Quotients of MGL, Their Slices and Their Geometric Parts

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Abstract. Let $x_1, x_2, \ldots$ be a system of homogeneous polynomial generators for the Lazard ring $\mathbb{L}^* = MU^{2*}$ and let MGL_S denote Voevodsky's algebraic cobordism spectrum in the motivic stable homotopy category over a base-scheme S [Vo98]. Relying on Hopkins-Morel-Hoyois isomorphism [Hoy] of the 0th slice s_0MGL_S for Voevodsky's slice tower with $MGL_S/(x_1, x_2, \ldots)$ (after inverting all residue characteristics of S), Spitzweck [S10] computes the remaining slices of MGL_S as $s_nMGL_S = \Sigma^n_T H\mathbb{Z} \otimes \mathbb{L}^{-n}$ (again, after inverting all residue characteristics of S). We apply Spitzweck's method to compute the slices of a quotient spectrum $MGL_S/(\{x_i : i \in I\})$ for I an arbitrary subset of $\mathbb{N}$, as well as the mod p version $MGL_S/(\{p, x_i : i \in I\})$ and localizations with respect to a system of homogeneous elements in $\mathbb{Z}[\{x_j : j \notin I\}]$. In case $S = \operatorname{Spec} k$, k a field of characteristic zero, we apply this to show that for $\mathcal{E}$ a localization of a quotient of MGL as above, there is a natural isomorphism for the theory with support

$$\Omega_*(X) \otimes_{\mathbb{L}^{-*}} \mathcal{E}^{-2*,-*}(k) \to \mathcal{E}_X^{2m-2*,m-*}(M)$$

for X a closed subscheme of a smooth quasi-projective k-scheme M, $m = \dim_k M$.

To Sasha Merkurjev with warmest regards on his 60th birthday

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Introduction

This paper has a two-fold purpose. We consider Voevodsky's slice tower on the motivic stable homotopy category $\mathcal{SH}(S)$ over a base-scheme[2] S [Vo00]. For $\mathcal{E}$ in $\mathcal{SH}(S)$, we have the nth layer $s_n\mathcal{E}$ in the slice tower for $\mathcal{E}$. Let MGL denote Voevodsky's algebraic cobordism spectrum in $\mathcal{SH}(S)$ [Vo98] and let $x_1, x_2, \ldots$ be a system of homogeneous polynomial generators for the Lazard ring $\mathbb{L}_*$. Via the classifying map for the formal group law for MGL, we may consider x_i as an element of $MGL^{-2i,-i}(S)$, and thereby as a map $x_i : \Sigma^{2i,i}MGL \to MGL$, giving the quotient $MGL/(x_1, x_2, \ldots)$. Spitzweck [S10] shows how to build on the Hopkins-Morel-Hoyois isomorphism [Hoy]

$$MGL/(x_1, x_2, \ldots) \cong s_0 MGL$$

to compute all the slices $s_n MGL$ of MGL. Our first goal here is to extend Spitzweck's method to handle quotients of MGL by a subset of $\{x_1, x_2, \ldots\}$, as well as localizations with respect to a system of homogeneous elements in the ring generated by the remaining variables; we also consider quotients of such spectra by an integer. Some of these spectra are Landweber exact, and the slices are thus computable by the results of Spitzweck on the slices of Landweber exact spectra [S12], but many of these, such as the truncated Brown-Peterson spectra or Morava K-theory, are not.

The second goal is to extend results of [DL14, L09, L15], which consider the "geometric part" $X \mapsto \mathcal{E}^{2*,*}(X)$ of the bi-graded cohomology defined by an oriented weak commutative ring T-spectrum $\mathcal{E}$ and raise the question: is the classifying map

$$\mathcal{E}^*(k) \otimes_{\mathbb{L}^*} \Omega^* \to \mathcal{E}^*$$

an isomorphism of oriented cohomology theories, that is, is the theory $\mathcal{E}^*$ a theory of *rational type* in the sense of Vishik [Vi12]? Starting with the case $\mathcal{E} = MGL$, discussed in [L09], which immediately yields the Landweber exact case, we have answered this affirmatively for "slice effective" algebraic K-theory in [DL14], and extended to the case of slice-effective covers of a Landweber exact theory in [L15]. In this paper, we use our computation of the slices of a quotient of MGL to show that the classifying map is an isomorphism for the quotients and localizations of MGL described above.

The paper is organized as follows: in §1 and §2, we abstract Spitzweck's method from [S10] to a more general setting. In §1 we give a description of quotients in a suitable symmetric monoidal model category in terms of a certain homotopy colimit. In §2 we begin by recalling some basic facts and the slice tower and its construction. We then apply the results of §1 to the category of $\mathcal{R}$-modules for $\mathcal{R} \in \mathcal{SH}(S)$ a commutative ring T-spectrum (with some additional technical assumptions), developing a method for computing the slices of an $\mathcal{R}$-module $\mathcal{M}$, assuming that $\mathcal{R}$ and $\mathcal{M}$ are effective and that the 0th slice $s_0\mathcal{M}$ is of the form $\mathcal{M}/(\{x_i : i \in I\})$ for some collection $\{[x_i] \in \mathcal{R}^{-2d_i,-d_i}(S), d_i < 0\}$

[2] In this paper a "scheme" will mean a noetherian separated scheme of finite Krull dimension.

of elements in $\mathcal{R}$-cohomology of the base-scheme S; see theorem 2.3. We also discuss localizations of such $\mathcal{R}$-modules and the mod p case (corollary 2.4 and corollary 2.5). We discuss the associated slice spectral sequence for such $\mathcal{M}$ and its convergence properties in §3, and apply these results to our examples of interest: truncated Brown-Peterson spectra, Morava K-theory and connective Morava K-theory, as well as the Landweber exact examples, the Brown-Peterson spectra BP and the Johnson-Wilson spectra $E(n)$, in §4.
The remainder of the paper discusses the classifying map from algebraic cobordism Ω_* and proves our results on the rationality of certain theories. This is essentially taken from [L15], but we need to deal with a technical problem, namely, that it is not at present clear if the theories $[MGL/(\{x_i : i \in I\})]^{2*,*}$ have a multiplicative structure. For this reason, we extend the setting used in [L15] to theories that are modules over ring-valued theories. This extension is taken up in §5 and we apply this theory to quotients and localizations of MGL in §6.
We are grateful to the referee for suggesting a number of improvements to an earlier version of this paper, especially for pointing out to us how to use works of Spitzweck to extend many of our results to an arbitrary base-scheme.

1. Quotients and homotopy colimits in a model category

In this section we consider certain quotients in a model category and give a description of these quotients as a homotopy colimit (see proposition 1.9). This is an abstraction of the methods developed in [S10] for computing the slices of MGL.
In what follows, we use the term "fibrant replacement" of an object x in a model category $\mathcal{C}$ to mean a morphism $\alpha : x \to x^f$ in $\mathcal{C}$, where x^f is fibrant and α is a cofibration and a weak equivalence. A cofibrant replacement of x is similarly a morphism $\beta : x^c \to x$ in $\mathcal{C}$ with x^c cofibrant and β a fibration and a weak equivalence.
Let $(\mathcal{C}, \otimes, 1)$ be a closed symmetric monoidal simplicial pointed model category with cofibrant unit 1. We assume that 1 admits a fibrant replacement $\alpha : 1 \to \mathbf{1}$ such that $\mathbf{1}$ is a 1-algebra in $\mathcal{C}$, that is, there is an associative multiplication map $\mu_{\mathbf{1}} : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ such that $\mu_1 \circ (\alpha \otimes \mathrm{id})$ and $\mu_1 \circ (\mathrm{id} \otimes \alpha)$ are the respective multiplication isomorphisms $1 \otimes \mathbf{1} \to \mathbf{1}$, $\mathbf{1} \otimes 1 \to \mathbf{1}$. We assume in addition that the functor $K \mapsto 1 \otimes K$, giving part of the simplicial structure, is a symmetric monoidal left Quillen functor.
For a cofibrant object T in $\mathcal{C}$, the map $T \cong T \otimes 1 \xrightarrow{\mathrm{id} \otimes \alpha} T \otimes \mathbf{1}$ is a cofibration and weak equivalence. Indeed, the functor $T \otimes (-)$ preserves cofibrations, and also maps that are both a cofibration and a weak equivalence, whence the assertion.

Remark 1.1. We will be applying the results of this section to the following situation: $\mathcal{M}$ is a cofibrantly generated symmetric monoidal simplicial model category satisfying the monoid axiom [ScSh, definition 3.3]; ; we assume in addition that the functor $K \mapsto e \wedge K$, e the unit in $\mathcal{M}$, giving part of the simplicial structure, is a symmetric monoidal left Quillen functor. We fix in addition a commutative monoid $\mathcal{R}$ in $\mathcal{M}$, cofibrant in $\mathcal{M}$, and $\mathcal{C}$ is the category of $\mathcal{R}$-modules in $\mathcal{M}$, with model structure as in [ScSh, §4], that is, a map is a fibration or a weak equivalence in $\mathcal{C}$ if and only if it is so as a map in $\mathcal{M}$, and cofibrations are determined by the LLP with respect to acyclic fibrations. By [ScSh, theorem 4.1(3)], the category $\mathcal{R}$-Alg of monoids in $\mathcal{C}$ has the structure of a cofibrantly generated model category, with fibrations and weak equivalence those maps which become a fibration or weak equivalence in $\mathcal{M}$, and each cofibration in $\mathcal{R}$-Alg is a cofibration in $\mathcal{C}$. The unit 1 in $\mathcal{C}$ is just $\mathcal{R}$ and we may take $\alpha : 1 \to \mathbf{1}$ to be a fibrant replacement in $\mathcal{R}$-Alg.

Let $\{x_i : T_i \to \mathbf{1} \mid i \in I\}$ be a set of maps with cofibrant sources T_i. We assign each T_i an integer degree $d_i > 0$.
Let $\mathbf{1}/(x_i)$ be the homotopy cofiber (i.e., mapping cone) of the map $x_i : \mathbf{1} \otimes T_i \to \mathbf{1}$ and let $p_i : \mathbf{1} \to \mathbf{1}/(x_i)$ be the canonical map.
Let $A = \{i_1, \ldots, i_k\}$ be a finite subset of I and define $\mathbf{1}/(\{x_i : i \in A\})$ as

$$\mathbf{1}/(\{x_i : i \in A\}) := \mathbf{1}/(x_{i_1}) \otimes \ldots \otimes \mathbf{1}/(x_{i_k}).$$

Of course, the object $\mathbf{1}/(\{x_i : i \in A\})$ depends on a choice of ordering of the elements in A, but only up to a canonical symmetry isomorphism. We could for example fix the particular choice by fixing a total order on A and taking the product in the proper order. The canonical maps p_i, $i \in I$ composed with the map $1 \to \mathbf{1}$ give rise to the canonical map

$$p_I : 1 \to \mathbf{1}/(\{x_i : i \in A\})$$

defined as the composition

$$1 \xrightarrow{\mu^{-1}} 1^{\otimes k} \to \mathbf{1}^{\otimes k} \xrightarrow{p_{i_1} \otimes \ldots \otimes p_{i_k}} \mathbf{1}/(\{x_i : i \in A\}).$$

For finite subsets $A \subset B \subset I$, define the map

$$\rho_{A \subset B} : \mathbf{1}/(\{x_i : i \in A\}) \to \mathbf{1}/(\{x_i : i \in B\})$$

as the composition

$$\begin{aligned} &\mathbf{1}/(\{x_i : i \in A\}) \xrightarrow{\mu^{-1}} \mathbf{1}/(\{x_i : i \in A\}) \otimes 1 \\ &\xrightarrow{\mathrm{id} \otimes p_{B \setminus A}} \mathbf{1}/(\{x_i : i \in A\}) \otimes \mathbf{1}/(\{x_i : i \in B \setminus A\}) \cong \mathbf{1}/(\{x_i : i \in B\}). \end{aligned}$$

where the last isomorphism is again the symmetry isomorphism.
Because $\mathcal{C}$ is a symmetric monoidal category with unit 1, we have a well-defined functor from the category $\mathcal{P}_{\mathrm{fin}}(I)$ of finite subsets of I to $\mathcal{C}$:

$$\mathbf{1}/(-) : \mathcal{P}_{\mathrm{fin}}(I) \to \mathcal{C}$$

sending $A \subset I$ to $\mathbf{1}/(\{x_i : i \in A\})$ and sending each inclusion $A \subset B$ to $\rho_{A \subset B}$.

