)+2) \ .$$

Next, we give another bound which will lead to further improvements.

Proposition 3.5. *Suppose that $u(F) < \infty$ and that F has ED (or, equivalently, that $\widetilde{u}(F) < \infty$). Let $\rho = \langle 1 \rangle \perp \rho'$ be a t.p.d. m-fold Pfister form, $m \geq 1$, such that its pure part ρ' is sgn-universal. Then*

$$\widetilde{u}(F) \leq 2^{m-2}(u(F)+6) \ .$$

If $m = 2$ then $\widetilde{u}(F) \leq u(F)+4$.

Proof. If $m = 1$, then $\dim \rho' = 1$ and the assumptions imply that F is pythagorean, hence $\widetilde{u} = u = 0$ and there is nothing to show. So we may assume $m \geq 2$. Furthermore, if d is an integer such that $2^d \leq p(F) = p \leq 2^{d+1}-1$, then we may assume that $m \leq d+1$. For we have that $(2^{d+1}-1) \times \langle 1 \rangle$ is the pure part of $\langle\langle -1, \ldots, -1 \rangle\rangle \in P_{d+1}F$ and it is totally positive definite and sgn-universal. We proceed similarly as before, but this time we put $\widetilde{u} = \widetilde{u}(F) = r2^m + k + 1$ with $r \geq 0$ and $0 \leq k \leq 2^m - 1$.
If $r = 0$ then we have $\widetilde{u} \leq 2^m$. If $2^d + 1 \leq p \leq 2^{d+1}-1$ then $u \geq 2^{d+1} \geq 2^m$ by Remark 3.2, and thus necessarily $u = \widetilde{u}$ and there is nothing to show.

Suppose that $p = 2^d$ so that in particular $u \geq 2^d$. Our previous bound yields $\widetilde{u} \leq 2^{d-1}(u+2)$. If $m = d+1$, then $2^{d-1}(u+2) < 2^{m-2}(u+6)$ and there is nothing to show. If $m \leq d$, then we have $\widetilde{u} = k+1 \leq 2^m \leq 2^d \leq u$ and thus $\widetilde{u} = u$, again there is nothing to show. So we may assume that $r \geq 1$.
Let φ_0 be an anisotropic t.i. form of dimension $\widetilde{u}$. As before, we may this time assume that $\dim\varphi_0 - 2 = r2^m + k - 1 \geq \mathrm{sgn}_P\,\varphi_0 \geq 0$ for all orderings P.
We claim that $i_W(\varphi_0 \perp -\rho) \leq 2^m - 2$. Indeed, otherwise φ_0 would contain a subform $\tilde{\rho}$ of dimension $2^m - 1$ with $\tilde{\rho} \subset \rho$. Now it is well known that all codimension 1 subforms of a Pfister form are similar to its pure part. Hence, φ_0 would contain a subform similar to ρ', and since φ_0 is t.i. and by ED, φ_0 would contain a subform similar to $\rho' \perp \langle -x\rangle$ for some $x \in D_F(\infty)$. By assumption, $\rho' \perp \langle -x\rangle$ is isotropic, a contradiction.
Thus, we obtain as in the proof of the previous lemma an anisotropic t.i. form φ_1 such that

$$\dim\varphi_1 \geq (r-1)2^m + k + 1 + 4\ ,$$
$$0 \leq \mathrm{sgn}_P\,\varphi_1 \leq \max\{(r-1)2^m + k - 1, 2^m\}\ ,$$

and reiterating this construction $r-1$ times, we get an anisotropic t.i. form φ_{r-1} such that

$$\dim\varphi_{r-1} \geq 2^m + k + 1 + 4(r-1)\ ,$$
$$0 \leq \mathrm{sgn}_P\,\varphi_{r-1} \leq \max\{2^m + k - 1, 2^m\} \text{ for all orderings } P.$$

