On Suslin's Singular Homology and Cohomology

Dedicated to A. A. Suslin on his 60th birthday

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Abstract. We study properties of Suslin homology and cohomology over non-algebraically closed base fields, and their p-part in characteristic p. In the second half we focus on finite fields, and consider finite generation questions and connections to tamely ramified class field theory.

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1 Introduction

Suslin and Voevodsky defined Suslin homology (also called singular homology) $H_i^S(X,A)$ of a scheme of finite type over a field k with coefficients in an abelian group A as $\mathrm{Tor}_i(\mathrm{Cor}_k(\Delta^*,X),A)$. Here $\mathrm{Cor}_k(\Delta^i,X)$ is the free abelian group generated by integral subschemes Z of $\Delta^i\times X$ which are finite and surjective over Δ^i, and the differentials are given by alternating sums of pull-back maps along face maps. Suslin cohomology $H^i_S(X,A)$ is defined to be $\mathrm{Ext}^i_{\mathrm{Ab}}(\mathrm{Cor}_k(\Delta^*,X),A)$. Suslin and Voevodsky showed in [22] that over a separably closed field in which m is invertible, one has

$$H^i_S(X,\mathbb{Z}/m)\cong H^i_{\mathrm{et}}(X,\mathbb{Z}/m) \tag{1}$$

(see [2] for the case of fields of characteristic p).
In the first half of this paper, we study both the situation that m is a power of the characteristic of k, and that k is not algebraically closed. In the second

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half, we focus on finite base fields and discuss a modified version of Suslin homology, which is closely related to etale cohomology on the one hand, but is also expected to be finitely generated. Moreover, its zeroth homology is $\mathbb{Z}^{\pi_0(X)}$, and its first homology is expected to be an integral model of the abelianized tame fundamental group.

We start by discussing the p-part of Suslin homology over an algebraically closed field of characteristic p. We show that, assuming resolution of singularities, the groups $H_i^S(X,\mathbb{Z}/p^r)$ are finite abelian groups, and vanish outside the range $0\leq i\leq \dim X$. Thus Suslin cohomology with finite coefficients is etale cohomology away from the characteristic, but better behaved than etale cohomology at the characteristic (for example, $H^1_{\mathrm{et}}(\mathbb{A}^1,\mathbb{Z}/p)$ is not finite). Moreover, Suslin homology is a birational invariant in the following strong sense: If X has a resolution of singularities $p: X'\to X$ which is an isomorphism outside of the open subset U, then $H_i^S(U,\mathbb{Z}/p^r)\cong H_i^S(X,\mathbb{Z}/p^r)$. It was pointed out to us by N.Otsubo that this can be applied to generalize a theorem of Spiess-Szamuely [20] to include p-torsion:

Theorem 1.1 *Let X be a smooth, connected, quasi-projective variety over an algebraically closed field and assume resolution of singularities. Then the albanese map*

$$alb_X: H_0^S(X,\mathbb{Z})^0\to Alb_X(k)$$

from the degree-0-part of Suslin homology to the k-valued points of the Albanese variety induces an isomorphism on torsion groups.

Next we examine the situation over non-algebraically closed fields. We redefine Suslin homology and cohomology by imposing Galois descent. Concretely, if G_k is the absolute Galois group of k, then we define Galois-Suslin homology to be

$$H_i^{GS}(X,A)=H^{-i}R\Gamma(G_k,\operatorname{Cor}_{\bar k}(\Delta^*_{\bar k},\bar X)\times A),$$

and Galois-Suslin cohomology to be

$$H^i_{GS}(X,A)=\operatorname{Ext}^i_{G_k}(\operatorname{Cor}_{\bar k}(\Delta^*_{\bar k},\bar X),A).$$

Ideally one would like to define Galois-Suslin homology via Galois homology, but we are not aware of such a theory. With rational coefficients, the newly defined groups agree with the original groups. On the other hand, with finite coefficients prime to the characteristic, the proof of (1) in [22] carries over to show that $H^i_{GS}(X,\mathbb{Z}/m)\cong H^i_{\mathrm{et}}(X,\mathbb{Z}/m)$. As a corollary, we obtain an isomorphism between $H_0^{GS}(X,\mathbb{Z}/m)$ and the abelianized fundamental group $\pi_1^{ab}(X)/m$ for any separated X of finite type over a finite field and m invertible. The second half of the paper focuses on the case of a finite base field. We work under the assumption of resolution of singularities in order to see the picture of the properties which can expected. The critical reader can view our statements as theorems for schemes of dimension at most three, and conjectures in general. A theorem of Jannsen-Saito [11] can be generalized to show that Suslin

homology and cohomology with finite coefficients for any X over a finite field is finite. Rationally, $H_0^S(X,\mathbb{Q}) \cong H_S^0(X,\mathbb{Q}) \cong \mathbb{Q}^{\pi_0(X)}$. Most other properties are equivalent to the following Conjecture P_0 considered in [7]: For X smooth and proper over a finite field, $CH_0(X,i)$ is torsion for $i \neq 0$. This is a particular case of Parshin's conjecture that $K_i(X)$ is torsion for $i \neq 0$. Conjecture P_0 is equivalent to the vanishing of $H_i^S(X,\mathbb{Q})$ for $i \neq 0$ and all smooth X. For arbitrary X of dimension d, Conjecture P_0 implies the vanishing of $H_i^S(X,\mathbb{Q})$ outside of the range $0 \leq i \leq d$ and its finite dimensionality in this range. Combining the torsion and rational case, we show that $H_i^S(X,\mathbb{Z})$ and $H_S^i(X,\mathbb{Z})$ are finitely generated for all X if and only if Conjecture P_0 holds.
Over a finite field and with integral coefficients, it is more natural to impose descent by the Weil group G generated by the Frobenius endomorphism φ instead of the Galois group [14, 3, 4, 7]. We define arithmetic homology

$$H_i^{\mathrm{ar}}(X,A) = \mathrm{Tor}_i^G(\mathrm{Cor}_{\bar{k}}(\Delta_{\bar{k}}^*,\bar{X}),A)$$

and arithmetic cohomology

$$H^i_{\mathrm{ar}}(X,\mathbb{Z}) = \mathrm{Ext}^i_G(\mathrm{Cor}_{\bar{k}}(\Delta_{\bar{k}}^*,\bar{X}),\mathbb{Z}).$$

We show that $H_0^{\mathrm{ar}}(X,\mathbb{Z}) \cong H^0_{\mathrm{ar}}(X,\mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$ and that arithmetic homology and cohomology lie in long exact sequences with Galois-Suslin homology and cohomology, respectively. They are finitely generated abelian groups if and only if Conjecture P_0 holds.
The difference between arithmetic and Suslin homology is measured by a third theory, which we call Kato-Suslin homology, and which is defined as $H_i^{KS}(X,A) = H_i((\mathrm{Cor}_{\bar{k}}(\Delta_{\bar{k}}^*,\bar{X}) \otimes A)_G)$. By definition there is a long exact sequence

$$\cdots \to H_i^S(X,A) \to H_{i+1}^{\mathrm{ar}}(X,A) \to H_{i+1}^{KS}(X,A) \to H_{i-1}^S(X,A) \to \cdots .$$

It follows that $H_0^{KS}(X,\mathbb{Z}) = \mathbb{Z}^{\pi_0(X)}$ for any X. As a generalization of the integral version [7] of Kato's conjecture [12], we propose

Conjecture 1.2 *The groups $H_i^{KS}(X,\mathbb{Z})$ vanish for all smooth X and $i > 0$.*

Equivalently, there are short exact sequences

$$0 \to H_{i+1}^S(\bar{X},\mathbb{Z})_G \to H_i^S(X,\mathbb{Z}) \to H_i^S(\bar{X},\mathbb{Z})^G \to 0$$

for all $i \geq 0$ and all smooth X. We show that this conjecture, too, is equivalent to Conjecture P_0. This leads us to a conjecture on abelian tamely ramified class field theory:

Conjecture 1.3 *For every X separated and of finite type over $\mathbb{F}_q$, there is a canonical injection*

$$H_1^{\mathrm{ar}}(X,\mathbb{Z}) \to \pi_1^t(X)^{ab}$$

with dense image.

It might even be true that the relative group $H_1^{\mathrm{ar}}(X,\mathbb{Z})^\circ := \ker(H_1^{\mathrm{ar}}(X,\mathbb{Z}) \to \mathbb{Z}^{\pi_0(X)})$ is isomorphic to the geometric part of the abelianized fundamental group defined in SGA 3X§6. To support our conjecture, we note that the generalized Kato conjecture above implies $H_0^S(X,\mathbb{Z}) \cong H_1^{\mathrm{ar}}(X,\mathbb{Z})$ for smooth X, so that in this case our conjecture becomes a theorem of Schmidt-Spiess [19]. In addition, we show (independently of any conjectures)

Proposition 1.4 *If $1/l \in \mathbb{F}_q$, then $H_1^{\mathrm{ar}}(X,\mathbb{Z})^{\wedge l} \cong \pi_1^t(X)^{ab}(l)$ for arbitrary X.*

In particular, the conjectured finite generation of $H_1^{\mathrm{ar}}(X,\mathbb{Z})$ implies the conjecture away from the characteristic. We also give a conditional result at the characteristic.

Notation: In this paper, scheme over a field k means separated scheme of finite type over k. The separable algebraic closure of k is denoted by $\bar{k}$, and if X is a scheme over k, we sometimes write $\bar{X}$ or $X_{\bar{k}}$ for $X \times_k \bar{k}$.

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2 Motivic homology

Suslin homology $H_i^S(X,\mathbb{Z})$ of a scheme X over a field k is defined as the homology of the global sections $C_*^X(k)$ of the complex of etale sheaves $C_*^X(-) = \mathrm{Cor}_k(- \times \Delta^*, X)$. Here $\mathrm{Cor}_k(U,X)$ is the group of universal relative cycles of $U \times Y/U$ [23]. If U is smooth, then $\mathrm{Cor}_k(U,X)$ is the free abelian group generated by closed irreducible subschemes of $U \times X$ which are finite and surjective over a connected component of U. Note that $C_*^X(-) = C_*^{X^{red}}(-)$, and we will use that all contructions involving C_*^X agree for X and X^{red} without further notice.

