Finite u Invariant and Bounds on Cohomology Symbol Lengths

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Abstract. In this note we answer a question of Parimala's, showing that fields with finite u invariant have bounds on the symbol lengths in their μ_2 cohomology in all degrees.

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Introduction

At the AIM Workshop on Period/Index problems in January 2011, Prof. Parimala asked whether fields of finite u invariant necessarily had bounded symbol length in their μ_2 cohomology. Parimala presented a proof that this was true in degrees one through three, using generic splitting constructions. In the subsequent breakout session on this problem, the first ideas of a proof were presented by myself, and I greatly benefited from the constructive comments by Prof. Parimala and Prof. Merkurjev. The note below is my write up of the argument. Similar results were simultaneously found by Daniel Krashen [K].

Much of the notation and definitions below are standard in quadratic form books (e.g. [Sc]), and we also assume familiarity with group and Galois cohomology and the Hochschild Serre spectral sequence (e.g. [NSW]). As notation which is perhaps not standard, for any field F let G_F be its absolute Galois group. That is, if F_s is the separable closure of F, then G_F is the Galois group of F_s/F. If R is a commutative domain, we let $q(R)$ be its field of fractions. Let me also add an extended discussion about Galois extensions which is also less standard ([Sa] p. 253). We list some conventions we use when talking about

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Galois extensions of rings. When we say S/R is a G Galois extension of rings, we include in this a fixed action of G on S. Similarly, if S/R and S'/R are G and G' Galois extensions respectively, and we write $S \subset S'$, we assume the inclusion preserves the Galois action. That is, there is a surjection $\rho : G' \to G$ such that the G' action on S' restricts to the G action on S via ρ.

Let S/R be an H Galois extension of commutative rings, where H is a subgroup of the finite group G. Then $\mathrm{Hom}_H(\mathbb{Z}[G], S)$ can be given the structure of an R algebra by pointwise operations in S, and has the coinduced G action. Together this defines $\mathrm{Ind}_H^G(S/R)$ which is a G Galois extension of R. Note that $\mathrm{Ind}_H^G(S/R)$ is of the form $S \oplus \ldots \oplus S$ as an R algebra. We will most often use this in the case $R = F$ is a field. Recall that any G Galois L/F (L is not necessarily a field) has the form $\mathrm{Ind}_H^G(K/F)$ where K/F is an H Galois extension of fields. In this case, suppose $K' \supset K \supset F$ is a tower of fields where K'/F is H' Galois and induces the surjection $\rho : H' \to H$ (we say $K' \supset K \supset F$ is Galois tower). Then there is a Galois tower of rings $\mathrm{Ind}_{H'}^{G'}(K') \supset L = \mathrm{Ind}_H^G(K) \supset F$ if and only if ρ extends to a surjection $\rho : G' \to G$ (as stated above, ρ is implicit in the inclusion $\mathrm{Ind}_H^G(K) \subset \mathrm{Ind}_{H'}^{G'}(K')$. Conversely, if $\mathrm{Ind}_{H'}^{G'}(K') \supset \mathrm{Ind}_H^G(K) \supset F$ is a Galois tower where K, K' are fields then there is an induced Galois tower $K' \supset K \supset F$. Finally, if S is a ring on which the group B acts we denote by S^B the subring of B fixed elements.

Underlying this work is the remarkable result of [OVV], based on the underlyng work of Voevodsky, that shows the maps $I_K^i/I_K^{i+1} \to H^i(K, \mu_2)$ are isomorphisms for all i. Since the paper [OVV] assumes all fields have characteristic 0 we also make this assumption, but really this work applies to any fields (characteristic not 2) where Milnor's Conjecture holds. We set μ_2 to be the group $\{1, -1\}$ of 2 roots of 1. Given a field F, then $H^i(F, \mu_2) = H^i(G_F, \mu_2)$ is the Galois cohomology group. If $a \in F^*$ then we abuse notation and write $a \in H^1(F, \mu_2)$ to be the character $\mathrm{Hom}(G_F, \mu_2) = H^1(F, \mu_2)$ determined by the field extension $F(a^{1/2})/F$. The cup product $a_1 \cup \ldots \cup a_i \in H^i(F, \mu_2)$ is called a SYMBOL and the symbol length of an element $\alpha \in H^i(F, \mu_2)$ is the least i such that α is a sum of i symbols. The canonical map $I_K^i/I_K^{i+1} \to H^i(K, \mu_2)$ of Milnor's Conjecture is determined by mapping the Pfister form (e.g., [Sc] p. 72) $\langle\langle a_1, \ldots, a_i \rangle\rangle \to -a_1 \cup \ldots \cup -a_i$ and this was shown in [OVV] to be an isomorphism. In particular, every element of $H^i(F, \mu_2)$ is a sum of symbols and so has a symbol length. Recall that the u invariant of a field is the integer or ∞, $u(F)$, such that any quadratic form over F of rank bigger than $u(F)$ is isotropic.

