Infinitesimal Deformations and the ℓ-Invariant

To Andrei Alexandrovich Suslin, for his 60th birthday

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Abstract. We give a formula for the generalized Greenberg's ℓ-invariant which was constructed in [Ben2] in terms of derivatives of eigenvalues of Frobenius.

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Introduction

0.1. Let M be a pure motive over $\mathbb{Q}$ with coefficients in a number field E. Assume that the L-function $L(M,s)$ is well defined. Fixinig an embedding $\iota\,:\,E\hookrightarrow \mathbb{C}$ we can consider it as a complex-valued Dirichlet series $L(M,s)=\sum\limits_{n=0}^{\infty} a_n n^{-s}$ which converges for $s\gg 0$ and is expected to admit a meromorphic continuation to $\mathbb{C}$ with a functional equation of the form

$$\Gamma(M,s)\,L(M,s)\,=\,\varepsilon(M,s)\,\Gamma(M^*(1),-s)\,L(M^*(1),-s)$$

where $\Gamma(M,s)$ is the product of some Γ-factors and the ε-factor has the form $\varepsilon(M,s)=ab^s$.

Assume that M is critical and that $L(M,0)\neq 0$. Fix a finite place $\lambda|p$ of E and assume that the λ-adic realization M_λ of M is semistable in the sense of Fontaine [Fo3]. The (φ,N)-module $\mathbf{D}_{\mathrm{st}}(M_\lambda)$ associated to M_λ is a finite dimensional E_λ-vector space equipped with an exhaustive decreasing filtration $\mathrm{Fil}^i\mathbf{D}_{\mathrm{st}}(M_\lambda)$, a E_λ-linear bijective frobenius $\varphi\,:\,\mathbf{D}_{\mathrm{st}}(M_\lambda)\,\rightarrow\,\mathbf{D}_{\mathrm{st}}(M_\lambda)$ and a nilpotent monodromy operator N such that $N\,\varphi\,=\,p\,\varphi\,N$. We say that a (φ,N)-submodule D of $\mathbf{D}_{\mathrm{st}}(M_\lambda)$ is regular if

$$\mathbf{D}_{\mathrm{st}}(M_\lambda)\,=\,D\oplus \mathrm{Fil}^0\mathbf{D}_{\mathrm{st}}(M_\lambda)$$

as E_λ-vector spaces. The theory of Perrin-Riou [PR] suggests that to any regular D one can associate a p-adic L-function $L_p(M,D,s)$ interpolating rational parts of special values of $L(M,s)$. In particular, the interpolation formula at $s=0$ should have the form

$$L_p(M,D,0) \,=\, \mathcal{E}(M,D)\,\frac{L(M,0)}{\Omega_\infty(M)}$$

where $\Omega_\infty(M)$ is the Deligne period of M and $\mathcal{E}(M,D)$ is a certain product of Euler-like factors. Therefore one can expect that $L_p(M,D,0)=0$ if and only if $\mathcal{E}(M,D)=0$ and in this case one says that $L_p(M,D,s)$ has a trivial zero at $s=0$.

0.2. According to the conjectures of Bloch and Kato [BK], the E_λ-adic representation M_λ should have the following properties:

C1) The Selmer groups $H^1_f(M_\lambda)$ and $H^1_f(M^*_\lambda(1))$ are zero.
C2) $H^0(M_\lambda) = H^0(M^*_\lambda(1)) = 0$ where we write H^* for the global Galois cohomology.
Moreover one expects that
C3) $\varphi : \mathbf{D}_{\mathrm{st}}(M_\lambda) \to \mathbf{D}_{\mathrm{st}}(M_\lambda)$ is semisimple (semisimplicity conjecture).
We also make the following assumption which is a direct generalization of the hypothesis **U)** from [G].
C4) The (φ,Γ)-module $\mathbf{D}^\dagger_{\mathrm{rig}}(M_\lambda)$ has no saturated subquotients of the form $U_{m,n}$ where $U_{m,n}$ is the unique crystalline (φ,Γ)-module sitting in a non split exact sequence

$$0\to \mathcal{R}_L(|x|x^m)\to U_{m,n}\to \mathcal{R}_L(x^{-n})\to 0, \quad L=E_\lambda$$

(see §1 for unexplained notations).

In [Ben2], we extended the theory of Greenberg [G] to L-adic pseudo geometric representations which are semistable at p and satisfy **C1-4)**. Namely to any regular $D\subset \mathbf{D}_{\mathrm{st}}(V)$ of a reasonably behaved representation V we associated an integer $e\geqslant 0$ and an element $\mathcal{L}(V,D)\in L$ which can be seen as a vast generalization of the $\mathcal{L}$-invariants constructed in [Mr] and [G]. If $V=M_\lambda$ we set $\mathcal{L}(M,D)=\mathcal{L}(M_\lambda,D)$. A natural formulation of the trivial zero conjecture states as follows:

Conjecture. $L_p(M,D,s)$ has a zero of order e at $s=0$ and

$$\lim_{s\to 0}\frac{L_p(M,D,s)}{s^e} \,=\, \mathcal{E}^+(M,D)\,\mathcal{L}(M^*(1),D^*)\,\frac{L(M,0)}{\Omega_\infty(M)}, \tag{0.1}$$

where $\mathcal{E}^+(M,D)$ is the subproduct of $\mathcal{E}(M,D)$ obtained by "excluding zero factors" and $D^* = \mathrm{Hom}(\mathbf{D}_{\mathrm{st}}(V)/D, \mathbf{D}_{\mathrm{st}}(L(1)))$ is the dual regular module (see [Ben2] for more details). We refer to this statement as Greenberg's conjecture because if M_λ is ordinary at p it coincides with the conjecture formulated in [G], p.166. Remark that if M_λ is crystalline at p, Greenberg's conjecture is compatible with Perrin-Riou's theory of p-adic L-functions [Ben3].

0.3. Consider the motive M_f attached to a normalized newform $f = \sum\limits_{n=1}^{\infty} a_n q^n$ of weight $2k$ on $\Gamma_0(Np)$ with $(N,p)=1$. The complex L-function of M_f is $L(f,s) = \sum\limits_{n=1}^{\infty} a_n n^{-s}$. The twisted motive $M_f(k)$ is critical. The eigenvalues of φ acting on $\mathbf{D}_{\mathrm{st}}(M_{f,\lambda}(k))$ are $\alpha = p^{-k}a_p$ and $\beta = p^{1-k}a_p$ with $v_p(a_p) = k-1$. The unique regular submodule of $\mathbf{D}_{\mathrm{st}}(M_{f,k}(k))$ is $D = E_\lambda d$ where $\varphi(d) = \alpha\, d$ and $L_p(M_f(k), D, s) = L_p(f, s+k)$ where $L_p(f,s)$ is the classical p-adic L-function associated to a_p via the theory of modular symbols [Mn], [AV]. If $a_p = p^{k-1}$, the function $L_p(f,s)$ vanishes at $s=k$. In this case several constructions of the $\mathcal{L}$-invariant based on different ideas were proposed (see [Co1], [Tm], [Mr], [O], [Br]). Thanks to the work of many people it is known that they are all equal and we refer to [Cz3] and [BDI] for further information. As $M_f(k)$ is self-dual (i.e. $M_f(k) \simeq M_f^*(1-k)$) one has $\mathcal{L}(M_f^*(1-k), D^*) = \mathcal{L}(M_f(k), D)$ (see also section 0.4 below). Moreover it is not difficult to prove that $\mathcal{L}(M_f(k), D)$ coincides with the $\mathcal{L}$-invariant of Fontaine-Mazur $\mathcal{L}_{\mathrm{FM}}(f)$ [Mr] ([Ben2], Proposition 2.3.7) and (0.1) takes the form of the Mazur-Tate-Teitelbaum conjecture

$$L_p'(f,k) = \mathcal{L}(f)\, \frac{L(f,k)}{\Omega_\infty(f)}$$

where we write $\mathcal{L}(f)$ for an unspecified $\mathcal{L}$-invariant and $\Omega_\infty(f)$ for the Shimura period of f [MTT]. This conjecture was first proved by Greenberg and Stevens in the weight two case [GS1] [GS2]. In the unpublished note [St], Stevens generalized this approach to the higher weights. Other proofs were found by Kato, Kurihara and Tsuji (unpublished but see [Cz2]), Orton [O], Emerton [E] and by Bertolini, Darmon and Iovita [BDI]. The approach of Greenberg and Stevens is based on the study of families of modular forms and their p-adic L-functions. Namely, Hida (in the ordinary case) and Coleman [Co1] (in general) constructed an analytic family $f_x = \sum\limits_{n=1}^{\infty} a_n(x) q^n$ of p-adic modular forms for $x \in \mathbb{C}_p$ passing through f with $f = f_{2k}$. Next, Panchishkin [Pa] and independently Stevens (unpublished) constructed a two-variable p-adic L-function L-function $L_p(x,s)$ satisfying the following properties:

- $L_p(2k,s) = L_p(f,s)$.
- $L_p(x, x-s) = -\langle N \rangle^{s-x} L_p(x,s)$.
- $L_p(x,k) = (1 - p^{k-1}a_p(x)^{-1})\, L^*(x)$ where $L_p^*(x)$ is a p-adic analytic function such that $L_p^*(2k) = L(f,k)/\Omega_\infty(f)$.

From these properties it follows easily that

$$L_p'(f,k) = -2\, d\log a_p(2k)\, \frac{L(f,k)}{\Omega_\infty(f)},$$

where $d\log a_p(x) = a_p(x)^{-1}\dfrac{da_p(x)}{dx}$. Thus the Mazur-Tate-Teitelbaum conjecture is equivalent to the assertion that

$$\mathcal{L}(f) = -2\,d\log a_p(2k). \tag{0.2}$$

This formula was first proved for weight two by Greenberg and Stevens. In the higher weight case several proofs of (0.2) have been proposed:
1. By Stevens [St], working with Coleman's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{C}}(f)$ defined in [Co1].
2. By Colmez [Cz5], working with the Fontaine-Mazur's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{FM}}(f)$ defined in [Mr].
3. By Colmez [Cz6], working with Breuil's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{Br}}(f)$ defined in [Br].
4. By Bertolini, Darmon and Iovita [BDI], working with Teitelbaum's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{T}}(f)$ [Tm] and Orton's $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{O}}(f)$ [O].