This yields $\dim\varphi_{r-1} - \mathrm{sgn}_P\,\varphi_{r-1} \geq 4r-2$ for all orderings P, and thus, by ED, the existence of an anisotropic torsion subform φ_t of φ_{r-1} with $\dim\varphi_t \geq 4r-2$. In particular, $u + 6 \geq 4(r+1)$. On the other hand, $\widetilde{u} \leq 2^m(r+1)$ and thus $\widetilde{u} \leq 2^{m-2}(u+6)$.
Now if $m = 2$, we have $\dim\varphi_{r-1} \geq 4r + k + 1 = \dim\varphi_0$ and $0 \leq \mathrm{sgn}_P\,\varphi_{r-1} \leq \max\{4 + k - 1, 4\}$. In particular, since all the forms φ_i are anisotropic and t.i., it follows readily from the construction and the fact that $\widetilde{u} = 4r + k + 1$ that $\dim\varphi_0 = \dim\varphi_1 = \ldots = \varphi_{r-1} = \widetilde{u}$. Note also that $0 \leq k \leq 3$, so that by repeating our construction one more time, we obtain an anisotropic t.i. form φ_r such that $\dim\varphi_r = \widetilde{u}$ and $\mathrm{sgn}_P\,\varphi_r \leq 4$ for all orderings P. Thus, φ_r contains a torsion subform of dimension $\geq \widetilde{u} - 4$ and therefore $\widetilde{u} \leq u + 4$. □

Proposition 3.6. *Suppose that $I_t^3F = 0$, and that $u(F) < \infty$ and F has ED (or, equivalently, that $\widetilde{u}(F) < \infty$). If there exists a t.p.d. sgn-universal binary form ρ over F, then $u(F) = \widetilde{u}(F)$.*

Proof. By [ELP, Th. H], $I_t^3F = 0$ implies that $\widetilde{u} = \widetilde{u}(F)$ is even. By Proposition 3.3, $\widetilde{u} \leq u + 2$. So let us assume that $\widetilde{u} \neq u$, i.e. $\widetilde{u} = u + 2$. The proof of Proposition 3.3 then shows that there exists an anisotropic t.i. form φ (which is nothing but the form φ_{r-1} in the proof) with $\dim\varphi = \widetilde{u}$ and which contains a torsion subform φ_t, $\dim\varphi_t = \dim\varphi - 2 = u$. After scaling, we may assume that $\varphi_t \perp \langle 1\rangle \subset \varphi$. Let $d = d_\pm\varphi_t$. Then $\varphi_t \perp \langle 1, -d\rangle \in I^2F$, and since $\mathrm{sgn}_P\,\varphi_t = 0$ and $\mathrm{sgn}_P\,\varphi_t \perp \langle 1, -d\rangle \in 4\mathbb{Z}$, it follows that $\varphi_t \perp \langle 1, -d\rangle \in I_t^2F$. As $\dim\varphi_t \perp \langle 1, -d\rangle = u + 2$, this form must be isotropic. Thus, $\varphi_t \perp \langle 1\rangle \cong \psi \perp \langle d\rangle$. Comparing discriminants and signatures, it follows that $\psi \in I_t^2F$. So $\langle 1, -x\rangle \otimes \psi \in I_t^3F = 0$ for all $x \in F^*$, thus $\psi \cong x\psi$ which implies that ψ is universal, hence the subform $\psi \perp \langle d\rangle$ of φ is isotropic, a contradiction. $\square$

The following is an immediate consequence.

Corollary 3.7. *Suppose that $p(F) = 2$ and $\widetilde{u}(F) < \infty$. If $I_t^3F = 0$ then $u(F) = \widetilde{u}(F)$. In particular, if $u(F) \leq 6$ or $\widetilde{u}(F) \leq 8$, then $\widetilde{u}(F) = u(F)$.*

Remark 3.8. Let F be a real field with $\widetilde{u}(F) < \infty$. Suppose that d is an integer with $2^d + 1 \leq p \leq 2^{d+1} - 1$. The Pfister form $\langle\langle -1, \dots, -1\rangle\rangle \in P_{d+1}F$ is t.p.d. and its pure part is sgn-universal, so we can use Proposition 3.5 for $m = d+1$. For $p = 2^d + 1$, $d \geq 1$, we get $2^{d-1}(u+6) - \frac{p}{2}(u+2) = 2^{d+1} - \frac{1}{2}u - 1$. In this case, Proposition 3.3 gives a better bound when $u \leq 2^{d+2} - 4$ (note that we will have $u \geq 2^{d+1}$), the bounds are the same for $u = 2^{d+2} - 2$, and for $u \geq 2^{d+2}$ Proposition 3.5 gives a sharper bound.
Summarizing our best bounds in the various cases, we obtain