Suppose K/F is H Galois and $\beta \in H^i(H, \mu_2)$. Then there is a natural image of β in $H^i(F, \mu_2) = H^i(G_F, \mu_2)$ via inflation. In detail, a choice of embedding $K \subset F_s$ determines a surjection $\rho : G_F \to H$ and we inflate via ρ. This is well defined because a different choice of embedding changes ρ via conjugation by an element of H, and cohomology is invariant under conjugation. More generally, if $L = \mathrm{Ind}_H^G(K/F)$, K is a field, and $\beta \in H^i(G, \mu_2)$ then β has a natural image in Galois cohomology by first forming the restriction $\beta_H \in H^i(H, \mu_2)$ and then taking the image of β_H in $H^i(F, \mu_2) = H^i(G_F, \mu_2)$ via inflation as above. Of course, the subgroup $H \subset G$ is only determined by L up to conjugation, but again cohomology is invariant under conjugation so the map $H^i(G, \mu_2) \to H^i(F, \mu_2)$ is well defined. Note that this map commutes with inflation. That is, If $G' \to G$ is defined by a tower $F \subset L \subset L'$ of Galois extensions, and $L' = \mathrm{Ind}_{H'}^{G'}(K'/F)$ then we can choose H' such that we have the diagram:

$$\begin{array}{ccc} H' & \subset & G' \\ \downarrow & & \downarrow \\ H & \subset & G \end{array}$$

and the restriction of β to H inflated to H' is the same as the inflation of β to G' restricted to H'.

The result.

We say that a field F is u-bounded if there is an integer function $N(n)$, depending only on $u(F)$, such that $u(L) \leq N(n)$ for all extensions L/F of degree dividing n. Note that this is equivalent (e.g., [Sc] p. 104) to just saying that $u(F) < \infty$ but we phrase it in this way to emphasize that we are considering properties closed under finite extension. We say that F has bounded symbol length in degree d if there is an integer $M_d(n)$, depending only on $u(F)$, such that every element in $H^d(L, \mu_2)$ is a sum of $M_d(n)$ symbols for every L/F finite of degree dividing n. The point of this note is to show:

Theorem 1. *Every field with finite u invariant has bounded symbol length in degree i for all i.*

We prove this by induction and we note that every field has bounded symbol length of degree 1. Also, since the premise of this result is preserved by finite extensions, we may assume we have shown for such F that $M_j(n)$ exists for all $j < d$ and n, and show $M_d(1)$ exists. That is, we show that every element of $H^d(F, \mu_2)$ has bounded symbol length.

We remark that a partial converse of Theorem 1 is true, as was pointed out to us by Hoffmann and Garibaldi. Suppose F is non-real and $H^i(F, \mu_2)$ has bounded symbol length for all i. By [Ka] Proposition 1, F has finite 2 cohomological dimension. By [OVV], there is a d such that in the Witt ring of F, $I^d = 0$. Moreover, in each I^r/I^{r+1}, every element is represented by a Pfister form of bounded rank. If $[q]$ is the class of a quadratic form, then q is Witt equivalent to a sum of Pfister forms of bounded rank. In particular, if q is anisotropic, it has bounded rank.

The idea of the proof of Theorem 1 is the following. Since $u(F)$ is finite, we can write down a generic quadratic form which specializes to all anisotropic quadratic forms. We would like to modify this generic form so that it is generic and lies in the Witt Ring fundamental ideal I^i and specializes to all Witt classes of forms in that ideal. That generic form, defined over some $F' \supset F$, maps to $H^i(F', \mu_2)$ and there it is a sum of some number of symbols. By specializing, all elements of $H^i(F, \mu_2)$ are the sum of that many or fewer symbols, and we would be done.

At the moment, there is no construction for such a generic form. The difficulty can be illustrated as follows. Suppose $\alpha = \sum_{j=1}^{m} a_{1,j} \cup \ldots \cup a_{i,j}$ is an element of $H^i(F, \mu_2)$. Let $L = F(a_{k,l}^{1/2} |$ all $k, l)$ and let A be the Galois group of L/F. Then our form for α defines an element of $H^i(A, \mu_2)$. If this element maps to 0 in $H^i(G_F, \mu_2) = H^i(F, \mu_2)$, then there must be a finite extension $L' \supset L$ with L'/F Galois with group B and so an induced surjection $B \to A$ such that α maps to 0 in $H^i(B, \mu_2)$. However, we see no way to, in general, bound the size of B and if the size of B is not bounded there can be no generic way to force a cohomology class of α's form to be 0 because the B that works for such a generic construction would then give you a bound. However, we can construct a generic zeroing of α for a fixed B. We will call this generic with the limitation B (formal definition to follow).

Let us note that the argument of this paper does construct a finite set of generic elements of I^d, when $u(F) < \infty$. We make this explicit in:

Corollary 2. *Suppose $u(F) < \infty$ and $d \geq 1$. Then there are finitely many field extensions $F_i \supset F$ and α_i in $I(F_i)^d$ such that any $\alpha \in I(F)^d$ is the specialization of one of the α_i.*

To begin, we mention the following way of thinking about writing a cohomology element as a sum of symbols.

Lemma 3. *An element of $\alpha \in H^i(F, \mu_2)$ is a sum of symbols if and only if α is the image of $H^i(A, \mu_2)$ where $A = \mathrm{Gal}(L/F)$ is an elementary abelian 2 group. Moreover, that all α have bounded symbol length is equivalent to bounding the size of such A.*

Proof. The basic equivalence is immediate from (e.g., [E] p. 33), which says the well-known fact that any element of such an $H^i(A, \mu_2)$ is the sum of i degree monomials of elements in $H^1(A, \mu_2) = \mathrm{Hom}(A, \mu_2)$. Moreover, if $\alpha = \sum_{j=1}^{M} (a_{j,1} \cup \ldots \cup a_{j,i})$ then A can be taken of order dividing 2^{im}. Conversely, for A of order 2^M, $H^i(A, \mu_2)$ has a basis consisting of $\binom{M}{i}$ elements. □

In what follows A will always be an elementary abelian 2 group.