Definition 1.2. The object $\mathbf{1}/(\{x_i : i \in I\})$ of $\mathcal{C}$ is defined by

$$\mathbf{1}/(\{x_i\}) = \operatorname*{hocolim}_{A \in \mathcal{P}_{\mathrm{fin}}(I)} \mathbf{1}/(\{x_i : i \in A\}).$$

More generally, for $M \in \mathcal{C}$, we define $M/(\{x_i : i \in I\})$ as

$$M/(\{x_i : i \in I\}) := \mathbf{1}/(\{x_i : i \in I\}) \otimes QM,$$

where $QM \to M$ is a cofibrant replacement for M. In case the index set I is understood, we often write these simply as $\mathbf{1}/(\{x_i\})$ or $M/(\{x_i\})$.

Remark 1.3. 1. The object $\mathbf{1}/(x_i)$ is cofibrant and hence the objects $\mathbf{1}/(\{x_i : i \in A\})$ are cofibrant for all finite sets A. As a pointwise cofibrant diagram has cofibrant homotopy colimit [Hir03, corollary 14.8.1, example 18.3.6, corollary 18.4.3], $\mathbf{1}/(\{x_i : i \in I\})$ is cofibrant. Thus $M/(\{x_i : i \in I\}) := \mathbf{1}/(\{x_i : i \in I\}) \otimes QM$ is also cofibrant.
2. We often select a single cofibrant object T and take $T_i := T^{\otimes d_i}$ for certain integers $d_i > 0$. As T is cofibrant, so is $T^{\otimes d_i}$. In this case we set $\deg T = 1$, $\deg T^{\otimes d_i} = d_i$.

We let $[n]$ denote the set $\{0, \dots, n\}$ with the standard order and Δ the category with objects $[n]$, $n = 0, 1, \dots$, and morphisms the order-preserving maps of sets. For a small category A and a functor $F : A \to \mathcal{C}$, we let $\mathrm{hocolim}_A F_*$ denote the standard simplicial object of $\mathcal{C}$ whose geometric realization is $\mathrm{hocolim}_A F$, that is

$$\operatorname*{hocolim}_A F_n = \bigvee_{\sigma:[n]\to A} F(\sigma(0)).$$

LEMMA 1.4. *Let $\{x_i : T_i \to \mathbf{1} : i \in I_1\}$, $\{x_i : T_i \to \mathbf{1} : i \in I_2\}$ be two sets of maps in $\mathcal{C}$, with cofibrant sources T_i, and with I_1, I_2 disjoint index sets. Then there is a canonical isomorphism*

$$\mathbf{1}/(\{x_i : i \in I_1 \amalg I_2\}) \cong \mathbf{1}/(\{x_i : i \in I_1\}) \otimes \mathbf{1}/(\{x_i : i \in I_2\}).$$

Proof. The category $\mathcal{P}_{\mathrm{fin}}(I_1 \amalg I_2)$ is clearly equal to $\mathcal{P}_{\mathrm{fin}}(I_1) \times \mathcal{P}_{\mathrm{fin}}(I_2)$. For functors $F_i : \mathcal{A}_i \to \mathcal{C}$, $i = 1, 2$, $[\mathrm{hocolim}_{\mathcal{A}_1 \times \mathcal{A}_2} F_1 \otimes F_2]_*$ is the diagonal simplicial space associated to the bisimplicial space $(n, m) \mapsto [\mathrm{hocolim}_{\mathcal{A}_1} F_1]_n \otimes [\mathrm{hocolim}_{\mathcal{A}_2} F_2]_m$. Thus

$$\operatorname*{hocolim}_{\mathcal{A}_1 \times \mathcal{A}_2} F_1 \otimes F_2 \cong \operatorname*{hocolim}_{\mathcal{A}_2} [\operatorname*{hocolim}_{\mathcal{A}_1} F_1] \otimes F_2.$$

This gives us the isomorphism

$$\begin{aligned}
&\mathbf{1}/(\{x_i : i \in I_1 \amalg I_2\}) \\
&\quad = \operatorname*{hocolim}_{(A_1, A_2) \in \mathcal{P}_{\mathrm{fin}}(I_1) \times \mathcal{P}_{\mathrm{fin}}(I_2)} \mathbf{1}/(\{x_i : i \in A_1\}) \otimes \mathbf{1}/(\{x_i : i \in A_2\}) \\
&\quad \cong \operatorname*{hocolim}_{A_1 \in \mathcal{P}_{\mathrm{fin}}(I_1)} \mathbf{1}/(\{x_i : i \in A_1\}) \otimes \operatorname*{hocolim}_{A_2 \in \mathcal{P}_{\mathrm{fin}}(I_2)} \mathbf{1}/(\{x_i : i \in A_2\}) \\
&\quad = \mathbf{1}/(\{x_i : i \in I_1\}) \otimes \mathbf{1}/(\{x_i : i \in I_2\}).
\end{aligned}$$

□

Remark 1.5. Via this lemma, we have the isomorphism for all $M \in \mathcal{C}$,

$$M/(\{x_i : i \in I_1 \amalg I_2\}) \cong (M/(\{x_i : i \in I_1\})/(\{x_i : i \in I_2\}).$$

Let $\mathcal{I}$ be the category of formal monomials in $\{x_i\}$, that is, the category of maps $N : I \to \mathbb{N}$, $i \mapsto N_i$, such that $N_i = 0$ for all but finitely many $i \in I$, and with a unique map $N \to M$ if $N_i \geq M_i$ for all $i \in I$. As usual, the monomial in the x_i corresponding to a given N is $\prod_{i\in I} x_i^{N_i}$, written x^N. The index $N = 0$, corresponding to $x^0 = 1$, is the final object of $\mathcal{I}$.

Take an $i \in I$. For $m > k \geq 0$ integers, define the map

$$\times x_i^{m-k} : \mathbf{1} \otimes T_i^{\otimes m} \to \mathbf{1} \otimes T_i^{\otimes k}$$

as the composition

$$\mathbf{1}\otimes T_i^{\otimes m} = \mathbf{1}\otimes T_i^{\otimes m-k}\otimes T_i^{\otimes k} \xrightarrow{\mathrm{id}_{\mathbf{1}}\otimes x_i^{\otimes m-k}\otimes \mathrm{id}_{T_i^{\otimes k}}} \mathbf{1}^{\otimes m-k+1}\otimes T_i^{\otimes k} \xrightarrow{\mu\otimes\mathrm{id}} \mathbf{1}\otimes T_i^{\otimes k}.$$

In case $k = 0$, we use $\mathbf{1}$ instead of $\mathbf{1} \otimes 1$ for the target; we define $\times x^0$ to be the identity map. The associativity of the maps $\mu_{\mathbf{1}}$ shows that $\times x_i^{m-k} \circ \times x_i^{n-m} = \times x_i^{n-k}$, hence the maps $\times x_i^n$ all commute with each other.

Now suppose we have a monomial in the x_i; to simplify the notation, we write the indices occurring in the monomial as $\{1, \ldots, r\}$ rather than $\{i_1, \ldots, i_r\}$. This gives us the monomial $x^N := x_1^{N_1} \cdot \ldots \cdot x_r^{N_r}$. Define

$$T_*^N := \mathbf{1} \otimes T_1^{\otimes N_1} \otimes \ldots \otimes \mathbf{1} \otimes T_r^{\otimes N_r} \otimes \mathbf{1};$$

in case $N_i = 0$, we replace $\ldots\otimes\mathbf{1}\otimes 1\otimes\mathbf{1}\otimes T_{i+1}^{\otimes M_{i+1}}\otimes\ldots$ with $\ldots\otimes\mathbf{1}\otimes T_{i+1}^{\otimes M_{i+1}}\otimes\ldots$, and we set $T_*^0 := \mathbf{1}$.

Let $N \to M$ be a map in $\mathcal{I}$, that is $N_i \geq M_i \geq 0$ for all i. We again write the relevant index set as $\{1, \ldots, r\}$. Define the map

$$\times x^{N-M} : T_*^N \to T_*^M$$

as the composition

$$T_*^N \xrightarrow{\bigotimes_{j=1}^r \times x_j^{N_j-M_j}} \mathbf{1} \otimes T_1^{\otimes M_1} \otimes \ldots \otimes \mathbf{1} \otimes T_r^{\otimes M_r} \otimes \mathbf{1} \xrightarrow{\mu_M} T_*^{\otimes M};$$

the map μ_M is a composition of $\otimes$-product of multiplication maps $\mu_{\mathbf{1}} : \mathbf{1}\otimes\mathbf{1} \to \mathbf{1}$, with these occurring in those spots with $M_j = 0$. In case $N_i = M_i = 0$, we simply delete the term $\times x_i^0$ from the expression.

The fact that the maps $\mu_{\mathbf{1}}$ satisfy associativity yields the relation

$$\times x^{M-K} \circ \times x^{N-M} = \times x^{N-K}$$

and thus the maps $\times x^{N-M}$ all commute with each other.

Defining $\mathcal{D}_x(N) := T_*^N$ and $\mathcal{D}_x(N \to M) = \times x^{N-M}$ gives us the $\mathcal{I}$-diagram

$$\mathcal{D}_x : \mathcal{I} \to \mathcal{C}.$$

We consider the following full subcategories of $\mathcal{I}$. For a monomial M let $\mathcal{I}_{\geq M}$ denote the subcategory of monomials which are divisible by M, and for a positive integer n, recalling that we have assigned each T_i a positive integral degree d_i, let $\mathcal{I}_{\deg\geq n}$ denote the subcategory of monomials of degree at least n, where the degree of $N := (N_1, \ldots, N_k)$ is $N_1d_1 + \cdots + N_kd_k$. One defines similarly the full subcategories $\mathcal{I}_{>M}$ and $\mathcal{I}_{\deg>n}$.