More generally [1], motivic homology of weight n are the extension groups in Voevodsky's category of geometrical mixed motives

$$H_i(X,\mathbb{Z}(n)) = \mathrm{Hom}_{DM_k^-}(\mathbb{Z}(n)[i], M(X)),$$

and are isomorphic to

$$H_i(X,\mathbb{Z}(n)) = \begin{cases} H^{2n-i}_{(0)}(\mathbb{A}^n, C_*^X) & n \geq 0 \\ H_{i-2n-1}(C_*\big(\frac{c_0(X\times(\mathbb{A}^n-\{0\}))}{c_0(X\times\{1\})}\big)(k)) & n < 0. \end{cases}$$

Here cohomology is taken for the Nisnevich topology. There is an obvious version with coefficients. Motivic homology is a covariant functor on the category of schemes of finite type over k, and has the following additional properties, see [1] (the final three properties require resolution of singularities)

a) It is homotopy invariant.

b) It satisfies a projective bundle formula

$$H_i(X\times\mathbb{P}^1,\mathbb{Z}(n)) = H_i(X,\mathbb{Z}(n))\oplus H_{i-2}(X,\mathbb{Z}(n-1)).$$

c) There is a Mayer-Vietoris long exact sequence for open covers.

d) Given an abstract blow-up square

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

there is a long exact sequence

$$\cdots\to H_{i+1}(X,\mathbb{Z}(n))\to H_i(Z',\mathbb{Z}(n))\to \\ H_i(X',\mathbb{Z}(n))\oplus H_i(Z,\mathbb{Z}(n))\to H_i(X,\mathbb{Z}(n))\to\cdots \quad (2)$$

e) If X is proper, then motivic homology agrees with higher Chow groups indexed by dimension of cycles, $H_i(X,\mathbb{Z}(n))\cong CH_n(X,i-2n)$.

f) If X is smooth of pure dimension d, then motivic homology agrees with motivic cohomology with compact support,

$$H_i(X,\mathbb{Z}(n))\cong H_c^{2d-i}(X,\mathbb{Z}(d-n)).$$

In particular, if Z is a closed subscheme of a smooth scheme X of pure dimension d, then we have a long exact sequence

$$\cdots\to H_i(U,\mathbb{Z}(n))\to H_i(X,\mathbb{Z}(n))\to H_c^{2d-i}(Z,\mathbb{Z}(d-n))\to\cdots. \quad (3)$$

In order to remove the hypothesis on resolution of singularities, it would be sufficient to find a proof of Theorem 5.5(2) of Friedlander-Voevodsky [1] that does not require resolution of singularities. For all arguments in this paper (except the p-part of the Kato conjecture) the sequences (2) and (3) and the existence of a smooth and proper model for every function field are sufficient.

2.1 Suslin cohomology

Suslin cohomology is by definition the dual of Suslin homology, i.e. for an abelian group A it is defined as

$$H^i_S(X,A):=\operatorname{Ext}^i_{\mathrm{Ab}}(C^X_*(k),A).$$

We have $H^i_S(X,\mathbb{Q}/\mathbb{Z})\cong\operatorname{Hom}(H_i^S(X,\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$, and a short exact sequence of abelian groups gives a long exact sequence of cohomology groups, in particular long exact sequences

$$\cdots\to H^i_S(X,\mathbb{Z})\to H^i_S(X,\mathbb{Z})\to H^i_S(X,\mathbb{Z}/m)\to H^{i+1}_S(X,\mathbb{Z})\to\cdots. \quad (4)$$

and

$$\cdots \to H_S^{i-1}(X,\mathbb{Q}/\mathbb{Z}) \to H_S^i(X,\mathbb{Z}) \to H_S^i(X,\mathbb{Q}) \to H_S^i(X,\mathbb{Q}/\mathbb{Z}) \to \cdots .$$

Consequently, $H_S^i(X,\mathbb{Z})_{\mathbb{Q}} \cong H_S^i(X,\mathbb{Q})$ if Suslin-homology is finitely generated. If A is a ring, then $H_S^i(X,A) \cong \operatorname{Ext}_A^i(C_*^X(k)\otimes A, A)$, and we get a spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_A^s(H_t^S(X,A),A) \Rightarrow H_S^{s+t}(X,A). \tag{5}$$

In particular, there are perfect pairings

$$H_i^S(X,\mathbb{Q}) \times H_S^i(X,\mathbb{Q}) \to \mathbb{Q}$$
$$H_i^S(X,\mathbb{Z}/m) \times H_S^i(X,\mathbb{Z}/m) \to \mathbb{Z}/m.$$

Lemma 2.1 *There are natural pairings*

$$H_S^i(X,\mathbb{Z})/\mathrm{tor} \times H_i^S(X,\mathbb{Z})/\mathrm{tor} \to \mathbb{Z}$$

and

$$H_S^i(X,\mathbb{Z})_{\mathrm{tor}} \times H_{i-1}^S(X,\mathbb{Z})_{\mathrm{tor}} \to \mathbb{Q}/\mathbb{Z}.$$

Proof. The spectral sequence (5) gives a short exact sequence

$$0 \to \operatorname{Ext}^1(H_{i-1}^S(X,\mathbb{Z}),\mathbb{Z}) \to H_S^i(X,\mathbb{Z}) \to \operatorname{Hom}(H_i^S(X,\mathbb{Z}),\mathbb{Z}) \to 0. \tag{6}$$

The resulting map $H_S^i(X,\mathbb{Z})/\mathrm{tor} \twoheadrightarrow \operatorname{Hom}(H_i^S(X,\mathbb{Z}),\mathbb{Z})$ induces the first pairing. Since $\operatorname{Hom}(H_i^S(X,\mathbb{Z}),\mathbb{Z})$ is torsion free, we obtain the map

$$\begin{aligned} H_S^i(X,\mathbb{Z})_{\mathrm{tor}} \hookrightarrow \operatorname{Ext}^1(H_{i-1}^S(X,\mathbb{Z}),\mathbb{Z}) \twoheadrightarrow& \\ \operatorname{Ext}^1(H_{i-1}^S(X,\mathbb{Z})_{\mathrm{tor}},\mathbb{Z}) &\xleftarrow{\sim} \operatorname{Hom}(H_{i-1}^S(X,\mathbb{Z})_{\mathrm{tor}},\mathbb{Q}/\mathbb{Z}) \end{aligned}$$

for the second pairing. □

2.2 Comparison to motivic cohomology

Recall that in the category DM_k^- of bounded above complexes of homotopy invariant Nisnevich sheaves with transfers, the motive $M(X)$ of X is the complex of presheaves with transfers C_*^X. Since a field has no higher Nisnevich cohomology, taking global sections over k induces a canonical map

$$\operatorname{Hom}_{DM_k^-}(M(X),A[i]) \to \operatorname{Hom}_{DM^-(\mathrm{Ab})}(C_*^X(k),A[i]),$$

hence a natural map

$$H_M^i(X,A) \to H_S^i(X,A). \tag{7}$$

If X is a schene over $L \supseteq k$, then even though the cohomology groups do not depend on the base field, the map does. For example, if L/k is an extension of degree d, then the diagram of groups isomorphic to $\mathbb{Z}$,

$$\begin{tikzcd} H^0_M(\operatorname{Spec} k, \mathbb{Z}) \arrow[r, equal] \arrow[d, equal] & H^0_S(\operatorname{Spec} k, \mathbb{Z}) \arrow[d, "\times d"] \\ H^0_M(\operatorname{Spec} L, \mathbb{Z}) \arrow[r] & H^0_S(\operatorname{Spec} L, \mathbb{Z}) \end{tikzcd}$$

shows that the lower horizontal map is multiplication by d. We will see below that conjecturally (7) is a map between finitely generated groups which is rationally an isomorphism, and one might ask if its Euler characteristic has any interpretation.

3 The mod p Suslin homology in characteristic p

We examine the p-part of Suslin homology in characteristic p. We assume that k is perfect and resolution of singularities exists over k in order to obtain stronger results. We first give an auxiliary result on motivic cohomology with compact support:

Proposition 3.1 *Let $d = \dim X$.*
a) We have $H^i_c(X, \mathbb{Z}/p^r(n)) = 0$ for $n > d$.
b) If k is algebraically closed, then $H^i_c(X, \mathbb{Z}/p^r(d))$ is finite, $H^i_c(X, \mathbb{Q}_p/\mathbb{Z}_p(d))$ is of cofinite type, and the groups vanish unless $d \le i \le 2d$.

Proof. By induction on the dimension and the localization sequence, the statement for X and a dense open subset of X are equivalent. Hence replacing X by a smooth subscheme and then by a smooth and proper model, we can assume that X is smooth and proper. Then a) follows from [8]. If k is algebraically closed, then

$$H^i(X, \mathbb{Z}/p(d)) \cong H^{i-d}(X_{Nis}, \nu^d) \cong H^{i-d}(X_{\mathrm{et}}, \nu^d),$$

by [8] and [13]. That the latter group is finite and of cofinite type, respectively, can be derived from [16, Thm.1.11], and it vanishes outside of the given range by reasons of cohomological dimension. □

Theorem 3.2 *Let X be separated and of finite type over k.*
a) The groups $H_i(X, \mathbb{Z}/p^r(n))$ vanish for all $n < 0$.
b) If k is algebraically closed, then the groups $H^S_i(X, \mathbb{Z}/p^r)$ are finite, the groups $H^S_i(X, \mathbb{Q}_p/\mathbb{Z}_p)$ are of cofinite type, and both vanish unless $0 \le i \le d$.