(i) $p(F) = 1$ if and only if $\widetilde{u}(F) = u(F) = 0$.
(ii) If $p(F) = 2$ then $\widetilde{u}(F) \leq u(F) + 2$. If in addition $I_t^3F = 0$ then $\widetilde{u}(F) = u(F) = 2n$ for some integer $n \geq 1$.
(iii) If $p(F) = 3$ then $\widetilde{u}(F) \leq u(F) + 4$.
(iv) If $p(F) = 2^m$ then $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 2)$.
(v) If $p(F) = 2^m + 1$ then $\widetilde{u}(F) \leq (2^{m-1} + \frac{1}{2})(u(F) + 2)$ if $u(F) \leq 2^{m+2} - 2$, and $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 6)$ if $u(F) \geq 2^{m+2} - 2$.
(vi) If $2^m + 2 \leq p(F) \leq 2^{m+1} - 1$, then $\widetilde{u}(F) \leq 2^{m-1}(u(F) + 6)$.

Remark 3.9. It is difficult to say at this point how good our bounds really are. In fact, we know extremely little about fields with $u(F) < \widetilde{u}(F) < \infty$. The only values which could be realized so far are fields where $u(F) = 2n$ and $\widetilde{u}(F) = 2n + 2$ for any $n \geq 2$ (see [L2], [Hor2], [H3]), and fields with $u(F) = 8$ and $\widetilde{u}(F) = 12$, see [H2, Cor. 6.4].

For the balance of this section, we finish with stating results about *all* possible pairs of values for $(p(F), u(F))$ for real fields, in particular real fields satisfying SAP but not S_1 or vice versa (such fields will always have $\widetilde{u} = \infty$). The construction of such fields with prescribed values (p, u) uses Merkurjev's method of iterated function fields and is rather technical. We omit the proof and refer the interested reader to [H4].

Theorem 3.10. *Let $\mathcal{N}'$ be the set of pairs of integers (p, u) such that either $p = 1$ and $u = 0$ or $u = 2n \geq 2^m \geq p \geq 2$ for some integers m and n. Let $\mathcal{N} = \mathcal{N}' \cup \{(p, \infty);\ p \geq 2$ or $p = \infty\}$.*

(i) *If F is a real field, then $(p(F), u(F)) \in \mathcal{N}$.*
(ii) *Let E be a real field and let $(p, u) \in \mathcal{N}$. Then there exists a real field extension F/E such that F is non-SAP, F has property S_1 and $(p(F), u(F)) = (p, u)$. In particular, $\widetilde{u}(F) = \infty$.*
(iii) *If F is a real SAP field with $\widetilde{u}(F) = \infty$, then $u(F) \geq 4$ and $(p(F), u(F)) \in \mathcal{N}$.*

(iv) *Let E be a real field and let $(p,u) \in \mathcal{N}$ with $u \geq 4$. Then there exists a real field extension F/E such that F is SAP, F does not have property S_1 and $(p(F), u(F)) = (p, u)$. In particular, $\widetilde{u}(F) = \infty$.*

4. Linkage of fields and the Pfister neighbor property

The purpose of this section is to derive a criterion for the finiteness of the Hasse number. Real fields with finite Hasse number are relatively scarce but interesting nonetheless. But our results are just as valid for nonreal fields, we thus get also a criterion for the finiteness of u for nonreal fields.
Recall that the field F is said to have the Pfister neighbor propery $PN(n)$, $n \geq 0$, if every form of dimension $2^n + 1$ over F is a Pfister neighbor. This property is a somewhat stronger version of the notion of n-linkage whose definition we now recall:

Definition 4.1. Let $n \geq 1$ be an integer. A field F is called *n-linked* if to any n-fold Pfister forms π_1 and π_2 over F there exist $a_1, a_2 \in F^*$ and an $(n-1)$-fold Pfister form σ such that $\pi_i \cong \langle\langle a_i \rangle\rangle \otimes \sigma$, $i = 1, 2$. F is called *linked* if F is 2-linked.