Let $\mathcal{I}^\circ$ be the full subcategory of $\mathcal{I}$ of monomials $N \neq 0$ and $\mathcal{I}^\circ_{\leq 1} \subset \mathcal{I}^\circ$ be the full subcategory of monomials N for which $N_i \leq 1$ for all i. We have the corresponding subdiagrams $\mathcal{D}_x : \mathcal{I}^\circ \to \mathcal{C}$ and $\mathcal{D}_x : \mathcal{I}^\circ_{\leq 1} \to \mathcal{C}$ of $\mathcal{D}_x$. For $J \subset I$ a subset, we have the corresponding full subcategories $\mathcal{J} \subset \mathcal{I}$, $\mathcal{J}^\circ \subset \mathcal{I}^\circ$ and $\mathcal{J}^\circ_{\leq 1} \subset \mathcal{I}^\circ_{\leq 1}$ and corresponding subdiagrams $\mathcal{D}_x$. If the collection of maps x_i is understood, we write simply $\mathcal{D}$ for $\mathcal{D}_x$.
Let $F : A \to \mathcal{C}$ be a functor, a an object in $\mathcal{C}$, $c_a : A \to \mathcal{C}$ the constant functor with value a and $\varphi : F \to c_a$ a natural transformation. Then φ induces a canonical map $\tilde{\varphi} : \operatorname{hocolim}_A F \to a$ in $\mathcal{C}$. As in the proof of [S10, Proposition 4.4], let $C(A)$ be the category A with a final object $*$ adjoined and $C(F,\varphi) : C(A) \to \mathcal{C}$ the functor with value a on $*$, with restriction to A being F, and which sends the unique map $y \to *$ in $C(A)$, $y \in A$, to $\varphi(y)$. Let $[0,1]$ be the category with objects $0, 1$ and a unique non-identity morphism $0 \to 1$, and let $C(A)^\Gamma$ be the full subcategory of $C(A) \times [0,1]$ formed by removing the object $* \times 1$. We extend $C(F,\varphi)$ to a functor $C(F,\varphi)^\Gamma : C(A)^\Gamma \to \mathcal{C}$ by $C(F,\varphi)^\Gamma(y \times 1) = pt$, where pt is the initial/final object in $\mathcal{C}$.

LEMMA 1.6. *There is a natural isomorphism in $\mathcal{C}$*

$$\operatorname*{hocolim}_{C(A)^\Gamma} C(F,\varphi)^\Gamma \cong \operatorname{hocofib}(\tilde{\varphi} : \operatorname*{hocolim}_A F \to a).$$

Proof. For a category $\mathcal{A}$ we let $\mathcal{N}(\mathcal{A})$ denote the simplicial nerve of $\mathcal{A}$. We have an isomorphism of simplicial sets $\mathcal{N}(C(A)) \cong \operatorname{Cone}(\mathcal{N}(A), *)$, where $\operatorname{Cone}(\mathcal{N}(A), *)$ is the cone over $\mathcal{N}(A)$ with vertex $*$. Similarly, the full subcategory $A \times [0,1]$ of $C(A)^\Gamma$ has nerve isomorphic to $\mathcal{N}(A) \times \Delta[1]$. This gives an isomorphism of $\mathcal{N}(C(A)^\Gamma)$ with the push-out in the diagram

$$\begin{tikzcd} \mathcal{N}(A) \arrow[r, hook] \arrow[d, hook, "\mathrm{id}\times\delta_0"'] & \operatorname{Cone}(\mathcal{N}(A), *) \\ \mathcal{N}(A) \times \Delta[1]. & \end{tikzcd}$$

This in turn gives an isomorphism of the simplicial object $\operatorname{hocolim}_{C(A)^\Gamma} C(F,\varphi)^\Gamma_*$ with the pushout in the diagram

$$\begin{tikzcd} \operatorname{hocolim}_A F \arrow[r, hook] \arrow[d, hook] & C(\operatorname{hocolim}_A F, a) \\ C(\operatorname{hocolim}_A F, pt). & \end{tikzcd}$$

This gives the desired isomorphism. $\square$

LEMMA 1.7. *Let $J \subset K \subset I$ be finite subsets of I. Then the map*

$$\operatorname*{hocolim}_{\mathcal{J}^\circ_{\leq 1}} \mathcal{D}_x \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\leq 1}} \mathcal{D}_x$$

induced by the inclusion $J \subset K$ is a cofibration in $\mathcal{C}$.

Proof. We give the category of simplicial objects in $\mathcal{C}$, $\mathcal{C}^{\Delta^{\mathrm{op}}}$, the Reedy model structure, using the standard structure of a Reedy category on Δ^{op}. By [Hir03, theorem 19.7.2(1), definition 19.8.1(1)], it suffices to show that

$$\operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_* \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_*$$

is a cofibration in $\mathcal{C}^{\Delta^{\mathrm{op}}}$, that is, for each n, the map

$$\varphi_n : \operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_n \amalg_{L^n \operatorname{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_*} L^n \operatorname*{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_* \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_n$$

is a cofibration in $\mathcal{C}$, where L^n is the nth latching space.
We note that

$$\operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_n = \bigvee_{\sigma \in \mathcal{N}(\mathcal{J}^\circ_{\le 1})_n} D(\sigma(0)),$$

where we view $\sigma \in \mathcal{N}(\mathcal{J}^\circ_{\le 1})_n$ as a functor $\sigma : [n] \to \mathcal{J}^\circ_{\le 1}$; we have a similar description of $\operatorname{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_n$. The latching space is

$$L^n \operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_* = \bigvee_{\sigma \in \mathcal{N}(\mathcal{J}^\circ_{\le 1})^{deg}_n} D(\sigma(0)),$$

where $\mathcal{N}(\mathcal{J}^\circ_{\le 1})^{deg}_n$ is the subset of $\mathcal{N}(\mathcal{J}^\circ_{\le 1})_n$ consisting of those σ which contain an identity morphism; $L^n \operatorname{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_*$ has a similar description. The maps

$$L^n \operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_* \to \operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_n, L^n \operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_* \to L^n \operatorname*{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}*,$$
$$L^n \operatorname*{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_* \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_n, \operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_n \to \operatorname*{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_n$$

are the unions of identity maps on $D(\sigma(0))$ over the respective inclusions of the index sets. As $\mathcal{N}(\mathcal{K}^\circ_{\le 1})^{deg}_n \cap \mathcal{N}(\mathcal{J}^\circ_{\le 1})_n = \mathcal{N}(\mathcal{J}^\circ_{\le 1})^{deg}_n$, we have

$$\operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_n \amalg_{L^n \operatorname{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_*} L^n \operatorname*{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_* \cong \operatorname*{hocolim}_{\mathcal{J}^\circ_{\le 1}} \mathcal{D}_n \bigvee C,$$

where

$$C = \bigvee_{\sigma \in \mathcal{N}(\mathcal{K}^\circ_{\le 1})^{deg}_n \setminus \mathcal{N}(\mathcal{J}^\circ_{\le 1})^{deg}_n} D(\sigma(0)),$$

and the map to $\operatorname{hocolim}_{\mathcal{K}^\circ_{\le 1}} \mathcal{D}_n$ is the evident inclusion. As $D(N)$ is cofibrant for all N, this map is clearly a cofibration, completing the proof. $\square$

We have the n-cube $\square^n$, the category associated to the partially ordered set of subsets of $\{1, \ldots, n\}$, ordered under inclusion, and the punctured n-cube $\square^n_0$ of proper subsets. We have the two inclusion functors $i^+_n, i^-_n : \square^{n-1} \to \square^n$, $i^+_n(I) := I \cup \{n\}$, $i^-_n(I) = I$ and the natural transformation $\psi_n : i^-_n \to i^+_n$ given as the collection of inclusions $I \subset I \cup \{n\}$. The functor i^-_n induces the functor $i^-_{n0} : \square^{n-1} \to \square^n_0$.
For a functor $F : \square^n \to \mathcal{C}$, we have the iterated homotopy cofiber, $\operatorname{hocofib}_n F$, defined inductively as the homotopy cofiber of $\operatorname{hocofib}_{n-1}(F(\psi_n)) : \operatorname{hocofib}(F \circ i^-_n) \to \operatorname{hocofib}(F \circ i^+_n)$. Using this inductive construction, it is easy to define

a natural isomorphism $\mathrm{hocofib}_n F \cong \mathrm{hocolim}_{\square_0^{n+1}} \hat{F}$, where $\hat{F} \circ i^-_{n+10} = F$ and $\hat{F}(I) = pt$ if $n \in I$.
The following result, in the setting of modules over a model of MGL as a commutative $\mathbb{S}$-algebra, is proven in [S10, Lemma 4.3 and Proposition 4.4]. We give here a somewhat different proof in our context, which allows for a wider application.

LEMMA 1.8. *Assume that I is countable. Then there is a canonical isomorphism in* $\mathbf{Ho}\,\mathcal{C}$

$$\mathbf{1}/(\{x_i \mid i \in I\}) \cong \mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x \to \operatorname*{hocolim}_{\mathcal{I}} \mathcal{D}_x].$$

Proof. As 1 is the final object in $\mathcal{I}$, the collection of maps $\times x^N : T_*^N \to \mathbf{1}$ defines a weak equivalence $\pi : \mathrm{hocolim}_{\mathcal{I}} \mathcal{D}_x \to \mathbf{1}$. In addition, for each $N \in \mathcal{I}^\circ$, the comma category $N/\mathcal{I}^\circ_{\le 1}$ has initial object the map $N \to \bar{N}$, where $\bar{N}_i = 1$ if $N_i > 0$, and $\bar{N}_i = 0$ otherwise. Thus $\mathcal{I}^\circ_{\le 1}$ is homotopy right cofinal in $\mathcal{I}^\circ$ (see e.g. [Hir03, definition 19.6.1]). Since $\mathcal{D}_x$ is a diagram of cofibrant objects in $\mathcal{C}$, it follows from [Hir03, theorem 19.6.7] that the map $\mathrm{hocolim}_{\mathcal{I}^\circ_{\le 1}} \mathcal{D}_x \to \mathrm{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x$ is a weak equivalence. This reduces us to identifying $\mathbf{1}/(\{x_i\})$ with the homotopy cofiber of $\pi^\circ_{\le 1} : \mathrm{hocolim}_{\mathcal{I}^\circ_{\le 1}} \mathcal{D}_x \to \mathbf{1}$, where $\pi^\circ_{\le 1}$ is the composition of π with the natural map : $\mathrm{hocolim}_{\mathcal{I}^\circ_{\le 1}} \mathcal{D}_x \to \mathrm{hocolim}_{\mathcal{I}} \mathcal{D}_x$.
Next, we reduce to the case of a finite set I. Take $I = \mathbb{N}$. Let $\mathcal{P}_{fin}(I)$ be the category of finite subsets of I, ordered by inclusion, consider the full subcategory $\mathcal{P}^O_{fin}(I)$ of $\mathcal{P}_{fin}(I)$ consisting of the subsets $I_n := \{1, \ldots, n\}$, $n = 1, 2 \ldots$, and let $\mathcal{I}^\circ_{n,\le 1} \subset \mathcal{I}^\circ_{\le 1}$ be the full subcategory with all indices in I_n. As $\mathcal{P}^O_{fin}(I)$ is cofinal in $\mathcal{P}_{fin}(I)$, we have

$$\operatorname*{colim}_n \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\le 1}} \mathcal{D}_x \cong \operatorname*{hocolim}_{\mathcal{I}^\circ_{\le 1}} \mathcal{D}_x.$$