Proof. If X is smooth, then $H_i(X, \mathbb{Z}/p^r(n)) \cong H^{2d-i}_c(X, \mathbb{Z}/p^r(d-n))$ and we conclude by the Proposition. In general, we can assume by (2) and induction on the number of irreducible components that X is integral. Proceeding by induction on the dimension, we choose a resolution of singularities X' of X, let Z be the closed subscheme of X where the map $X' \to X$ is not an isomorphism, and let $Z' = Z \times_X X'$. Then we conclude by the sequence (2). □

Example. If X' is the blow up of a smooth scheme X in a smooth subscheme Z, then the strict transform $Z' = X' \times_X Z$ is a projective bundle over Z, hence by the projective bundle formula $H_i^S(Z, \mathbb{Z}/p^r) \cong H_i^S(Z', \mathbb{Z}/p^r)$ and $H_i^S(X, \mathbb{Z}/p^r) \cong H_i^S(X', \mathbb{Z}/p^r)$. More generally, we have

Proposition 3.3 *Assume X has a desingularization $p : X' \to X$ which is an isomorphism outside of the dense open subset U. Then $H_i^S(U, \mathbb{Z}/p^r) \cong H_i^S(X, \mathbb{Z}/p^r)$. In particular, the p-part of Suslin homology is a birational invariant.*

The hypothesis is satisfied if X is smooth, or if U contains all singular points of X and a resolution of singularities exists which is an isomorphism outside of the singular points.

Proof. If X is smooth, then this follows from Proposition 3.1a) and the localization sequence (3). In general, let Z be the set of points where p is not an isomorphism, and consider the cartesian diagram

$$\begin{array}{ccccc} Z' & \longrightarrow & U' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & U & \longrightarrow & X. \end{array}$$

Comparing long exact sequence (2) of the left and outer squares,

$$\begin{array}{ccccc} \to H_i^S(Z', \mathbb{Z}/p^r) & \longrightarrow & H_i^S(U', \mathbb{Z}/p^r) \oplus H_i^S(Z, \mathbb{Z}/p^r) & \longrightarrow & H_i^S(U, \mathbb{Z}/p^r) \to \\ \| & & \| & & \downarrow \\ \to H_i^S(Z', \mathbb{Z}/p^r) & \longrightarrow & H_i^S(X', \mathbb{Z}/p^r) \oplus H_i^S(Z, \mathbb{Z}/p^r) & \longrightarrow & H_i^S(X, \mathbb{Z}/p^r) \to \end{array}$$

we see that $H_i^S(U', \mathbb{Z}/p^r) \cong H_i^S(X', \mathbb{Z}/p^r)$ implies $H_i^S(U, \mathbb{Z}/p^r) \cong H_i^S(X, \mathbb{Z}/p^r)$. □

Example. If X is a node, then the blow-up sequence gives $H_i^S(X, \mathbb{Z}/p^r) \cong H_{i-1}^S(k, \mathbb{Z}/p^r) \oplus H_i^S(k, \mathbb{Z}/p^r)$, which is $\mathbb{Z}/p^r$ for $i = 0, 1$ and vanishes otherwise. Reid constructed a normal surface with a singular point whose blow-up is a node, showing that the p-part of Suslin homology is not a birational invariant for normal schemes.

Corollary 3.4 *The higher Chow groups $CH_0(X, i, \mathbb{Z}/p^r)$ and the logarithmic de Rham-Witt cohomology groups $H^i(X_{\text{et}}, \nu_r^d)$, for $d = \dim X$, are birational invariants.*

Proof. Suslin homology agrees with higher Chow groups for proper X, and with motivic cohomology for smooth and proper X. □

Note that integrally $CH_0(X)$ is a birational invariant, but the higher Chow groups $CH_0(X,i)$ are generally not.
Suslin and Voevodsky [22, Thm.3.1] show that for a smooth compactification $\bar{X}$ of the smooth curve X, $H_0^S(X,\mathbb{Z})$ is isomorphic to the relative Picard group $\mathrm{Pic}(\bar{X},Y)$ and that all higher Suslin homology groups vanish. Proposition 3.3 implies that the kernel and cokernel of $\mathrm{Pic}(\bar{X},Y) \to \mathrm{Pic}(\bar{X})$ are uniquely p-divisible. Given U with compactification $j : U \to X$, the normalization $X^\sim$ of X in U is the affine bundle defined by the integral closure of $\mathcal{O}_X$ in $j_*\mathcal{O}_U$. We call X normal in U if $X^\sim \to X$ is an isomorphism.

Proposition 3.5 *If X is normal in the curve U, then $H_i^S(U,\mathbb{Z}/p) \cong H_i^S(X,\mathbb{Z}/p)$.*

Proof. This follows by applying the argument of Proposition 3.3 to X' the normalization of X, Z the closed subset where $X' \to X$ is not an isomorphism, $Z' = X' \times_X Z$ and $U' = X' \times_X U$. Since X is normal in U, we have $Z \subseteq U$ and $Z' \subseteq U'$. □

3.1 The albanese map

The following application was pointed out to us by N.Otsubo. Let X be a smooth connected quasi-projective variety over an algeraically closed field k of characteristic p. Then Spiess and Szamuely defined in [20] an albanese map

$$alb_X : H_0^S(X,\mathbb{Z})^0 \to Alb_X(k)$$

from the degree-0-part of Suslin homology to the k-valued points of the Albanese variety in the sense of Serre. They proved that if X is a dense open subscheme in a smooth projective scheme over k, then alb_X induces an isomorphism of the prime-to-p-torsion subgroups. We can remove the last hypothesis:

Theorem 3.6 *Assuming resolution of singularities, the map alb_X induces an isomorphism on torsion groups for any smooth, connected, quasi-projective variety over an algebraically closed field.*

Proof. In view of the result of Spiess and Szamuely, it suffices to consider the p-primary groups. Let T be a smooth and projective model of X. Since both sides are covariantly functorial and alb_X is functorial by construction, we obtain a commutative diagram

$$\begin{array}{ccc}
H_0^S(X,\mathbb{Z})^0 & \xrightarrow{alb_X} & Alb_X(k) \\
\downarrow & & \downarrow \\
H_0^S(T,\mathbb{Z})^0 & \xrightarrow{alb_T} & Alb_T(k)
\end{array}$$

The lower map is an isomorphism on torsion subgroups by Milne [15]. To show that the left vertical map is an isomorphism, consider the map of coefficient sequences

$$\begin{array}{ccccccc} H_1^S(X,\mathbb{Z})\otimes\mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & H_1^S(X,\mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & {}_pH_0^S(X,\mathbb{Z}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_1^S(T,\mathbb{Z})\otimes\mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & H_1^S(T,\mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & {}_pH_0^S(T,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

The right vertical map is an isomorphism because the middle map vertical map is an isomorphism by Proposition 3.3, and because $H_1^S(T,\mathbb{Z})\otimes\mathbb{Q}_p/\mathbb{Z}_p \cong CH_0(T,1)\otimes\mathbb{Q}_p/\mathbb{Z}_p$ vanishes by [6, Thm.6.1]. Finally, the map $Alb_X(k)\to Alb_T(k)$ is an isomorphism on p-torsion groups because by Serre's description [18], the two Albanese varieties differ by a torus, which does not have any p-torsion k-rational points in characteristic p, □

4 Galois properties

Suslin homology is covariant, i.e. a separated map $f : X \to Y$ of finite type induces a map $f_* : \mathrm{Cor}_k(T,X)\to\mathrm{Cor}_k(T,Y)$ by sending a closed irreducible subscheme Z of $T\times X$, finite over T, to the subscheme $[k(Z):k(f(Z))]\cdot f(Z)$ (note that $f(Z)$ is closed in $T\times Y$ and finite over T). On the other hand, Suslin homology is contravariant for finite flat maps $f : X\to Y$, because f induces a map $f^* : \mathrm{Cor}_k(T,Y)\to\mathrm{Cor}_k(T,X)$ by composition with the graph of f in $\mathrm{Cor}_k(Y,X)$ (note that the graph is a universal relative cycle in the sense of [23]). We examine the properties of Suslin homology under change of base-fields.

Lemma 4.1 *Let L/k be a finite extension of fields, X a scheme over k and Y a scheme over L. Then $\mathrm{Cor}_L(Y,X_L)=\mathrm{Cor}_k(Y,X)$ and if X is smooth, then $\mathrm{Cor}_L(X_L,Y)=\mathrm{Cor}_k(X,Y)$. In particular, Suslin homology does not depend on the base field.*

Proof. The first statement follows because $Y\times_L X_L\cong Y\times_k X$. The second statement follows because the map $X_L\to X$ is finite and separated, hence a closed subscheme of $X_L\times_L Y\cong X\times_k Y$ is finite and surjective over X_L if and only if it is finite and surjective over X. □

Given a scheme over k, the graph of the projection $X_L\to X$ in $X_L\times X$ gives elements $\Gamma_X\in\mathrm{Cor}_k(X,X_L)$ and $\Gamma_X^t\in\mathrm{Cor}_k(X_L,X)$.

4.1 Covariance

Lemma 4.2 *a) If X and Y are separated schemes of finite type over k, then the two maps*

$$\mathrm{Cor}_L(X_L,Y_L)\to\mathrm{Cor}_k(X,Y)$$

induced by composition and precomposition, respectively, with Γ_Y^t *and* Γ_X *agree. Both maps send a generator* $Z \subseteq X_L \times_k Y \cong X \times_k Y_L$ *to its image in* $X \times Y$ *with multiplicity* $[k(Z) : k(f(Z))]$*, a divisor of* $[L : k]$.
b) If F/k *is an infinite algebraic extension, then* $\lim_{L/k} \mathrm{Cor}_L(X_L, Y_L) = 0$.