Remark 4.2. (i) Trivially, every field is 1-linked and satisfies $PN(0)$ and $PN(1)$.
(ii) Let $n \geq 2$. Every isotropic form of dimension $2^n + 1$ is a Pfister neighbor. In fact, if $\dim \varphi = 2^n + 1$ and φ is isotropic, then $\varphi \cong \mathbb{H} \perp \psi$ with $\dim \psi = 2^n - 1$. Then $\varphi \perp -\psi \cong \pi \in P_{n+1}F$, where π denotes the hyperbolic $(n+1)$-fold Pfister form. So in particular, if F is nonreal and $u(F) \leq 2^n$, then F has property $PN(n)$

Lemma 4.3. *Let $n \geq 2$.*

(i) *If F is n-linked then F is m-linked for all $m \geq n$ and $I_t^{n+2}F = 0$.*
(ii) *F is n-linked iff to each form $\varphi \in I^nF$ there exists a form $\pi \in P_nF$ such that $\varphi \equiv \pi \bmod I^{n+1}F$ iff to each anisotropic $\varphi \in I^nF$ there exist $\tau \in P_{n-1}F$ and an even-dimensional form σ such that $\varphi \cong \tau \otimes \sigma$.*
(iii) *F has property $PN(n)$ if and only if there exists to every form φ over F a form ψ such that $\dim \psi \leq 2^n$ if $\dim \varphi$ even (resp. $\dim \psi \leq 2^n - 1$ if $\dim \varphi$ odd) such that $\varphi \equiv \psi \bmod I^{n+1}F$.*
(iv) *If F has property $PN(n)$ then F is n-linked. In particular, $I_t^{n+2}F = 0$. Furthermore, F is ED.*
(v) *F has property $PN(2)$ iff F is linked.*

Proof. (i) and (ii) are well known, see [EL2, § 2], [H1].
(iii) 'only if': If $\dim \varphi \leq 2^n$, then put $\psi \cong \varphi$. So suppose $\dim \varphi \geq 2^n + 1$. Write $\varphi \cong \psi \perp \tau$ with $\dim \psi = 2^n + 1$. By $PN(n)$, ψ is a Pfister neighbor and there exists ψ', $\dim \psi' = 2^n - 1$ such that $\psi \perp -\psi' \cong \pi \in GP_{n+1}F$. Then, in WF, we have

$$\varphi \equiv \varphi - \pi \equiv \psi' \perp \tau \bmod I^{n+1}F\ .$$

Now $\dim \psi' \perp \tau = \dim \varphi - 2$ and the result follows by an easy induction on the dimension.

'if': Let $\dim\varphi = 2^n+1$. By assumption, there exists a form ψ, $\dim\psi = 2^n-1$ (possibly after adding hyperbolic planes) such that $\varphi \perp -\psi \in I^{n+1}F$. Then $\dim(\varphi \perp -\psi) = 2^{n+1}$ and thus $\varphi \perp -\psi \in GP_{n+1}F$ by APH, which implies that φ is a Pfister neighbor.
(iv) To show that F is n-linked, let $\varphi \in I^nF$. By (iii), there exists ψ such that $\dim\psi = 2^n$ (possibly after adding hyperbolic planes) and $\varphi \equiv \psi \bmod I^{n+1}F$. But clearly $\psi \in I^nF$, and thus $\psi \in GP_nF$ by APH. Let $x \in F^*$ be such that $x\psi \in P_nF$. We then have $\psi \equiv x\psi \bmod I^{n+1}F$, and n-linkage together with $I_t^{n+2}F = 0$ follows from (i) and (ii).
Now n-linked fields, $n \geq 2$, are easily seen to be SAP. So to establish ED, it suffices to establish property S_1 by Theorem 2.1. Let $\langle a,b\rangle$ be any torsion form. Let $\gamma \cong \langle\underbrace{1,\ldots,1}_{2^n-1}\rangle$. Then by $PN(n)$, the form $\gamma \perp \langle -a,-b\rangle$ is a t.i. Pfister neighbor of a Pfister form $\pi \in P_{n+1}F$. Since π contains γ which is a Pfister neighbor (and in fact subform) of $\sigma_n \cong \langle 1,1\rangle^{\otimes n}$, one necessarily has that σ_n divides π, so there exists $c \in F^*$ such that $\pi \cong \sigma_n \otimes \langle 1,c\rangle$. Now π contains a t.i. Pfister neighbor and is therefore also t.i. and hence torsion. But then $\rho \cong \langle 1,1\rangle \otimes \sigma_n \otimes \langle 1,c\rangle \in P_{n+2}F$ is torsion as well and therefore hyperbolic by (i). Now $\sigma_n \perp \gamma \perp \langle -a,-b\rangle$ is a Pfister neighbor of ρ. Since ρ is hyperbolic, its neighbor $\sigma_n \perp \gamma \perp \langle -a,-b\rangle$ is isotropic. Hence there exists $x \in D_F(\langle a,b\rangle) \cap D_F(\sigma_n \perp \gamma)$. But clearly, $D_F(\sigma_n \perp \gamma) \subset D_F(\infty)$ which shows that the binary torsion form $\langle a,b\rangle$ represents the totally positive element x.
(v) This follows immediately from the fact that a field is linked iff the classes of quaternion algebras form a subgroup in $\mathrm{Br}(F)$ together with the characterization of 5-dimensional Pfister neighbors by their Clifford invariant (see [Kn, p. 10]). ☐

