Divisible Abelian Groups are Brauer Groups

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It is well known that the Brauer group of a field is an abelian torsion group. Examples where the Brauer group of a field can be explicitly computed show that this group is close to being divisible. However, for a long time there was not a single known example of an abelian torsion group A such that $A \not\simeq \mathrm{Br}(F)$ for any field F. First examples of this type were constructed in [3], where it was shown that for $p = 2$ or 3 the p-component of the Brauer group of any field either is an elementary 2-group or contains a non-trivial divisible subgroup. In [1] Fein and Schacher conjectured that any abelian divisible torsion group is isomorphic to the Brauer group of some field. We will now give a proof of this conjecture.

Theorem. *For every abelian divisible torsion group A there exists a field F such that* $\mathrm{Br}(F) \simeq A$.

Proof. We will construct, inductively, a tower of fields $F_1 \subset F_2 \subset F_3 \subset \cdots$ and subgroups $A_i, B_i \subset \mathrm{Br}(F_i)$ satisfying the following conditions:

1. A is isomorphic to A_1.
2. $\mathrm{Br}(F_i) = A_i \oplus B_i$ $(i = 1, 2, \ldots)$.
3. The kernel of the natural homomorphism $\mathrm{Br}(F_i) \to \mathrm{Br}(F_{i+1})$ induced by the inclusion of fields $F_i \subset F_{i+1}$ is B_i. Moreover, this homomorphism restricts to an isomorphism between A_i and A_{i+1}.

[1]Translation given here with the kind permission of Uspekhi Mat. Nauk.

Let us begin with $i = 1$. By [2, Theorem 2] there exists a field F_1 such that A is isomorphic to some subgroup A_1 of $\mathrm{Br}(F_1)$. Since A_1 is divisible, it is a direct summand in $\mathrm{Br}(F_1)$, i.e., there exists a subgroup $B_1 \subset \mathrm{Br}(F_1)$ such that $\mathrm{Br}(F_1) = A_1 \oplus B_1$.

Now suppose we have constructed the fields $F_1 \subset F_2 \subset \cdots \subset F_n$ and subgroups $A_i, B_i \subset \mathrm{Br}(F_i)$ for $i = 1, \ldots, n$. By [2, Theorem 1] there exists a field F_{n+1} such that $F_n \subset F_{n+1}$, and the kernel of the homomorphism $\mathrm{Br}(F_n) \to \mathrm{Br}(F_{n+1})$ induced by this inclusion is B_n. Denote by A_{n+1} the image of A_n under this homomorphism, and by B_{n+1} any complement to A_{n+1} in $\mathrm{Br}(F_{n+1})$ (a complement to A_{n+1} exists because $A_{n+1} \simeq A$ is divisible). This completes the construction of the tower of fields $F_1 \subset F_2 \subset F_3 \subset \cdots$.

Now denote the union of the fields F_i $(i = 1, 2, \ldots)$ by F. Clearly

$$\mathrm{Br}(F) = \varinjlim \mathrm{Br}(F_i) = \varinjlim (A_i \oplus B_i) = \varinjlim A_i \simeq A\,,$$

as desired. □

References

[1] B. Fein and M. Schacher, Brauer groups of fields, in *Ring theory and algebra, III (Proc. Third Conf., Univ. Oklahoma, Norman, Okla., 1979)*, 345–356, Lecture Notes in Pure and Appl. Math., 55, Dekker, New York. MR0584617 (81k:12025)

[2] B. Fein and M. Schacher, Relative Brauer groups. I, J. Reine Angew. Math. 321 (1981), 179–194. MR0597988 (82f:12027)

[3] A. S. Merkurjev, Brauer groups of fields, Comm. Algebra 11 (1983), no. 22, 2611–2624. MR0733345 (85f:12006)

Editorial remarks

The above note is a translation of one presented at the October 4, 1983, meeting of the Leningrad Mathematical Society on the occasion of Merkurjev winning the Society's Young Mathematician Prize. It was originally published in Russian in 1985 and has not previously appeared in English.

The prehistory of the note was explained to us by Burt Fein: "Merkurjev made a tour of the US in the early 1980s and visited Oregon State. While he was here, Bill Jacob and I took him to the University of Oregon to give a seminar talk; we also took him to Cafe Zenon to sample their wonderful cream puffs. Over cream puffs I told him about the conjecture from [1] and asked him specifically about whether there was a field with Brauer group $\mathbb{Z}/3$. He solved it on the spot, first using K_2 and then coming up with a more traditional proof that it could not. I wrote up that proof and circulated it to the experts in the field under the title 'Merkurjev's Cream Puff Theorem'. That was the start of reference [3] and eventually to the note itself."