0.4. In this paper, working with the $\mathcal{L}$-invariant defined in [Ben2] we generalize (0.2) to some infinitesimal deformations of pseudo geometric representations. Our result is purely algebraic and is a direct generalization of Theorem 2.3.4 of [GS2] using the cohomology of (φ,Γ)-modules instead Galois cohomology. Let V be a pseudo-geometric representation with coefficients in $L/\mathbb{Q}_p$ which satisfies **C1-4)**. Fix a regular submodule D. In view of (0.1) it is convenient to set

$$\ell(V,D) = \mathcal{L}(V^*(1),D^*).$$

Suppose that $e=1$. Conjecturally this means that the p-adic L-function has a simple trivial zero. Then either $D^{\varphi=p^{-1}}$ or $(D^*)^{\varphi=p^{-1}}$ has dimension 1 over L. To fix ideas, assume that $\dim_L D^{\varphi=p^{-1}}=1$. Otherwise, as one expects a functional equation relating $L_p(M,D,s)$ and $L_p(M^*(1),D^*,-s)$ one can consider $V^*(1)$ and D^* instead V and D. We distinguish two cases. In each case one can express $\ell(V,D)$ directly in terms of V and D.

• *The crystalline case:* $D^{\varphi=p^{-1}}\cap N\left(\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\right)=\{0\}$. Let $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ be the (φ,Γ)-module over the Robba ring $\mathcal{R}_L$ associated to V [Ber1], [Cz1]. Set $D_{-1}=(1-p^{-1}\varphi^{-1})D$ and $D_0=D$. The two step filtration $D_{-1}\subset D_0\subset \mathbf{D}_{\mathrm{st}}(V)$ induces a filtration

$$F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\subset F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$$

such that $\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\simeq \mathcal{R}_L(\delta)$ is the (φ,Γ)-module of rank 1 associated to a character $\delta:\mathbb{Q}_p^*\to L^*$ of the form $\delta(x)=|x|x^m$ with $m\geqslant 1$. The cohomology of (φ,Γ)-modules of rank 1 is studied in details in [Cz4]. Let $\eta:\mathbb{Q}_p^*\to L^*$ be a continuous character. Colmez proved that $H^1(\mathcal{R}_L(\eta))$ is a one dimensional L-vector space except for $\eta(x)=|x|x^m$ with $m\geqslant 1$ and $\eta(x)=x^{-n}$ with $n\leqslant 0$. In the exceptional cases $H^1(\mathcal{R}_L(\eta))$ has dimension 2 and can be canonically decomposed into direct sum of one dimensional subspaces

$$H^1(\mathcal{R}_L(\eta))\simeq H^1_f(\mathcal{R}_L(\eta))\oplus H^1_c(\mathcal{R}_L(\eta)),\quad \eta(x)=|x|x^m \text{ or } \eta(x)=x^{-n} \tag{0.3}$$

([Ben2], Theorem 1.5.7). The condition **C1)** implies that

$$H^1(V) \simeq \bigoplus_{l\in S} \frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)} \tag{0.4}$$

for a finite set of primes S. This isomorphism defines a one dimensional subspace $H^1(D,V)$ of $H^1(V)$ together with an injective localisation map $\kappa_D\,:\,H^1(D,V) \to H^1(\mathcal{R}_L(\delta))$. Then $\ell(V,D)$ is the slope of $\mathrm{Im}(\kappa_D)$ with respect to the decomposition of $H^1(\mathcal{R}_L(\delta))$ into direct sum (0.3). Let

$$0 \to V \to V_x \to L \to 0$$

be an extension in the category of global Galois representations such that $\mathrm{cl}(x) \in H^1(D,V)$ is non zero. We equip $\mathbf{D}^\dagger_{\mathrm{rig}}(V_x)$ with a canonical filtration

$$\{0\} \subset F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V_x)$$

such that $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) = F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ for $i=-1,0$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_x) \simeq \mathcal{R}_L$. Let $V_{A,x}$ be an infinitesimal deformation of V_x over $A=L[T]/(T^2)$ endowed with a filtration $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_{A,x})$ such that $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V) = F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_{A,x}) \otimes_A L$. Write

$$\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_{A,x}) \simeq \mathcal{R}_A(\delta_{A,x}), \qquad \mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_{A,x}) \simeq \mathcal{R}_A(\psi_{A,x})$$

with $\delta_{A,x},\,\psi_{A,x}\,:\,\mathbb{Q}_p^* \to A^*$.

Theorem 1. *Assume that* $\left.\dfrac{d(\delta_{A,x}\psi_{A,x}^{-1})(u)}{dT}\right|_{T=0} \neq 0$ *for* $u \equiv 1 \pmod{p^2}$. *Then*

$$\ell(V,D) = -\log(u)\left.\frac{d\log(\delta_{A,x}\psi^{-1}_{A,x})(p)}{d\log(\delta_{A,x}\psi^{-1}_{A,x})(u)}\right|_{T=0}$$

(note that the right hand side does not depend on the choice of u*).*

• *The semistable case:* $D^{\varphi=p^{-1}} \subset N\left(\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\right)$. Set $D_{-1}=(1-p^{-1}\varphi^{-1})D$, $D_0=D$ and $D_1=N^{-1}(D^{\varphi=p^{-1}}) \cap \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$. The filtration

$$D_{-1} \subset D_0 \subset D_1 \subset \mathbf{D}_{\mathrm{st}}(V)$$

induces a filtration

$$F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V) \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)$$

Then $\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V) \simeq \mathcal{R}_L(\delta)$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V) \simeq \mathcal{R}_L(\psi)$ where the characters δ and ψ are such that $\delta(x)=|x|x^m$ and $\psi(x)=x^{-n}$ for some $m \geqslant 1$ and $n \geqslant 0$. Set $M=F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ and consider the map $\kappa_D\,:\,H^1(M) \to H^1(\mathcal{R}_L(\psi))$ induced by the projection $M \to \mathcal{R}_L(\psi)$. The image of κ_D is a one dimensional L-subspace of $H^1(\mathcal{R}_L(\psi))$ and $\ell(V,D)$ is the slope of $\mathrm{Im}(\kappa_D)$ with respect to (0.3).

Assume that V_A is an infinitesimal deformation of V equipped with a filtration $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)$ such that $F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V) = F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \otimes_A L$. Write $\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \simeq \mathcal{R}_A(\delta_A)$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \simeq \mathcal{R}_A(\psi_A)$.

THEOREM 2. *Assume that*

$$\left.\frac{d(\delta_A\psi_A^{-1})(u)}{dT}\right|_{T=0} \neq 0 \ \textit{for } u \equiv 1 \pmod{p^2}. \tag{0.5}$$

Then

$$\ell(V,D) = -\log(u)\left.\frac{d\log(\delta_A\psi_A^{-1})\,(p)}{d\log(\delta_A\psi_A^{-1})(u)}\right|_{T=0}.$$

Remark that in the semistable case $\ell(V,D) = \mathcal{L}(V,D)$.

For classical modular forms the existence of deformations having the above properties follows from the theory of Coleman-Mazur [CM] together with deep results of Saito and Kisin [Sa], [Ki]. Applying Theorem 2 to the representation $M_{f,\lambda}(k)$ we obtain a new proof of (0.2) with the Fontaine-Mazur $\mathcal{L}$-invariant. Remark that the local parameter T corresponds to the weight of a p-adic modular form and (0.5) holds automatically. In the general case the existence of deformations satisfying the above conditions should follow from properties of eigenvarieties of reductive groups [BC].

The formulations of Theorems 1 and 2 look very similar and the proof is essentially the same in the both cases. The main difference is that in the crystalline case the ℓ-invariant is global and contains information about the localisation map $H^1(V) \to H^1(\mathbb{Q}_p, V)$. In the proof of Theorem 1 we consider V_x as a representation of the local Galois group but the construction of V_x depends on the isomorphism (0.4). In the semistable case the definition of $\ell(V,D)$ is purely local and the hypothesis **C1-2)** can be omitted. However **C1-2)** are essential for the formulation of Greenberg conjecture because (0.1) is meaningless if $L(M,0) = 0$. One can compare our results with Hida's paper [Hi] where the case of ordinary representations over totally real ground field is studued.

Here goes the organization of this paper. The §1 contains some background material. In section 1.1 we review the theory of (φ,Γ)-modules and in section 1.2 recall the definition of the ℓ-invariant following [Ben2]. The crystalline and semistable cases of trivial zeros are treated in §2 and §3 respectively. I would like to thank Pierre Parent for several very valuable discussions which helped me with the formulation of Theorem 1 and the referee for pointing out several inaccuracies in the first version of this paper.

It is a great pleasure to dedicate this paper to Andrei Alexandrovich Suslin on the occasion of his 60th birthday.