Take $n \le m$. By lemma 1.7 the the map $\mathrm{hocolim}_{\mathcal{I}^\circ_{n,\le 1}} \mathcal{D}_x \to \mathrm{hocolim}_{\mathcal{I}^\circ_{m,\le 1}} \mathcal{D}_x$ is a cofibration in $\mathcal{C}$. Thus, using the Reedy model structure on $\mathcal{C}^{\mathbb{N}}$ with $\mathbb{N}$ considered as a direct category, the $\mathbb{N}$-diagram in $\mathcal{C}$, $n \mapsto \mathrm{hocolim}_{\mathcal{I}^\circ_{n,\le 1}} \mathcal{D}_x$, is a cofibrant object in $\mathcal{C}^{\mathbb{N}}$. As $\mathbb{N}$ is a direct category, the fibrations in $\mathcal{C}^{\mathbb{N}}$ are the pointwise ones, hence $\mathbb{N}$ has pointwise constants [Hir03, definition 15.10.1] and therefore [Hir03, theorem 19.9.1] the canonical map

$$\operatorname*{hocolim}_{n \in \mathbb{N}} \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\le 1}} \mathcal{D}_x \to \operatorname*{colim}_{n \in \mathbb{N}} \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\le 1}} \mathcal{D}_x$$

is a weak equivalence in $\mathcal{C}$. This gives us the weak equivalence in $\mathcal{C}$

$$\operatorname*{hocolim}_n \operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\le 1}} \mathcal{D}_x \to \operatorname*{hocolim}_{\mathcal{I}^\circ_{\le 1}} \mathcal{D}_x.$$

Since $\mathbb{N}$ is contractible, the canonical map $\mathrm{hocolim}_{\mathbb{N}}\,\mathbf{1}\to\mathbf{1}$ is a weak equivalence in $\mathcal{C}$, giving us the weak equivalences

$$\begin{aligned}\mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ_{\le 1}}\mathcal{D}_x\to\mathbf{1}]&\\ &\sim \mathrm{hocofib}[\operatorname*{hocolim}_{n\in\mathbb{N}}\operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\le 1}}\mathcal{D}_x\to\operatorname*{hocolim}_{n\in\mathbb{N}}\mathbf{1}]\\ &\qquad\sim \operatorname*{hocolim}_{n\in\mathbb{N}}[\mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\le 1}}\mathcal{D}_x\to\mathbf{1}]].\end{aligned}$$

Thus, we need only exhibit isomorphisms in $\mathbf{Ho}\,\mathcal{C}$

$$\rho_n:\mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\le 1}}\mathcal{D}_x\to\mathbf{1}]\to\mathbf{1}/(x_1,\ldots,x_n):=\mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_n),$$

which are natural in $n\in\mathbb{N}$.
By lemma 1.6 we have a natural isomorphism in $\mathcal{C}$,

$$\mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ_{n,\le 1}}\mathcal{D}_x\to\mathbf{1}]\cong\operatorname*{hocolim}_{C(\mathcal{I}^\circ_{n,\le 1})^\Gamma}C(\mathcal{D}_x,\pi)^\Gamma.$$

However, $\mathcal{I}^\circ_{n,\le 1}$ is isomorphic to $\square^n_0$ by sending $N=(N_1,\ldots,N_n)$ to $I(N):=\{i\mid N_i=0\}$. Similarly, $C(\mathcal{I}^\circ_{n,\le 1})$ is isomorphic to $\square^n$, and $C(\mathcal{I}^\circ_{n,\le 1})^\Gamma$ is thus isomorphic to $\square^{n+1}_0$. From our discussion above, we see that $\mathrm{hocolim}_{C(\mathcal{I}^\circ_{n,\le 1})^\Gamma}\,C(\mathcal{D}_x,\pi)^\Gamma$ is isomorphic to $\mathrm{hocofib}_n C(\mathcal{D}_x,\pi)$, so we need only exhibit isomorphisms in $\mathbf{Ho}\,\mathcal{C}$

$$\rho_n:\mathrm{hocofib}_n C(\mathcal{D}_x,\pi)\to\mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_n)$$

which are natural in $n\in\mathbb{N}$.
We do this inductively as follows. To include the index n in the notation, we write $C(\mathcal{D}_x,\pi)_n$ for the functor $C(\mathcal{D}_x,\pi):\square^n\to\mathcal{C}$. For $n=1$, $\mathrm{hocofib}_1 C(\mathcal{D}_x,\pi)_1$ is the mapping cone of $\mu_1\circ(\times x_1\otimes\mathrm{id}):\mathbf{1}\otimes T_1\otimes\mathbf{1}\to\mathbf{1}$, which is isomorphic in $\mathbf{Ho}\,\mathcal{C}$ to the homotopy cofiber of $\times x_1:\mathbf{1}\otimes T_1\to\mathbf{1}$. As this latter homotopy cofiber is equal to $\mathbf{1}/(x_1)$, so we take $\rho_1:\mathrm{hocofib}_1 C(\mathcal{D}_x,\pi)_1\to\mathbf{1}/(x_1)$ to be this isomorphism. We note that $C(\mathcal{D}_x,\pi)_n\circ i^+_n=C(\mathcal{D}_x,\pi)_{n-1}$ and $C(\mathcal{D}_x,\pi)_n\circ i^-_n=C(\mathcal{D}_x,\pi)_{n-1}\otimes T_n\otimes\mathbf{1}$.
Define $C(\mathcal{D}_x,\pi)'_n$ by $C(\mathcal{D}_x,\pi)'_n\circ i^-_n=C(\mathcal{D}_x,\pi)_{n-1}\otimes\mathbf{1}\otimes T_n\otimes\mathbf{1}$, $C(\mathcal{D}_x,\pi)'_n\circ i^+_n=C(\mathcal{D}_x,\pi)_{n-1}\otimes\mathbf{1}$, with the natural transformation $C(\mathcal{D}_x,\pi)'_n\circ\psi_n$ given as

$$C(\mathcal{D}_x,\pi)_{n-1}\otimes\mathbf{1}\otimes T_n\otimes\mathbf{1}\xrightarrow{(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\times x_n\otimes\mathrm{id}_1)}C(\mathcal{D}_x,\pi)_{n-1}\otimes\mathbf{1}.$$

The evident multiplication maps give a weak equivalence $C(\mathcal{D}_x,\pi)'_n\to C(\mathcal{D}_x,\pi)_n$, giving us the isomorphism in $\mathbf{Ho}\,\mathcal{C}$

$$\rho_n:\mathrm{hocofib}_n C(\mathcal{D}_x,\pi)_n\to\mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_n)$$

defined as the composition

$$\begin{aligned}
\mathrm{hocofib}_n C(\mathcal{D}_x,\pi)_n &\cong \mathrm{hocofib}_n C(\mathcal{D}_x,\pi)'_n \\
&\cong \mathrm{hocofib}(\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}\otimes \mathbf{1}\otimes T_n) \\
&\qquad \xrightarrow{\mathrm{hocofib}_{n-1}(\mathrm{id}\otimes\times x_n)} \mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}\otimes\mathbf{1})) \\
&\cong \mathrm{hocofib}(\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1})\otimes \mathbf{1}\otimes T_n \\
&\qquad \xrightarrow{\mathrm{id}\otimes\times x_n} \mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1})\otimes\mathbf{1}) \\
&\cong \mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1})\otimes \mathrm{hocofib}(\times x_n : \mathbf{1}\otimes T_n\to\mathbf{1}) \\
&= \mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1})\otimes \mathbf{1}/(x_n) \\
&\xrightarrow{\rho_{n-1}\otimes\mathrm{id}} \mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_{n-1})\otimes\mathbf{1}/(x_n).
\end{aligned}$$

Via the definition of $\mathrm{hocofib}_n$,

$$\begin{aligned}
\mathrm{hocofib}_n C(\mathcal{D}_x,\pi)_n = \mathrm{hocofib}[&\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_n\circ i_n^-) \\
&\xrightarrow{\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}(\psi_n))} \mathrm{hocofib}_n(C(\mathcal{D}_x,\pi)_n\circ i_n^+]
\end{aligned}$$

and the identification $C(\mathcal{D}_x,\pi)_n\circ i_n^+ = C(\mathcal{D}_x,\pi)_{n-1}$, we have the canonical map $\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}) \to \mathrm{hocofib}_n(C(\mathcal{D}_x,\pi)_n)$. One easily sees that the diagram

$$\begin{tikzcd}
\mathrm{hocofib}[\mathrm{hocolim}_{\mathcal{I}^\circ_{n-1,\le 1}}\mathcal{D}_x\to\mathbf{1}] \arrow[r] \arrow[d, "\sim"'] & \mathrm{hocofib}[\mathrm{hocolim}_{\mathcal{I}^\circ_{n,\le 1}}\mathcal{D}_x\to\mathbf{1}] \arrow[d, "\sim"] \\
\mathrm{hocofib}_{n-1}(C(\mathcal{D}_x,\pi)_{n-1}) \arrow[r] \arrow[d, "\rho_{n-1}"'] & \mathrm{hocofib}_n(C(\mathcal{D}_x,\pi)_n) \arrow[d, "\rho_n"] \\
\mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_{n-1}) \arrow[r, "\rho_{\{1,\ldots,n-1\}\subset\{1,\ldots,n\}}"'] & \mathbf{1}/(x_1)\otimes\ldots\otimes\mathbf{1}/(x_n)
\end{tikzcd}$$

commutes in $\mathbf{Ho}\,\mathcal{C}$, giving the desired naturality in n. □

Now let M be an object in $\mathcal{C}$, let $QM\to M$ be a cofibrant replacement and form the $\mathcal{I}$-diagram $\mathcal{D}_x\otimes QM:\mathcal{I}\to\mathcal{C}$, $(\mathcal{D}_x\otimes QM)(N)=\mathcal{D}_x(N)\otimes QM$.

PROPOSITION 1.9. *Assume that I is countable. Let M be an object in $\mathcal{C}$. Then there is a canonical isomorphism in* $\mathbf{Ho}\,\mathcal{C}$

$$M/(\{x_i\mid i\in I\})\cong \mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ}\mathcal{D}_x\otimes QM\to \operatorname*{hocolim}_{\mathcal{I}}\mathcal{D}_x\otimes QM].$$

Proof. This follows directly from lemma 1.8, noting the definition of $M/(\{x_i\mid i\in I\})$ as $[\mathbf{1}/(\{x_i\mid i\in I\})]\otimes QM$ and the canonical isomorphism

$$\begin{aligned}
\mathrm{hocofib}[&\operatorname*{hocolim}_{\mathcal{I}^\circ}\mathcal{D}_x\otimes QM\to \operatorname*{hocolim}_{\mathcal{I}}\mathcal{D}_x\otimes QM] \\
&\cong \mathrm{hocofib}[\operatorname*{hocolim}_{\mathcal{I}^\circ}\mathcal{D}_x\to \operatorname*{hocolim}_{\mathcal{I}}\mathcal{D}_x]\otimes QM.
\end{aligned}$$

□

Proposition 1.10. *Let $\mathcal{F} : \mathcal{I}_{\deg\geq n} \to \mathcal{C}$ be a diagram in a cofibrantly generated model category $\mathcal{C}$. Suppose for every monomial M of degree n the natural map* $\operatorname{hocolim} \mathcal{F}|_{\mathcal{I}_{>M}} \to \mathcal{F}(M)$ *is a weak equivalence. Then the natural map*

$$\operatorname{hocolim} \mathcal{F}|_{\deg\geq n+1} \to \operatorname{hocolim} \mathcal{F}$$

is a weak equivalence.

Proof. This is just [S10, lemma 4.5], with the following corrections: the statement of the lemma in *loc. cit.* has "$\operatorname{hocolim} \mathcal{F}|_{\mathcal{I}_{\geq M}} \to \mathcal{F}(M)$ is a weak equivalence" rather than the correct assumption "$\operatorname{hocolim} \mathcal{F}|_{\mathcal{I}_{>M}} \to \mathcal{F}(M)$ is a weak equivalence" and in the proof, one should replace the object $Q(M)$ with $\operatorname{colim} Q|_{I>M}$ rather than with $\operatorname{colim} Q|_{I\geq M}$. □

2. Slices of effective motivic module spectra

In this section we will describe the slices for modules for a commutative and effective ring T-spectrum $\mathcal{R}$, assuming certain additional conditions. We adapt the constructions used in describing slices of MGL in [S10].

