Note on the Counterexamples for the Integral Tate Conjecture over Finite Fields

Alena Pirutka and Nobuaki Yagita

Received: March 17, 2014

Abstract. In this note we discuss some examples of non-torsion and non-algebraic cohomology classes for varieties over finite fields. The approach follows the construction of Atiyah-Hirzebruch and Totaro.

A.S. Merkurjev die natali oblatum.

2010 Mathematics Subject Classification: Primary 14C15; Secondary 14L30, 55R35
Keywords and Phrases: Chow groups, classifying spaces, cohomology operations, integral Tate conjecture

1. Introduction

Let k be a finite field and let X be a smooth and projective variety over k. Let ℓ be a prime, $\ell \neq char(k)$. The Tate conjecture [20] predicts that the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_H H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^H,$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$, is surjective.
In the integral version one is interested in the cokernel of the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^H. \tag{1.1}$$

This map is not surjective in general: the counterexamples of Atiyah-Hirzebruch [1], revisited by Totaro [21], to the integral version of the Hodge conjecture, provide also counterexamples to the integral Tate conjecture [3]. More precisely, one constructs an ℓ-torsion class in $H^4_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is not algebraic, for some smooth and projective variety X. However, one then wonders if there exists an example of a variety X over a finite field, such that the map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^{2i}_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^H/torsion \tag{1.2}$$

is not surjective ([13, 3]). In the context of an integral version of the Hodge conjecture, Kollár [12] constructed such examples of curve classes. Over a finite field, Schoen [18] has proved that the map (1.2) is always surjective for curve classes, if the Tate conjecture holds for divisors on surfaces.
In this note we follow the approach of Atiyah-Hirzebruch and Totaro and we produce examples where the map (1.2) is not surjective for $\ell = 2, 3$ or 5.

Theorem 1.1. *Let ℓ be a prime from the following list: $\ell = 2, 3$ or 5. There exists a smooth and projective variety X over a finite field k, $char k \neq \ell$, such that the cycle class map*

$$CH^2(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^4_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^H/torsion,$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$, is not surjective.

As in the examples of Atiyah-Hirzebruch and Totaro, our counterexamples are obtained as a projective approximation of the cohomology of classifying spaces of some simple simply connected groups, having ℓ-torsion in its cohomology. The non-algebraicity of a cohomology class is obtained by means of motivic cohomology operations: the operation Q_1 always vanishes on the algebraic classes and one establishes that it does not vanish on some class of degree 4. This is discussed in section 2. Next, in section 3 we investigate some properties of classifying spaces in our context and finally, following a suggestion of B. Totaro, we construct a projective variety approximating the cohomology of these spaces in small degrees in section 4.

Acknowledgements. This work has started during the Spring School and Workshop on Torsors, Motives and Cohomological Invariants in the Fields Institute, Toronto, as a part of a Thematic Program on Torsors, Nonassociative Algebras and Cohomological Invariants (January-June 2013), organized by V. Chernousov, E. Neher, A. Merkurjev, A. Pianzola and K. Zainoulline. We would like to thank the organizers and the Institute for their invitation, hospitality and support. We are very grateful to B. Totaro for his interest and for generously communicating his construction of a projective algebraic approximation in theorem 1.1. The first author would like to thank B. Kahn and J. Lannes for useful discussions. We are also very grateful to the referee for the comments and corrections.

2. Motivic version of Atiyah-Hirzebruch arguments, revisited

2.1. Operations. Let k be a perfect field with $char(k) \neq \ell$ and let $\mathcal{H}.(k)$ be the motivic homotopy category of pointed k-spaces (see [15]). For $X \in \mathcal{H}.(k)$,

denote by $H^{*,*'}(X,\mathbb{Z}/\ell)$ the motivic cohomology groups with $\mathbb{Z}/\ell$-coefficients (*loc.cit.*). If X is a smooth variety over k (viewed as an object of $\mathcal{H}.(k)$), note that one has an isomorphism $CH^*(X)/\ell \xrightarrow{\sim} H^{2*,*}(X,\mathbb{Z}/\ell)$.
Voevodsky ([23], see also [17]) defined the reduced power operations P^i and the Milnor's operations Q_i on $H^{*,*'}(X,\mathbb{Z}/\ell)$:

$$P^i : H^{*,*'}(X,\mathbb{Z}/\ell) \to H^{*+2i(\ell-1),*'+i(\ell-1)}(X,\mathbb{Z}/\ell), i \geq 0$$

$$Q_i : H^{*,*'}(X,\mathbb{Z}/\ell) \to H^{*+2\ell^i-1,*'+(\ell^i-1)}(X,\mathbb{Z}/\ell), i \geq 0,$$

where $Q_0 = \beta$ is the Bockstein operation of degree $(1,0)$ induced from the short exact sequence $0 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$.
One of the key ingredients for this construction is the following computation of the motivic cohomology of the classifying space $B_{\acute{e}t}\mu_\ell \in \mathcal{H}.(k)$:

Lemma 2.1. *([23, §6]) For each object $X \in \mathcal{H}.(k)$, the graded algebra $H^{*,*'}(X \times B_{\acute{e}t}\mu_\ell, \mathbb{Z}/\ell)$ is generated over $H^{*,*'}(X,\mathbb{Z}/\ell)$ by elements x and y, $deg(x) = (1,1)$ and $deg(y) = (2,1)$, with $\beta(x) = y$ and* $x^2 = \begin{cases} 0 & \ell \text{ is odd} \\ \tau y + \rho x & \ell = 2 \end{cases}$
where τ is a generator of $H^{0,1}(Spec(k),\mathbb{Z}/2) \cong \mu_2$ and ρ is the class of (-1) in $H^{1,1}(Spec(k),\mathbb{Z}/2) \simeq k^*/(k^*)^2$.

For what follows, we assume that k contains a primitive ℓ^2-th root of unity ξ, so that $B_{\acute{e}t}\mathbb{Z}/\ell \xrightarrow{\sim} B_{\acute{e}t}\mu_\ell$ and $\beta(\tau) = \xi^\ell$ $(= \rho$ for $p = 2)$ is zero in $k^*/(k^*)^\ell = H^{1,1}_{\acute{e}t}(Spec(k);\mathbb{Z}/\ell)$.

We will need the following properties:

Proposition 2.2. *Let $X \in \mathcal{H}.(k)$.*

(i) $P^i(x) = 0$ *for $i > m-n$ and $i \geq n$ and $x \in H^{m,n}(X,\mathbb{Z}/\ell)$;*
(ii) $P^i(x) = x^\ell$ *for $x \in H^{2i,i}(X,\mathbb{Z}/\ell)$;*
(iii) *if X is a smooth variety over k, the operation*

$$Q_i : CH^m(X)/\ell = H^{2m,m}(X,\mathbb{Z}/\ell) \to H^{2m+2\ell^i-1,m+(\ell^i-1)}(X,\mathbb{Z}/\ell)$$

is zero ;
(iv) $Op.(\tau x) = \tau Op.(x)$ *for $Op. = \beta, Q_i$ or P^i;*
(v) $Q_i = [P^{\ell^{i-1}}, Q_{i-1}]$.

Proof. See [23, §9]. For (iii) one uses that $H^{m,n}(X,\mathbb{Z}/\ell) = 0$ if $m - 2n > 0$ and X is a smooth variety over k, (iv) follows from the Cartan formula for the motivic cohomology.

2.2. Computations for $B_{ét}\mathbb{Z}/\ell$. The computations in this section are similar to [1, 21, 22].

Lemma 2.3. *In $H^{*,*'}(B_{ét}\mathbb{Z}/\ell, \mathbb{Z}/\ell)$, we have $Q_i(x) = y^{\ell^i}$ and $Q_i(y) = 0$.*

Proof. By definition $Q_0(x) = \beta(x) = y$. Using induction and Proposition 2.2, we compute

$$Q_i(x) = P^{\ell^{i-1}}Q_{i-1}(x) - Q_{i-1}P^{\ell^{i-1}}(x) = P^{\ell^{i-1}}Q_{i-1}(x) \\ = P^{\ell^{i-1}}(y^{\ell^{i-1}}) = y^{\ell^i}.$$

Then $Q_1(y) = -Q_0P^1(y) = -\beta(y^\ell) = 0$. For $i > 1$, using induction and Proposition 2.2 again, we conclude that $Q_i(y) = -Q_{i-1}P^{\ell^{i-1}}(y) = 0$. □