§1. The ℓ-invariant

1.1. (φ,Γ)-modules. ([Fo1], [Ber1], [Cz1])
1.1.1. Let p be a prime number. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and set $G_{\mathbb{Q}_p}=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We denote by $\mathbb{C}_p$ the p-adic completion of $\overline{\mathbb{Q}}_p$ and write $|\cdot|$ for the absolute value on $\mathbb{C}_p$ normalized by $|p|=1/p$. For any $0\leqslant r<1$ set

$$B(r,1)=\{z\in\mathbb{C}_p\mid p^{-1/r}\leqslant |z|<1\}.$$

Let $\chi\,:\,G_{\mathbb{Q}_p}\to\mathbb{Z}_p^*$ denote the cyclotomic character. Set $H_{\mathbb{Q}_p}=\ker(\chi)$ and $\Gamma=G_{\mathbb{Q}_p}/H_{\mathbb{Q}_p}$. The character χ will be often considered as an isomorphism $\chi\,:\,\Gamma\xrightarrow{\sim}\mathbb{Z}_p^*$. Let L be a finite extension of $\mathbb{Q}_p$. For any $0\leqslant r<1$ we denote by $\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$ the ring of p-adic functions $f(\pi)=\sum\limits_{k\in\mathbb{Z}}a_k\pi^k$ $(a_k\in L)$ which are holomorphic on the annulus $B(r,1)$. The Robba ring over L is defined as $\mathcal{R}_L=\bigcup\limits_r\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$. Recall that $\mathcal{R}_L$ is equipped with commuting, L-linear, continuous actions of Γ and a frobenius φ which are defined by

$$\begin{aligned}&\gamma(f(\pi))=f((1+\pi)^{\chi(\gamma)}-1), &&\gamma\in\Gamma,\\ &\varphi(f(\pi))=f((1+\pi)^p-1).\end{aligned}$$

Set $t=\log(1+\pi)=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{\pi^n}{n}$. Remark that $\gamma(t)=\chi(\gamma)\,t$ and $\varphi(t)=p\,t$. A finitely generated free $\mathcal{R}_L$-module $\mathbf{D}$ is said to be a (φ,Γ)-module if it is equipped with commuting semilinear actions of Γ and φ and such that $\mathcal{R}_L\varphi(\mathbf{D})=\mathbf{D}$. The last condition means simply that $\varphi(e_1),\ldots,\varphi(e_d)$ is a basis of $\mathbf{D}$ if $e_1,\ldots,e_d$ is.
Let $\delta\,:\,\mathbb{Q}_p^*\to L^*$ be a continuous character. We will write $\mathcal{R}_L(\delta)$ for the (φ,Γ)-module $\mathcal{R}_Le_\delta$ of rank 1 defined by

$$\varphi(e_\delta)=\delta(p)\,e_\delta,\qquad \gamma(e_\delta)=\delta(\chi(\gamma))\,e_\delta,\quad \gamma\in\Gamma.$$

For any $\mathbf{D}$ we let $\mathbf{D}(\chi)$ denote the φ-module $\mathbf{D}$ endowed with the action of Γ twisted by the cyclotomic character χ.
Fix a topological generator $\gamma\in\Gamma$. For any (φ,Γ)-module $\mathbf{D}$ we denote by $C_{\varphi,\gamma}(\mathbf{D})$ the complex

$$0\to\mathbf{D}\xrightarrow{f}\mathbf{D}\oplus\mathbf{D}\xrightarrow{g}\mathbf{D}\to 0$$

with $f(x)=((\varphi-1)x,(\gamma-1)x)$ and $g(y,z)=(\gamma-1)y-(\varphi-1)z$ ([H1], [Cz4]). We shall write $H^*(\mathbf{D})$ for the cohomology of $C_{\varphi,\gamma}(\mathbf{D})$. The main properties of these groups are the following

1) Long cohomology sequence. A short exact sequence of (φ,Γ)-modules

$$0 \to \mathbf{D}' \to \mathbf{D} \to \mathbf{D}'' \to 0$$

gives rise to an exact sequence

$$0 \to H^0(\mathbf{D}') \to H^0(\mathbf{D}) \to H^0(\mathbf{D}) \xrightarrow{\Delta^0} H^1(\mathbf{D}') \to \cdots \to H^2(\mathbf{D}'') \to 0.$$

2) Euler-Poincaré characteristic. $H^i(\mathbf{D})$ are finite dimensional L-vector spaces and

$$\chi(\mathbf{D}) = \sum_{i=0}^{2} (-1)^i \dim_L H^i(\mathbf{D}) \;=\; -\mathrm{rg}(\mathbf{D}).$$

(see [H1] and [Li]).

3) Computation of the Brauer group. The map

$$\mathrm{cl}(x) \mapsto -\left(1-\frac{1}{p}\right)^{-1} (\log \chi(\gamma))^{-1} \mathrm{res}(xdt)$$

is well defined and induces an isomorphism $\mathrm{inv}\,:\, H^2(\mathcal{R}_L(\chi)) \xrightarrow{\sim} L$ (see [H2] [Ben1] and [Li]).

4) The cup-products. Let $\mathbf{D}$ and $\mathbf{M}$ be two (φ,Γ)-modules. For all i and j such that $i+j \leqslant 2$ define a bilinear map

$$\cup\,:\, H^i(\mathbf{D}) \times H^j(\mathbf{M}) \to H^{i+j}(\mathbf{D} \otimes \mathbf{M})$$

by

$$\begin{aligned}
&\mathrm{cl}(x) \cup \mathrm{cl}(y) = \mathrm{cl}(x \otimes y) \qquad \text{if } i=j=0,\\
&\mathrm{cl}(x) \cup \mathrm{cl}(y_1,y_2) = \mathrm{cl}(x \otimes y_1, x \otimes y_2) \qquad \text{if } i=0,\ j=1,\\
&\mathrm{cl}(x_1,x_2) \cup \mathrm{cl}(y_1,y_2) = \mathrm{cl}(x_2 \otimes \gamma(y_1) - x_1 \otimes \varphi(y_2)) \qquad \text{if } i=1,\ j=1,\\
&\mathrm{cl}(x) \cup \mathrm{cl}(y) = \mathrm{cl}(x \otimes y) \qquad \text{if } i=0,\ j=2.
\end{aligned}$$

These maps commute with connecting homomorphisms in the usual sense.

5) Duality. Let $\mathbf{D}^* = \mathrm{Hom}_{\mathcal{R}_L}(\mathbf{D}, \mathcal{R}_L)$. For $i=0,1,2$ the cup product

$$H^i(\mathbf{D}) \times H^{2-i}(\mathbf{D}^*(\chi)) \xrightarrow{\cup} H^2(\mathcal{R}_L(\chi)) \simeq L \tag{1.1}$$

is a perfect pairing ([H2], [Li]).

1.1.2. Recall that a filtered (φ,N)-module with coefficients in L is a finite dimensional L-vector space M equipped with an exhaustive decreasing filtration $\mathrm{Fil}^i M$, a linear bijective map $\varphi\,:\, M \to M$ and a nilpotent operator $N\,:\, M \to M$ such that $\varphi N = p\,\varphi N$. Filtered (φ,N)-modules form a $\otimes$-category which we denote by $\mathbf{MF}^{\varphi,N}$. A filtered (φ,N)-module M is said to be a Dieudonné module if $N = 0$ on M. Filtered Dieudonné modules form a full subcategory $\mathbf{MF}^{\varphi}$ of $\mathbf{MF}^{\varphi,N}$. It is not difficult to see that the series $\log(\varphi(\pi)/\pi^p)$ and $\log(\gamma(\pi)/\pi)$ $(\gamma\in\Gamma)$ converge in $\mathcal{R}_L$. Let $\log\pi$ be a transcendental element over the field of fractions of $\mathcal{R}_L$ equipped with actions of φ and Γ given by

$$\varphi(\log\pi) = p\log\pi + \log\left(\frac{\varphi(\pi)}{\pi^p}\right), \qquad \gamma(\log\pi) = \log\pi + \log\left(\frac{\gamma(\pi)}{\pi}\right).$$

Thus the ring $\mathcal{R}_{L,\log} = \mathcal{R}_L[\log\pi]$ is equipped with natural actions of φ and Γ and the monodromy operator $N = -\left(1-\frac{1}{p}\right)^{-1}\frac{d}{d\log\pi}$. For any (φ,Γ)-module $\mathbf{D}$ set

$$\mathcal{D}_{\mathrm{st}}(\mathbf{D}) = (\mathbf{D}\otimes_{\mathcal{R}_L}\mathcal{R}_{L,\log}[1/t])^{\Gamma}$$

with $t=\log(1+\pi)$. Then $\mathcal{D}_{\mathrm{st}}(\mathbf{D})$ is a finite dimensional L-vector space equipped with natural actions of φ and N such that $N\varphi = p\,\varphi N$. Moreover, it is equipped with a canonical exhaustive decreasing filtration $\mathrm{Fil}^i\mathcal{D}_{\mathrm{st}}(\mathbf{D})$ which is induced by the embeddings $\iota_n\,:\,\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}\hookrightarrow L_\infty[[t]]$, $n\gg 0$ constructed in [Ber1] (see [Ber2] for more details). Set

$$\mathcal{D}_{\mathrm{cris}}(\mathbf{D}) = \mathcal{D}_{\mathrm{st}}(\mathbf{D})^{N=0} = (\mathbf{D}[1/t])^{\Gamma}.$$

Then

$$\dim_L\mathcal{D}_{\mathrm{cris}}(\mathbf{D})\leqslant\dim_L\mathcal{D}_{\mathrm{st}}(\mathbf{D})\leqslant\mathrm{rg}(\mathbf{D})$$

and one says that $\mathbf{D}$ is semistable (resp. crystalline) if $\dim_L\mathcal{D}_{\mathrm{cris}}(\mathbf{D})=\mathrm{rg}(\mathbf{D})$ (resp. if $\dim_L\mathcal{D}_{\mathrm{st}}(\mathbf{D})=\mathrm{rg}(\mathbf{D})$). If $\mathbf{D}$ is semistable, the jumps of the filtration $\mathrm{Fil}^i\mathbf{D}_{\mathrm{st}}(\mathbf{D})$ are called the Hodge-Tate weights of $\mathbf{D}$ and the tangent space of $\mathbf{D}$ is defined as $t_{\mathbf{D}}(L)=\mathcal{D}_{\mathrm{st}}(\mathbf{D})/\mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(\mathbf{D})$.
We let denote by $\mathbf{M}^{\varphi,\Gamma}_{\mathrm{pst}}$ and $\mathbf{M}^{\varphi,\Gamma}_{\mathrm{cris}}$ the categories of semistable and crystalline representations respectively. In [Ber2] Berger proved that the functors

$$\mathcal{D}_{\mathrm{st}}\,:\,\mathbf{M}^{\varphi,\Gamma}_{\mathrm{pst}}\to\mathbf{MF}^{\varphi,N}, \qquad \mathcal{D}_{\mathrm{cris}}\,:\,\mathbf{M}^{\varphi,\Gamma}_{\mathrm{cris}}\to\mathbf{MF}^{\varphi} \tag{1.2}$$

are equivalences of $\otimes$-categories.