Let us first recall from [Vo00] the definition of the slice tower in $\mathcal{SH}(S)$. We will use the standard model category $Mot := Mot(S)$ of symmetric T-spectra over S, $T := \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$, with the motivic model structure as in [J00], for defining the triangulated tensor category $\mathcal{SH}(S) := \mathbf{Ho}\, Mot(S)$.

For an integer q, let $\Sigma^q_T\mathcal{SH}^{eff}(S)$ denote the localizing subcategory of $\mathcal{SH}(S)$ generated by $\mathcal{S}_q := \{\Sigma^q_T\Sigma^\infty_T X_+ \mid p \geq q, X \in \mathbf{Sm}/S\}$, that is, $\Sigma^q_T\mathcal{SH}^{eff}(S)$ is the smallest triangulated subcategory of $\mathcal{SH}(S)$ which contains $\mathcal{S}_q$ and is closed under direct sums and isomorphisms in $\mathcal{SH}(S)$. This gives a filtration on $\mathcal{SH}(S)$ by full localizing subcategories

$$\cdots \subset \Sigma^{q+1}_T\mathcal{SH}^{eff}(S) \subset \Sigma^q_T\mathcal{SH}^{eff}(S) \subset \Sigma^{q-1}_T\mathcal{SH}^{eff}(S) \subset \cdots \subset \mathcal{SH}(S).$$

The set $\mathcal{S}_q$ is a set of compact generators of $\Sigma^q_T\mathcal{SH}(S)$ and the set $\cup_q\mathcal{S}_q$ is similarly a set of compact generators for $\mathcal{SH}(S)$. By Neeman's triangulated version of Brown representability theorem [N97], the inclusion $i_q : \Sigma^q_T\mathcal{SH}^{eff}(S) \to \mathcal{SH}(S)$ has a right adjoint $r_q : \mathcal{SH}(S) \to \Sigma^q_T\mathcal{SH}^{eff}(S)$. We let $f_q := i_q\circ r_q$. The inclusion $\Sigma^{q+1}_T\mathcal{SH}^{eff}(S) \to \Sigma^q_T\mathcal{SH}^{eff}(S)$ induces a canonical natural transformation $f_{q+1} \to f_q$. Putting these together forms the slice tower

$$\cdots \to f_{q+1} \to f_q \to \cdots \to \mathrm{id}. \tag{2.1}$$

For each q there exists a triangulated functor $s_q : \mathcal{SH}(S) \to \mathcal{SH}(S)$ and a canonical and natural distinguished triangle

$$f_{q+1}(\mathcal{E}) \to f_q(\mathcal{E}) \to s_q(\mathcal{E}) \to \Sigma f_{q+1}(\mathcal{E})$$

in $\mathcal{SH}(S)$. In particular, $s_q(\mathcal{E})$ is in $\Sigma^q_T\mathcal{SH}^{eff}(S)$ for each $\mathcal{E} \in \mathcal{SH}(S)$.

Pelaez has given a lifting of the construction of the functors f_q to the model category level. For this, he starts with the model category Mot and forms for each n the right Bousfield localization of Mot with respect to the objects

$\Sigma_T^m F_n X_+$ with $m-n \geq q$ and $X \in \mathbf{Sm}/S$. Here $F_n X_+$ is the shifted T-suspension spectrum, that is, $\Sigma_T^{m-n} X_+$ in degree $m \geq n$, pt in degree $m < n$, and with identity bonding maps. Calling this Bousfield localization Mot_q, the functor r_q is given by taking a functorial cofibrant replacement in Mot_q. As the underlying categories are all the same, this gives liftings $\tilde{f}_q$ of f_q to endofunctors on Mot. The technical condition on Mot invoked by Pelaez is that of *cellularity* and right properness, which ensures that the right Bousfield localization exists; this follows from the work of Hirschhorn [Hir03]. Alternatively, one can use the fact that Mot is a *combinatorial* right proper model category, following work of J. Smith, detailed for example in [B10].

The combinatorial property passes to module categories, and so this approach will be useful here. The category Mot is a closed symmetric monoidal simplicial model category, with cofibrant unit the sphere (symmetric) spectrum $\mathbb{S}_S$ and product $\wedge$. Let $\mathcal{R}$ be a commutative monoid in Mot. We have the model category $\mathcal{C} := \mathcal{R}$-Mod of $\mathcal{R}$-modules, as constructed in [ScSh]. The fibrations and weak equivalences are the morphisms which are fibrations, resp. weak equivalences, after applying the forgetful functor to Mot; cofibrations are those maps having the left lifting property with respect to trivial fibrations. This makes $\mathcal{C}$ into a pointed closed symmetric monoidal simplicial model category; $\mathcal{C}$ is in addition cofibrantly generated and combinatorial. Assuming that $\mathcal{R}$ is a cofibrant object in Mot, the free $\mathcal{R}$-module functor, $\mathcal{E} \mapsto \mathcal{R} \wedge \mathcal{E}$, gives a left adjoint to the forgetful functor and gives rise to a Quillen adjunction. For details as to these facts and a general construction of this model category structure on module categories, we refer the reader to [ScSh]; another source is [Hov], especially theorem 1.3, proposition 1.9 and proposition 1.10.

The model category $\mathcal{R}$-Mod inherits right properness from Mot. We may therefore form the right Bousfield localization $\mathcal{C}_q$ with respect to the free $\mathcal{R}$-modules $\mathcal{R} \wedge \Sigma_T^m F_n X_+$ with $m-n \geq q$ and $X \in \mathbf{Sm}/S$, and define the endofunctor $\tilde{f}_q^{\mathcal{R}}$ on $\mathcal{C}$ by taking a functorial cofibrant replacement in $\mathcal{C}_q$. By the adjunction, one sees that $\mathbf{Ho}\,\mathcal{C}_q$ is equivalent to the localizing subcategory of $\mathbf{Ho}\,\mathcal{C}$ (compactly) generated by $\{\mathcal{R} \wedge \Sigma_T^m F_n X_+ \mid m-n \geq q, X \in \mathbf{Sm}/S\}$. We denote this localizing subcategory by $\Sigma_T^q \mathbf{Ho}\,\mathcal{C}^{eff}$, or $\mathbf{Ho}\,\mathcal{C}^{eff}$ for $q = 0$. We call an object $\mathcal{M}$ of $\mathcal{C}$ *effective* if the image of $\mathcal{M}$ in $\mathbf{Ho}\,\mathcal{C}$ is in $\mathbf{Ho}\,\mathcal{C}^{eff}$, and denote the full subcategory of effective objects of $\mathcal{C}$ by $\mathcal{C}^{eff}$.

Just as above, Neeman's results give a right adjoint $r_q^{\mathcal{R}}$ to the inclusion $i_q^{\mathcal{R}} : \Sigma_T^q \mathcal{C}^{eff} \to \mathcal{C}$ and the composition $f_q^{\mathcal{R}} := i_q^{\mathcal{R}} \circ r_q^{\mathcal{R}}$ is represented by $\tilde{f}_q^{\mathcal{R}}$. One recovers the functors f_q and $\tilde{f}_q$ by taking $\mathcal{R} = \mathbb{S}_S$.

LEMMA 2.1. *Let $\mathcal{R}$ be a cofibrant commutative monoid in Mot. The functors $f_q^{\mathcal{R}} : \mathbf{Ho}\,\mathcal{C} \to \mathbf{Ho}\,\mathcal{C}$ and their liftings $\tilde{f}_q^{\mathcal{R}}$ have the following properties.*

1. *Each $f_n^{\mathcal{R}}$ is idempotent, i.e., $(f_n^{\mathcal{R}})^2 = f_n^{\mathcal{R}}$.*
2. *$f_n^{\mathcal{R}} \Sigma_T^1 = \Sigma_T^1 f_{n-1}^{\mathcal{R}}$ for $n \in \mathbb{Z}$.*
3. *Each $\tilde{f}_n^{\mathcal{R}}$ commutes with homotopy colimits.*

(4) *Suppose that $\mathcal{R}$ is in $\mathcal{SH}^{eff}(S)$. Then the forgetful functor $U : \mathbf{Ho}\,\mathcal{R}$-Mod $\to \mathcal{SH}(S)$ induces an isomorphism $U \circ f_q^{\mathcal{R}} \cong f_q \circ U$ as well as an isomorphism $U \circ s_q^{\mathcal{R}} \cong s_q \circ U$, for all $q \in \mathbb{Z}$.*

Proof. (1) and (2) follow from universal property of triangulated functors $f_n^{\mathcal{R}}$. In case $\mathcal{R} = \mathbb{S}_S$, (3) is proved in [S10, Cor 4.6]; the proof for general $\mathcal{R}$ is the same. For (4), it suffices to prove the result for f_q and $f_q^{\mathcal{R}}$. Take $\mathcal{M} \in \mathcal{C}$. We check the universal property of $Uf_q^{\mathcal{R}}\mathcal{M} \to U\mathcal{M}$: Since $\mathcal{R}$ is in $\mathcal{SH}^{eff}(S)$ and the functor $-\wedge\mathcal{R}$ is compatible with homotopy cofiber sequences and direct sums, $-\wedge\mathcal{R}$ maps $\Sigma_T^q\mathcal{SH}^{eff}(S)$ into itself for each $q \in \mathbb{Z}$. As $U(\mathcal{R}\wedge\mathcal{E}) = \mathcal{R}\wedge\mathcal{E}$, it follows that $U(\Sigma_T^q\mathbf{Ho}\,\mathcal{R}\text{-Mod}^{eff}) \subset \Sigma_T^q\mathcal{SH}^{eff}(S)$ for each q. In particular, $U(f_q^{\mathcal{R}}(\mathcal{M}))$ is in $\Sigma_T^q\mathcal{SH}^{eff}(S)$. For $p \geq q$, $X \in \mathbf{Sm}/S$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma_T^p\Sigma_T^\infty X_+, U(f_q^{\mathcal{R}}(\mathcal{M}))) &\cong \mathrm{Hom}_{\mathbf{Ho}\,\mathcal{C}}(\mathcal{R}\wedge\Sigma_T^p\Sigma_T^\infty X_+, f_q^{\mathcal{R}}(\mathcal{M}))\\ &\cong \mathrm{Hom}_{\mathbf{Ho}\,\mathcal{C}}(\mathcal{R}\wedge\Sigma_T^p\Sigma_T^\infty X_+, \mathcal{M})\\ &\cong \mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma_T^p\Sigma_T^\infty X_+, U(\mathcal{M})),\end{aligned}$$

so the canonical map $U(f_q^{\mathcal{R}}(\mathcal{M})) \to f_q(U(\mathcal{M}))$ is therefore an isomorphism. □