Let $G = (\mathbb{Z}/\ell)^3$. As above, we view $B_{ét}G$ as an object of the category $\mathcal{H}.(k)$ and we assume that k contains a primitive ℓ^2-th root of unity. From Lemma 2.1, we have an isomorphism of modules over $H^{*,*'}(Spec(k), \mathbb{Z}/\ell)$:

$$H^{*,*'}(B_{ét}G, \mathbb{Z}/\ell) \cong H^{*,*'}(Spec(k), \mathbb{Z}/\ell)[y_1, y_2, y_3] \otimes \Lambda(x_1, x_2, x_3)$$

where $\Lambda(x_1, x_2, x_3)$ is isomorphic to the $\mathbb{Z}/\ell$-module generated by 1 and $x_{i_1}...x_{i_s}$ for $i_1 < ... < i_s$, with relations $x_ix_j = -x_jx_i$ $(i \leq j)$, $\beta(x_i) = y_i$ and $x_i^2 = \tau y_i$ for $\ell = 2$.

Lemma 2.4. *Let $x = x_1x_2x_3$ in $H^{3,3}(B_{ét}G, \mathbb{Z}/\ell)$. Then*

$$Q_iQ_jQ_k(x) \neq 0 \in H^{2*,*}(B_{ét}G, \mathbb{Z}/\ell) \quad for\ i < j < k.$$

Proof. Using Proposition 2.2(v) and Cartan formula for the operations on cup-products ([23] Proposition 9.7 and Proposition 13.4), we first get $Q_k(x) = y_1^{\ell^k}x_2x_3 - y_2^{\ell^k}x_1x_3 + y_3^{\ell^k}x_1x_2$ and one then deduces

$$Q_iQ_jQ_k(x) = \sum_{\sigma \in S_3} \pm y_{\sigma(1)}^{\ell^k} y_{\sigma(2)}^{\ell^j} y_{\sigma(3)}^{\ell^i} \neq 0 \in \mathbb{Z}/\ell[y_1, y_2, y_3].$$

□

3. exceptional Lie groups

Let (G, ℓ) be a simple simply connected Lie group and a prime number from the following list:

$$(G, \ell) = \begin{cases} G_2, \ell = 2, \\ F_4, \ell = 3, \\ E_8, \ell = 5. \end{cases} \tag{3.1}$$

Then G is 2-connected and we have $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ for its (singular) cohomology group in degree 3. Hence BG, viewed as a topological space, is 3-connected and $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$ (see [14] for example). We write $x_4(G)$ for a generator of $H^4(BG, \mathbb{Z})$.

Given a field k with $char(k) \neq \ell$, let us denote by G_k the (split) reductive algebraic group over k corresponding to the Lie group G.
The Chow ring $CH^*(BG_k)$ has been defined by Totaro [22]. More precisely, one has

$$BG_k = \varinjlim(U/G_k), \tag{3.2}$$

where $U \subset W$ is an open set in a linear representation W of G_k, such that G_k acts freely on U. One can then identify $CH^i(BG_k)$ with the group $CH^i(U/G_k)$ if $\mathrm{codim}_W(W \setminus U) > i$, the group $CH^i(BG_k)$ is then independent of a choice of such U and W. Similarly, one can define the étale cohomology groups $H^i_{ét}(BG_k, \mathbb{Z}_\ell(j))$ and the motivic cohomology groups $H^{*,*'}(BG_k, \mathbb{Z}/\ell)$ (see [8]), the latter coincide with the motivic cohomology groups of $B_{ét}G$ as in [15] (cf. [8, Proposition 2.29 and Proposition 3.10]). We also have the cycle class map

$$cl : CH^*(BG_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_H H^{2*}_{ét}(BG_{\bar{k}}, \mathbb{Z}_\ell(*))^H, \tag{3.3}$$

where the union is over all open subgroups H of $Gal(\bar{k}/k)$.
The following proposition is known.