Proof. The first part is easy. If Z is of finite type over k, then $k(Z)$ is a finitely generated field extension of k. For every component Z_i of Z_F, we obtain a map $F \to F \otimes_k k(Z) \to k(Z_i)$, and since F is not finitely generated over k, neither is $k(Z_i)$. Hence going up the tower of finite extensions L/k in F, the degree of $[k(W_L) : k(Z)]$, for W_L the component of Z_L corresponding to Z_i, goes to infinity. □

4.2 Contravariance

Lemma 4.3 *a) If* X *and* Y *are schemes over* k*, then the two maps*

$$\mathrm{Cor}_k(X, Y) \to \mathrm{Cor}_L(X_L, Y_L)$$

induced by composition and precomposition, respectively with Γ_Y *and* Γ_X^t *agree. Both maps send a generator* $Z \subseteq X \times Y$ *to the cycle associated to* $Z_L \subseteq X \times_k Y_L \cong X_L \times_k Y$*. If* L/k *is separable, this is a sum of the integral subschemes lying over* Z *with multiplicity one. If* L/k *is Galois with group* G*, then the maps induce an isomorphism*

$$\mathrm{Cor}_k(X, Y) \cong \mathrm{Cor}_L(X_L, Y_L)^G.$$

b) Varying L*,* $\mathrm{Cor}_L(X_L, Y_L)$ *forms an etale sheaf on* $\mathrm{Spec}\, k$ *with stalk* $M = \mathrm{colim}_L \mathrm{Cor}_L(X_L, Y_L) \cong \mathrm{Cor}_{\bar{k}}(X_{\bar{k}}, Y_{\bar{k}})$*, where* L *runs through the finite extensions of* k *in a separable algebraic closure* $\bar{k}$ *of* k*. In particular,* $\mathrm{Cor}_L(X_L, Y_L) \cong M^{\mathrm{Gal}(\bar{k}/L)}$.

Proof. Again, the first part is easy. If L/k is separable, Z_L is finite and etale over Z, hence $Z_L \cong \sum_i Z_i$, a finite sum of the integral cycles lying over Z with multiplicity one each. If L/k is moreover Galois, then $\mathrm{Cor}_k(X, Y) \cong \mathrm{Cor}_L(X_L, Y_L)^G$ and $\mathrm{Cor}_{\bar{k}}(X_{\bar{k}}, Y_{\bar{k}}) \cong \mathrm{colim}_{L/k} \mathrm{Cor}_L(X_L, Y_L)$ by EGA IV Thm. 8.10.5. □

The proposition suggests to work with the complex C_*^X of etale sheaves on $\mathrm{Spec}\, k$ given by

$$C_*^X(L) := \mathrm{Cor}_L(\Delta_L^*, X_L) \cong \mathrm{Cor}_k(\Delta_L^*, X).$$

Corollary 4.4 *If* $\bar{k}$ *is a separable algebraic closure of* k*, then* $H_i^S(X_{\bar{k}}, A) \cong \mathrm{colim}_{L/k} H_i^S(X_L, A)$*, and there is a spectral sequence*

$$E_2^{s,t} = \lim_{L/k}{}^s H_S^t(X_L, A) \Rightarrow H_S^{s+t}(X_{\bar{k}}, A).$$

The direct and inverse system run through finite separable extensions L/k, and the maps in the systems are induced by contravariant functoriality of Suslin homology for finite flat maps.

Proof. This follows from the quasi-isomorphisms

$$R\operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}),\mathbb{Z}) \cong R\operatorname{Hom}_{\mathrm{Ab}}(\operatorname*{colim}_L C_*^X(L),\mathbb{Z}) \cong R\lim_L R\operatorname{Hom}_{\mathrm{Ab}}(C_*^X(L),\mathbb{Z}).$$

□

4.3 Coinvariants

If G_k is the absolute Galois group of k, then $\mathrm{Cor}_{\bar{k}}(\bar{X},\bar{Y})_{G_k}$ can be identified with $\mathrm{Cor}_k(X,Y)$ by associating orbits of points of $\bar{X}\times_{\bar{k}}\bar{Y}$ with their image in $X\times_k Y$. However, this identification is neither compatible with covariant nor with contravariant functoriality, and in particular not with the differentials in the complex $C_*^X(k)$. But the obstruction is torsion, and we can remedy this problem by tensoring with $\mathbb{Q}$: Define an isomorphism

$$\tau : (\mathrm{Cor}_{\bar{k}}(\bar{X},\bar{Y})_{\mathbb{Q}})_{G_k} \to \mathrm{Cor}_k(X,Y)_{\mathbb{Q}}.$$

as follows. A generator $1_{\bar{Z}}$ corresponding to the closed irreducible subscheme $\bar{Z}\subseteq\bar{X}\times\bar{Y}$ is sent to $\frac{1}{g_Z}1_Z$, where Z is the image of $\bar{Z}$ in $X\times Y$ and g the number of irreducible components of $Z\times_k\bar{k}$, i.e. g_Z is the size of the Galois orbit of $\bar{Z}$.

Lemma 4.5 *The isomorphism τ is functorial in both variables, hence it induces an isomorphism of complexes*

$$(C_*^X(\bar{k})_{\mathbb{Q}})_{G_k} \cong C_*^X(k)_{\mathbb{Q}}.$$

Proof. This can be proved by explicit calculation. We give an alternate proof. Consider the composition

$$\mathrm{Cor}_k(X,Y) \to \mathrm{Cor}_{\bar{k}}(\bar{X},\bar{Y})^{G_k} \to \mathrm{Cor}_{\bar{k}}(\bar{X},\bar{Y})_{G_k} \xrightarrow{\tau} \mathrm{Cor}_k(X,Y)_{\mathbb{Q}}.$$

The middle map is induced by the identity, and is multiplication by g_Z on the component corresponding to Z. All maps are isomorphisms upon tensoring with $\mathbb{Q}$. The first map, the second map, and the composition are functorial, hence so is τ. □

5 Galois descent

Let $\bar{k}$ be the algebraic closure of k with Galois group G_k, and let A be a continuous G_k-module. Then $C_*^X(\bar{k})\otimes A$ is a complex of continuous G_k-modules, and if k has finite cohomological dimension we define Galois-Suslin homology to be

$$H_i^{GS}(X,A) = H^{-i}R\Gamma(G_k, C_*^X(\bar{k}) \otimes A).$$

By construction, there is a spectral sequence

$$E^2_{s,t} = H^{-s}(G_k, H_t^S(\bar{X},A)) \Rightarrow H^{GS}_{s+t}(X,A).$$

The case $X = \operatorname{Spec} k$ shows that Suslin homology does not agree with Galois-Suslin homology, i.e. Suslin homology does not have Galois descent. We define Galois-Suslin cohomology to be

$$H^i_{GS}(X,A) = \operatorname{Ext}^i_{G_k}(C_*^X(\bar{k}), A). \tag{8}$$

This agrees with the old definition if k is algebraically closed. Let τ_* be the functor from G_k-modules to continuous G_k-modules which sends M to $\operatorname{colim}_L M^{G_L}$, where L runs through the finite extensions of k. It is easy to see that $R^i\tau_* M = \operatorname{colim}_H H^i(H,M)$, with H running through the finite quotients of G_k.

Lemma 5.1 *We have* $H^i_{GS}(X,A) = H^iR\Gamma G_k R\tau_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}),A)$. *In particular, there is a spectral sequence*

$$E_2^{s,t} = H^s(G_k, R^t\tau_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}),A)) \Rightarrow H^{s+t}_{GS}(X,A). \tag{9}$$

Proof. This is [17, Ex. 0.8]. Since $C_*^X(\bar{k})$ is a complex of free $\mathbb{Z}$-modules, $\operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}),-)$ is exact and preserves injectives. Hence the derived functor of $\tau_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}),-)$ is $R^t\tau_*$ applied to $\operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}),-)$. $\square$

Lemma 5.2 *For any abelian group A, the natural inclusion $C_*^X(k)\otimes A \to (C_*^X(\bar{k})\otimes A)^{G_k}$ is an isomorphism.*

Proof. Let Z be a cycle corresponding to a generator of $C_*(k)$. If $Z\otimes_k \bar{k}$ is the union of g irreducible components, then the corresponding summand of $C_*(\bar{k})$ is a free abelian group of rank g on which the Galois group permutes the summands transitively. The claim is now easy to verify. $\square$

Proposition 5.3 *We have*

$$H_i^{GS}(X,\mathbb{Q}) \cong H_i^S(X,\mathbb{Q})$$
$$H^i_{GS}(X,\mathbb{Q}) \cong H^i_S(X,\mathbb{Q}).$$

Proof. By the Lemma, $H_i^S(X,\mathbb{Q}) = H_i(C_*^X(k)\otimes\mathbb{Q}) \cong H_i((C_*^X(\bar{k})\otimes\mathbb{Q})^{G_k})$. But the latter agrees with $H_i^{GS}(X,\mathbb{Q})$ because higher Galois cohomology is torsion. Similarly, we have $R^t\tau_* \operatorname{Hom}(C_i^X(\bar{k}),\mathbb{Q}) = 0$ for $t>0$, and

$H^s(G_k, \tau_* \operatorname{Hom}(C_*^X(\bar{k}), \mathbb{Q})) = 0$ for $s > 0$. Hence $H^i_{GS}(X, \mathbb{Q})$ is isomorphic to the ith cohomology of

$$\operatorname{Hom}_{G_k}(C_*^X(\bar{k}), \mathbb{Q}) \cong \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k})_{G_k}, \mathbb{Q}) \cong \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(k), \mathbb{Q}).$$

The latter equality follows with Lemma 4.5. □

Theorem 5.4 *If m is invertible in k and A is a finitely generated m-torsion G_k-module, then*

$$H^i_{GS}(X, A) \cong H^i_{\mathrm{et}}(X, A).$$

Proof. This follows with the argument of Suslin-Voevodsky [22]. Indeed, let $f : (Sch/k)_h \to Et_k$ be the canonical map from the large site with the h-topology of k to the small etale site of k. Clearly $f_* f^* \mathcal{F} \cong \mathcal{F}$, and the proof of Thm.4.5 in loc.cit. shows that the cokernel of the injection $f^* f_* \mathcal{F} \to \mathcal{F}$ is uniquely m-divisible, for any homotopy invariant presheaf with transfers (like, for example, $C_i^X : U \mapsto \mathrm{Cor}_k(U \times \Delta^i, X)$). Hence

$$\operatorname{Ext}^i_h(\mathcal{F}_h^\sim, f^*A) \cong \operatorname{Ext}^i_h(f^* f_* \mathcal{F}_h^\sim, f^*A) \cong \operatorname{Ext}^i_{Et_k}(f_* \mathcal{F}_h^\sim, A) \cong \operatorname{Ext}^i_{G_k}(\mathcal{F}(\bar{k}), A).$$

Then the argument of section 7 in loc.cit. together with Theorem 6.7 can be descended from the algebraic closure of k to k. □

Duality results for the Galois cohomology of a field k lead via theorem 5.4 to duality results between Galois-Suslin homology and cohomology over k.