From the adjunction $\mathrm{Hom}_{\mathcal{C}}(\mathcal{R}, \mathcal{M}) \cong \mathrm{Hom}_{Mot}(\mathbb{S}_S, \mathcal{M})$ and the fact that $\mathbb{S}_S$ is a cofibrant object of Mot, we see that $\mathcal{R}$ is a cofibrant object of $\mathcal{C}$. Thus $\mathcal{C}$ is a closed symmetric monoidal simplicial model category with cofibrant unit $1 := \mathcal{R}$ and monoidal product $\otimes = \wedge_{\mathcal{R}}$. Similarly, $T_{\mathcal{R}} := \mathcal{R}\wedge T$ is a cofibrant object of $\mathcal{C}$. Abusing notation, we write $\Sigma_T(-)$ for the endofunctor $A \mapsto A \otimes T_{\mathcal{R}}$ of $\mathcal{C}$. The compatibility of the simplicial monoidal structure with monoidal structure of $\mathcal{C}$ follows directly from the construction of $\mathcal{C}$.
We recall that the category Mot satisfies the monoid axiom of Schwede-Shipley [ScSh, definition 3.3]; the reader can see for example the proof of [Hoy, lemma 4.2]. Following remark 1.1, there is a fibrant replacement $\mathcal{R} \to \mathbf{1}$ in $\mathcal{C}$ such that $\mathbf{1}$ is an $\mathcal{R}$-algebra; in particular, $\mathcal{R} \to \mathbf{1}$ is a cofibration and a weak equivalence in both $\mathcal{C}$ and in Mot, and $\mathbf{1}$ is fibrant in in both $\mathcal{C}$ and in Mot.
For each $\bar{x} \in \mathcal{R}^{-2d,-d}(S)$, we have the corresponding element $\bar{x} : T_{\mathcal{R}}^{\otimes d} \to \mathcal{R}$ in $\mathbf{Ho}\,\mathcal{C}$, which we may lift to a morphism $x : T_{\mathcal{R}}^{\otimes d} \to \mathbf{1}$ in $\mathcal{C}$. Thus, for a collection of elements $\{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S) \mid i \in I\}$, we have the associated collection of maps in $\mathcal{C}$, $\{x_i : T_{\mathcal{R}}^{\otimes d_i} \to \mathbf{1} \mid i \in I\}$ and thereby the quotient object $\mathbf{1}/(\{x_i\})$ in $\mathcal{C}$. Similarly, for $\mathcal{M}$ an $\mathcal{R}$-module, we have the $\mathcal{R}$-module $\mathcal{M}/(\{x_i\})$, which is a cofibrant object in $\mathcal{C}$. We often write $\mathcal{R}/(\{x_i\})$ for $\mathbf{1}/(\{x_i\})$.

Lemma 2.2. *Suppose that $\mathcal{R}$ is in $\mathcal{SH}^{eff}(S)$. Then for any set*

$$\{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S) \mid i \in I, d_i > 0\}$$

of elements of $\mathcal{R}$-cohomology, the object $\mathcal{R}/(\{x_i\})$ is effective. If in addition $\mathcal{M}$ is an $\mathcal{R}$-module and is effective, then $\mathcal{M}/(\{x_i\})$ is effective.

Proof. This follows from lemma 2.1 since $f_n^{\mathcal{R}}$ is a triangulated functor and $\mathcal{C}^{eff}$ is closed under homotopy colimits. □

Let A be an abelian group and SA the topological sphere spectrum with A-coefficients. For a T-spectrum $\mathcal{E}$ let us denote the spectrum $\mathcal{E} \wedge SA$ by $\mathcal{E} \otimes A$. Of course, if A is the free abelian group on a set S, then $\mathcal{E} \otimes A = \oplus_{s \in S}\mathcal{E}$.
Let $\{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S) \mid i \in I, d_i > 0\}$ be a set of elements of $\mathcal{R}$-cohomology, with I countable. Suppose that $\mathcal{R}$ is cofibrant as an object in Mot and is in $\mathcal{SH}^{eff}(S)$. Let $\mathcal{M}$ be in $\mathcal{C}^{eff}$ and let $Q\mathcal{M} \to \mathcal{M}$ be a cofibrant replacement. By lemma 1.8, we have a homotopy cofiber sequence in $\mathcal{C}$,

$$\operatorname*{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x \otimes Q\mathcal{M} \to Q\mathcal{M} \to \mathcal{M}/(\{x_i\}).$$

Clearly $\operatorname{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x \otimes Q\mathcal{M}$ is in $\Sigma^1_T \mathbf{Ho}\,\mathcal{C}^{eff}$, hence the above sequence induces an isomorphism in $\mathbf{Ho}\,\mathcal{C}$

$$s_0^{\mathcal{R}}\mathcal{M} \xrightarrow{\sigma_{\mathcal{M}}} s_0^{\mathcal{R}}(\mathcal{M}/(\{x_i\})).$$

Composing the canonical map $\mathcal{M}/(\{x_i\}) \to s_0^{\mathcal{R}}(\mathcal{M}/(\{x_i\}))$ with $\sigma_{\mathcal{M}}^{-1}$ gives the canonical map

$$\pi^{\mathcal{R}}_{\mathcal{M}} : \mathcal{M}/(\{x_i\}) \to s_0^{\mathcal{R}}\mathcal{M}$$

in $\mathbf{Ho}\,\mathcal{C}$. Applying the forgetful functor gives the canonical map in $\mathcal{SH}(S)$

$$\pi_{\mathcal{M}} : U(\mathcal{M}/(\{x_i\})) \to U(s_0^{\mathcal{R}}\mathcal{M}) \cong s_0(U\mathcal{M}).$$

This equal to the canonical map $U(\mathcal{M}/(\{x_i\})) \to s_0(U(\mathcal{M}/(\{x_i\})))$ composed with the inverse of the isomorphism $s_0(U\mathcal{M}) \to s_0(U(\mathcal{M}/(\{x_i\})))$.

THEOREM 2.3. *Let $\mathcal{R}$ be a commutative monoid in $Mot(S)$, cofibrant as an object in $Mot(S)$, such that $\mathcal{R}$ is in $\mathcal{SH}^{eff}(S)$. Let $X = \{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S) \mid i \in I, d_i > 0\}$ be a countable set of elements of $\mathcal{R}$-cohomology. Let $\mathcal{M}$ be an $\mathcal{R}$-module in $\mathcal{C}^{eff}$ and suppose that the canonical map $\pi_{\mathcal{M}} : U(\mathcal{M}/(\{x_i\})) \to s_0(U\mathcal{M})$ is an isomorphism. Then for each $n \geq 0$, we have a canonical isomorphism in $\mathbf{Ho}\,\mathcal{C}$,*

$$s_n^{\mathcal{R}}\mathcal{M} \cong \Sigma^n_T s_0^{\mathcal{R}}\mathcal{M} \otimes \mathbb{Z}[X]_n,$$

where $\mathbb{Z}[X]_n$ is the abelian group of weighted-homogeneous degree n polynomials over $\mathbb{Z}$ in the variables $\{x_i, i \in I\}$, $\deg x_i = d_i$. Moreover, for each n, we have a canonical isomorphism in $\mathcal{SH}(S)$,

$$s_n U\mathcal{M} \cong \Sigma^n_T s_0 U\mathcal{M} \otimes \mathbb{Z}[X]_n.$$

Proof. Replacing $\mathcal{M}$ with a cofibrant model, we may assume that $\mathcal{M}$ is cofibrant in $\mathcal{C}$; as $\mathcal{R}$ is cofibrant in Mot, it follows that $U\mathcal{M}$ is cofibrant in Mot. Since $\pi_{\mathcal{M}} = U(\pi^{\mathcal{R}}_{\mathcal{M}})$, our assumption on $\pi_{\mathcal{M}}$ is the same as assuming that $\pi^{\mathcal{R}}_{\mathcal{M}}$ is an isomorphism in $\mathbf{Ho}\,\mathcal{C}$. By construction, $\pi^{\mathcal{R}}_{\mathcal{M}}$ extends to a map of distinguished triangles

$$\begin{tikzcd}
(\operatorname{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x) \otimes \mathcal{M} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{M} \arrow[r] \arrow[d, equal] & \mathcal{M}/(\{x_i\}) \arrow[r] \arrow[d, "\pi^{\mathcal{R}}_{\mathcal{M}}"] & \Sigma(\operatorname{hocolim}_{\mathcal{I}^\circ} \mathcal{D}_x) \otimes \mathcal{M} \arrow[d, "\Sigma\alpha"] \\
f_1^{\mathcal{R}}\mathcal{M} \arrow[r] & \mathcal{M} \arrow[r] & s_0^{\mathcal{R}}\mathcal{M} \arrow[r] & \Sigma f_1^{\mathcal{R}}\mathcal{M},
\end{tikzcd}$$

and thus the map α is an isomorphism. We note that α is equal to the canonical map given by the universal property of $f_1^{\mathcal{R}}\mathcal{M} \to \mathcal{M}$.
We will now identify $f_n^{\mathcal{R}}\mathcal{M}$ in terms of the diagram $\mathcal{D}_x|_{\mathcal{I}_{\deg\geq n}} \otimes \mathcal{M}$, proving by induction on $n \geq 1$ that the canonical map $\operatorname{hocolim} \mathcal{D}_x \otimes \mathcal{M}|_{\deg \geq n} \to f_n^{\mathcal{R}}\mathcal{M}$ in $\mathbf{Ho}\,\mathcal{C}$ is an isomorphism.
As $\mathcal{I}^\circ = \mathcal{I}_{\deg\geq 1}$, the case $n = 1$ is settled. Assume the result for n. We claim that the diagram

$$\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n}] : \mathcal{I}_{\deg\geq n} \to \mathcal{C}$$

satisfies the hypotheses of proposition 1.10. That is, we need to verify that for every monomial M of degree n the natural map

$$\operatorname{hocolim} \tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{>M} \otimes \mathcal{M}] \to \tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}(M) \otimes \mathcal{M}]$$

is a weak equivalence in $\mathcal{C}$. This follows by the string of isomorphisms in $\mathbf{Ho}\,\mathcal{C}$

$$\begin{aligned}
\operatorname{hocolim} \tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{>M} \otimes \mathcal{M}] &\cong \operatorname{hocolim} \tilde{f}_{n+1}^{\mathcal{R}}[\Sigma_T^n \mathcal{D}_{\deg\geq 1} \otimes \mathcal{M}]\\
&\cong \operatorname{hocolim} \Sigma_T^n \tilde{f}_1^{\mathcal{R}}[\mathcal{D}_{\deg\geq 1} \otimes \mathcal{M}]\\
&\cong \Sigma_T^n f_1^{\mathcal{R}} \operatorname{hocolim}[\mathcal{D}_{\deg\geq 1} \otimes \mathcal{M}]\\
&\cong \Sigma_T^n f_1^{\mathcal{R}} f_1^{\mathcal{R}} \mathcal{M}\\
&\cong \Sigma_T^n f_1^{\mathcal{R}} \mathcal{M}\\
&\cong f_{n+1}^{\mathcal{R}} \Sigma_T^n \mathcal{M}\\
&\cong f_{n+1}^{\mathcal{R}}[\mathcal{D}(M) \otimes \mathcal{M}].
\end{aligned}$$