Proposition 3.1. *Let (G, ℓ) be a group and a prime number from the list (3.1). Then*

(i) *the group G has a maximal elementary non toral subgroup of rank 3:*
$$i : A \simeq (\mathbb{Z}/\ell)^3 \subset G;$$
(ii) *$H^4(BG, \mathbb{Z}/\ell) \simeq \mathbb{Z}/\ell$, generated by the image x_4 of the generator $x_4(G)$ of $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$;*
(iii) *$Q_1(i^*x_4) = Q_1Q_0(x_1x_2x_3)$, in the notations of Lemma 2.4. In particular, $Q_1(i^*x_4)$ is nonzero.*

Proof. For (i) see [5], for the computation of the cohomology groups with $\mathbb{Z}/\ell$-coefficients in (ii) see [14] VII 5.12; (iii) follows from [11] for $\ell = 2$ and [9, Proposition 3.2] for $\ell = 3, 5$ (see [10] as well). The class $Q_1(i^*x_4)$ is nonzero by Lemma 2.4 (see also [8, Théorème 4.1]). $\square$

4. Algebraic approximation of BG

Write

$$BG_k = \varinjlim(U/G_k) \tag{4.1}$$

as in the previous section. Using proposition 3.1 and a specialization argument, we will first construct a quasi-projective algebraic variety X over a finite field k as a quotient $X = U/G_k$ (where $codim_W(W \setminus U)$ is big enough), such that the cycle class map (1.2) is not surjective for such X. However, if one is interested only in quasi-projective counterexamples for the surjectivity of the map (1.2), one can produce more naive examples, for instance as a complement of some smooth hypersurfaces in a projective space. Hence we are interested to find an approximation of Chow groups and the étale cohomology of $BG_{\bar{k}}$ as a smooth and projective variety. In the case when the group G is finite, this is done in [3, Théorème 2.1]. In this section we give such an approximation for the groups we consider here, this construction is suggested by B. Totaro. We will proceed in three steps. First, we construct a quasi-projective approximation in a family parametrized by $Spec\,\mathbb{Z}$. Then, for the geometric generic fibre we produce a projective approximation, by a topological argument. We finish the construction by specialization.

Let G be a compact Lie group as in (3.1). Let $\mathcal{G}$ be a split reductive group over $Spec\,\mathbb{Z}$ corresponding to G, such a group exists by [SGA3] XXV 1.3.

Lemma 4.1. *For any fixed integer $s \geq 0$ there exists a projective scheme $\mathcal{Y}/\mathrm{Spec}\,\mathbb{Z}$ and an open subscheme $\mathcal{W} \subset \mathcal{Y}$ such that*

(i) *$W \to \mathrm{Spec}\,\mathbb{Z}$ is smooth and the complement of $\mathcal{W}$ is of codimension at least s in each fiber of $\mathcal{Y} \to \mathrm{Spec}\,\mathbb{Z}$;*

(ii) *for any point $t \in \mathrm{Spec}\,\mathbb{Z}$ with residue field $\kappa(t)$ there is a natural map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ inducing an isomorphism*

$$H^i_{\acute{e}t}(\mathcal{W}_{\bar{t}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{\acute{e}t}(B(\mathbb{G}_m \times \mathcal{G})_{\bar{t}}, \mathbb{Z}_\ell) \text{ for } i \leq s, \ell \neq char\,\kappa(t). \tag{4.2}$$

Proof. Write $T = Spec\,\mathbb{Z}$, as it is an affine scheme of dimension 1, we can embed $\mathcal{G}$ as a closed subgroup of $\mathcal{H} = GL_{d,T}$ for some d (see [SGA3] VI_B 13.2). Moreover, it induces an embedding $\mathcal{G} \hookrightarrow PGL_{d,T}$, as the center of $\mathcal{G}$ is trivial for groups we consider here.
By a construction of [22, Remark 1.4] and [2, Lemme 9.2], there exists $n > 0$, a linear $\mathcal{H}$-representation $\mathcal{O}_T^{\oplus n}$ and an $\mathcal{H}$-invariant open subset $\mathcal{U} \subset \mathcal{O}_T^{\oplus n}$, which one can assume flat over T, such that the action of $\mathcal{H}$ is free on $\mathcal{U}$. Let $\mathcal{V}_N = \mathcal{O}_T^{\oplus Nn}$. Then the group $PGL_{n,T}$ acts on $\mathbb{P}(\mathcal{V}_N)$ and, taking N sufficiently large, one can assume that the action is free outside a subset S of high codimension (with respect to s).
By restriction, the group $\mathcal{G}$ acts on $\mathbb{P}(\mathcal{V}_N)$ as well, let $\mathcal{Y} = \mathbb{P}(\mathcal{V}_N)//\mathcal{G}$ be the GIT quotient for this action [16, 19]. The scheme $\mathcal{Y}$ is projective over T and we fix an embedding $\mathcal{Y} \subset \mathbb{P}^M_T$. Let