1.1.3. As usually, $H^1(\mathbf{D})$ can be interpreted in terms of extensions. Namely, to any cocycle $\alpha=(a,b)\in Z^1(C_{\varphi,\gamma}(\mathbf{D}))$ one associates the extension

$$0\to\mathbf{D}\to\mathbf{D}_\alpha\to\mathcal{R}_L\to 0$$

such that $\mathbf{D}_\alpha=\mathbf{D}\oplus\mathcal{R}_Le$ with $\varphi(e)=e+a$ and $\gamma(e)=e+b$. This defines a canonical isomorphism

$$H^1(\mathbf{D})\simeq\mathrm{Ext}^1(\mathcal{R}_L,\mathbf{D}).$$

We say that $\mathrm{cl}(\alpha) \in H^1(\mathbf{D})$ is crystalline if $\dim_L \mathcal{D}_{\mathrm{cris}}(\mathbf{D}_\alpha) = \dim_L \mathcal{D}_{\mathrm{cris}}(\mathbf{D})+1$ and define

$$H^1_f(\mathbf{D}) = \{\mathrm{cl}(\alpha) \in H^1(\mathbf{D}) \mid \mathrm{cl}(\alpha) \text{ is crystalline}\}.$$

It is easy to see that $H^1_f(\mathbf{D})$ is a subspace of $H^1(\mathbf{D})$. If $\mathbf{D}$ is semistable (even potentially semistable), one has

$$H^0(\mathbf{D}) = \mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(\mathbf{D})^{\varphi=1,N=0},$$

$$\dim_L H^1_f(\mathbf{D}) = \dim_L t_{\mathbf{D}}(L) + \dim_L H^0(\mathbf{D}) \tag{1.3}$$

(see [Ben2], Proposition 1.4.4 and Corollary 1.4.5). Moreover, $H^1_f(\mathbf{D})$ and $H^1_f(\mathbf{D}^*(\chi))$ are orthogonal complements to each other under duality (1.1) ([Ben2], Corollary 1.4.10).

1.1.4. Let $\mathbf{D}$ be semistable (φ,Γ)-module of rank d. Assume that $\mathcal{D}_{\mathrm{st}}(\mathbf{D})^{\varphi=1} = \mathcal{D}_{\mathrm{st}}(\mathbf{D})$ and that the all Hodge-Tate weights of $\mathbf{D}$ are $\geqslant 0$. Since $N\varphi = p\varphi N$ this implies that $N=0$ on $\mathcal{D}_{\mathrm{st}}(\mathbf{D})$ and $\mathbf{D}$ is crystalline. The results of this section are proved in [Ben2] (see Proposition 1.5.9 and section 1.5.10). The canonical map $\mathbf{D}^\Gamma \to \mathcal{D}_{\mathrm{cris}}(\mathbf{D})$ is an isomorphism and therefore $H^0(\mathbf{D}) \simeq \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) = \mathbf{D}^\Gamma$ has dimension d over L. The Euler-Poincaré characteristic formula gives

$$\dim_L H^1(\mathbf{D}) = d + \dim_L H^0(\mathbf{D}) + \dim_L H^0(\mathbf{D}^*(\chi)) = 2d.$$

On the other hand $\dim_L H^1_f(\mathbf{D}) = d$ by (1.3). The group $H^1(\mathbf{D})$ has the following explicit description. The map

$$i_{\mathbf{D}} \,:\, \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) \oplus \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) \to H^1(\mathbf{D}),$$
$$i_{\mathbf{D}}(x,y) = \mathrm{cl}(-x,\, \log\chi(\gamma)\, y)$$

is an isomorphism. (Remark that the sign -1 and $\log\chi(\gamma)$ are normalizing factors.) We let denote $i_{\mathbf{D},f}$ and $i_{\mathbf{D},c}$ the restrictions of $i_{\mathbf{D}}$ on the first and second summand respectively. Then $\mathrm{Im}(i_{\mathbf{D},f}) = H^1_f(\mathbf{D})$ and we set $H^1_c(\mathbf{D}) = \mathrm{Im}(i_{\mathbf{D},c})$. Thus we have a canonical decomposition

$$H^1(\mathbf{D}) \simeq H^1_f(\mathbf{D}) \oplus H^1_c(\mathbf{D})$$

([Ben2], Proposition 1.5.9).

Now consider the dual module $\mathbf{D}^*(\chi)$. It is crystalline, $\mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))^{\varphi=p^{-1}} = \mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))$ and the all Hodge-Tate weights of $\mathbf{D}^*(\chi)$ are $\leqslant 0$. Let

$$[\ ,\ ]_{\mathbf{D}} \,:\, \mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi)) \times \mathcal{D}_{\mathrm{cris}}(\mathbf{D}) \to L$$

denote the canonical pairing. Define

$$i_{\mathbf{D}^*(\chi)} \,:\, \mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi)) \oplus \mathcal{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi)) \to H^1(\mathbf{D}^*(\chi))$$

by

$$i_{\mathbf{D}^*(\chi)}(\alpha,\beta) \cup i_{\mathbf{D}}(x,y) = [\beta,x]_{\mathbf{D}} - [\alpha,y]_{\mathbf{D}}.$$

As before, let $i_{\mathbf{D}*(\chi),\,f}$ and $i_{\mathbf{D}^*(\chi),\,c}$ denote the restrictions of $i_{\mathbf{D}}$ on the first and second summand respectively. From $H^1_f(\mathbf{D}^*(\chi)) = H^1_f(\mathbf{D})^\perp$ it follows that $\mathrm{Im}(i_{\mathbf{D}^*(\chi),\,f}) = H^1_f(\mathbf{D}^*(\chi))$ and we set $H^1_c(\mathbf{D}^*(\chi)) = \mathrm{Im}(i_{\mathbf{D}^*(\chi),\,c})$.

Write ∂ for the differential operator $(1+\pi)\dfrac{d}{d\pi}$.

Proposition 1.1.5. *Let $\mathcal{R}_L(|x|x^m)$ be the (φ,Γ)-module $\mathcal{R}_L e_\delta$ associated to the character $\delta(x)=|x|x^m$ $(m\geqslant 1)$. Then*

i) $\mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(|x|x^m))$ is the one-dimensional L-vector space generated by $t^{-m}e_\delta$. Moreover $\mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(|x|x^m))=\mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(|x|x^m))^{\varphi=p^{-1}}$ and the unique Hodge-Tate weight of $\mathcal{R}_L(|x|x^m)$ is $-m$.

ii) $H^0(\mathcal{R}_L(|x|x^m))=0$ and $H^1(\mathcal{R}_L(|x|x^m))$ is the two-dimensional L-vector space generated by $\alpha_m^*=-\left(1-\frac{1}{p}\right)\mathrm{cl}(\alpha_m)$ and $\beta_m^*=\left(1-\frac{1}{p}\right)\log\chi(\gamma)\,\mathrm{cl}(\beta_m)$ where

$$\alpha_m=\frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2},a\right)e_\delta$$

with $a\in\mathcal{R}_L^+=\mathcal{R}_L\cap L[[\pi]]$ such that $(1-\varphi)\,a=\,(1-\chi(\gamma)\gamma)\left(\frac{1}{\pi}+\frac{1}{2}\right)$ and

$$\beta_m=\frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(b,\frac{1}{\pi}\right)e_\delta$$

with $b\in\mathcal{R}_L$ such that $(1-\varphi)\left(\frac{1}{\pi}\right)=(1-\chi(\gamma)\gamma)\,b$. Moreover $i_{m,f}(1)=\alpha_m^*$ and $i_{m,c}(1)=\beta_m^*$ where i_m denotes the map i defined in 1.1.4 for $\mathcal{R}_L(|x|x^m)$. In particular, $H^1_f(\mathcal{R}_L(|x|x^m))$ is generated by α_m^* and $H^1_c(\mathcal{R}_L(|x|x^m))$ is generated by β_m^*.

iii) Let $x=\mathrm{cl}(u,v)\in H^1(\mathcal{R}_L(|x|x^m))$. Then

$$x=a\,\mathrm{cl}(\alpha_m)+b\,\mathrm{cl}(\beta_m)$$

with $a=\mathrm{res}(ut^{m-1}dt)$ and $b=\mathrm{res}(vt^{m-1}dt)$.

iv) The map

$$\begin{aligned}&\mathrm{Res}_m\,:\,\mathcal{R}_L(|x|x^m)\to L,\\ &\mathrm{Res}_m(\alpha)=-\left(1-\frac{1}{p}\right)^{-1}(\log\chi(\gamma))^{-1}\mathrm{res}\left(\alpha t^{m-1}dt\right)\end{aligned}$$

induces an isomorphism $\mathrm{inv}_m\,:\,H^2(\mathcal{R}_L(|x|x^m))\simeq L$. Moreover

$$\mathrm{inv}_m\,(\omega_m)=1\qquad\text{where}\;\;\omega_m=(-1)^m\left(1-\frac{1}{p}\right)\frac{\log\chi(\gamma)}{(m-1)!}\,\mathrm{cl}\left(\partial^{m-1}(1/\pi)\right)$$

Proof. The assertions i) and ii) are proved in [Cz4], sections 2.3-2.5 and [Ben2], Theorem 1.5.7 and (16). The assertions iii) and iv) are proved in [Ben2], Proposition 1.5.4 iii) Corollary 1.5.5.

1.1.6. In [Fo1], Fontaine worked out a general approach to the classification of p-adic representations in terms of (φ,Γ)-modules. Thanks to the work of Cherbonnier-Colmez [CC] and Kedlaya [Ke] this approach allows to construct an equivalence

$$\mathbf{D}^{\dagger}_{\mathrm{rig}}\,:\,\mathbf{Rep}_L(G_{\mathbb{Q}_p}) \to \mathbf{M}^{\varphi,\Gamma}_{\text{ét}}$$

between the category of L-adic representations of $G_{\mathbb{Q}_p}$ and the category $\mathbf{M}^{\varphi,\Gamma}_{\text{ét}}$ of étale (φ,Γ)-modules in the sense of [Ke]. If V is a L-adic representation of $G_{\mathbb{Q}_p}$, define

$$\mathbf{D}_{\mathrm{st}}(V)=\mathcal{D}_{\mathrm{st}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)), \qquad \mathbf{D}_{\mathrm{cris}}(V)=\mathcal{D}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)).$$

Then $\mathbf{D}_{\mathrm{st}}$ and $\mathbf{D}_{\mathrm{cris}}$ are canonically isomorphic to classical Fontaine's functors [Fo2], [Fo3] defined using the rings $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{cris}}$ ([Ber1], Theorem 0.2). The continuous Galois cohomology $H^*(\mathbb{Q}_p,V)=H^*_{\mathrm{cont}}(G_{\mathbb{Q}_p},V)$ is functorially isomorphic to $H^*(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$ ([H1], [Li]). and under this isomorphism

$$H^1_f(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \simeq H^1_f(\mathbb{Q}_p,V)$$

where $H^1_f(\mathbb{Q}_p,V)=\ker(H^1(\mathbb{Q}_p,V)\to H^1(\mathbb{Q}_p,V\otimes \mathbf{B}_{\mathrm{cris}}))$ is H^1_f of Bloch and Kato [BK].