Theorem 5.5 *Let k be a finite field, A a finite G_k-module, and $A^* = \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$. Then there is a perfect pairing of finite groups*

$$H^{GS}_{i-1}(X, A) \times H^i_{GS}(X, A^*) \to \mathbb{Q}/\mathbb{Z}.$$

Proof. According to [17, Example 1.10] we have

$$\operatorname{Ext}^r_{G_k}(M, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}^{r+1}_{G_k}(M, \mathbb{Z}) \cong H^{1-r}(G_k, M)^*$$

for every finite G_k-module M, and the same holds for any torsion module by writing it as a colimit of finite modules. Hence

$$\begin{aligned}\operatorname{Ext}^r_{G_k}(C_*^X(\bar{k}), \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})) &\cong \operatorname{Ext}^r_{G_k}(C_*^X(\bar{k}) \otimes A, \mathbb{Q}/\mathbb{Z}) \cong \\ &H^{1-r}(G_k, C_*^X(\bar{k}) \otimes A)^* = H^{GS}_{r-1}(X, A)^*.\end{aligned}$$

□

The case of non-torsion sheaves is discussed below.

Theorem 5.6 *Let k be a local field with finite residue field and separable closure k^s. For a finite G_k-module A let $A^D = \mathrm{Hom}(A,(k^s)^\times)$. Then we have isomorphisms*

$$H^i_{GS}(X, A^D) \cong \mathrm{Hom}(H^{GS}_{i-2}(X, A), \mathbb{Q}/\mathbb{Z}).$$

Proof. According to [17, Thm.2.1] we have

$$\mathrm{Ext}^r_{G_k}(M,(k^s)^\times) \cong H^{2-r}(G_k, M)^*$$

for every finite G_k-module M. This implies the same statement for torsion modules, and the rest of the proof is the same as above. □

6 Finite base fields

From now on we fix a finite field $\mathbb{F}_q$ with algebraic closure $\bar{\mathbb{F}}_q$. To obtain the following results, we assume resolution of singularities. This is needed to use the sequences (2) and (3) to reduce to the smooth and projective case on the one hand, and the proof of Jannsen-Saito [11] of the Kato conjecture on the other hand (however, Kerz and Saito announced a proof of the prime to p-part of the Kato conjecture which does not require resolution of singularities). The critical reader is invited to view the following results as conjectures which are theorems in dimension at most 3.
We first present results on finite generation in the spirit of [11] and [7].

Theorem 6.1 *For any $X/\mathbb{F}_q$ and any integer m, the groups $H_i^S(X,\mathbb{Z}/m)$ and $H^i_S(X,\mathbb{Z}/m)$ are finitely generated.*

Proof. It suffices to consider the case of homology. If X is smooth and proper of dimension d, then $H_i^S(X,\mathbb{Z}/m) \cong CH_0(X,i,\mathbb{Z}/m) \cong H_c^{2d-i}(X,\mathbb{Z}/m(d))$, and the result follows from work of Jannsen-Saito [11]. The usual devisage then shows that $H_c^j(X,\mathbb{Z}/m(d))$ is finite for all X and $d \geq \dim X$, hence $H_i^S(X,\mathbb{Z}/m)$ is finite for smooth X. Finally, one proceeds by induction on the dimension of X with the blow-up long-exact sequence to reduce to the case X smooth. □

6.1 Rational Suslin-homology

We have the following unconditional result:

Theorem 6.2 *For every connected X, the map $H_0^S(X,\mathbb{Q}) \to H_0^S(\mathbb{F}_q,\mathbb{Q}) \cong \mathbb{Q}$ is an isomorphism.*

Proof. By induction on the number of irreducible components and (2) we can first assume that X is irreducible and then reduce to the situation where X is smooth. In this case, we use (3) and the following Proposition to reduce to the smooth and proper case, where $H_0^S(X,\mathbb{Q}) = CH_0(X)_{\mathbb{Q}} \cong CH_0(\mathbb{F}_q)_{\mathbb{Q}}$. □

Proposition 6.3 *If $n > \dim X$, then $H^i_c(X, \mathbb{Q}(n)) = 0$ for $i \geq n + \dim X$.*

Proof. By induction on the dimension and the localization sequence for motivic cohomology with compact support one sees that the statement for X and a dense open subscheme of X are equivalent. Hence we can assume that X is smooth and proper of dimension d. Comparing to higher Chow groups, one sees that this vanishes for $i > d + n$ for dimension (of cycles) reasons. For $i = d + n$, we obtain from the niveau spectral sequence a surjection

$$\bigoplus_{X_{(0)}} H^{n-d}_M(k(x), \mathbb{Q}(n-d)) \twoheadrightarrow H^{d+n}_M(X, \mathbb{Q}(n)).$$

But the summands vanish for $n > d$ because higher Milnor K-theory of finite fields is torsion. □

By definition, the groups $H_i(X, \mathbb{Q}(n))$ vanish for $i < n$. We will consider the following conjecture P_n of [5]:

Conjecture P_n: *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, the groups $H_i(X, \mathbb{Q}(n))$ vanish for $i \neq 2n$.*

This is a special case of Parshin's conjecture: If X is smooth and projective of dimension d, then

$$H_i(X, \mathbb{Q}(n)) \cong H^{2d-i}_M(X, \mathbb{Q}(d-n)) \cong K_{i-2n}(X)^{(d-n)}$$

and, according to Parshin's conjecture, the latter group vanishes for $i \neq 2n$. By the projective bundle formula, P_n implies P_{n-1}.

Proposition 6.4 *a) Let U be a curve. Then $H^S_i(U, \mathbb{Q}) \cong H^S_i(X, \mathbb{Q})$ for any X normal in U.*
b) Assume conjecture P_{-1}. Then $H_i(X, \mathbb{Q}(n)) = 0$ for all X and $n < 0$, and if X has a desingularization $p : X' \to X$ which is an isomorphism outside of the dense open subset U, then $H^S_i(U, \mathbb{Q}) \cong H^S_i(X, \mathbb{Q})$. In particular, Suslin homology and higher Chow groups of weight 0 are birational invariant.
c) Under conjecture P_0, the groups $H^S_i(X, \mathbb{Q})$ are finite dimensional and vanish unless $0 \leq i \leq d$.
d) Conjecture P_0 is equivalent to the vanishing of $H^S_i(X, \mathbb{Q})$ for all $i \neq 0$ and all smooth X.

Proof. The argument is the same as in Theorem 3.2. To prove b), we have to show that $H^i_c(X, \mathbb{Q}(n)) = 0$ for $n > d = \dim X$ under P_{-1}, and for c) we have to show that $H^i_c(X, \mathbb{Q}(d))$ is finite dimensional and vanishes unless $d \leq i \leq 2d$ under P_0. By induction on the dimension and the localization sequence we can assume that X is smooth and projective. In this case, the statement is Conjecture P_{-1} and P_0, respectively, plus the fact that $H^S_0(X, \mathbb{Q}) \cong CH_0(X)_{\mathbb{Q}}$ is a finite dimensional vector space. The final statement follows from the exact sequence (3) and the vanishing of $H^i_c(X, \mathbb{Q}(n)) = 0$ for $n > d = \dim X$ under P_{-1}. $\square$

Proposition 6.5 *Conjecture P_0 holds if and only if the map $H^i_M(X, \mathbb{Q}) \to H^i_S(X, \mathbb{Q})$ of (7) is an isomorphism for all $X/\mathbb{F}_q$ and i.*

Proof. The second statement implies the first, because if the map is an isomorphism, then $H^i_S(X, \mathbb{Q}) = 0$ for $i \neq 0$ and X smooth and proper, and hence so is the dual $H^S_i(X, \mathbb{Q})$. To show that P_0 implies the second statement, first note that because the map is compatible with long exact blow-up sequences, we can by induction on the dimension assume that X is smooth of dimension d. In this case, motivic cohomology vanishes above degree 0, and the same is true for Suslin cohomology in view of Proposition 6.4d). To show that for connected X the map (7) is an isomorphism of $\mathbb{Q}$ in degree zero, we consider the commutative diagram induced by the structure map

$$\begin{tikzcd}
H^0_M(\mathbb{F}_q, \mathbb{Q}) \arrow[r] \arrow[d] & H^0_S(\mathbb{F}_q, \mathbb{Q}) \arrow[d] \\
H^0_M(X, \mathbb{Q}) \arrow[r] & H^0_S(X, \mathbb{Q})
\end{tikzcd}$$

This reduces the problem to the case $X = \operatorname{Spec} \mathbb{F}_q$, where it can be directly verified. $\square$

6.2 Integral coefficients

Combining the torsion results [11] with the rational results, we obtain the following

Proposition 6.6 *Conjecture P_0 is equivalent to the finite generation of $H^S_i(X, \mathbb{Z})$ for all $X/\mathbb{F}_q$.*

Proof. If X is smooth and proper, then according to the main theorem of Jannsen-Saito [11], the groups $H^S_i(X, \mathbb{Q}/\mathbb{Z}) = CH_0(X, i, \mathbb{Q}/\mathbb{Z})$ are isomorphic to etale homology, and hence finite for $i > 0$ by the Weil-conjectures. Hence finite generation of $H^S_i(X, \mathbb{Z})$ implies that $H^S_i(X, \mathbb{Q}) = 0$ for $i > 0$.
Conversely, we can by induction on the dimension assume that X is smooth and has a smooth and proper model. Expressing Suslin homology of smooth schemes in terms of motivic cohomology with compact support and again using induction, it suffices to show that $H^i_M(X, \mathbb{Z}(n))$ is finitely generated for smooth and proper X and $n \geq \dim X$. Using the projective bundle formula we can assume that $n = \dim X$, and then the statement follows because $H^i_M(X, \mathbb{Z}(n)) \cong CH_0(X, 2n - i)$ is finitely generated according to [7, Thm 1.1]. $\square$

Recall the pairings of Lemma 2.1. We call them perfect if they identify one group with the dual of the other group. In the torsion case, this implies that the groups are finite, but in the free case this is not true: For example, $\oplus_I\mathbb{Z}$ and $\prod_I \mathbb{Z}$ are in perfect duality.