Applying proposition 1.10 and our induction hypothesis gives us the string of isomorphisms in $\mathbf{Ho}\,\mathcal{C}$

$$\begin{aligned}
f_{n+1}^{\mathcal{R}}\mathcal{M} &\cong f_{n+1}^{\mathcal{R}} f_n^{\mathcal{R}}\mathcal{M} \cong f_{n+1}^{\mathcal{R}} \operatorname{hocolim}[\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n}]\\
&\cong \operatorname{hocolim} \tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n}] \cong \operatorname{hocolim} \tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n+1}]\\
&\cong \operatorname{hocolim} \mathcal{D}_x \otimes \mathcal{M}|_{\deg\geq n+1},
\end{aligned}$$

the last isomorphism following from the fact that $\mathcal{D}_x(x^N) \otimes \mathcal{M}$ is in $\Sigma_T^{|N|}\mathcal{C}^{eff}$, and hence the canonical map $\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_x \otimes \mathcal{M}] \to \mathcal{D}_x \otimes \mathcal{M}$ is an objectwise weak equivalence on $\mathcal{I}_{\deg\geq n+1}$.
For the slices s_n we have

$$\begin{aligned}
s_n^{\mathcal{R}}\mathcal{M} := \operatorname{hocofib}(\tilde{f}_{n+1}^{\mathcal{R}}\mathcal{M} \to \tilde{f}_n^{\mathcal{R}}\mathcal{M}) &\cong \operatorname{hocofib}(\tilde{f}_{n+1}^{\mathcal{R}}\tilde{f}_n^{\mathcal{R}}\mathcal{M} \to \tilde{f}_n^{\mathcal{R}}\mathcal{M})\\
&\cong \operatorname{hocofib}(\operatorname{hocolim} \tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{\deg\geq n} \otimes \mathcal{M}] \to \operatorname{hocolim} \mathcal{D}_{\deg\geq n} \otimes \mathcal{M})\\
&\cong \operatorname{hocolim} \operatorname{hocofib}(\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{\deg\geq n} \otimes \mathcal{M}] \to \mathcal{D}_{\deg\geq n} \otimes \mathcal{M}).
\end{aligned}$$

At a monomial of degree greater than n, the canonical map $\tilde{f}_{n+1}^{\mathcal{R}}[\mathcal{D}_{\deg\geq n} \otimes \mathcal{M}] \to \mathcal{D}_{\deg\geq n} \otimes \mathcal{M}$ is a weak equivalence, and at a monomial M of degree n

the homotopy cofiber is given by

$$\begin{aligned}\mathrm{hocofib}(\tilde{f}^{\mathcal{R}}_{n+1}[\mathcal{D}(M)\otimes\mathcal{M}]\to\mathcal{D}(M)\otimes\mathcal{M}) &= \mathrm{hocofib}((\tilde{f}^{\mathcal{R}}_{n+1}[\Sigma^n_T\mathcal{M}]\to\Sigma^n_T\mathcal{M})\\ &\cong \mathrm{hocofib}(\Sigma^n_T\tilde{f}^{\mathcal{R}}_1\mathcal{M}\to\Sigma^n_T\mathcal{M})\cong\Sigma^n_T s^{\mathcal{R}}_0\mathcal{M}\end{aligned}$$

Let $\tilde{s}^{\mathcal{R}}_0$ be the functor on $\mathcal{C}^{eff}$, $\mathcal{N}\mapsto\mathrm{hocofib}(\tilde{f}^{\mathcal{R}}_1\mathcal{N}\to\mathcal{N})$, and let $F_n\mathcal{M}:\mathcal{I}_{\deg\geq n}\to\mathcal{C}^{eff}$ be the diagram

$$F_n(M)=\begin{cases}pt & \text{for }\deg M>n\\ \Sigma^n_T\tilde{s}^{\mathcal{R}}_0\mathcal{M} & \text{for }\deg M=n.\end{cases}$$

We thus have a weak equivalence of pointwise cofibrant functors

$$\mathrm{hocofib}(\tilde{f}^{\mathcal{R}}_{n+1}[\mathcal{D}_{\deg\geq n}\otimes\mathcal{M}]\to\mathcal{D}_{\deg\geq n}\otimes\mathcal{M})\to F_n:\mathcal{I}_{\deg\geq n}\to\mathcal{C},$$

and therefore a weak equivalence on the homotopy colimits. As we have the evident isomorphism in $\mathbf{Ho}\,\mathcal{C}$

$$\operatorname*{hocolim}_{\mathcal{I}_{\deg\geq n}}F_n\cong\oplus_{M,\deg M=n}\Sigma^n_T s^{\mathcal{R}}_0\mathcal{M},$$

this gives us the desired isomorphism $s^{\mathcal{R}}_n\mathcal{M}\cong\Sigma^n_T s^{\mathcal{R}}_0\mathcal{M}\otimes\mathbb{Z}[X]_n$ in $\mathbf{Ho}\,\mathcal{C}$. Applying the forgetful functor and using lemma 2.1 gives the isomorphism $s_nU\mathcal{M}\cong\Sigma^n_T s_0U\mathcal{M}\otimes\mathbb{Z}[X]_n$ in $\mathcal{SH}(S)$. □

Corollary 2.4. *Let $\mathcal{R}$, X and $\mathcal{M}$ be as in theorem 2.3. Let $Z=\{z_j\in\mathbb{Z}[X]_{e_j}\}$ be a collection of homogeneous elements of $\mathbb{Z}[X]$, and let $\mathcal{M}[Z^{-1}]\in\mathcal{C}$ be the localization of $\mathcal{M}$ with respect to the collection of maps $\times z_j:\mathcal{M}\to\Sigma^{-e_j}_T\mathcal{M}$. Then there are natural isomorphisms*

$$s^{\mathcal{R}}_n\mathcal{M}[Z^{-1}]\cong\Sigma^n_T s^{\mathcal{R}}_0\mathcal{M}\otimes\mathbb{Z}[X][Z^{-1}]_n,$$

$$s_nU\mathcal{M}[Z^{-1}]\cong\Sigma^n_T s_0U\mathcal{M}\otimes\mathbb{Z}[X][Z^{-1}]_n.$$

Proof. Each map $\times z_j:\mathcal{M}\to\Sigma^{-e_j}_T\mathcal{M}$ induces the isomorphism $\times z_j:\mathcal{M}[Z^{-1}]\to\Sigma^{-e_j}_T\mathcal{M}[Z^{-1}]$ in $\mathbf{Ho}\,\mathcal{C}$, with inverse $\times z_j^{-1}:\Sigma^{-e_j}_T\mathcal{M}[Z^{-1}]\to\mathcal{M}[Z^{-1}]$. Applying $f^{\mathcal{R}}_q$ gives us the map in $\mathbf{Ho}\,\mathcal{C}$

$$\times z_j:f^{\mathcal{R}}_q\mathcal{M}\to f^{\mathcal{R}}_q\Sigma^{-e_j}_T\mathcal{M}\cong\Sigma^{-e_j}_T f^{\mathcal{R}}_{q+e_j}\mathcal{M}.$$

As $f^{\mathcal{R}}_{q+e_j}\mathcal{M}$ is in $\Sigma^{q+e_j}_T\mathbf{Ho}\,\mathcal{C}^{eff}$, both $\Sigma^{-e_j}_T f^{\mathcal{R}}_{q+e_j}\mathcal{M}$ and $f^{\mathcal{R}}_q\mathcal{M}$ are in $\Sigma^q_T\mathbf{Ho}\,\mathcal{C}^{eff}$. The composition

$$\Sigma^{-e_j}_T f^{\mathcal{R}}_{q+e_j}\mathcal{M}\to\Sigma^{-e_j}_T\mathcal{M}\xrightarrow{\times z_j^{-1}}\mathcal{M}[Z^{-1}]$$

gives via the universal property of $f^{\mathcal{R}}_q$ the map $\Sigma^{-e_j}_T f^{\mathcal{R}}_{q+e_j}\mathcal{M}\to f^{\mathcal{R}}_q\mathcal{M}[Z^{-1}]$. Setting $|N|=\sum_j N_je_j$, this extends to give a map of the system of monomial multiplications

$$\times z^{N-M}:\Sigma^{-|N|}_T f^{\mathcal{R}}_{q+|N|}\mathcal{M}\to\Sigma^{-|M|}_T f^{\mathcal{R}}_{q+|M|}\mathcal{M}$$

to $f_q^{\mathcal{R}}\mathcal{M}[Z^{-1}]$; the universal property of the truncation functors f_n and of localization shows that this system induces an isomorphism

$$\operatorname*{hocolim}_{N\in\mathcal{I}^{\mathrm{op}}}\Sigma_T^{-|N|}f^{\mathcal{R}}_{q+|N|}\mathcal{M}\cong f_q^{\mathcal{R}}\mathcal{M}[Z^{-1}]$$

in $\mathbf{Ho}\,\mathcal{C}$. As the slice functors s_q are exact and commute with hocolim, we have a similar collection of isomorphisms

$$\operatorname*{hocolim}_{N\in\mathcal{I}^{\mathrm{op}}}\Sigma_T^{-|N|}s^{\mathcal{R}}_{q+|N|}\mathcal{M}\cong s_q(\mathcal{M}[Z^{-1}]).$$

Theorem 2.3 gives us the natural isomorphisms

$$\Sigma_T^{-|N|}s^{\mathcal{R}}_{q+|N|}\mathcal{M}\cong\Sigma_T^q s_0^{\mathcal{R}}\mathcal{M}\otimes\mathbb{Z}[X]_{q+|N|};$$

via this isomorphism, the map $\times z_j$ goes over to $\mathrm{id}_{\Sigma_T^q s_0^{\mathcal{R}}\mathcal{M}}\otimes\times z_j$, which yields the result. $\square$

Corollary 2.5. *Let $\mathcal{R}$, X and $\mathcal{M}$ be as in theorem 2.3. Let $Z=\{z_j\in\mathbb{Z}[X]_{e_j}\}$ be a collection of homogeneous elements of $\mathbb{Z}[X]$, and let $\mathcal{M}[Z^{-1}]\in\mathcal{C}$ be the localization of $\mathcal{M}$ with respect to the collection of maps $\times z_j:\mathcal{M}\to\Sigma_T^{-e_j}\mathcal{M}$. Let $m\geq 2$ be an integer. We let $\mathcal{M}[Z^{-1}]/m:=\mathrm{hocofib}\times m:\mathcal{M}[Z^{-1}]\to\mathcal{M}[Z^{-1}]$. Then there are natural isomorphisms*

$$s_n^{\mathcal{R}}\mathcal{M}[Z^{-1}]/m\cong\Sigma_T^n s_0^{\mathcal{R}}\mathcal{M}/m\otimes\mathbb{Z}[X][Z^{-1}]_n,$$

$$s_nU\mathcal{M}[Z^{-1}]/m\cong\Sigma_T^n s_0U\mathcal{M}/m\otimes\mathbb{Z}[X][Z^{-1}]_n.$$

This follows directly from corollary 2.4, noting that $s_n^{\mathcal{R}}$ and s_n are exact functors.

Remark 2.6. Let P be a multiplicatively closed subset of $\mathbb{Z}$. We may replace Mot with its localization $Mot[P^{-1}]$ with respect to P in theorem 2.3, corollary 2.4 and corollary 2.5, and obtain a corresponding description of $s_n^{\mathcal{R}}\mathcal{M}$ and $s_nU\mathcal{M}$ for a commutative monoid $\mathcal{R}$ in $Mot[P^{-1}]$ and an effective $\mathcal{R}$-module $\mathcal{M}$.

For $P=\mathbb{Z}\setminus\{p^n,n=1,2,\ldots\}$, we write $Mot\otimes\mathbb{Z}_{(p)}$ for $Mot[P^{-1}]$ and $\mathcal{SH}(S)\otimes\mathbb{Z}_{(p)}$ for $\mathbf{Ho}\,Mot\otimes\mathbb{Z}_{(p)}$.