1.2. THE ℓ-INVARIANT.

1.2.1. The results of this section are proved in [Ben2], 2.1-2.2. Denote by $\overline{\mathbb{Q}}^{(S)}/\mathbb{Q}$ the maximal Galois extension of $\mathbb{Q}$ unramified outside $S\cup\{\infty\}$ and set $G_S=\mathrm{Gal}(\overline{\mathbb{Q}}^{(S)}/\mathbb{Q})$. If V is a L-adic representation of G_S we write $H^*(V)$ for the continuous cohomology of G_S with coefficients in V. If V is potentially semistable at p, set

$$H^1_f(\mathbb{Q}_l,V)=\begin{cases} \ker(H^1(\mathbb{Q}_l,V)\to H^1(\mathbb{Q}_l^{\mathrm{nr}},V) & \text{if } l\neq p,\\ H^1_f(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) & \text{if } l=p.\end{cases}$$

The Selmer group of Bloch and Kato is defined by

$$H^1_f(V)=\ker\left(H^1(V)\to \bigoplus_{l\in S}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}\right).$$

Assume that V satisfies the condition **C1-4)** of 0.2.
The Poitou-Tate exact sequence together with **C1)** gives an isomorphism

$$H^1(V)\simeq \bigoplus_{l\in S}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}. \tag{1.4}$$

Recall that a (φ, N)-submodule D of $\mathbf{D}_{\mathrm{st}}(V)$ is said to be regular if the canonical projection $D \to t_V(L)$ is an isomorphism. To any regular D we associate a filtration on $\mathbf{D}_{\mathrm{st}}(V)$

$$\{0\} \subset D_{-1} \subset D_0 \subset D_1 \subset \mathbf{D}_{\mathrm{st}}(V)$$

setting

$$D_i = \begin{cases} (1-p^{-1}\varphi^{-1})\, D + N(D^{\varphi=1}) & \text{if } i=-1, \\ D & \text{if } i=0, \\ D + \mathbf{D}_{\mathrm{st}}(V)^{\varphi=1} \cap N^{-1}(D^{\varphi=p^{-1}}) & \text{if } i=1. \end{cases}$$

By (1.2) this filtration induces a filtration on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ by saturated (φ, Γ)-submodules

$$\{0\} \subset F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V).$$

Set $W = F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. In [Ben2], Proposition 2.1.7 we proved that

$$W \simeq W_0 \oplus W_1 \oplus M, \tag{1.5}$$

where W_0 and W_1 are direct summands of $\mathrm{gr}_0\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)$ and $\mathrm{gr}_1\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)$ of ranks $\dim_L H^0(W^*(\chi))$ and $\dim_L H^0(W)$ respectively. Moreover M seats in a non split exact sequence

$$0 \to M_0 \xrightarrow{f} M \xrightarrow{g} M_1 \to 0$$

with $\mathrm{rg}(M_0) = \mathrm{rg}(M_1)$, $\mathrm{gr}_0\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right) = M_0 \oplus W_0$ and $\mathrm{gr}_1\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right) = M_1 \oplus W_1$. Set

$$e = \mathrm{rg}(W_0) + \mathrm{rg}(W_1) + \mathrm{rg}(M_0).$$

Generalizing [G] we expect that the p-adic L-function $L_p(V,D,s)$ has a zero of order e at $s=0$.

If $W_0 = 0$, the main construction of [Ben2] associates to V and D an element $\mathcal{L}(V,D) \in L$ which can be viewed as a generalization of Greenberg's $\mathcal{L}$-invariant to semistable representations. Now assume that $W_1 = 0$. Let $D^* = \mathrm{Hom}(\mathbf{D}_{\mathrm{st}}(V)/D, \mathbf{D}_{\mathrm{st}}(\mathbb{Q}_p(1)))$ be the dual regular space. As the decompositions (1.5) for the pairs (V,D) and $(V^*(1), D^*)$ are dual to each other, one can define

$$\ell(V,D) = \mathcal{L}(V^*(1), D^*).$$

In this paper we do not review the construction of the $\mathcal{L}$-invariant but give a direct description of $\ell(V,D)$ in terms of V and D in two important particular cases.

1.2.2. THE CRYSTALLINE CASE: $W = W_0$ (see [Ben2], 2.2.6-2.2.7 and 2.3.3). In this case W is crystalline, $W_1 = M = 0$ and $F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) = F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. From the decomposition (1.5) it is not difficult to obtain the following description of $H^1_f(\mathbb{Q}_p, V)$ in the spirit of Greenberg's local conditions:

$$H^1_f(\mathbb{Q}_p,V) \,=\, \ker\left(H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to \frac{H^1(W)}{H^1_f(W)}\right). \tag{1.6}$$

Let $H^1(D,V)$ denote the inverse image of $H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))/H^1_f(\mathbb{Q}_p,V)$ under the isomorphism (1.4). Thus one has a commutative diagram

$$\begin{tikzcd} H^1(D,V) \arrow[r] \arrow[dr] & H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \arrow[d] \\ & H^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \end{tikzcd} \tag{1.7}$$

where the vertical map is injective ([Ben2], section 2.2.1). From (1.6) it follows that the composition map

$$\kappa_D \,:\, H^1(D,V) \to H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1(W)$$

is injective. By construction, $\mathcal{D}_{\mathrm{cris}}(W) = D/D_{-1} = D^{\varphi=p^{-1}}$. As D is regular, the Hodge-Tate weights of W are $\leqslant 0$. Thus one has a decomposition

$$i_W \,:\, \mathcal{D}_{\mathrm{cris}}\,(W) \oplus \mathcal{D}_{\mathrm{cris}}\,(W) \simeq H^1_f\,(W) \oplus H^1_c\,(W) \simeq H^1\,(W)\,.$$

Denote by $p_{D,f}$ and $p_{D,c}$ the projection of $H^1\,(W)$ on the first and the second direct summand respectively. We have a diagram

$$\begin{tikzcd} & \mathcal{D}_{\mathrm{cris}}(W) \\ H^1(D,V) \arrow[ur, "\rho_{D,f}"] \arrow[r, "\kappa_D"] \arrow[dr, "\rho_{D,c}"'] & H^1(W) \arrow[u, "p_{D,f}"'] \arrow[d, "p_{D,c}"] \\ & \mathcal{D}_{\mathrm{cris}}(W) \end{tikzcd}$$

where $\rho_{D,c}$ is an isomorphism. Then

$$\ell(V,D) = \det{}_L\left(\rho_{D,f} \circ \rho_{D,c}^{-1} \mid \mathcal{D}_{\mathrm{cris}}(W)\right).$$

1.2.3. The semistable case: $W = M$ (see [Ben2], 2.2.3-2.2.4 and 2.3.3). In this case W is semistable , $W_0 = W_1 = 0$ and

$$H^1_f(\mathbb{Q}_p, V) = \ker\left(H^1(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \to H^1(M_1)\right). \tag{1.8}$$

Let $H^1(D,V)$ be the inverse image of $H^1(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))/H^1_f(\mathbb{Q}_p,V)$ under the isomorphism (1.4). Consider the exact sequence

$$\begin{tikzcd} H^1(M_0) \arrow[r, "h_1(f)"] & H^1(M) \arrow[r, "h_1(g)"] & H^1(M_1) \arrow[r, "\Delta^1"] & H^2(M_0) \arrow[r] & 0\,. \\ & H^1(D,V) \arrow[u, "\kappa_D"] \arrow[ur, "\bar\kappa_D"'] & & & \end{tikzcd}$$

By (1.8), the map $\bar\kappa_D$ is injective and it is not difficult to prove that the image of $H^1(D,V)$ in $H^1(M_1)$ coincides with $\mathrm{Im}(h_1(g))$ ([Ben2], section 2.2.3). Thus in the semistable case the position of $H^1(D,V)$ in $H^1(M_1)$ is completely determined by the the restriction of V on the decomposition group at p. By construction, $\mathcal{D}_{\mathrm{st}}(M_1) = D_1/D$ where $(D_1/D)^{\varphi=1} = D_1/D$ and the Hodge-Tate weights of M_1 are $\geqslant 0$. Again, one has an isomorphism

$$i_{M_1}\,:\, \mathcal{D}_{\mathrm{cris}}(M_1) \oplus \mathcal{D}_{\mathrm{cris}}(M_1) \simeq H^1_f(M_1) \oplus H^1_c(M_1) \simeq H^1(M_1)$$

which allows to construct a diagram

$$\begin{tikzcd} & \mathcal{D}_{\mathrm{st}}(M_1) \\ \mathrm{Im}(h_1(g)) \arrow[ur, "\rho_{D,f}"] \arrow[r, "\kappa_D"] \arrow[dr, "\rho_{D,c}"'] & H^1(M_1) \arrow[u, "p_{D,f}"'] \arrow[d, "p_{D,c}"] \\ & \mathcal{D}_{\mathrm{st}}(M_1). \end{tikzcd}$$

Then

$$\ell(V,D) = \mathcal{L}(V,D) = \det{}_L\left(\rho_{D,f} \circ \rho_{D,c}^{-1} \mid \mathcal{D}_{\mathrm{st}}(M_1)\right). \tag{1.9}$$

From (1.5) it is clear that if $e = 1$ then either $W = W_0$ with $\mathrm{rg}(W_0) = 1$ or $W = M$ with $\mathrm{rg}(M_0) = \mathrm{rg}(M_1) = 1$. We consider these cases separately in the rest of the paper.