Proposition 6.7 *Let X be a separated scheme of finite type over a finite field. Then the following statements are equivalent:*

a) The groups $H_i^S(X,\mathbb{Z})$ are finitely generated for all i.

b) The groups $H^i_S(X,\mathbb{Z})$ are finitely generated for all i.

c) The groups $H^i_S(X,\mathbb{Z})$ are countable for all i.

d) The pairings of Lemma 2.1 are perfect for all i.

Proof. a) $\Rightarrow$ b) $\Rightarrow$ c) are clear, and c) $\Rightarrow$ a) follows from [9, Prop.3F.12], which states that if A is not finitely generated, then either $\operatorname{Hom}(A,\mathbb{Z})$ or $\operatorname{Ext}(A,\mathbb{Z})$ is uncountable.
Going through the proof of Lemma 2.1 it is easy to see that a) implies d). Conversely, if the pairing is perfect, then ${}_{\mathrm{tor}}H_i^S(X,\mathbb{Z})$ is finite. Let $A = H^i_S(X,\mathbb{Z})/\mathrm{tor}$ and fix a prime l. Then A/l is a quotient of $H^i_S(X,\mathbb{Z})/l \subseteq H^i_S(X,\mathbb{Z}/l)$, and which is finite by Theorem 6.1. Choose lifts $b_i \in A$ of a basis of A/l and let B be the finitely generated free abelian subgroup of A generated by the b_i. By construction, A/B is l-divisible, hence $H_i^S(X,\mathbb{Z})/\mathrm{tor} = \operatorname{Hom}(A,\mathbb{Z}) \subseteq \operatorname{Hom}(B,\mathbb{Z})$ is finitely generated. □

6.3 The algebraically closure of a finite field

Suslin homology has properties similar to a Weil-cohomology theory. Let X_1 be separated and of finite type over $\mathbb{F}_q$, $X_n = X \times_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and $X = X_1 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. From Corollary 4.4, we obtain a short exact sequence

$$0 \to \lim{}^1 H^{t+1}_S(X_n,\mathbb{Z}) \to H^t_S(X,\mathbb{Z}) \to \lim H^t_S(X_n,\mathbb{Z}) \to 0.$$

The outer terms can be calculated with the 6-term lim-lim^1-sequence associated to (6). The theorem of Suslin and Voevodsky implies that

$$\lim H^i_S(X,\mathbb{Z}/l^r) \cong H^i_{\mathrm{et}}(X,\mathbb{Z}_l)$$

for $l \neq p = \operatorname{char}\mathbb{F}_q$. For X is proper and $l = p$, we get the same result from [6]

$$H^i_S(X,\mathbb{Z}/p^r) \cong \operatorname{Hom}(CH_0(X,i,Z/p^r),\mathbb{Z}/p^r) \cong H^i_{\mathrm{et}}(X,\mathbb{Z}/p^r).$$

We show that this is true integrally:

Proposition 6.8 *Let X be a smooth and proper curve over the algebraic closure of a finite field k of characteristic p. Then the non-vanishing cohomology groups are*

$$H^i_S(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \lim_r \operatorname{Hom}_{GS}(\mu_{p^r}, \operatorname{Pic} X) \times \prod_{l\neq p} T_l \operatorname{Pic} X(-1) & i=1 \\ \prod_{l\neq p} \mathbb{Z}_l(-1) & i=2. \end{cases}$$

Here Hom_{GS} *denotes homomorphisms of group schemes.*

Proof. By properness and smoothness we have

$$H_i^S(X,\mathbb{Z}) \cong H^{2-i}_M(X,\mathbb{Z}(1)) \cong \begin{cases} \operatorname{Pic} X & i=0; \\ k^\times & i=1; \\ 0 & i\neq 0,1. \end{cases}$$

Now

$$\operatorname{Ext}^1(k^\times,\mathbb{Z}) = \operatorname{Hom}(\operatorname*{colim}_{p\nmid m} \mu_m, \mathbb{Q}/\mathbb{Z}) \cong \prod_{l\neq p} \mathbb{Z}_l(-1)$$

and since $\operatorname{Pic} X$ is finitely generated by torsion,

$$\begin{aligned} \operatorname{Ext}^1(\operatorname{Pic} X,\mathbb{Z}) &\cong \operatorname{Hom}(\operatorname*{colim}_m {}_m \operatorname{Pic} X, \mathbb{Q}/\mathbb{Z}) \cong \\ &\qquad \lim \operatorname{Hom}_{GS}({}_m \operatorname{Pic} X, \mathbb{Z}/m) \cong \lim_m \operatorname{Hom}_{GS}(\mu_m, {}_m \operatorname{Pic} X) \end{aligned}$$

by the Weil-pairing. □

Proposition 6.9 *Let X be smooth, projective and connected over the algebraic closure of a finite field. Assuming conjecture P_0, we have*

$$H^i_S(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \prod_l H^i_{\mathrm{et}}(X,\mathbb{Z}_l) & i\geq 1. \end{cases}$$

In particular, the l-adic completion of $H^i_S(X,\mathbb{Z})$ is l-adic cohomology $H^i_{\mathrm{et}}(X,\mathbb{Z}_l)$ for all l.

Proof. Let $d=\dim X$. By properness and smoothness we have

$$H_i^S(X,\mathbb{Z}) \cong H^{2d-i}_M(X,\mathbb{Z}(d)).$$

Under hypothesis P_0, the groups $H_i^S(X,\mathbb{Z})$ are torsion for $i>0$, and $H_0^S(X,\mathbb{Z}) = CH_0(X)$ is the product of a finitely generated group and a torsion group. Hence for $i\geq 1$ we get by (6) that

$$\begin{aligned} H^i_S(X,\mathbb{Z}) &\cong \operatorname{Ext}^1(H^S_{i-1}(X,\mathbb{Z}),\mathbb{Z}) \cong \operatorname{Hom}(H^S_{i-1}(X,\mathbb{Z})_{\mathrm{tor}}, \mathbb{Q}/\mathbb{Z}) \\ &\cong \operatorname{Hom}(H^{2d-i+1}_M(X,\mathbb{Z}(d))_{\mathrm{tor}}, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(H^{2d-i}_{\mathrm{et}}(X,\mathbb{Q}/\mathbb{Z}(d)),\mathbb{Q}/\mathbb{Z}) \\ &\cong \operatorname{Hom}(\operatorname*{colim}_m H^{2d-i}_{\mathrm{et}}(X,\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}) \cong \lim_m \operatorname{Hom}(H^{2d-i}_{\mathrm{et}}(X,\mathbb{Z}/m(d)),\mathbb{Z}/m). \end{aligned}$$

By Poincare-duality, the latter agrees with $\lim H^i_{\mathrm{et}}(X,\mathbb{Z}/m)\cong\prod_l H^i_{\mathrm{et}}(X,\mathbb{Z}_l)$.
□

7 Arithmetic homology and cohomology

We recall some definitions and results from [3]. Let X be separated and of finite type over a finite field $\mathbb{F}_q$, $\bar{X}=X\times_{\mathbb{F}_q}\bar{\mathbb{F}}_q$ and G be the Weil-group of $\mathbb{F}_q$. Let $\gamma:\mathcal{T}_G\to\mathcal{T}_{\hat{G}}$ be the functor from the category of G-modules to the category of continuous $\hat{G}=\mathrm{Gal}(\mathbb{F}_q)$-modules which associated to M the module $\gamma_*=\mathrm{colim}_m M^{mG}$, where the index set is ordered by divisibility. It is easy to see that the forgetful functor is a left adjoint of γ_*, hence γ_* is left exact and preserves limits. The derived functors γ_*^i vanish for $i>1$, and $\gamma_*^1M=R^1\gamma_*M=\mathrm{colim}\, M_{mG}$, where the transition maps are given by $M_{mG}\to M_{mnG}, x\mapsto\sum_{s\in mG/mnG}sx$. Consequently, a complex $M^\cdot$ of G-modules gives rise to an exact triangle of continuous $\hat{G}$-modules

$$\gamma_*M^\cdot\to R\gamma_*M^\cdot\to\gamma_*^1M^\cdot[-1]. \tag{10}$$

If $M=\gamma^*N$ is the restriction of a continuous $\hat{G}$-module, then $\gamma_*M=N$ and $\gamma_*^1M=N\otimes\mathbb{Q}$. In particular, Weil-etale cohomology and etale cohomology of torsion sheaves agree. Note that the derived functors γ_* restricted to the category of $\hat{G}$-modules does not agree with the derived functors of τ_* considered in Lemma 5.1. Indeed, $R^i\tau_*M=\mathrm{colim}_L H^i(G_L,M)$ is the colimit of Galois cohomology groups, whereas $R^i\gamma_*M=\mathrm{colim}_m H^i(mG,M)$ is the colimit of cohomology groups of the discrete group $\mathbb{Z}$.