Let C be our ground field so all rings and fields will be C algebras. We suppose $\alpha \in H^i(F, \mu_2)$ is the image of $\beta \in H^i(A, \mu_2)$ where $A = \mathrm{Gal}(L/F)$ is as above. Further, suppose S/R is a A Galois extension with $q(R) = F$, R affine over C, and $L = S \otimes_R F$ as A Galois extensions. We call S/R, A and β a PRESENTATION of α. Since we are usually not interested in the specific rings S/R in a presentation, we define S'/R', A, β' EQUIVALENT to S/R, A, β if and only if $A = A'$, $\beta = \beta'$, and there are nonzero $r \in R$ and $r' \in R'$ such that $R(1/r) = R'(1/r')$ and $S(1/r)/R(1/r)$ and $S'(1/r')/R'(1/r')$ are isomorphic as A Galois extensions of $R(1/r)$. Obviously equivalent presentations have the same induced cohomology element. In discussing presentations up to equivalence, the ring extension S/R can often be surpressed and we can just say the presentation β, L/F where L/F is A Galois.

Presentations will be important to us because they allow specializations of cohomology classes as follows. In fact, we will be defining specializations of presentations. Let β, S/R, and A be a presentation of α. Suppose $\phi : R \to R_1 \subset F_1$ for a ring and field $R_1, F_1 \supset C$ with $q(R_1) = F_1$. If we set $S_1 = S \otimes_\phi R_1$ then β, S_1/R_1, and A is the specialization with respect to ϕ. This specialized presentation defines an $\alpha_1 \in H^i(F_1, \mu_2)$ which we can call a specialization of α.

Note that we have defined the notion of presentation without assuming $L = S \otimes_R F$ is a field. In fact, suppose $L = \mathrm{Ind}_{A_1}^A(L_1/F)$. Then there is some $0 \neq r \in R$ such that $S(1/r) = \mathrm{Ind}_{A_1}^A(S_1/R(1/r))$ and we can define (up to equivalence) β_1, $S_1/R(1/r)$, A_1 to be a restriction of the original presentation, and this restriction presents the same cohomology element.

Next we must talk about presentations that represent 0 and their so called limitations. Note that this is a key idea in the argument that follows. We will frequently be talking about β and L/F where β maps to $0 \in H^i(F, \mu_2)$ and so perhaps seem not to be important. In fact, we are very interested in why β maps to 0 and more specifically we will be bounding the reason why β maps to 0. To be precise, let $\beta \in H^i(A, \mu_2)$ be as above and suppose L/F is an A Galois extension. We say β and L/F PRESENTS 0 if the image of β in $H^i(F, \mu_2)$ is 0. We say that β and L/F has LIMITATION $B \to A$ (sometimes we write only B) if and only if there is a Galois extension L'/F with Galois group B such that L'/F contains L/F and induces $B \to A$ where β maps to 0 in $H^i(B, \mu_2)$. Suppose $L = \mathrm{Ind}_{A_1}^A(L_1/F)$, β restricts to $\beta_1 \in H^i(A_1, \mu_2)$, and β has limitation B realized by a B Galois extension L'/F. Write $L' = \mathrm{Ind}_{B_1}^B(K')$ where K' is a field. Then $K' \supset K \supset F$ is a Galois tower inducing a surjection $B_1 \to A_1$. We have the following commutative diagram:

$$\begin{array}{ccc} H^i(A, \mu_2) & \longrightarrow & H^i(B, \mu_2) \\ \downarrow & & \downarrow \\ H^i(A_1, \mu_2) & \longrightarrow & H^i(B_1, \mu_2) \end{array}$$

from which it follows that β_1 has limitation B_1. In particular, if β has some limitation B then β presents 0.

If β and L/F presents 0 and L is a field then there is a L'/F Galois extension containing L/F and an associated surjection of Galois groups $B \to A$ such that β presents 0 with limitation B. More generally, if $L = \mathrm{Ind}_{A_1}^A(L_1/F)$ with L_1 a field there is a Galois L_1'/F containing L_1/F and associated surjection $B_1 \to A_1$ such that β maps to 0 in $H^i(B_1, \mu_2)$.

For symmetry, though we do not need this fact, we observe that there is a Galois $L' \supset L \supset F$ and associated surjection $B \to A$ such that β maps to 0 in $H^i(B, \mu_2)$ and so β has limitation B. Write $A = A_1 \oplus A'$ and so $H^1(A, \mu_2) = H^1(A_1, \mu_2) \oplus H^1(A', \mu_2)$. We view any $\alpha \in H^1(A', \mu_2)$ as an element of $H^1(A, \mu_2)$ by setting $\alpha(A_1) = 1$. Write $\beta = \beta_1 + \beta'$ where all the symbols in β_1 have entries from $H^1(A_1, \mu_2)$, all the symbols in β' have have at least one entry from $H^1(A', \mu_2)$. Thus each symbol in β' has a subsymbol $\alpha \cup \alpha'$ which maps to $0 \in H^2(F, \mu_2)$ (since one of the α's is trivial). It follows that we can form $1 \to M \to B' \to A$ where each symbol of β' splits in B' and M is a direct sum of $\mathbb{Z}/2\mathbb{Z}$'s, one for each symbol. Form $B = B_1 \oplus B'$ and $L' = \mathrm{Ind}_{B_1}^B(L_1'/F)$. Note that the order of B is bounded in terms of the order of B_1, i, and the order of A.