§2. The crystalline case

2.1. Let $A = L[T]/(T^2)$ and let V_A be a free finitely generated A-module equipped with a A-linear action of G_S. One says that V_A is an infinitesimal deformation of a p-adic representation V if $V \simeq V_A \otimes_A L$. Write $\mathcal{R}_A = A \otimes_L \mathcal{R}_L$ and extend the actions of φ and Γ to $\mathcal{R}_A$ by linearity. A (φ, Γ)-module over $\mathcal{R}_A$ is a free finitely generated $\mathcal{R}_A$-module $\mathbf{D}_A$ equipped with commuting semilinear actions of φ and Γ and such that $\mathcal{R}_A \varphi(\mathbf{D}_A) = \mathbf{D}_A$. We say that $\mathbf{D}_A$ is an infinitesimal deformation of a (φ, Γ)-module $\mathbf{D}$ over $\mathcal{R}_L$ if $\mathbf{D} = \mathbf{D}_A \otimes_A L$.

2.2. Let V be a p-adic representation of G_S which satisfies the conditions **C1-4)** and such that $W = W_0$. Moreover we assume that $\mathrm{rg}(W) = 1$. Thus W is a crystalline (φ, Γ)-module of rank 1 with $\mathcal{D}_{\mathrm{cris}}(W) = \mathcal{D}_{\mathrm{cris}}(W)^{\varphi = p^{-1}}$ and such that $\mathrm{Fil}^0 \mathcal{D}_{\mathrm{cris}}(W) = 0$. This implies that

$$W \simeq \mathcal{R}_L(\delta) \quad \text{with} \quad \delta(x) = |x| x^m, \quad m \geqslant 1. \tag{2.1}$$

(see for example [Ben2], Proposition 1.5.8). Note that the Hodge-Tate weight of W is $-m$. The L-vector space $H^1(D, V)$ is one dimensional. Fix a basis $\mathrm{cl}(x) \in H^1(D, V)$. We can associate to $\mathrm{cl}(x)$ a non trivial extension

$$0 \to V \to V_x \to L \to 0.$$

This gives an exact sequence of (φ, Γ)-modules

$$0 \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x) \to \mathcal{R}_L \to 0.$$

From (1.7) it follows that there exists an extension in the category of (φ, Γ)-modules

$$0 \to F_0 \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to \mathbf{D}_x \to \mathcal{R}_L \to 0$$

which is inserted in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_0 \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \arrow[r] \arrow[d] & \mathbf{D}_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \arrow[r] & \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x) \arrow[r] & \mathcal{R}_L \arrow[r] & 0.
\end{tikzcd}$$

Define a filtration

$$\{0\} \subset F_{-1} \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x) \subset F_0 \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x) \subset F_1 \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x) \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x)$$

by $F_i \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x) = F_i \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ for $i = -1, 0$ and $F_1 \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x) = \mathbf{D}_x$. Set

$$W_x = F_1 \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x) / F_{-1} \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x).$$

Thus one has a diagram

$$\begin{tikzcd}
0 \arrow[r] & F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \arrow[r] \arrow[d] & \mathbf{D}_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & W \arrow[r] & W_x \arrow[r] & \mathcal{R}_L \arrow[r] & 0.
\end{tikzcd}$$

2.3. Let $V_{A,x}$ be an infinitesimal deformation of V_x. Assume that $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$ is equipped with a filtration by saturated (φ,Γ)-modules over $\mathcal{R}_A$:

$$\{0\} \subset F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \subset F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \subset F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$$

such that $F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})\otimes_A L \simeq F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x)$ for all i. The quotients $\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$ and $\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$ are (φ,Γ)-modules of rank 1 over $\mathcal{R}_A$ and by [BC], Proposition 2.3.1 there exists unique characters $\delta_{A,x},\psi_{A,x} : \mathbb{Q}_p^* \to A^*$ such that $\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \simeq \mathcal{R}_A(\delta_{A,x})$ and $\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \simeq \mathcal{R}_A(\psi_{A,x})$. It is clear that $\delta_{A,x} \pmod{T} = \delta$ and $\psi_{A,x} \pmod{T} = 1$. One has a diagram

$$\begin{tikzcd}
0 \arrow[r] & F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A) \arrow[r] \arrow[d] & F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x}) \arrow[r] \arrow[d] & \mathcal{R}_A(\psi_A) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & W_A \arrow[r] & W_{A,x} \arrow[r] & \mathcal{R}_A(\psi_A) \arrow[r] & 0
\end{tikzcd}$$

with $W_A = \mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$ and $W_{A,x} = F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})/F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{A,x})$. Assume that

$$\left.\frac{d(\delta_{A,x}\psi_{A,x}^{-1})(u)}{dT}\right|_{T=0} \neq 0, \qquad u \equiv 1 \pmod{p^2}$$

(as the multiplicative group $1+p^2\mathbb{Z}_p$ is procyclic it is enough to assume that this holds for $u = 1+p^2$.)

Theorem 1. *Let $V_{A,x}$ be an infinitesimal deformation of V_x which satisfies the above conditions. Then*

$$\ell(V,D) = -\log\chi(\gamma)\left.\frac{d\log(\delta_{A,x}\psi_{A,x}^{-1})(p)}{d\log(\delta_{A,x}\psi_{A,x}^{-1})(\chi(\gamma))}\right|_{T=0}.$$

This theorem will be proved in section 2.5. We start with an auxiliary result which plays a key role in the proof. Set $\delta(x) = |x|x^m$ $(m \geqslant 1)$ and fix a character $\delta_A : \mathbb{Q}_p^* \to A^*$ such that $\delta_A \pmod{T} = \delta$. Consider the exact sequence

$$0 \to \mathcal{R}_L(\delta) \to \mathcal{R}_A(\delta_A) \to \mathcal{R}_L(\delta) \to 0$$

and denote by B^i_δ the connecting maps $H^i(\mathcal{R}_L(\delta)) \to H^{i+1}(\mathcal{R}_L(\delta))$.

Proposition 2.4. *One has*

$$\mathrm{inv}_m\big(\mathrm{B}_\delta^1(\alpha_m^*)\big) = (\log\chi(\gamma))^{-1} d\log\delta_A(\chi(\gamma))\big|_{T=0},$$
$$\mathrm{inv}_m\big(\mathrm{B}_\delta^1(\beta_m^*)\big) = d\log\delta_A(p)\big|_{T=0}.$$

Proof. a) Recall that

$$\alpha_m^* = -\left(1-\frac{1}{p}\right)\frac{(-1)^{m-1}}{(m-1)!}\,\mathrm{cl}\left(\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2},a\right)e_\delta\right).$$

Let $e_{A,\delta}$ be a generator of $\mathcal{R}_A(\delta_A)$ such that $e_\delta = e_{A,\delta} \pmod{T}$. Directly from the definition of the connecting map

$$\mathrm{B}_\delta^1(\alpha_m^*) \;=\; -\left(1-\frac{1}{p}\right)\frac{(-1)^{m-1}}{(m-1)!}\,\mathrm{cl}\left(\frac{1}{T}\left((\gamma-1)\left(\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)e_{A,\delta}\right)-\right.\right.$$
$$\left.\left. -\;(\varphi-1)\left(\partial^{m-1}(a)e_{A,\delta}\right)\right)\right).$$

Write

$$(\gamma-1)\left(\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)e_{A,\delta}\right)-(\varphi-1)\left(\partial^{m-1}(a)e_{A,\delta}\right)=$$
$$=\big(\chi(\gamma)^{-m}\delta_A(\chi(\gamma))-1\big)\,\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)e_{A,\delta}+z$$

where

$$z=\big(\gamma-\chi(\gamma)^{-m}\big)\,\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)\delta_A(\chi(\gamma))e_{A,\delta}\;-\;\big(\delta_A(p)\varphi-1\big)\,\partial^{m-1}(a)e_{A,\delta}.$$

Since $\delta_A(\chi(\gamma))\equiv\chi(\gamma)^m \pmod{T}$, from the definition of a it follows that $z\equiv 0 \pmod{T}$. On the other hand, as $a\in\mathcal{R}_L^+$ and

$$\big(\gamma-\chi(\gamma)^{-m}\big)\,\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)\in\mathcal{R}_L^+$$

we obtain that $z/T\in\mathcal{R}_L^+e_\delta$. Thus the class of z/T in $H^2(\mathcal{R}_L(\delta))$ is zero. On the other hand, writing δ_A in the form

$$\delta_A(u)=u^m+T\frac{d\delta_A(u)}{dT}\bigg|_{T=0}$$

one finds that

$$\frac{\chi(\gamma)^{-m}\delta_A(\chi(\gamma))-1}{T}\;=\;d\log\delta_A(\chi(\gamma))\big|_{T=0}$$

and the first formula follows from Proposition 1.1.5 iv).
b) By the definition of B^1_δ

$$\mathrm{B}^1_\delta(\beta^*_m) = \left(1-\frac{1}{p}\right)\frac{(-1)^{m-1}\log\chi(\gamma)}{(m-1)!}\,\mathrm{cl}\left(\frac{1}{T}\left((\gamma-1)\left(\partial^{m-1}(b)e_{A,\delta}\right)-\right.\right.$$
$$\left.\left. -\ (\varphi-1)\ \left(\partial^{m-1}\left(1/\pi\right)e_{A,\delta}\right)\right)\right).$$

As

$$\delta_A(p)\ \left(\varphi-\delta(p)^{-1}\right)\ \left(\partial^{m-1}\left(1/\pi\right)\right)\ =\ \frac{\delta_A(p)}{\delta(p)}\ \left(\delta(\chi(\gamma))\,\gamma-1\right)\ \partial^{m-1}(b)$$

we can write

$$(\gamma-1)\left(\partial^{m-1}(b)e_{A,\delta}\right)-(\varphi-1)\ \left(\partial^{m-1}\left(1/\pi\right)e_{A,\delta}\right)=$$
$$=-\left(\delta(p)^{-1}\delta_A(p)-1\right)\partial^{m-1}\left(1/\pi\right)+w$$

where

$$w=\left(\delta_A(\chi(\gamma))\,\gamma-1\right)\ \left(\partial^{m-1}b\right)e_{A,\delta}\ +\ \frac{\delta_A(p)}{\delta(p)}\ \left(\delta(\chi(\gamma))\,\gamma-1\right)\ \left(\partial^{m-1}b\right)e_{A,\delta}.$$

Remark that

$$\frac{\delta(p)^{-1}\delta_A(p)-1}{T}\ =\ -d\log\delta_A(p)\big|_{T=0}$$

On the other hand

$$\mathrm{res}\left(\partial^{m-1}(b)\,t^{m-1}dt\right)=0$$

(see [Ben2], proof of Corollary 1.5.6). As $\mathrm{res}\left((\chi(\gamma)^m\gamma-1)\,\partial^{m-1}(b)\,t^{m-1}dt\right)=0$, this implies that $\mathrm{res}\left(\gamma(\partial^{m-1}b)\,t^{m-1}dt\right)=0$ and we obtain that $\mathrm{Res}_m(w)=0$. Thus

$$\mathrm{inv}_m(\mathrm{B}^1_\delta(\beta^*_m))\ =\ -d\log\delta_A(p)\big|_{T=0}\mathrm{Res}_m(\omega_m)=d\log\delta_A(p)\big|_{T=0}$$

and the Proposition is proved.