7.1 Homology

We define arithmetic homology with coefficients in the G-module A to be

$$H_i^{\mathrm{ar}}(X,A):=\mathrm{Tor}_i^G(C_*^X(\bar{k}),A).$$

A concrete representative is the double complex

$$C_*^X(\bar{k})\otimes A\xrightarrow{1-\varphi}C_*^X(\bar{k})\otimes A,$$

with the left and right term in homological degrees one and zero, respectively, and with the Frobenius endomorphism φ acting diagonally. We obtain short exact sequences

$$0\to H_i^S(\bar{X},A)_G\to H_i^{\mathrm{ar}}(X,A)\to H_{i-1}^S(\bar{X},A)^G\to 0. \tag{11}$$

Lemma 7.1 *The groups $H_i^{\mathrm{ar}}(X,\mathbb{Z}/m)$ are finite. In particular, $H_i^{\mathrm{ar}}(X,\mathbb{Z})/m$ and ${}_mH_i^{\mathrm{ar}}(X,\mathbb{Z})$ are finite.*

Proof. The first statement follows from the short exact sequence (11). Indeed, if m is prime to the characteristic, then we apply (1) together with finite generation of etale cohomology, and if m is a power of the characteristic, we apply Theorem 3.2 to obtain finiteness of the outer terms of (11). The final statements follows from the long exact sequence

$$\cdots \to H_i^{\mathrm{ar}}(X,\mathbb{Z}) \xrightarrow{\times m} H_i^{\mathrm{ar}}(X,\mathbb{Z}) \to H_i^{\mathrm{ar}}(X,\mathbb{Z}/m) \to \cdots$$

□

If A is the restriction of a $\hat{G}$-module, then (10), applied to the complex of continuous $\hat{G}$-modules $C_*^X(\bar{k}) \otimes A$, gives upon taking Galois cohomology a long exact sequence

$$\cdots \to H_i^{GS}(X,A) \to H_{i+1}^{\mathrm{ar}}(X,A) \to H_{i+1}^{GS}(X,A_{\mathbb{Q}}) \to H_{i-1}^{GS}(X,A) \to \cdots$$

With rational coefficients this sequence breaks up into isomorphisms

$$H_i^{\mathrm{ar}}(X,\mathbb{Q}) \cong H_i^S(X,\mathbb{Q}) \oplus H_{i-1}^S(X,\mathbb{Q}). \tag{12}$$

7.2 Cohomology

In analogy to (8), we define arithmetic cohomology with coefficients in the G-module A to be

$$H^i_{\mathrm{ar}}(X,A) = \mathrm{Ext}^i_G(C_*^X(\bar{k}), A). \tag{13}$$

Note the difference to the definition in [14], which does not give well-behaved (i.e. finitely generated) groups for schemes which are not smooth and proper. A concrete representative is the double complex

$$\mathrm{Hom}(C_*^X(\bar{k}), A) \xrightarrow{1-\varphi} \mathrm{Hom}(C_*^X(\bar{k}), A),$$

where the left and right hand term are in cohomological degrees zero and one, respectively. There are short exact sequences

$$0 \to H_S^{i-1}(\bar{X},A)_G \to H^i_{\mathrm{ar}}(X,A) \to H^i_S(\bar{X},A)^G \to 0. \tag{14}$$

The proof of Lemma 7.1 also shows

Lemma 7.2 *The groups $H^i_{\mathrm{ar}}(X,\mathbb{Z}/m)$ are finite. In particular, ${}_mH^i_{\mathrm{ar}}(X,\mathbb{Z})$ and $H^i_{\mathrm{ar}}(X,\mathbb{Z})/m$ are finite.*

Lemma 7.3 *For every G-module A, we have an isomorphism*

$$H^i_{\mathrm{ar}}(X,A) \cong H^i_{GS}(X, R\gamma_*\gamma^* A).$$

Proof. Since $M^G = (\gamma_* M)^{\hat{G}}$, arithmetic cohomology is the Galois cohomology of the derived functor of $\gamma_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), -)$ on the category of G-modules. By Lemma 5.1, it suffices to show that this derived functor agrees with the derived functor of $\tau_* \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}), \gamma_* -)$ on the category of G-modules. But given a continuous $\hat{G}$-modules M and a G-module N, the inclusion

$$\tau_* \operatorname{Hom}_{\mathrm{Ab}}(M, \gamma_* N) \subseteq \gamma_* \operatorname{Hom}_{\mathrm{Ab}}(\gamma^* M, N)$$

induced by the inclusion $\gamma_* N \subseteq N$ is an isomorphism. Indeed, if $f : M \to N$ is H-invariant and $m \in M$ is fixed by H', then $f(m)$ is fixed by $H \cap H'$, hence f factors through $\gamma_* N$. □

Corollary 7.4 *If A is a continuous $\hat{G}$-module, then there is a long exact sequence*

$$\cdots \to H^i_{GS}(X, A) \to H^i_{\mathrm{ar}}(X, A) \to H^{i-1}_{GS}(X, A_{\mathbb{Q}}) \to H^{i+1}_{GS}(X, A) \to \cdots .$$

Proof. This follows from the Lemma by applying the long exact $\operatorname{Ext}^*_{\hat{G}}(C_*^X(\bar{k}), -)$-sequence to (10). □

7.3 Finite generation and duality

Lemma 7.5 *There are natural pairings*

$$H^i_{\mathrm{ar}}(X, \mathbb{Z})/\mathrm{tor} \times H_i^{\mathrm{ar}}(X, \mathbb{Z})/\mathrm{tor} \to \mathbb{Z}$$

and

$$H^i_{\mathrm{ar}}(X, \mathbb{Z})_{\mathrm{tor}} \times H_{i-1}^{\mathrm{ar}}(X, \mathbb{Z})_{\mathrm{tor}} \to \mathbb{Q}/\mathbb{Z}.$$

Proof. From the adjunction $\operatorname{Hom}_G(M, \mathbb{Z}) \cong \operatorname{Hom}_{\mathrm{Ab}}(M_G, \mathbb{Z})$ and the fact that $L(-)_G = R(-)^G[-1]$, we obtain by deriving a quasi-isomorphism

$$R \operatorname{Hom}_G(C_*^X(\bar{k}), \mathbb{Z}) \cong R \operatorname{Hom}_{\mathrm{Ab}}(C_*^X(\bar{k}) \otimes^L_G \mathbb{Z}, \mathbb{Z}).$$

Now we obtain the pairing as in Lemma 2.1 using the resulting spectral sequence

$$\operatorname{Ext}^s_{\mathrm{Ab}}(H_t^{\mathrm{ar}}(X, \mathbb{Z}), \mathbb{Z}) \Rightarrow H^{s+t}_{\mathrm{ar}}(X, \mathbb{Z}).$$

□

Proposition 7.6 *For a given separated scheme X of finite type over $\mathbb{F}_q$, the following statements are equivalent:*

a) The groups $H_i^{\mathrm{ar}}(X, \mathbb{Z})$ are finitely generated.

b) The groups $H^i_{\mathrm{ar}}(X, \mathbb{Z})$ are finitely generated.

c) The groups $H^i_{\mathrm{ar}}(X,\mathbb{Z})$ are countable.

d) The pairings of Lemma 7.5 are perfect.

Proof. This is proved exactly as Proposition 6.7, with Theorem 6.1 replaced by Lemma 7.1. □

We need a Weil-version of motivic cohomology with compact support. We define $H^i_c(X_W,\mathbb{Z}(n))$ to be the ith cohomology of $R\Gamma(G,R\Gamma_c(\bar{X},\mathbb{Z}(n)))$, where the inner term is a complex defining motivic cohomology with compact support of $\bar{X}$. We use this notation to distinguish it from arithmetic homology with compact support considered in [4], which is the cohomology of $R\Gamma(G,R\Gamma_c(\bar{X}_{\mathrm{et}},\mathbb{Z}(n)))$. However, if $n\geq \dim X$, which is the case of most importance for us, both theories agree.
Similar to (3) we obtain for a closed subscheme Z of a smooth scheme X of pure dimension d with open complement U a long exact sequence

$$\cdots\to H_i^{\mathrm{ar}}(U,\mathbb{Z})\to H_i^{\mathrm{ar}}(X,\mathbb{Z})\to H_c^{2d+1-i}(Z_W,\mathbb{Z}(d))\to\cdots. \tag{15}$$

The shift by 1 in degrees occurs because arithmetic homology is defined using homology of G, whereas cohomology with compact support is defined using cohomology of G.

Proposition 7.7 *The following statements are equivalent:*

a) Conjecture P_0.

b) The groups $H_i^{\mathrm{ar}}(X,\mathbb{Z})$ are finitely generated for all X.