Frequently when we specialize as above there will be many ways to do it and this is important. Given a presentation S/R, and $\beta \in H^i(A, \mu_2)$, we say β densely specializes to a presentation β_1 of $\alpha_1 \in H^i(F_1, \mu_2)$ if the following holds. For any $0 \neq r \in R$, there is a $\phi : R \to F_1$ such that $\phi(r) \neq 0$ and ϕ causes β to specialize to β_1 inducing the same presentation β_1. If R and R' are affine C algebras with $q(R) = F = q(R')$, then $R(1/r) = R'(1/r')$ for some $0 \neq r \in R$ and $0 \neq r' \in R'$ (e.g. [Sw] p. 152). Thus when β densely specializes to β_1 this is well defined up to equivalence.

Lemma 4. *Suppose $\beta \in H^i(A, \mu_2)$ and L/F, A are a presentation of* 0 *which densely specializes to β_1, L_1/F_1, A. Then if β has limitation $B \to A$ so does β_1 and in particular β_1 presents 0.*

Proof. Assume $L' \supset L \supset F$ is $B \to A$ Galois and β maps to 0 in $H^i(B, \mu_2)$. If S/R is A Galois and $q(R) = F$, there is a $0 \neq r \in R$ and a $S'/R(1/r)$ which is B Galois, contains $S(1/r)$, with $S' \otimes_{R(1/r)} F = L'$. Choose $\phi : R(1/r) \to F_1$ realizing the specialization and set $\tilde{L}_1' = S' \otimes_\phi F'$. Then $L_1' \supset L_1 \supset F_1$ is $B \to A$ Galois. □

Let $\alpha \in H^i(F, \mu_2)$ have presentation β, $\mathrm{Gal}(L/F)$. Assume $B \to A$ is a surjection of finite groups and β maps to 0 in $H^i(B, \mu_2)$. The following result is routine and we only include the proof for ease of the reader.

Proposition 5. *There is a field extension $F_B \supset F$ with the following properties (where we set $L_B = L \otimes_F F_B$).*

a) The extension L_B/F_B and β presents 0 with limitation B.

b) Suppose β, L/F densely specializes to β_1 and $A_1 = \mathrm{Gal}(K_1/F_1)$. If β_1 has limitation B, then the presentation β, L_B/F_B densely specializes to β_1.

Proof. All this really means is that we are constructing generically the B Galois extension extending L/F. To achieve this let V be the faithful B module $F[B]$. Let B act on the field of fractions $L(V)$ as follows. B acts on V as usual, and B acts on L via $B \to A$. Set $F_B = L(V)^B$. It is clear that $L(V)/F_B$ is B Galois. If N is the kernel of $B \to A$, then $L(V)^N = L_B = L \otimes_F F_B$ so $L(V) \supset L_B \supset F_B$ induces $B \to A$. This proves a).

Let S/R be A Galois such that $q(R) = F$. Then we can choose $t \in S[V]^B$ with the property that if $S_B = S[V](1/t)$ then S_B/R_B is B Galois with $R_B = (S_B)^B$. Suppose $0 \neq s \in S[V]^B$. It suffices to show that there is a $\phi : R_B \to F_1$ with $\phi(s) \neq 0$. By assumption there is a $\phi : R \to F_1$ specializing β to β_1. Set $L_1 = S \otimes_\phi F_1$ which is A Galois over F_1 and has the form $\mathrm{Ind}_{A_1}^A(K_1)$. It follows that ϕ extends to an A morphism $\phi : S \to L_1$. Since β_1 has limitation B, there is a B Galois extension $L_1'/F \supset L_1/F$ inducing $B \to A$. Since V has basis $\{x_g | g \in B\}$ with the obvious action, algebraic independence of Galois group elements (e.g. [BAI] p. 295) shows that we can define $\phi(x_g) = g(a) \in L_1'$ for some a such that $\phi(st) \neq 0$. Then ϕ extends to a B morphism and restricts to the needed ϕ on R_B. □

If β and $B \to A$ are as in Proposition 5, we say that F_B is the generic splitting field of β with limitation B.

Let's outline our argument a bit. We start with a generic quadratic form $\gamma = \sum_{i=1}^N a_i x_i^2$ (N is even) meaning that the ground field has the form $F_1 = C(a_1, \ldots, a_N)$ and the a_i are a transcendence base. Note that for any field $F \supset C$, this specializes to all Witt classes in the fundamental ideal I as long as $u(F) \leq N$. We want to write down a generic element in I^n with a fixed so called history as follows. Let F_2/F_1 be the extension defined by taking the square root of the determinant of γ. The extension γ_2 of γ to the Witt ring $W(F_2)$ is in $I_{F_2}^2$ and so defines an element $\alpha_2 \in H^2(F_2, \mu_2)$. We take F_3/F_2 to be a generic splitting field of α_2 and so the extension, $\gamma_3 \in W(F_3)$ is in $I_{F_3}^3$. So far there has been no limitations. However, if $\alpha_3 \in H^3(F_3, \mu_2)$ is the image of γ_3 then we can write γ_3 as a sum of Pfister forms and thereby write α_3 as a sum of symbols. Given that, we can choose a presentation β_3, $A_3 = \mathrm{Gal}(L_3/F_3)$ of α_3. For any $B_3 \to A_3$ that splits β_3, we form the generic splitting field of β_3 with limitation B_3 and call that F_4. We proceed by induction until the extension, $\gamma_n \in I^n_{F_n}$ is defined. The choice of presentations β_i and limitations B_i is the HISTORY of this construction.