$$f : \mathcal{W} \to T \tag{4.3}$$

be the open set of $\mathcal{Y}$ corresponding to the quotient of the open set $\mathcal{U}$ as above where $\mathcal{G}_T$ acts freely. From the construction, one can assume that $\mathcal{W}$ has codimension at least s in $\mathcal{Y}$ in each fibre over T.
For any point $t \in T$ the fibre $\mathcal{W}_t$ is a smooth quasi-projective variety and if N is big enough, we have isomorphisms (cf. p. 263 in [22])

$$\mathcal{W}_t \cong (\mathbb{P}(\mathcal{V}_N) - S)_t/\mathcal{G}_t \cong ((\mathcal{V}_N - \{0\})/\mathbb{G}_m - S)_t)/\mathcal{G}_t \cong (\mathcal{V}_N - S')_t/(\mathbb{G}_m \times \mathcal{G})_t$$

where $S' = pr^{-1}S \cup \{0\}$ for the projection $pr : (\mathcal{V}_N - \{0\}) \to \mathbb{P}(\mathcal{V}_N)$. Hence we have isomorphisms

$$H^i_{\acute{e}t}(\mathcal{W}_{\bar{t}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{\acute{e}t}(B(\mathbb{G}_m \times \mathcal{G})_{\bar{t}}, \mathbb{Z}_\ell) \text{ for } i \leq s, \ell \neq char\,\kappa(b),$$

induced by a natural map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ from the presentation (4.1). $\square$

Remark 4.2. More generally, in the statement above the map $\mathcal{W}_t \to B(\mathbb{G}_m \times \mathcal{G})_t$ induces an isomorphism $H^i_{\acute{e}t}(\mathcal{W}_F, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{\acute{e}t}(B(\mathbb{G}_m \times \mathcal{G})_F, \mathbb{Z}_\ell)$, $i \leq s, \ell \neq char\, \kappa(t)$ for any F-point of T over t.

Lemma 4.3. *Let $Y \subset \mathbb{P}^M_{\mathbb{C}}$ be a projective variety over $\mathbb{C}$ and let $W \subset Y$ be a dense open in Y. Assume that W is smooth. Then for a general linear subspace L in $\mathbb{P}^M$ of codimension equal to $1 + dim(Y - W)$, the scheme $X = L \cap W$ is smooth and projective and the natural maps $H^i(W, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ are isomorphisms for $i < dim\, X$.*

Proof. We apply a version of the Lefschetz hyperplane theorem for quasi-projective varieties, established by Hamm (as a special case of Theorem II.1.2 in [4]): for $V \subset \mathbb{P}^M$ a closed complex subvariety of dimension d, not necessarily smooth, $Z \subset V$ a closed subset, and H a hyperplane in $\mathbb{P}^M$, if $V - (Z \cup H)$ is local complete intersection (e.g. $V - Z$ is smooth) then

$$\pi_i((V - Z) \cap H) \to \pi_i(V - Z)$$

is an isomorphism for $i < d - 1$ and surjective for $i = d - 1$. In particular, $H^i(V - Z, \mathbb{Z}) \to H^i((V - Z) \cap H, \mathbb{Z})$ is an isomorphism for $i < d - 1$ and surjective for $i = d - 1$ by the Whitehead theorem.
Applying this statement to W and to successive intersections of W with linear forms defining L, we then deduce that $H^i(W, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i < dim\, X$.

$\square$

Proposition 4.4. *Let G be a compact Lie group as in (3.1).*
For all but finitely many primes p there exists a smooth and projective variety X_k over a finite field k with $char\, k = p$, an element $x_{4,\bar{k}} \in H^4_{\acute{e}t}(B(\mathbb{G}_m \times G_{\bar{k}}), \mathbb{Z}_\ell(2))$, invariant under the action of $Gal(\bar{k}/k)$ and a map $\iota : X_k \to B(\mathbb{G}_m \times G_k)$ in the category $\mathcal{H}.(k)$ such that

(i) *$\alpha_{\bar{k}} = \iota^* x_{4,\bar{k}}$ is a nonzero class in $H^4_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(2))/torsion$;*
(ii) *the operation $Q_1(\bar{\alpha}_{\bar{k}})$ is nonzero, where we write $\bar{\alpha}_{\bar{k}}$ for the image of $\alpha_{\bar{k}}$ in $H^4_{\acute{e}t}(X_{\bar{k}}, \mu_\ell^{\otimes 2})$.*