Proof. a) ⇒ b): By induction on the dimension of X and the blow-up square, we can assume that X is smooth of dimension d, where

$$H_i^{\mathrm{ar}}(X,\mathbb{Z})\cong H_c^{2d+1-i}(X_W,\mathbb{Z}(d)).$$

By localization for $H_c^*(X_W,\mathbb{Z}(d))$ and induction on the dimension we can reduce the question to X smooth and projective. In this case $\mathbb{Z}(d)$ has etale hypercohomological descent over an algebraically closed field by [6], hence $H^j_c(X_W,\mathbb{Z}(d))$ agrees with the Weil-etale cohomology $H^j_W(X,\mathbb{Z}(d))$ considered in [3]. These groups are finitely generated for $i>2d$ by [3, Thm.7.3,7.5]. By conjecture P_0, and the isomorphism $H^i_W(X,\mathbb{Z}(d))_{\mathbb{Q}}\cong CH_0(X,2d-i)_{\mathbb{Q}}\oplus CH_0(X,2d-i+1)_{\mathbb{Q}}$ of Thm.7.1c) loc.cit., these groups are torsion for $i<2d$, so that the finite group $H^{i-1}(X_{\mathrm{et}},\mathbb{Q}/\mathbb{Z}(d))$ surjects onto $H^i_W(X,\mathbb{Z}(d))$. Finally, $H^{2d}_W(X,\mathbb{Z}(d))$ is an extension of the finitely generated group $CH_0(\bar{X})^G$ by the finite group $H^{2d-1}(\bar{X}_{\mathrm{et}},\mathbb{Z}(d))_G\cong H^{2d-2}(\bar{X}_{\mathrm{et}},\mathbb{Q}/\mathbb{Z}(d))_G$.
b) ⇒ a) Consider the special case that X is smooth and projective. Then as above, $H_i^{\mathrm{ar}}(X,\mathbb{Z})\cong H_W^{2d+1-i}(X,\mathbb{Z}(d))$. If this group is finitely generated, then we obtain from the coefficient sequence that $H_W^{2d+1-i}(X,\mathbb{Z}(d))\otimes\mathbb{Z}_l\cong\lim H^{2d+1-i}(X_{\mathrm{et}},\mathbb{Z}/l^r(d))$, and the latter group is torsion for $i>1$ by the Weil-conjectures. Now use (12). □

Theorem 7.8 *For connected X, the map $H_0^{\mathrm{ar}}(X,\mathbb{Z}) \to H_0^{\mathrm{ar}}(\mathbb{F}_q,\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism. In particular, we have $H_0^{\mathrm{ar}}(X,\mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$.*

Proof. The proof is similar to the proof of Theorem 6.2. Again we use induction on the dimension and the blow-up sequence to reduce to the situation where X is irreducible and smooth. In this case, we can use (15) and the following Proposition to reduce to the smooth and proper case, where we have $H_0^{\mathrm{ar}}(X,\mathbb{Z}) = CH_0(\bar{X})_G \cong \mathbb{Z}$. □

Proposition 7.9 *If $n > \dim X$, then $H_c^i(X_W,\mathbb{Z}(n)) = 0$ for $i > n + \dim X$.*

Proof. By induction on the dimension and the localization sequence for motivic cohomology with compact support one sees that the statement for X and a dense open subscheme of X are equivalent. Hence we can assume that X is smooth and proper of dimension d. In this case, $H_c^i(X_W,\mathbb{Z}(n))$ is an extension of $H_M^i(\bar{X},\mathbb{Z}(n))^G$ by $H_M^{i-1}(\bar{X},\mathbb{Z}(n))_G$. These groups vanish for $i-1 > d+n$ for dimension (of cycles) reasons. For $i = d+n+1$, we have to show that $H_M^{d+n}(\bar{X},\mathbb{Z}(n))_G$ vanishes. From the niveau spectral sequence for motivic cohomology we obtain a surjection

$$\bigoplus_{\bar{X}_{(0)}} H_M^{n-d}(k(x),\mathbb{Z}(n-d)) \twoheadrightarrow H_M^{d+n}(\bar{X},\mathbb{Z}(n)).$$

The summands are isomorphic to $K_{n-d}^M(\bar{\mathbb{F}}_q)$. If $n > d+1$, then they vanish because higher Milnor K-theory of the algebraical closure of a finite field vanishes. If $n = d+1$, then the summands are isomorphic to $(\bar{\mathbb{F}}_q)^\times$, whose coinvariants vanish. □

8 A Kato type homology

We construct a homology theory measuring the difference between Suslin homology and arithmetic homology. The cohomological theory can be defined analogously. Kato-Suslin-homology $H_i^{KS}(X,A)$ with coefficients in the G-module A is defined as the ith homology of the complex of coinvariants $(C_*^X(\bar{k})\otimes A)_G$. If A is trivial as a G-module, then Lemma 5.2 gives a short exact sequence of double complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*^X(k)\otimes A & \longrightarrow & C_*^X(\bar{k})\otimes A & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow{\scriptstyle 1-\varphi} & & \downarrow & & \\
 & & 0 & \longrightarrow & C_*^X(\bar{k})\otimes A & \longrightarrow & (C_*^X(\bar{k})\otimes A)_G & \longrightarrow & 0
\end{array}$$

and hence a long exact sequence

$$\cdots \to H_i^S(X,A) \to H_{i+1}^{\mathrm{ar}}(X,A) \to H_{i+1}^{KS}(X,A) \to H_{i-1}^S(X,A) \to \cdots.$$

By Theorem 7.8 we have $H_0^{KS}(X,\mathbb{Z}) \cong H_0^{\mathrm{ar}}(X,\mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$. The following is a generalization of the integral version [7] of Kato's conjecture [12].

Conjecture 8.1 *(Generalized integral Kato-conjecture) If X is smooth, then $H_i^{KS}(X,\mathbb{Z}) = 0$ for $i > 0$.*

Equivalently, the canonical map $H_i^S(X,\mathbb{Z}) \cong H_{i+1}^{\mathrm{ar}}(X,\mathbb{Z})$ is an isomorphism for all smooth X and all $i \geq 0$, i.e. there are short exact sequences

$$0 \to H_{i+1}^S(\bar{X},\mathbb{Z})_G \to H_i^S(X,\mathbb{Z}) \to H_i^S(\bar{X},\mathbb{Z})^G \to 0.$$

Theorem 8.2 *Conjecture 8.1 is equivalent to conjecture P_0.*

Proof. If Conjecture 8.1 holds, then

$$H_i^S(X,\mathbb{Q}) \cong H_{i+1}^{\mathrm{ar}}(X,\mathbb{Q}) \cong H_{i+1}^S(X,\mathbb{Q}) \oplus H_i^S(X,\mathbb{Q})$$

implies the vanishing of $H_i^S(X,\mathbb{Q})$ for $i > 0$.
Conversely, we first claim that for smooth and proper Z, the canonical map $H_c^i(Z,\mathbb{Z}(n)) \to H_c^i(Z_W,\mathbb{Z}(n))$ is an isomorphism for all i if $n > \dim Z$, and for $i \leq 2n$ if $n = \dim Z$. Indeed, if $n \geq \dim Z$ then the cohomology of $\mathbb{Z}(n)$ agrees with the etale hypercohomology of $\mathbb{Z}(n)$, see [6], hence satisfies Galois descent. But according to (the proof of) Proposition 6.4b), these groups are torsion groups, so that the derived funtors $R\Gamma G_k$ and $R\Gamma G$ agree.
Using localization for cohomology with compact support and induction on the dimension, we get next that $H_c^i(Z,\mathbb{Z}(n)) \cong H_c^i(Z_W,\mathbb{Z}(n))$ for all i and all Z with $n > \dim Z$. Now choose a smooth and proper compactification C of X. Comparing the exact sequences (3) and (15), we see with the 5-Lemma that the isomorphism $H_i^S(C,\mathbb{Z}) \cong H_c^{2d-i}(C,\mathbb{Z}(d)) \to H_{i+1}^{\mathrm{ar}}(C,\mathbb{Z}) \cong H_c^{2d-i}(C_W,\mathbb{Z}(d))$ for C implies the same isomorphism for X and $i \geq 0$. □

9 Tamely ramified class field theory

We propose the following conjecture relating Weil-Suslin homology to class field theory:

Conjecture 9.1 *(Tame reciprocity) For any X separated and of finite type over a finite field, there is a canonical injection to the tame abelianized fundamental group with dense image*

$$H_1^{\mathrm{ar}}(X,\mathbb{Z}) \to \pi_1^t(X)^{ab}.$$

Note that the group $H_1^{\mathrm{ar}}(X,\mathbb{Z})$ is conjecturally finitely generated. At this point, we do not have an explicit construction (associating elements in the Galois groups to algebraic cycles) of the map. One might even hope that $H_1^{\mathrm{ar}}(X,\mathbb{Z})^\circ := \ker(H_1^{\mathrm{ar}}(X,\mathbb{Z}) \to \mathbb{Z}^{\pi_0(X)}$ is finitely generated and isomorphic to

the abelianized geometric part of the tame fundamental group defined in SGA 3X§6.
Under Conjecture 8.1, $H_0^S(X,\mathbb{Z}) \cong H_1^{\mathrm{ar}}(X,\mathbb{Z})$ for smooth X, and Conjecture 9.1 is a theorem of Schmidt-Spiess [19].

Proposition 9.2 *a) We have $H_1^{\mathrm{ar}}(X,\mathbb{Z})^{\wedge l} \cong \pi_1^t(X)^{ab}(l)$. In particular, the prime to p-part of Conjecture 9.1 holds if $H_1^{\mathrm{ar}}(X,\mathbb{Z})$ is finitely generated.*
b) The analog statement holds for the p-part if X has a compactification T which has a desingularization which is an isomorphism outside of X.

Proof. a) By Theorem 7.8, $H_0^{\mathrm{ar}}(X,\mathbb{Z})$ contains no divisible subgroup. Hence if $l \neq p$, we have by Theorems 5.4 and 5.5

$$\begin{aligned} H_1^{\mathrm{ar}}(X,\mathbb{Z})^{\wedge l} &\cong \lim H_1^{\mathrm{ar}}(X,\mathbb{Z}/l^r) \cong \lim H_0^{GS}(X,\mathbb{Z}/l^r) \\ &\cong \lim H^1_{\mathrm{et}}(X,\mathbb{Z}/l^r)^* \cong \pi_1^t(X)^{ab}(l). \end{aligned}$$

b) Under the above hypothesis, we can use the duality result of [6] for the proper scheme T to get with Proposition 3.3

$$\begin{aligned} H_1^{\mathrm{ar}}(X,\mathbb{Z}) \otimes \mathbb{Z}_p &\cong \lim H_0^{GS}(X,\mathbb{Z}/p^r) \cong \lim H_0^{GS}(T,\mathbb{Z}/p^r) \\ &\cong \lim H^1_{\mathrm{et}}(T,\mathbb{Z}/p^r)^* \cong \pi_1(T)^{ab}(p) \cong \pi_1^t(X)^{ab}(p). \end{aligned}$$

□

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