Now given a u bounded field K every element $\alpha' \in H^i(K, \mu_2)$ is the image of a quadratic form γ' which is in I^i_K. We show that we can bound the order and hence number of the limitations which enforce this property of γ''s and hence write α' as the specialization of one of finitely many of the generic contructions of α_n (as above) as we vary the histories among finitely many choices of the B_i. This proves the result.

To make this argument more formal, if β, L/K is a presentation α then the ORDER of β is the order of the group $A = \mathrm{Gal}(L/K)$. Obviously the order of a presentation cannot increase under specialization. We say that a field K is LIMITATION BOUNDED in degree i if and only if for all d, all field extensions K'/K of degree dividing d, and all degree i presentations of zero, β over K' of order less than or equal to N, there is a $L(N, d)$ such that β has a limitation B of order less than or equal to $L(N, d)$. The above argument is an outline of the proof of:

THEOREM 6. *Suppose K is limitation bounded in all degrees $j < d$ and is also u bounded. Then all finite extensions of K have bounded symbol length in degree d and the bound is a function of the degree and the assumed u and limitation bounds.*

Proof. This is perhaps already clear except for the fact we are choosing presentations of zero. For simplicity we only treat K itself, the extension to the K'/K being clear. We prove by induction that every $\gamma' \in I^d_K$ is the specialization of some $\gamma_i \in I^d_{F_d}$ as above with only finitely many choices of histories. By induction there are finitely many histories such that γ' is the specialization some γ_{d-1}. For this γ_{d-1} there is a presentation β_{d-1} and thus a degree $d-1$ presentation of γ' we call β'_{d-1}. Since $\gamma' \in I^d_K$ it follows that β'_{d-1} is a presentation of zero and so there are only finitely many further choices of limitations B_d. We are done by Proposition 5. ◻

Given Theorem 6, we need to prove these u bounded fields are limitation bounded. This is an involved argument using the Hochschild–Serre spectral sequence. Note that we feel that the limitation bound we obtain is far from optimal. For this reason we will not be particularly explicit about the bound, as in the definition of "predictable" below. However, there is a group structure bound in our argument that seems interesting and so we will endeavor to prove it and make it explicit. In fact, let G be a finite group (for us usually abelian). A d-ABELIAN G GROUP is an extension $1 \to N \to G' \to G \to 1$ such that N contains G' normal subgroups $N = N(0) \supset N(1) \supset \ldots \supset N(d) = 1$ with $N(i)/N(i+1)$ abelian. Given any d abelian group we will use obvious modifications of the $N(i)$ notation above to denote the corresponding tower of groups.

Fix a field K with an absolute Galois group G_K. We say G' is a d-abelian G Galois group over K to mean G' is also an image of G_K (so $G' = \mathrm{Gal}(L(d)/K)$ for a field $L(d)$). Set $L(i) = L(d)^{N(i)}$ and $L = L(0)$. Whenever we talk about d abelian G Galois groups we will use the $L(i)$ notation, or obvious variants of it, to indicate the associated tower of fields.

In the course of the proof we will alter G' in several ways. In all cases we will want to construct Galois groups so we will usually construct these further groups via field theory.

If L'/L is abelian with L'/K Galois, then $L'(d)/K = L(d)L'/K$ is Galois with group G'' which is still a d-abelian G group. To see this, set $L(i)' = L(i)L'$ for $i > 0$ and $L'(0) = L(0)$. Then $L'(i)/L'(i-1)$ is abelian Galois with group a subgroup of $N(i-1)/N(i)$ for $i > 1$, but $L'(1)/L$ is abelian with Galois group a subgroup of $N_0/N_1 \oplus \mathrm{Gal}(L'/L)$. We call this EXPANDING the d-abelian G group G'.

Another construction we will need is the following. Suppose $K'(d')/K$ is Galois with d' abelian Galois group so $K'(d') \supset \ldots \supset K'(1) = L(1) \supset L(0) = K'(0) = L \supset K$ the point being here is that the beginning of the series of fields for K' coincides with the beginning for $L(d)$ (and $K'(i)/K'(i-1)$ is, of course, abelian Galois). Let d'' be the maximum of d and d', and set $K'(j) = K'(d')$ for $j > d'$ and similarly $L(j) = L(d)$ for $j > d$. Then if $L'(i) = L(i)K'(i)$ we have that $G'' = \mathrm{Gal}(L'(d'')/K)$ is a d''-abelian G group. Moreover, the first $\mathrm{Gal}(L'(1)/L'(0))$ is unchanged but the rest of the abelian series is larger. We call this REFINING the group G'. Note that expanding G' increases $\mathrm{Hom}(N_0, \mu_2)$ and so increases the cohomology cup products in $H^q(N_0, \mu_2)$. On the other hand, we will see that by refining our d abelian G groups we will introduce more relations among these cup products.