The following observation is essentially due to Fitzgerald [F, Lemma 4.5(ii)].

Lemma 4.4. *Suppose that $\widetilde{u}(F) \leq 2^n$. Let φ be a form over F of dimension 2^n+1. Then φ is a Pfister neighbor. In particular, F has $PN(n)$.*

Proof. By Remark 4.2(ii) the result is clear if φ is isotropic. Thus, we may assume φ anisotropic, so necessarily F must be real. Since $\widetilde{u}(F) < \infty$ implies that F is SAP, we may assume that after scaling, $\mathrm{sgn}_P(\varphi) \geq 0$ for all $P \in X_F$, and that there exists $c \in F^*$ such that $H(c) = \{P \in X_F \mid \mathrm{sgn}_P(\varphi) = \dim\varphi\}$. In particular, the Pfister form $\langle\langle\underbrace{-1,\ldots,-1}_{n},-c\rangle\rangle \in P_{n+1}F$ is positive definite at all those $P \in X_F$ at which φ is positive definite, and it has signature zero at all those $P \in X_F$ at which φ is indefinite. Let $\psi \cong (\pi \perp -\varphi)_{\mathrm{an}}$. It follows that $|\mathrm{sgn}_P(\psi)| \leq 2^n-1$ for all $P \in X_F$. But since $\widetilde{u}(F) \leq 2^n$, the anisotropic form ψ must therefore have $\dim\psi \leq 2^n$, so in particular,

$$i_W(\pi \perp -\varphi) = \tfrac{1}{2}(\dim(\pi \perp -\varphi) - \dim\psi) \geq \tfrac{1}{2}(2^{n+1}+1)\ ,$$

and therefore $i_W(\pi \perp -\varphi) \geq 2^n+1 = \dim\varphi$, which implies that $\varphi \subset \pi$. In particular, φ is a Pfister neighbor of π. ☐

THEOREM 4.5. *If a field F has property $PN(n)$, $n \geq 2$, then either $u(F) \leq \widetilde{u}(F) \leq 2^n$, or $2^{n+1} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+1} + 2^n - 2$.*