Proof. Let $\mathcal{W} \subset \mathcal{Y} \subset \mathbb{P}^M_{\mathbb{Z}}$ be as in Lemma 4.1 for $s \geq 4$.
Let $Y = \mathcal{Y}_{\mathbb{C}}$ and $W = \mathcal{W}_{\mathbb{C}}$ be the geometric generic fibres of $\mathcal{Y}$ and $\mathcal{W}$. Consider a general linear space L in $\mathbb{P}^M$ of codimension equal to $1 + dim(Y - W)$. We deduce from Lemma 4.3 above, that the variety $X := L \cap W$ is smooth and projective, and

$$H^i(X, R) \simeq H^i(B(\mathbb{G}_m \times G), R) \text{ for } i \leq s \text{ and } R = \mathbb{Z} \text{ or } \mathbb{Z}/n. \tag{4.4}$$

Hence $H^i_{\acute{e}t}(X, \mathbb{Z}/n) \simeq H^i_{\acute{e}t}(B(\mathbb{G}_m \times G), \mathbb{Z}/n), i \leq s$. In particular, by functoriality of the isomorphisms $H^i_{\acute{e}t}(\cdot, \mathbb{Z}/n) \simeq H^i_{\acute{e}t}(\cdot, \mu_n^{\otimes j}), i \leq s, j > 0$, for $\cdot = X$ and

$B(\mathbb{G}_m \times G)$, we get

$$H^i_{\acute{e}t}(X, \mu_n^{\otimes j}) \simeq H^i_{\acute{e}t}(B(\mathbb{G}_m \times G), \mu_n^{\otimes j}), i \leq s. \tag{4.5}$$

We can assume that we have an isomorphism as above for $i = 4$ and $i = 2\ell + 3$. Note that the cohomology of BG is a direct factor in the cohomology of $B(\mathbb{G}_m \times G)$ (cf. [8, Lemme 2.23]). Using Proposition 3.1, we then get an element $x_{4,\mathbb{C}}$ generating a direct factor isomorphic to $\mathbb{Z}_\ell$ in the cohomology group $H^4_{\acute{e}t}(B(\mathbb{G}_m \times G), \mathbb{Z}_\ell(2))$. Denote $\alpha_{\mathbb{C}}$ its image in $H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2))$.

We can now specialize the construction above to obtain the statement over a finite field. Note that one can assume that L is defined over $\mathbb{Q}$. One can then find an open $T' \subset \operatorname{Spec} \mathbb{Z}$ and a linear space $\mathcal{L} \subset \mathbb{P}^M_{T'}$ such that $\mathcal{L}_{\mathbb{C}} \simeq L$ and such that for any $t \in T'$ the fibre $\mathcal{X}_t$ of $\mathcal{X} = \mathcal{L} \cap \mathcal{T}$ is smooth. After passing to an étale cover T'' of T', one can assume that the inclusion $(\mathbb{Z}/\ell)^3 \subset G_{\mathbb{C}}$ from proposition 3.1 extends to an inclusion $i : \mathcal{A} = (\mathbb{Z}/\ell)^3_{T''} \hookrightarrow \mathcal{G}_{T''}$ (cf. [SGA3] XI.5.8).
Let $t \in T''$ and let $k = \kappa(t)$. As the schemes $\mathcal{X}_{T''}$, $\mathcal{W}_{T''}$ and $\mathcal{U}/\mathcal{A}$ are smooth over T'', we have the following commutative diagram, where the vertical maps are induced by the specialization maps (cf. [SGA4 1/2] Arcata V.3):

$$\begin{tikzcd}
H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2)) \arrow[d] & H^4_{\acute{e}t}(W, \mathbb{Z}_\ell(2)) \arrow[l] \arrow[r] \arrow[d] & H^4_{\acute{e}t}(\mathcal{U}_{\mathbb{C}}/(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) \arrow[d] & H^4_{\acute{e}t}(B(\mathbb{Z}/\ell)^3_{\mathbb{C}}, \mathbb{Z}/\ell) \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2)) & H^4_{\acute{e}t}(\mathcal{W}_{\bar{k}}, \mathbb{Z}_\ell(2)) \arrow[l] \arrow[r] & H^4_{\acute{e}t}(\mathcal{U}_{\bar{k}}/(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & H^4_{\acute{e}t}(B(\mathbb{Z}/\ell)^3_{\bar{k}}, \mathbb{Z}/\ell) \arrow[l, "\simeq"']
\end{tikzcd}$$