2.5. We pass to the proof of Theorem 1. By Proposition 1.1.5, $H^1(W)$ is a two dimensional L-vector space generated by α^*_m and β^*_m. One has a commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_A \arrow[r] \arrow[d] & W_{A,x} \arrow[r] \arrow[d] & \mathcal{R}_A(\psi_{A,x}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Twisting the middle row by $\psi_{A,x}^{-1}$ and taking into account that $\psi_{A,x} \equiv 1 \pmod{T}$ we obtain

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_A(\psi_{A,x}^{-1}) \arrow[r] \arrow[d] & W_{A,x}(\psi_{A,x}^{-1}) \arrow[r] \arrow[d] & \mathcal{R}_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[d] & W_x \arrow[r] \arrow[d] & \mathcal{R}_L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.2}$$

The connecting map $\Delta^0 : H^0(\mathcal{R}_L) \to H^1(W)$ sends 1 to $y = \kappa_D(\mathrm{cl}(x))$ and we can write

$$y \;=\; a\,\alpha_m^* + b\,\beta_m^*$$

with $a, b \in L$. Directly from the definition of the ℓ-invariant one has

$$\ell(V, D) = b^{-1}a. \tag{2.3}$$

The diagram (2.2) gives rise to a commutative diagram

$$\begin{tikzcd}
H^0(\mathcal{R}_L) \arrow[r, "\Delta^0"] \arrow[d, "\mathrm{B}^0"] & H^1(W) \arrow[d, "\mathrm{B}_W^1"] \\
H^1(\mathcal{R}_L) \arrow[r, "\Delta^1"] & H^2(W).
\end{tikzcd}$$

Since the rightmost vertical row of (2.2) splits, the connecting map B^0 is zero and

$$a\mathrm{B}_W^1(\alpha_m^*) + b\mathrm{B}_W^1(\beta_m^*) \;=\; \mathrm{B}_W^1(y) = 0.$$

As $W_A(\psi_{A,x}^{-1}) \simeq \mathcal{R}_A(\delta_{A,x}\psi_{A,x}^{-1})$, Proposition 2.4 gives

$$\begin{aligned}
&\mathrm{inv}_m(\mathrm{B}_W^1(\alpha_m^*)) = (\log(\chi(\gamma))^{-1} d\log(\delta_{A,x}\psi_{A,x}^{-1})(\chi(\gamma))\big|_{T=0},\\
&\mathrm{inv}_m(\mathrm{B}_W^1(\beta_m^*)) = d\log(\delta_{A,x}\psi_{A,x}^{-1})(p)\big|_{T=0}.
\end{aligned}$$

Together with (2.3) this gives the Theorem.

§3. The semistable case

3.1. In this section we assume that V is a p-adic representation which satisfies the conditions **C1-4)** and such that $W=M$. Thus one has an exact sequence

$$0\to M_0\xrightarrow{f}W\xrightarrow{g}M_1\to 0 \tag{3.1}$$

where M_0 and M_1 are such that $e=\mathrm{rg}(M_0)=\mathrm{rg}(M_1)$. We will assume that $e=1$. Then

$$\begin{aligned} M_0&=\mathcal{R}_Le_\delta\simeq\mathcal{R}_L(\delta), &\quad \delta(x)=|x|x^m, &\quad m\geqslant 1,\\ M_1&=\mathcal{R}_Le_\psi\simeq\mathcal{R}_L(\psi), &\quad \psi(x)=x^{-n}, &\quad n\geqslant 0\end{aligned}$$

(see for example [Ben2], Lemma 1.5.2 and Proposition 1.5.8). Thus

$$\{0\}\subset F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V)\subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)$$

with $\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\simeq\mathcal{R}_L(\delta)$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\simeq\mathcal{R}_L(\psi)$. Assume that V_A is an infinitesimal deformation of V and that $\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)$ is equipped with a filtration by saturated (φ,Γ)-modules over $\mathcal{R}_A$

$$\{0\}\subset F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)\subset F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)\subset F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)\subset \mathbf{D}^\dagger_{\mathrm{rig}}(V_A)$$

such that

$$F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)\otimes_L A\simeq F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V),\qquad -1\leqslant i\leqslant 1.$$

Then

$$\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)\simeq\mathcal{R}_A(\delta_A),\qquad \mathrm{gr}_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)\simeq\mathcal{R}_A(\psi_A),$$

where $\delta_A,\,\psi_A\,:\,\mathbb{Q}_p^*\to A^*$ are such that $\delta_A \pmod{T}=\delta$ and $\psi_A \pmod{T}=\psi$. As before, assume that

$$\left.\frac{d(\delta_A\psi_A^{-1})(u)}{dT}\right|_{T=0}\neq 0,\qquad u\equiv 1 \pmod{p^2}.$$

Theorem 2. *Let V_A be an infinitesimal deformation of V which satisfies the above conditions. Then*

$$\ell(V,D)\;=\;-\log\chi(\gamma)\left.\frac{d\log(\delta_A\psi_A^{-1})(p)}{d\log(\delta_A\psi_A^{-1})(\chi(\gamma))}\right|_{T=0} \tag{3.2}$$

3.2. Proof of Theorem 2. The classes $x_n^*=-\mathrm{cl}(t^ne_\psi,0)$ and $y_n^*=\log\chi(\gamma)\,\mathrm{cl}(0,t^ne_\psi)$ form a basis of $H^1(M_1)$ and $H^1_f(M_1)$ is generated by x_n^* (see section 1.1.4). Consider the long cohomology sequence associated to (3.1):

$$\cdots\to H^1(M_0)\xrightarrow{h_1(f)}H^1(W)\xrightarrow{h_1(g)}H^1(M_1)\xrightarrow{\Delta^1}H^2(M_0)\to\cdots.$$

We can also consider the dual sequence $0 \to M_1^*(\chi) \to W^*(\chi) \to M_0^*(\chi) \to 0$ and write

$$\cdots \to H^0(M_0^*(\chi)) \xrightarrow{\Delta_*^0} H^1(M_1^*(\chi)) \to H^1(W^*(\chi)) \to H^1(M_0^*(\chi)) \to \cdots .$$

As $M_0^*(\chi) = \mathcal{R}_L e_{\delta^{-1}\chi}$ is isomorphic to $\mathcal{R}_L(x^{1-m})$, the cohomology $H^0(M_0^*(\chi))$ is the one dimensional L-vector space generated by $\xi = t^{m-1} e_{\delta^{-1}\chi}$. Write

$$\Delta_*^0(\xi) \;=\; a\,\alpha_{n+1}^* + b\,\beta_{n+1}^*,$$

where α_{n+1}^*, β_{n+1}^* is the canonical basis of $H^1(M_1^*(\chi)) \simeq \mathcal{R}_L(|x|x^{n+1})$. From the duality it follows that $\mathrm{Im}(\Delta_*^0)$ is orthogonal to $\ker(\Delta^1)$ under the pairing

$$H^1(\mathcal{R}_L(|x|x^{n+1})) \times H^1(\mathcal{R}_L(x^{-n})) \xrightarrow{\cup} L$$

Since

$$\alpha_{n+1}^* \cup x_n^* = \beta_{n+1}^* \cup y_n^* = 0, \quad \alpha_{n+1}^* \cup y_n^* = -1, \quad \beta_{n+1}^* \cup x_n^* = 1$$

(see Proposition 1.1.5 ii), we obtain that $\mathrm{Im}(h_1(g)) = \ker(\Delta^1)$ is generated by $ax_n^* + by_n^*$. By the definition of the $\mathcal{L}$-invariant

$$\mathcal{L}(V,D) = b^{-1}a. \tag{3.3}$$

Set $W_A = F_1\mathbf{D}_{\mathrm{rig}}^\dagger(V)/F_{-1}\mathbf{D}_{\mathrm{rig}}^\dagger(V)$. One has a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{R}_L(\psi^{-1}\chi) \arrow[r] \arrow[d] & W^*(\chi) \arrow[r] \arrow[d] & \mathcal{R}_L(\delta^{-1}\chi) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{R}_A(\psi_A^{-1}\chi) \arrow[r] \arrow[d] & W_A^*(\chi) \arrow[r] \arrow[d] & \mathcal{R}_A(\delta_A^{-1}\chi) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{R}_L(\psi^{-1}\chi) \arrow[r] \arrow[d] & W^*(\chi) \arrow[r] \arrow[d] & \mathcal{R}_L(\delta^{-1}\chi) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now the theorem can be proved either by twisting this diagram by $\delta_A\chi^{-1}$ and applying the argument used in the proof of Theorem 2.3 or by the following direct computation. One has an anticommutative square

$$\begin{tikzcd}
H^0(\mathcal{R}_L(\delta^{-1}\chi)) \arrow[r, "\Delta_*^0"] \arrow[d, "\mathrm{B}^0_{\delta^{-1}\chi}"] & H^1(\mathcal{R}_L(\psi^{-1}\chi)) \arrow[d, "\mathrm{B}^1_{\psi^{-1}\chi}"] \\
H^1(\mathcal{R}_L(\delta^{-1}\chi)) \arrow[r, "\Delta_*^1"] & H^2(\mathcal{R}_L(\psi^{-1}\chi)).
\end{tikzcd}$$