Proof. Let F be a field with property $PN(n)$ for some $n \geq 2$. Suppose that $\widetilde{u}(F) > 2^n$, i.e. there exists an anisotropic t.i. φ with $\dim \varphi = m > 2^n$. By Lemma 4.3(iv), F has ED and so φ can be diagonalized as $\varphi \cong \langle a_1, \ldots, a_m \rangle$ with $-a_1, a_m \in D_F(\infty)$. By removing some of the a_i, $2 \leq i \leq m-1$ if necessary, we will retain a t.i. form, so we may assume that φ is t.i. and $\dim \varphi = 2^n + 1$. But then, by $PN(n)$, φ is a Pfister neighbor of some $\pi \in P_{n+1}F$ which in turn is torsion and anisotropic as its Pfister neighbor φ is t.i. and anisotropic. This shows that $2^{n+1} \leq u(F) \leq \widetilde{u}(F)$.
Now suppose that $\widetilde{u}(F) > 2^{n+1} + 2^n - 2$. By a similar argument as above, we conclude that there exists an anisotropic t.i. form φ with $\dim \varphi = 2^{n+1} + 2^n - 1$. By Lemma 4.3(iii), there exists an anisotropic form ψ of dimension $\leq 2^n - 1$ such that $\varphi \equiv \psi \bmod I^{n+1}F$. Let $\pi \cong (\varphi \perp -\psi)_{\mathrm{an}} \in I^{n+1}F$. Then by dimension count and since φ is anisotropic, we have $2^{n+1} \leq \dim \pi \leq 2^{n+2} - 2$. Since F is $(n+1)$-linked, Lemma 4.3(ii) implies $\dim \pi = 2^{n+1}$, and thus, by APH, $\pi \in GP_{n+1}F$. Also, by dimension count, we have $\varphi \cong \pi \perp \psi$.
After scaling, we may assume that $\pi \in P_{n+1}F$, so that $\mathrm{sgn}_P(\pi) \in \{0, 2^{n+1}\}$. Now φ is t.i., and since F has ED by Lemma 4.3(iv), we can write $\psi \cong \langle a, \ldots \rangle$ with $a <_P 0$ whenever $\mathrm{sgn}_P(\pi) = 2^{n+1}$. But then $\pi \perp \langle a \rangle$ is a t.i. subform of φ. On the other hand, $\pi \perp \langle a \rangle$ is also a Pfister neighbor of $\pi \otimes \langle 1, a \rangle \in P_{n+2}F$. Since $\pi \perp \langle a \rangle$ is t.i., this implies that $\pi \otimes \langle 1, a \rangle$ is torsion and therefore hyperbolic since $I_t^{n+2}F = 0$ by Lemma 4.3(ii). But then the Pfister neighbor $\pi \perp \langle a \rangle$ is isotropic and therefore also φ, a contradiction. $\square$

Remark 4.6. (i) The above proof also shows that if F has $PN(n)$, $n \geq 2$, then the case $\widetilde{u}(F) \leq 2^n$ occurs iff there are no anisotropic torsion $(n+1)$-fold Pfister forms iff $I_t^{n+1}F = 0$.
(ii) If we were only considering nonreal fields then the proofs could be shortened by essentially deleting arguments referring to or making use of ED, signatures, etc..

COROLLARY 4.7. *$\widetilde{u}(F) < \infty$ if and only if F has $PN(n)$ for some $n \geq 2$. In particular, if F is nonreal then $u(F) < \infty$ if and only if F has $PN(n)$ for some $n \geq 2$*

Proof. The 'if'-part in the first statement follows from Theorem 4.5, the converse from Lemma 4.4. The statement for nonreal fields is then clear because in that case $u = \widetilde{u}$. $\square$

Remark 4.8. If F is real, then we still get a sufficient criterion for the finiteness of $u(F)$ even if $\widetilde{u}(F) = \infty$. Indeed, for real F, one has that if $u(F(\sqrt{-1}))$ is finite then $u(F)$ is finite, more precisely, one has $u(F) < 4u(F(\sqrt{-1}))$ (see [EKM, Th. 37.4]). Thus, we get the following: If $F(\sqrt{-1})$ has property $PN(n)$ for some $n \geq 2$, then $u(F) < 2^{n+3} + 2^{n+2} - 8$.

Conjecture 4.9. If a field F has property $PN(n)$, $n \geq 2$, then $u(F) \leq \widetilde{u}(F) \leq 2^n$, or $u(F) = \widetilde{u}(F) = 2^{n+1}$.

Corollary 4.10. *For $n \geq 2$, $PN(n)$ implies $PN(m)$ for all $m \geq n+2$. Furthermore, the following are equivalent:*

(i) *Conjecture 4.9 holds.*
(ii) *For $n \geq 2$, $PN(n)$ implies $PN(n+1)$.*

Proof. If $n \geq 2$, then $PN(n)$ implies that $\widetilde{u}(F) \leq 2^{n+2}$, and $PN(m)$ for $m \geq n+2$ follows from Lemma 4.4.
Now suppose that F has $PN(n)$ and that Conjecture 4.9 holds. Then $PN(n+1)$ follows from Lemma 4.4. Conversely, suppose that $n \geq 2$ and that $PN(n)$ implies $PN(n+1)$. Then we have $u(F) \leq \widetilde{u}(F) \leq 2^n$ or $2^{n+1} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+1} + 2^n - 2$ because of $PN(n)$, and also $u(F) \leq \widetilde{u}(F) \leq 2^{n+1}$ or $2^{n+2} \leq u(F) \leq \widetilde{u}(F) \leq 2^{n+2} + 2^{n+1} - 2$ because of $PN(n+1)$. Putting the two together, we obtain $u(F) \leq \widetilde{u}(F) \leq 2^n$ or $u(F) = \widetilde{u}(F) = 2^{n+1}$. $\square$

The only evidence we have as to the veracity of Conjecture 4.9 is the following.