The left vertical map is an isomorphism since $\mathcal{X}$ is proper, by a smooth-proper base change theorem. Hence we get a class $\alpha_{\bar{k}} \in H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2))$, corresponding to $\alpha_{\mathbb{C}} \in H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2))$. The map $H^4_{\acute{e}t}(W, \mathbb{Z}_\ell(2)) \to H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2))$ is an isomorphism by Lemma 4.3, so that $\alpha_{\bar{k}}$ comes from an element $x_{4,\bar{k}} \in H^4_{\acute{e}t}(\mathcal{W}_{\bar{k}}, \mathbb{Z}_\ell(2))$. Let $\bar{\alpha}_{\mathbb{C}} \in H^4_{\acute{e}t}(X, \mu_\ell^{\otimes 2})$ be the image of $\alpha_{\mathbb{C}}$ and let $\bar{\alpha}_{\bar{k}} \in H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mu_\ell^{\otimes 2})$ be the image of $\alpha_{\bar{k}}$. As the operation Q_1 commutes with the isomorphisms $H^i_{\acute{e}t}(X, \mathbb{Z}/\ell) \to H^i_{\acute{e}t}(X, \mu_\ell^{\otimes j})$, we get $Q_1(\bar{\alpha}_{\mathbb{C}}) \neq 0$ by proposition 3.1. The étale cohomology operation Q_1 also commutes with the specialization maps (cf. [7]), since these maps are obtained as composite of the natural maps $\phi \circ \psi^{-1}$ on the étale cohomology groups with torsion coefficients $H^i_{\acute{e}t}(X_{\mathbb{C}}) \stackrel{\psi}{\leftarrow} H^i_{\acute{e}t}(\mathcal{X}_S) \stackrel{\phi}{\to} H^i_{\acute{e}t}(\mathcal{X}_{\bar{k}})$, where S is the strict henselization of T'' at t and ϕ is an isomorphism since $\mathcal{X}$ is smooth. Hence $Q_1(\bar{\alpha}_{\bar{k}})$ is nonzero as well. From the construction, the class $\alpha_{\bar{k}}$ generates a subgroup of $H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is a direct factor isomorphic to $\mathbb{Z}_\ell$, and is Galois-invariant. Letting $X_k = \mathcal{X}_k$ this finishes the proof of the proposition.

□

Remark 4.5. For the purpose of this note, the proposition above is enough. See also [6] for a a general statement on a projective approximation of the cohomology of classifying spaces.

Theorem 1.1 now follows from the proposition above:

Proof of theorem 1.1.
For k a finite field and X_k as in the proposition above, we find a nontrivial class $\alpha_{\bar{k}}$ in its cohomology in degree 4 modulo torsion, which is not annihilated by the operation Q_1. This class cannot be algebraic by proposition 2.2(iii). □

Remark 4.6. We can also adapt the arguments of [3, Théorème 2.1] to produce projective examples with higher torsion non-algebraic classes, while in *loc.cit.* one constructs ℓ-torsion classes. Let $G(n)$ be the finite group $G(\mathbb{F}_{\ell^n})$, so that we have

$$\varprojlim H^*_{\acute{e}t}(BG(n), \mathbb{Z}_\ell) = H^*_{\acute{e}t}(BG_{\bar{k}}, \mathbb{Z}_\ell).$$

Then, following the construction in *loc.cit.* one gets

> *For any $n > 0$, there exists a positive integer i_n and a Godeaux-Serre variety $X_{n,\bar{k}}$ for the finite group $G(i_n)$ such that*
> (1) *there is an element $x \in H^4_{\acute{e}t}(X_{n,\bar{k}}; \mathbb{Z}_\ell(2))$ generating $\mathbb{Z}/\ell^{n'}$ for some $n' \geq n$;*
> (2) *x is not in the image of the cycle class map (1.1).*

References

[1] M. F. Atiyah, F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology 1 (1962), 25 – 45.