The above two constructions are special cases of the following. If $L(d)/K$ and $L_1(d')/K$ have d and d' abelian G Galois groups G' and G_1 respectively (with the same G Galois L/K) then the amalgamation $L(d)L'(d')/K$ has a d'' abelian G Galois group G'_1 where d'' is the maximum of d and d'. We call G'_1 the AMALGAMATION of G' and G_1.

Suppose $A' = \mathrm{Gal}(L'/L)$ is abelian and we have a d abelian A' Galois group over L, with associated field extensions $L'(d) \supset \ldots \supset L'(0) = L' \supset L$. Let L/K be G Galois as above. Then $L'(d)$ is not Galois over K, but if $L''(d)$ is the Galois closure of $L'(d)$ over K, then $L''(d)$ is the amalgamation of all the G conjugates of $L'(d)$ and so $\mathrm{Gal}(L''(d)/K)$ is a $d+1$ abelian G Galois group over K. We call this EXTENDING the d-abelian A' group to a $d+1$ abelian G group.

As another bit of terminology, if we have d and d_1 abelian G groups with a diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & N_1 & \rightarrow & G'_1 & \rightarrow & G & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \rightarrow & N & \rightarrow & G' & \rightarrow & G & \rightarrow & 1 \end{array}$$

where all vertical arrows are surjective then we say the d_1 abelian G group G'_1 is a COVER of G' and if these maps are induced by field extensions we call it a Galois cover. Clearly, expanding, refining, amalgamating and extending are ways of constructing Galois covers.

When we expand or refine or extend a d-abelian G group G' we say that the size of the new group is PREDICTABLY bounded if the bound is only a function of G', the degrees of the cohomology groups involved, and previously proven symbol length bounds for field extensions of bounded degree. Note how unspecific this notion is. Any function of predictable bounds, or functions of predictable bounds and $|G|$ etc., also would be a predictable bound. For example, when we expand, or refine, or extend or amalgamate predictably bounded groups we get other ones.

Next we need some notation to help us navigate through the complexities of the Hochschild-Serre spectral sequence. We will employ this spectral sequence for sequences $1 \to \bar{N} \to \bar{G} \to G \to 1$ and $1 \to \bar{N}/\bar{N}' \to G' \to G \to 1$ where $\bar{N}/\bar{N}'$ is finite. Of course the natural map defines a morphism from the second spectral sequence to the first. Let me define notation in the first case as extension to the second is obvious.

In this spectral sequence, the $E_2^{p,q}$ term is $H^p(G, H^q(\bar{N}, \mu_2))$. The differential of this spectral sequence is d_r so $d_2 : H^p(G, H^q(\bar{N}, \mu_2)) \to H^{p+2}(G, H^{q-1}(\bar{N}, \mu_2))$. We wish to treat each $E_r^{p,q}$ as a subquotient of $H^p(G, H^q(\bar{N}, \mu_2))$ and so write

$$E_r^{p,q} = H^p(G, H^q(\bar{N}, \mu_2))_r^u / H^p(G, H^q(\bar{N}, \mu_2))_r^l.$$

Thus $H^p(G, H^q(\bar{N}, \mu_2))_2^u = H^p(G, H^q(\bar{N}, \mu_2))$ and $H^p(G, H^q(\bar{N}, \mu_2))_2^l = 0$. Moreover, the differentials d_r can be viewed as morphisms

$$d_r^{p,q} : H^p(G, H^q(\bar{N}, \mu_2))_r^u \to H^{p+r}(G, H^{q-r+1}(\bar{N}, \mu_2))/ H^{p+r}(G, H^{q-r+1}(\bar{N}, \mu_2))_r^l$$

and the kernel of $d_r^{p,q}$ is $H^p(G, H^q(\bar{N}, \mu_2))_{r+1}^u$ while the image of $d_r^{p,q}$ is

$$H^{p+r}(G, H^{q-r+1}(\bar{N}, \mu_2))_{r+1}^l / H^{p+r}(G, H^{q-r+1}(\bar{N}, \mu_2))_r^l.$$

Since all $d_r^{d,0}$ are 0, the kernel of $H^d(G, H^0(\bar{N}, \mu_2)) \to H^d(\bar{G}, \mu_2)$ is the union of all the $H^d(G, \mu_2)_r^l$ for all r.

In the arguments to follow we are going to use the above notions and work through the details of the Hochschild–Serre spectral sequence showing we can stay "predictably bounded" all the way. Since the overall argument is by induction, we will be able to assume the following at the degree d cohomology step:

(*) For all $q < d$ the following two facts hold. First, for all finite extensions of L/K of degree dividing n there is a symbol length bound, $M_q(n)$, for $H^q(L, \mu_2)$ only depending on n. Second, for any abelian extension L'/L of predictably bounded degree and Galois group G_1, and any element $\beta \in H^q(G_1, \mu_2)$ which presents 0, there is a predictably bounded $q-1$ abelian group G_1 Galois group over L written $1 \to N_1 \to G'_1 \to G_1 \to 1$ such that β maps to 0 in $H^q(G'_1, \mu_2)$.