Lemma 4.11. *$PN(2)$ implies $PN(3)$. In particular, if F has $PN(2)$, then $u(F) \leq \widetilde{u}(F) \leq 4$ or $u(F) = \widetilde{u}(F) = 8$.*

Proof. Suppose F has $PN(2)$ and let φ be any 9-dimensional form over F. Write $\varphi \cong \alpha \perp \beta$ with $\dim \alpha = 5$. Since α is a Pfister neighbor, there exists $\pi \in GP_2F$ such that $\pi \subset \alpha \subset \varphi$ (see, e.g., [L3, Ch. X, Prop. 4.19]). Write $\varphi \cong \pi \perp \gamma$. Then $\dim \gamma = 5$ and γ is also a Pfister neighbor, so there exists $\rho \in GP_2F$ such that $\rho \subset \gamma$. Hence, there exist $a, b, c, d, e, f, g \in F^*$ such that $\varphi \cong a\langle\langle b, c\rangle\rangle \perp d\langle\langle e, f\rangle\rangle \perp \langle g\rangle$.
Since $PN(2)$ implies that F is linked by Lemma 4.3(v), we may assume that $b = e$, and after scaling (which doesn't change the property of being a Pfister neighbor), we may also assume $a = 1$, so

$$\varphi \cong \langle\langle b, c\rangle\rangle \perp d\langle\langle b, f\rangle\rangle \perp \langle g\rangle \subset \langle\langle b\rangle\rangle \otimes (\langle\langle c\rangle\rangle \perp d\langle\langle f\rangle\rangle \perp \langle g\rangle) .$$

Now $\delta \cong \langle\langle c\rangle\rangle \perp d\langle\langle f\rangle\rangle \perp \langle g\rangle$ has dimension 5 and is therefore again a Pfister neighbor, so as above there exist $h, k, l, m \in F^*$ such that $\delta \cong h\langle\langle k, l\rangle\rangle \perp \langle m\rangle$. We thus get that

$$\varphi \subset \langle\langle b\rangle\rangle \otimes \delta \cong h\langle\langle b, k, l\rangle\rangle \perp m\langle\langle b\rangle\rangle \subset h\langle\langle b, k, l, -hm\rangle\rangle \in GP_4F ,$$

which shows that φ is a Pfister neighbor.
The remaining statement now follows from Corollary 4.10. $\square$

Since linked fields are exactly the fields that have $PN(2)$, one readily recovers the following result due to Elman and Lam [EL2] and Elman [E, Th. 4.7]. We leave it as an exercise to the reader to fill in the details.

Corollary 4.12. *Let F be a linked field. Then $u(F) = \widetilde{u}(F) \in \{0, 1, 2, 4, 8\}$. In particular, $I_t^4 F = 0$. Furthermore, let $n \in \{0, 1, 2\}$. Then $\widetilde{u}(F) \leq 2^n$ iff $I_t^{n+1} F = 0$.*

Note that $u(F) = \widetilde{u}(F) = 0$ can only occur when F is real, whereas $u(F) = \widetilde{u}(F) = 1$ implies that F is nonreal.

Remark 4.13. It is not difficult to see that the iterated power series field $F = \mathbb{C}((X_1))((X_2))\dots((X_n))$ is a (nonreal) field with property $PN(n)$ and $u(F) = 2^{n+1}$.
Using Merkurjev's method of iterated function fields, it is also possible to construct to any $n \geq 2$ a real field F with property $PN(n)$ and $\widetilde{u}(F) = 2^{n+1}$. For details, see [H4].

Remark 4.14. Merkurjev [M] constructed to each positive integer n a field F with $I^3F = 0$ and $u(F) = 2n$ (resp. a field F with $I^3F = 0$ and $u(F) = \infty$). Trivially, such a (nonreal) field is 3-linked. So the n-linkage property, $n \geq 3$, does not give any indication on how large u might be, whereas the stronger property $PN(n)$ does.