Thus

$$\mathrm{B}^1_{\psi^{-1}\chi}\Delta^0_*(\xi) \;=\; -\Delta^1_*\mathrm{B}^0_{\delta^{-1}\chi}(\xi). \tag{3.4}$$

From Proposition 2.3 it follows that

(3.5)
$$\begin{aligned}\mathrm{inv}_{n+1}\bigl(\mathrm{B}^1_{\psi^{-1}\chi}\Delta^0_*(\xi)\bigr) \;&=\; a\,\mathrm{inv}_{n+1}\bigl(\mathrm{B}^1_{\psi^{-1}\chi}(\alpha^*_{n+1})\bigr) \;+\; b\,\bigl(\mathrm{B}^1_{\psi^{-1}\chi}(\beta^*_{n+1})\bigr) \;=\\ &= -a\log(\chi(\gamma))^{-1}d\log\psi_A(\chi(\gamma))\big|_{T=0} \;-\; b\,d\log\psi_A(p)\big|_{T=0}.\end{aligned}$$

Fix a generator $e_{A,\delta^{-1}\chi}$ of $\mathcal{R}_A(\delta_A^{-1}\chi)$. We can assume that $e_{A,\delta^{-1}\chi}$ is a lifting of $e_{\delta^{-1}\chi}$ and set $\xi_A = t^{m-1}e_{A,\delta^{-1}\chi}$. Directly by the definition of the connecting map

$$\begin{aligned}\mathrm{B}^0_{\delta^{-1}\chi}(\xi) \;&=\frac{1}{T}\,\mathrm{cl}((\varphi-1)\,\xi_A,(\gamma-1)\,\xi_A) \;=\\ &=\;\frac{1}{T}\mathrm{cl}((p^{m-1}\delta_A^{-1}(p)-1)\,\xi_A,(\chi(\gamma)^m\delta_A^{-1}(\chi(\gamma))-1)\,\xi_A) \;=\\ &=\;-\mathrm{cl}\bigl(d\log\delta_A(p)\,\xi,d\log\delta_A(\chi(\gamma))\,\xi\bigr)\big|_{T=0}.\end{aligned}$$

Let $\hat\xi$ be a lifting of ξ in $W^*(\chi)$. Then

$$\Delta^1_*\mathrm{B}^0_{\delta^{-1}\chi}(\xi) \;=\; -\mathrm{cl}\bigl(d\log\delta_A(p)\,(\gamma-1)\,\hat\xi - d\log\delta_A(\chi(\gamma))\,(\varphi-1)\,\hat\xi\bigr)\big|_{T=0}.$$

On the other hand, $\Delta^0_*(\xi)=\mathrm{cl}((\varphi-1)\,\hat\xi,(\gamma-1)\,\hat\xi)$ and by Proposition 1.1.5 iii)

$$\begin{aligned}\mathrm{res}\bigl((\varphi-1)\,(\hat\xi)\,t^n dt\bigr) \;&=\; \left(1-\frac{1}{p}\right)\,a,\\ \mathrm{res}\bigl((\gamma-1)\,(\hat\xi)\,t^n dt\bigr) \;&=\; \log(\chi(\gamma))\,\left(1-\frac{1}{p}\right)\,b.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{inv}_{n+1}\bigl(\Delta^1_*\mathrm{B}^0_{\delta^{-1}\chi}(\xi)\bigr) \;&=\\ &= b\,d\log\delta_A(p)\big|_{T=0} \;+\; a\log(\chi(\gamma))^{-1}d\log\delta_A(\chi(\gamma))\big|_{T=0}.\end{aligned} \tag{3.6}$$

From (3.4), (3.5) and (3.6) we obtain that

$$a\,(\log\chi(\gamma))^{-1}d\log(\delta_A\psi_A^{-1})(\chi(\gamma))\big|_{T=0} = -b\,d\log(\delta_A\psi_A^{-1})\,(p)\big|_{T=0}.$$

Together with (3.3) this prove the theorem.

3.4. Remark. It would be interesting to generalize Theorems 1 and 2 to the case $e>1$. For this one should first understand what kind of filtrations on

$\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ appears naturally if V comes from automorphic forms [BC].

3.5. Modular forms. Let f be a normalized newform of weight $x_0 = 2k$ which is split multiplicative at p. Let $V = M_{f,\lambda}$ be the λ-adic representation associated to f by Deligne [D]. The structure of $\mathbf{D}_{\mathrm{st}}(V)$ is well known (see for example [Cz2]) Namely, $\mathbf{D}_{\mathrm{st}}(V) = Ld_1 + Ld_2$ with $N(d_2) = d_1$, $N(d_1) = 0$, $\varphi(d_2) = p^k d_2$ and $\varphi(d_1) = p^{k-1}d_1$. Thus $\mathbf{D}_{\mathrm{st}}(V(k)) = Ld_1^{(k)} + Ld_2^{(k)}$ with $\varphi(d_2^{(k)}) = d_2^{(k)}$, $\varphi(d_1^{(k)}) = p^{-1}d_1^{(k)}$ and $D = \mathbf{D}_{\mathrm{cris}}(V(k)) = Ld_1^{(k)}$ is the unique regular subspace of $\mathbf{D}_{\mathrm{st}}(V(k))$. It is clear that $D_{-1} = 0$, $D_1 = \mathbf{D}_{\mathrm{st}}(V(k))$ and for the associated filtration on $\mathbf{D}^\dagger_{\mathrm{rig}}(V(k))$ we have $F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V(k)) = (D \otimes \mathcal{R}_L[1/t]) \cap \mathbf{D}^\dagger_{\mathrm{rig}}(V(k))$, $F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V(k)) = \mathbf{D}^\dagger_{\mathrm{rig}}(V(k))$. In [Ben2], Proposition 2.2.6 it is proved that $\mathcal{L}(V(k), D)$ coincides with the $\mathcal{L}$-invariant of Fontaine-Mazur $\mathcal{L}_{\mathrm{FM}}(f)$.

In [Co2], Coleman constructed an analytic family of overconvergent modular forms $f_x = \sum\limits_{n=1}^{\infty} a_n(x)q^n$ on an affinoid disk U containing $2k$ which satisfies the following conditions

- For any $x \in \mathbb{N} \cap U$ the form f_x is classical.
- $f_{x_0} = f$.

Moreover, one can interpolate the p-adic representations associated to classical forms f_x ($x \in \mathbb{N} \cap U$) and construct a two dimensional representation $\mathcal{V}$ of $G_{\mathbb{Q}}$ over the Tate algebra $\mathcal{O}(U)$ of U such that

- For any integer $x \in \mathbb{N}$ in U the Galois representation $\mathcal{V}_x$ obtained by specialization of $\mathcal{V}$ at x is isomorphic to the λ-adic representation associated to f_x [CM]. In particular, it is semistable with the Hodge-Tate weights $(0, x-1)$ [Fa]. By continuity this implies that for all $x \in U$ the Hodge-Tate-Sen weights of $\mathcal{V}_x$ are $(0, x-1)$.
- $\wedge^2 \mathcal{V}_x \simeq L_x\left(\chi^{1-2k}\left<\chi\right>^{2k-x}\right)$ where as usually $\left<\chi\right>$ denotes the projection of χ and L_x is the field of coefficients of $\mathcal{V}_x$.
- $\left(\mathbf{B}_{\mathrm{cris}}^{\varphi = a_p(x)} \hat{\otimes} \mathcal{V}\right)^{G_{\mathbb{Q}_p}}$ is locally free of rank 1 on U [Sa], [Ki].

Let $\mathcal{O}_{x_0}$ denote the local ring of U at x_0 and let $A = \mathcal{O}_{x_0}/(T^2)$ where $T = x - x_0$ is a local parameter at x_0. Then $V_A = \mathcal{V} \otimes_{\mathcal{O}(U)} \mathcal{O}_{x_0}$ of $V = \mathcal{V}_{x_0}$ is an infinitesimal deformation of $V = \mathcal{V}_{x_0}$. It is not difficult to see that

$$F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) = \mathcal{R}_A \otimes_L \mathcal{D}_{\mathrm{cris}}(\mathbf{D}^\dagger_{\mathrm{rig}}(V_A))^{\varphi = a_p(x)}$$

is a saturated (φ, Γ)-submodule of $\mathbf{D}^\dagger_{\mathrm{rig}}(V_A)$ ([BC], Lemma 2.5.2 iii)). We see immediately that $F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) \simeq \mathcal{R}_A(\delta_A)$ where $\delta_A(u) = 1$ for $u \in \mathbb{Z}_p^*$ and $\delta_A(p) = a_p(2k) + a_p'(2k)T \pmod{T^2}$ with $a_p(2k) = p^{k-1}$. Set $F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_A) = \mathbf{D}^\dagger_{\mathrm{rig}}(V_A)$. As

$$\left<\chi(\gamma)\right> = \exp\bigl((2k-x)\log\chi(\gamma)\bigr) = 1 - (\log\chi(\gamma))T \pmod{T^2}$$

we obtain that

$$(\psi_A\delta_A)(p)=1,\qquad (\psi_A\delta_A)(\chi(\gamma))=1-(\log\chi(\gamma))T \pmod{T^2}$$

Thus $\psi_A(\chi(\gamma)) = 1 - \log\chi(\gamma)\,T \pmod{T^2}$ and $d\log\psi_A(\chi(\gamma))\big|_{T=0} = -\log\chi(\gamma)$. Twisting V_A by χ^k we obtain an infinitesimal deformation $V_A(k)$ of $V(k)$. The formula (3.2) writes

$$\mathcal{L}(V(k),D)=-2\,d\log a_p(2k).$$

In particular we obtain that $\mathcal{L}_{\mathrm{FM}}(f) = -2\,d\log a_p(2k)$. The first direct proof of this result was done in [Cz5] using Galois cohomology computations inside the rings of p-adic periods. Remark that in [Cz6], Colmez used the theory of (φ,Γ)-modules to prove this formula with Breuil's $\mathcal{L}$-invariant. His approach is based on the local Langlands correspondence for two-dimensional trianguline representations.

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