LEMMA 7. *Suppose $1 \to N \to G' \to G \to 1$ is a t-abelian G Galois group over K and $\gamma \in H^p(G, H^q(N, \mu_2))$ maps to 0 in $H^p(G, H^q(\bar{N}, \mu_2))$ where $q < d$. Assume (*). Let q' be the maximum of t and q. Then there is a predictably bounded q' abelian Galois cover of G' where γ maps to 0.*

Proof. The element γ is represented by a p-cocycle $c(g_1, \ldots, g_p) \in H^q(N, \mu_2)$ whose image in $H^q(\bar{N}, \mu_2))$ is the coboundary of a $p-1$ cochain $b(g_1, \ldots, g_{p-1})$. Since each $b(g_1, \ldots, g_{p-1})$ is a sum of $M_q([L : K])$ symbols, there is a 1-abelian G Galois group $1 \to \bar{N}/\bar{N}_1 \to G'' \to G \to 1$ such that the b's are the image of elements of $H^q(\bar{N}/\bar{N}_1, \mu_2)$ where $\bar{N}/\bar{N}_1$ is a predictably bounded elementary abelian 2 group. Expanding G' by this G'', and calling the result G' again, we have that the c's and b's are both in $H^p(G, H^q(N, \mu_2))$. There are $|G|^p$ relations in $H^q(N, \mu_2)$ that must be satisfied in order that the coboundary $\delta(b)$ equals c. Now let $G'' = N/N(1)$ be the abelian group. By assumption, we can iteratively refine the $t-1$ abelian G'' Galois group $1 \to N(1) \to N \to G'' \to 1$ to force these relations, and result is a $q'-1$ abelian G'' Galois group. Extending this to a q' abelian G Galois group we get the cover we need. Note that the size of the new group is again predictably bounded, though the bound is quite large. □

In a similar vein is:

LEMMA 8. *Assume (*). Suppose $p + q < d$ and $\gamma \in H^p(G, H^q(\bar{N}, \mu_2))$. Then there is a q-abelian G Galois group $1 \to N \to G' \to G \to 1$ of predictably bounded order and an element $\gamma' \in H^p(G, H^q(N, \mu_2))$ which inflates to γ.*

Proof. The element γ is represented by a p cocycle $c(g_1, \ldots, g_p)$ consisting of less than or equal to $|G|^p$ elements of $H^q(\bar{N}, \mu_2)$. Each of these elements can be written as a sum of $M_q(|G|)$ symbols so that there are at most $|G|^p M(|G|)$ symbols and hence $|G|^p M_q(|G|)q$ elements of $\mathrm{Hom}(\bar{N}, \mu_2)$ are involved in writing all the $c(g_1, \ldots, g_p)$'s. Said differently, there is a $\bar{N}' \subset \bar{N}$ of index dividing $2^{|G|^p M_q(|G|)q}$ such that all these $c(g_1, \ldots, g_{d-r})$'s are in the image of

$H^q(\bar{N}/\bar{N}', \mu_2)$. Now $\bar{N}'$ is not normal in $\bar{G}$ so we take the intersection of the $|G|$ conjugates of $\bar{N}'$ to define $\bar{N}_1$ such that $\bar{N}_1$ is normal in $\bar{G}$ and all the $c(g_1, \ldots, g_{d-r})$'s come from $c_1(g_1, \ldots, g_p) \in H^q(\bar{N}/\bar{N}_1, \mu_2)$. Note that N/N_1 has order dividing $2^{|G|^{p+1}M(|G|)q}$. Now the c_1's do not form a q cocycle neccessarily, but their image in $H^q(\bar{N}, \mu_2)$ is a cocycle. Being a cocycle means that there are less than or equal to $|G|^{p+1}$ relations that must be satisfied. Since $q - 1 < d$, by assumption (*) there is a $q - 1$ abelian G' group where each of the cocycle relations become true after inflation (of predictably bounded size). By extending we get an q abelian G group for each needed cocycle relation, and we can refine all these together to get an q abelian G group $1 \to N \to G' \to G \to 1$ of predictably bounded size such that the c_1's inflate to an element $\gamma' \in H^p(G, H^q(N, \mu_2))$ which inflates to γ. □

The previous results do not suffice, as we need to show that we can achieve, element by element, the spectral sequence filtration in a predictably bounded cover. This is the next result.

Lemma 9. *Suppose $p + q < d$, and $1 \to N \to G' \to G \to 1$ is a t abelian G Galois group over K and $\gamma' \in H^p(G, H^q(N, \mu_2))$ maps via inflation to an element $\gamma \in H^p(G, H^q(\bar{N}, \mu_2))^u_s$. Let t' be the maximum of t and $q - 1$. Assume (*). Then there is a t' abelian Galois cover $1 \to N_1 \to G_1 \to G \to 1$ of predictably bounded size such that the inflation of γ' in $H^p(G, H^q(N_1, \mu_2))$ lies in $H^p(G, H^q(N_1, \mu_2))^u_s$.*

Proof. We prove this by induction on s. The statement is vacuous for $s = 2$ and by way of illustration the fact that γ' lies in $H^p(G, H^q(N_1, \mu_2))^u_3$ is equivalent to $d_2(\gamma') = 0 \in H^{p+2}(G, H^{q-1}(N_1, \mu_2))$ which (by Lemma 7) we can achieve after refining and thereby creating a predictably bounded t'-abelian G Galois cover where t' is the maximum of t and $q - 1$.