We finish this paper with some remarks on a possible geometric interpretation of the property $PN(n)$ which can be formulated in the language of Chow groups. We refer to [Kar], [EKM, §80].
Let φ be a (nondegenerate) quadratic form of dimension $n + 2 \geq 3$, and let $X = X_\varphi$ be the smooth projective n-dimensional quadric $\{\varphi = 0\}$ over F. We call X (an)isotropic if φ is (an)isotropic. Let $\overline{F}$ denote the algebraic closure of F and let $\overline{X} = X_{\overline{F}}$. Let l_0 be the class of a rational point in $\mathrm{CH}^n(\overline{X})$, the Chow group of 0-dimensional cycles, and let $1 \in \mathrm{CH}^0(X)$ be the class of X. A *Rost correspondence* on X is an element $\rho \in \mathrm{CH}^n(X \times X)$ which, over $\overline{F}$, is equal to $l_0 \times 1 + 1 \times l_0 \in \mathrm{CH}^n(\overline{X} \times \overline{X})$. A *Rost projector* is a Rost correspondence that is also an idempotent in the ring of correspondences on X. It is known that if a quadric has a Rost correspondence, then it has in fact also a Rost projector (see [Kar, Rem. 1.4]). The study of Rost correspondences/projectors has proven to be crucial in the motivic theory of quadrics.
It is known that if X is isotropic, then $l_0 \times 1 + 1 \times l_0$ is actually the unique Rost projector on X (see [Kar, Lem. 5.1]). For anisotropic forms, the situation is much more complicated.
The following is known:

Theorem 4.15. *Let φ be an anisotropic form over F of dimension ≥ 3.*

(i) *If X_φ possesses a Rost projector, then* $\dim\varphi = 2^n + 1$ *for some* $n \geq 1$ *(see Karpenko* [Kar, Prop. 6.2, 6.4]*).*
(ii) *If φ is a Pfister neighbor of dimension* $2^n + 1$ *then X_φ has a unique Rost projector (considered as element in $CH^r(X_\varphi \times X_\varphi)$, $r = 2^n - 1$) (see Izhboldin-Vishik* [IV, Th. 1.12] *for* $\mathrm{char}(F) = 0$*, Elman-Karpenko-Merkurjev* [EKM, Cor. 80.11] *in the general case).*

In view of part (i), it is natural to ask whether or not the converse of part (ii) also holds. This is still an open problem (see also [Kar, Conj. 1.6]):

Conjecture 4.16. If an anisotropic quadric X_φ possesses a Rost correspondence, then φ is a Pfister neighbor of dimension $2^n + 1$ for some $n \geq 1$.

Of course, by Theorem 4.15(ii), to prove the conjecture, one may assume that $\dim\varphi = 2^n+1$ for some $n \geq 1$. Since 3-dimensional forms are always Pfister neighbors, trivially the conjecture holds in that case. The conjecture is also true in the cases $n = 2, 3$ as shown by Karpenko (see [Kar, Prop. 10.8, Th. 1.7]):

Theorem 4.17. *Let φ be an anisotropic form over F of dimension 2^n+1, $n = 2, 3$. If X_φ possesses a Rost correspondence, then φ is a Pfister neighbor.*

It is now natural to introduce the property $RP(n)$ for $n \geq 1$:

$RP(n)$: *F has the property $RP(n)$ for $n \geq 1$ if every form φ over F of dimension 2^n+1 has a Rost projector.*

In view of the above, we immediately get

Proposition 4.18. *Let $n \geq 1$.*

(i) *$PN(n)$ implies $RP(n)$.*
(ii) *If $n \leq 3$, then $RP(n)$ implies $PN(n)$.*
(iii) *If Conjecture 4.16 holds, then $RP(n)$ implies $PN(n)$ for all $n \in \mathbb{N}$.*

Conjecturally and in view of Theorem 4.5, we therefore get an 'algebro-geometric' criterion for the finiteness of the Hasse number:

Corollary 4.19. *If Conjecture 4.16 holds, then $\widetilde{u}(F) < \infty$ (resp. $u(F) < \infty$ for nonreal F) if and only if F has property $RP(n)$ for some $n \geq 2$.*

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