[2] J.-L. Colliot-Thélène et J.-J. Sansuc, *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*, Algebraic groups and homogeneous spaces, 113–186, Tata Inst. Fund. Res. Stud. Math., 19, Tata Inst. Fund. Res., Mumbai, 2007.

[3] J.-L. Colliot-Thélène et T. Szamuely, *Autour de la conjecture de Tate à coefficients $\mathbb{Z}_\ell$ sur les corps finis*, The Geometry of Algebraic Cycles (ed. Akhtar, Brosnan, Joshua), AMS/Clay Institute Proceedings (2010), 83–98.

[4] M. Goresky and R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 14. Springer-Verlag, Berlin, 1988.

[5] R. L. Griess, *Elementary abelian p-subgroups of algebraic groups*, Geom. Dedicata 39 (1991), no. 3, 253–305.

[6] T. Ekedahl, *Approximating classifying spaces by smooth projective varieties*, arXiv:0905.1538.

[7] D. B. A. Epstein, *Steenrod operations in homological algebra,* Invent. Math. 1 1966, 152–208.

[8] B. Kahn et T.-K.-Ngan Nguyen, *Modules de cycles et classes non ramifiées sur un espace classifiant*, arXiv:1211.0304.

[9] M. Kameko and N. Yagita, *Chern subrings*, Proc. Amer. Math. Soc. 138 (2010), no. 1, 367–373.

[10] R. Kane and D. Notbohn, *Elementary abelian p-subgroups of Lie groups*, Publ. Res. Inst. Math. Sci. 27 (1991), no. 5, 801–811.

[11] M. Kameko, M. Tezuka and N. Yagita, *Coniveau spectral sequences of classifying spaces for exceptional and Spin groups*, Math. Proc. Cambridge Phil. Soc. 98 (2012), 251–278.

[12] J. Kollár, *In Trento examples*, in *Classification of irregular varieties*, edited by E. Ballico, F. Catanese, C. Ciliberto, Lecture Notes in Math. 1515, Springer (1990).

[13] J. S. Milne, *The Tate conjecture over finite fields (AIM talk)*, 2007.

[14] M. Mimura and H. Toda, *Topology of Lie groups*, I and II, Translations of Math. Monographs, Amer. Math. Soc, 91 (1991).

[15] F.Morel and V.Voevodsky, $\mathbb{A}^1$*-homotopy theory of schemes*, Publ.Math. IHES, 90 (1999), 45–143.

[16] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34 Springer-Verlag, Berlin-New York 1965.

[17] J. Riou, *Opérations de Steenrod motiviques*, preprint, 2012.

[18] Ch. Schoen, *An integral analog of the Tate conjecture for one-dimensional cycles on varieties over finite fields*, Math. Ann. 311 (1998), no. 3, 493-500.

[19] C.S. Seshadri, *Geometric reductivity over arbitrary base*, Adv. Math., 26 (1977) 225–274.

[20] J. Tate, *Algebraic cycles and poles of zeta functions*, Arithmetical algebraic geometry (Proc. Conf. Purdue Univ. 1963), 93 – 110, Harper and Row, New York (1965).

[21] B. Totaro, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc. 10 (1997), no. 2, 467–493.

[22] B. Totaro, *The Chow ring of a classifying space*, in "Algebraic K-theory", ed. W. Raskind and C. Weibel, Proceedings of Symposia in Pure Mathematics, 67, American Mathematical Society (1999), 249–281.

[23] V.Voevodsky, *Reduced power operations in motivic cohomology*, Publ. Math. IHES 98 (2003),1–57.

[SGA3] M. Demazure et A. Grothendieck, *Schémas en groupes*, Séminaire de Géométrie Algébrique du Bois Marie SGA 3, Lecture Notes in Math. 151, 152, 153, Springer, Berlin-Heildelberg-New York, 1977, réédition Tomes I, III, Publications de la SMF, Documents mathématiques 7, 8 (2011).

[SGA4 1/2] P. Deligne, *Cohomologie étale*, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2. Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin-New York, 1977.

Alena Pirutka
Centre de Mathématiques
Laurent Schwartz
UMR 7640 de CNRS
École Polytechnique
91128 Palaiseau
France
alena.pirutka
@polytehcnique.edu

Nobuaki Yagita
Department of Mathematics
Faculty of Education
Ibaraki University
Mito
Ibaraki
Japan
yagita@mx.ibaraki.ac.jp