So assume the result for $s - 1$. We need to unpack the meaning when we say $\gamma \in H^p(G, H^q(\bar{N}, \mu_2))^u_s$. By definition, this is equivalent to $d_{s-1}(\gamma) \in H^{p+s-1}(G, H^{q-s+2}(\bar{N}, \mu_2))^l_{s-1}$ or $d_{s-1}(\gamma) - d_{s-2}(\gamma_1) \in H^{p+s-1}(G, H^{q-s+2}(\bar{N}, \mu_2))^l_{s-2}$ where $\gamma_1 \in H^{p+1}(G, H^{q-1}(\bar{N}, \mu_2))^u_{s-2}$. Proceeding by induction we have elements $\gamma_i \in H^{p+i}(G, H^{q-i}(\bar{N}, \mu_2))^u_{s-i-1}$ for bounded i such that $d_{s-1}(\gamma) = \sum_i d_{s-i-1}(\gamma_i) \in H^{p+s-1}(G, H^{q-s+2}(\bar{N}, \mu_2))$. By repeated use of Lemma 8 the γ_i are the image of $\gamma'_i \in H^{p+i}(G, H^{q-i}(N, \mu_2))$ for a $q - 1$ abelian G group G'. By induction we can assume the $\gamma'_i \in H^{p+i}(G, H^{q-i}(N, \mu_2))^u_{s-i-1}$ and one further refinement allows us to assert that $d_{s-1}(\gamma') = \sum_i d_{s-i-1}(\gamma'_i) \in H^{p+s-1}(G, H^{q-s+2}(N, \mu_2))^l_{s-1}$. □

Until now our spectral sequence notation has been unambiguous as to whether we are referring to the absolute sequence $1 \to \bar{N} \to \bar{G} \to G \to 1$ or some finite image of it, but we have to make such a distinction when we deal with $H^d(G, \mu_2)$

which appears in all these spectral sequences. Thus we will let $H^d(G,\mu_2)^l_r$ refer to the filtration induced by the absolute sequence and for the image sequence $1 \to \bar{N}/N' \to G' \to G \to 1$ we will use the notation $H^d_{G'}(G,\mu_2)^l_r$.

Proposition 10. *Assume (*). Suppose $\beta \in H^d(G,\mu_2)^l_{r+1}$ for some r. Then there is a $r-1$ abelian G Galois group $1 \to N \to G' \to G \to 1$ of predictably bounded order such that $\beta \in H^d_{G'}(G,\mu_2)^l_{r+1}$.*

Proof. There is a $\gamma \in H^{d-r}(G,H^{r-1}(\bar{N},\mu_2))^u_r$ which maps to β modulo $H^d(G,\mu_2)^l_r$. Let $\eta = d(\gamma) - \beta \in H^d(G,\mu_2)^l_r$. By induction on r there is a predictably bounded $r-2$ abelian G Galois group $1 \to N_1 \to G_1 \to G \to 1$ such that η is the inflation of $\eta' \in H^d_{G_1}(G,\mu_2)^l_r$. By Lemma 8 there is a predictably bounded $r-1$ abelian G Galois group $1 \to N \to G' \to G \to 1$ such that γ is the inflation of $\gamma' \in H^{d-r}(G,H^{r-1}(N,\mu_2))$. Since $\gamma \in H^{d-r}(G,H^{r-1}(\bar{N},\mu_2))^u_r$ we know from Lemma 9 that there is $r-1$ abelian G Galois group cover of G', which we also call G', such that $\gamma' \in H^{d-r}(G,H^{r-1}(N,\mu_2))^u_r$. We can amalgamate G_1 and G' to get a predictably bounded $r-1$ abelian G Galois group (we also call G') where $d_r(\gamma')$, β and η' are all defined and $\beta = d_r(\gamma') - \eta'$ so $\beta \in H^d_{G'}(G,\mu_2)^l_{r+1}$. □

Corollary 11. *Suppose $\beta \in H^d(G,\mu_2)$ maps to 0 in $H^d(\bar{G},\mu_2)$. Assume (*). Then there is a predictably bounded $d-1$ abelian G Galois group such that β maps to 0 in $H^d(G',\mu_2)$.*

Proof. In the spectral sequence the last nontrivial derivation with image $H^d(G,\mu_2)$ is $d_d : H^0(G,H^{d-1}(\bar{N},\mu_2))^u_d \to H^d(G,\mu_2)/H^d(G,\mu_2)^l_d$. That is, the ascending tower in $H^d(G,\mu_2)$ stabilizes at $H^d(G,\mu_2)^l_{d+1}$. The result follows from Proposition 10. □

Now we are in a position to prove Theorem 1, and we do it by noting it is a part of the following.

Theorem 12. *Let K have finite u invariant. Let K'/K be a any field extension of degree dividing n. For all degrees d, there is a symbol bound for $H^d(K',\mu_2)$ that only depends on n. Also, there is a limitation bound for K' that only depends on n. Moreover, given a presentation of zero $\beta \in H^d(G,\mu_2)$ and $G = \mathrm{Gal}(L'/K')$, this limitation bound is realized by $d-1$ abelian G Galois groups.*

Proof. As we have said all along, we prove this by induction and so we assume this statement for all degrees $j < d$. By Theorem 6 we have the symbol boundedness of K' in degree d. By Corollary 11, we have the limitation bound in degree d. □

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