3. The slice spectral sequence

The slice tower in $\mathcal{SH}(S)$ gives us the slice spectral sequence, for $\mathcal{E}\in\mathcal{SH}(S)$, $X\in\mathbf{Sm}/S$, $n\in\mathbb{Z}$,

$$E_2^{p,q}(n):=(s_{-q}(\mathcal{E}))^{p+q,n}(X)\Longrightarrow\mathcal{E}^{p+q,n}(X).\tag{3.1}$$

This spectral sequence is not always convergent, however, we do have a convergence criterion:

Lemma 3.1 ([L15, lemma 2.1]). *Suppose that $S=\operatorname{Spec}k$, k a perfect field. Take $\mathcal{E}\in\mathcal{SH}(S)$. Suppose that there is a non-decreasing function $f:\mathbb{Z}\to\mathbb{Z}$*

with $\lim_{n\to\infty} f(n) = \infty$*, such that* $\pi_{a+b,b}\mathcal{E} = 0$ *for* $a \le f(b)$*. Then the for all* Y*, and all* $n \in \mathbb{Z}$*, the spectral sequence* (3.1) *is strongly convergent.*[3]

This yields our first convergence result. For $\mathcal{E} \in \mathcal{SH}(S)$, $Y \in \mathbf{Sm}/S$, $p, q, n \in \mathbb{Z}$, define

$$H^{p-q}(Y, \pi^{\mu}_{-q}(\mathcal{E})(n-q)) := \mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma^{\infty}_T Y_+, \Sigma^{p+q,n} s_{-q}(\mathcal{E})).$$

Here $\Sigma^{a,b}$ is suspension with respect to the sphere $S^{a,b} \cong S^{a-b} \wedge \mathbb{G}_m^{\wedge b}$. This notation is justified by the case $S = \mathrm{Spec}\, k$, k a field of characteristic zero. In this case, there is for each q a canonically defined object $\pi^{\mu}_q(\mathcal{E})$ of Voevodsky's "big" triangulated category of motives $DM(k)$, and a canonical isomorphism

$$EM_{\mathbb{A}^1}(\pi^{\mu}_q(\mathcal{E})) \cong \Sigma^q_T s_q(\mathcal{E}),$$

where $EM_{\mathbb{A}^1} : DM(k) \to \mathcal{SH}(k)$ is the motivic Eilenberg-MacLane functor. The adjoint property of $EM_{\mathbb{A}^1}$ yields the isomorphism

$$\begin{aligned} H^{p-q}(Y, \pi^{\mu}_{-q}(\mathcal{E})(n-q)) &:= \mathrm{Hom}_{DM(k)}(M(Y), \pi^{\mu}_{-q}(\mathcal{E})(n-q)[p-q]) \\ &\cong \mathrm{Hom}_{\mathcal{SH}(S)}(\Sigma^{\infty}_T Y_+, \Sigma^{p+q,n} s_{-q}(\mathcal{E})). \end{aligned}$$

We refer the reader to [P11, RO08, Vo04] for details.

PROPOSITION 3.2. *Let* $\mathcal{R}$ *be a commutative monoid in* $Mot(S)$*, cofibrant as an object in* $Mot(S)$*, with* $\mathcal{R}$ *in* $\mathcal{SH}^{eff}(S)$*. Let* $X := \{\bar{x}_i \in \mathcal{R}^{-2d_i,-d_i}(S)\}$ *be a countable set of elements of* $\mathcal{R}$*-cohomology, with* $d_i > 0$*. Let* P *be a multiplicatively closed subset of* $\mathbb{Z}$ *and let* $\mathcal{M}$ *be an* $\mathcal{R}[P^{-1}]$*-module, with* $U\mathcal{M} \in \mathcal{SH}(S)^{eff}[P^{-1}]$*. Suppose that the canonical map*

$$U(\mathcal{M}/(\{x_i\})) \to s_0 U\mathcal{M}$$

is an isomorphism in $\mathcal{SH}(S)[P^{-1}]$*. Then*
1. The slice spectral sequence for $\mathcal{M}^{**}(Y)$ *has the following form:*

$$E_2^{p,q}(n) := H^{p-q}(Y, \pi^{\mu}_0(\mathcal{M})(n-q)) \otimes_{\mathbb{Z}} \mathbb{Z}[X]_{-q} \Longrightarrow \mathcal{M}^{p+q,n}(Y).$$

2. Suppose that $S = \mathrm{Spec}\, k$*,* k *a perfect field. Suppose further that there is an integer* a *such that* $\mathcal{M}^{2r+s,r}(Y) = 0$ *for all* $Y \in \mathbf{Sm}/S$*, all* $r \in \mathbb{Z}$ *and all* $s \ge a$*. Then the slice spectral sequence converges strongly for all* $Y \in \mathbf{Sm}/S$*,* $n \in \mathbb{Z}$*.*

Proof. The form of the slice spectral sequence follows directly from theorem 2.3, extended via remark 2.6 to the P-localized situation. The convergence statement follows directly from lemma 3.1, where one uses the function $f(r) = r - a$. □

We may extend the slice spectral sequence to the localizations $\mathcal{M}[Z^{-1}]$ as in corollary 2.4.

[3] As spectral sequence $\{E_r^{pq}\} \Rightarrow G^{p+q}$ *converges strongly* to G^* if for each n, the spectral sequence filtration F^*G^n on G^n is finite and exhaustive, there is an $r(n)$ such that for all p and all $r \ge r(n)$, all differentials entering and leaving $E_r^{p,n-p}$ are zero and the resulting maps $E_r^{p,n-p} \to E_\infty^{p,n-p} = \mathbf{Gr}^p_F G^n$ are all isomorphisms.

Proposition 3.3. *Let $\mathcal{R}$, X, P and $\mathcal{M}$ be as in proposition 3.3 and assume that all the hypotheses for (1) in that proposition hold. Let $Z = \{z_j \in \mathbb{Z}[X]_{e_j}\}$ be a collection of homogeneous elements of $\mathbb{Z}[X]$, and let $\mathcal{M}[Z^{-1}] \in \mathcal{C}$ be the localization of $\mathcal{M}$ with respect to the collection of maps $\times z_j : \mathcal{M} \to \Sigma_T^{-e_j}\mathcal{M}$. Then the slice spectral sequence for $\mathcal{M}[Z^{-1}]^{**}(Y)$ has the following form:*

$$E_2^{p,q}(n) := H^{p-q}(Y, \pi_0^\mu(\mathcal{M})(n-q)) \otimes_{\mathbb{Z}} \mathbb{Z}[X][Z^{-1}]_{-q} \Longrightarrow \mathcal{M}[Z^{-1}]^{p+q,n}(Y).$$

Suppose further that $S = \operatorname{Spec} k$, k a perfect field, and there is an integer a such that $\mathcal{M}^{2r+s,r}(Y) = 0$ for all $Y \in \mathbf{Sm}/S$ all $r \in \mathbb{Z}$ and all $s \geq a$. Then the slice spectral sequence converges strongly for all $Y \in \mathbf{Sm}/S$, $n \in \mathbb{Z}$.

The proof is same as for proposition 3.2, using corollary 2.4 to compute the slices of $\mathcal{M}[Z^{-1}]$.

Remark 3.4. Let $\mathcal{R}$ be a commutative monoid in Mot, with $\mathcal{R} \in \mathcal{SH}^{eff}(S)$. Suppose that there are elements $a_i \in \mathcal{R}^{2f_i,f_i}(S)$, $i = 1, 2, \ldots$, $f_i \leq 0$, so that $\mathcal{M}$ is the quotient module $\mathcal{R}/(\{a_i\})$. Suppose in addition that there is a constant c such that $\mathcal{R}^{2r+s,r}(Y) = 0$ for all $Y \in \mathbf{Sm}/S$, $r \in \mathbb{Z}$, $s \geq c$. Then $\mathcal{M}^{2r+s,r}(Y) = 0$ for all $Y \in \mathbf{Sm}/S$, $r \in \mathbb{Z}$, $s \geq c$. Indeed

$$\mathcal{M} := \operatorname*{hocolim}_n \mathcal{R}/(a_1, a_2, \ldots, a_n),$$

so it suffices to handle the case $\mathcal{M} = \mathcal{R}/(a_1, a_2, \ldots, a_n)$, for which we may use induction in n. Assuming the result for $\mathcal{N} := \mathcal{R}/(a_1, a_2, \ldots, a_{n-1})$, we have the long exact sequence ($f = f_n$)

$$\ldots \to \mathcal{N}^{p+2f,q+f}(Y) \xrightarrow{\times a_n} \mathcal{N}^{p,q}(Y) \to \mathcal{M}^{p,q}(Y) \to \mathcal{N}^{p+2f+1,q+f}(Y) \to \ldots .$$

Thus the assumption for $\mathcal{N}$ implies the result for $\mathcal{M}$ and the induction goes through.

4. Slices of quotients of MGL

The slices of a Landweber exact spectrum have been described by Spitzweck in [S12, S10], but a quotient of MGL or a localization of such is often not Landweber exact. We will apply the results of the previous section to describe the slices of the motivic truncated Brown-Peterson spectra $BP\langle n\rangle$, effective motivic Morava K-theory $k(n)$ and motivic Morava K-theory $K(n)$, as well as recovering the known computations for the Landweber examples [S12], such as the Brown-Peterson spectra BP and the Johnson-Wilson spectra $E(n)$.

Let MGL_p be the commutative monoid in $Mot \otimes \mathbb{Z}_{(p)}$ representing p-local algebraic cobordism, as constructed in [PPR, §2.1][4]. As noted in *loc. cit.*, MGL_p is a cofibrant object of $Mot \otimes \mathbb{Z}_{(p)}$. The motivic BP was first constructed by Vezzosi in [Ve01] as a direct summand of MGL_p by using Quillen's idempotent theorem. Here we construct BP and $BP\langle n\rangle$ as quotients of MGL_p; the effective Morava K-theory $k(n)$ is similarly a quotient of MGL_p/p. Our explicit

[4]This gives MGL as a symmetric spectrum, we take the image in the p-localized model structure to define MGL_p.

description of the slices allows us to describe the E_2-terms of slice spectral sequences for BP and $BP\langle n\rangle$.

The bigraded coefficient ring $\pi_{*,*}MGL_p(S)$ contains $\pi_{2*}MU \simeq \mathbb{L}_*$, localized at p, as a graded subring of the bi-degree $(2*,*)$ part, via the classifying map for the formal group law of MGL; see for example [Hoy, remark 6.3]. The ring $\mathbb{L}_{*p} := \mathbb{L}_* \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is isomorphic to polynomial ring $\mathbb{Z}_{(p)}[x_1, x_2, \cdots]$ [A95, Part II, theorem 7.1], where the element x_i has degree $2i$ in π_*MU, degree $(2i, i)$ in $\pi_{*,*}MGL_p$ and degree i in $\mathbb{L}_*$.

The following result of Hopkins-Morel-Hoyois [Hoy] is crucial for the application of the general results of the previous sections to quotients of MGL and MGL_p.

Theorem 4.1 ([Hoy, theorem 7.12]). *Let p be a prime integer, S an essentially smooth scheme over a field of characteristic prime to p. Then the canonical maps $MGL_p/(\{x_i : i = 1, 2, \ldots\}) \to s_0MGL_p \to H\mathbb{Z}_{(p)}$ are isomorphisms in $\mathcal{SH}(S)$. In case $S = \operatorname{Spec} k$, k a perfect field of characteristic prime to p, the inclusion $\mathbb{L}_{*p} \subset \pi_{2*,*}MGL_p(S)$ is an equality.*

This has been extended by Spitzweck. He has constructed [S13] a motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$ in $\mathbf{Spt}_{\mathbb{P}^1}(X)$ with a highly structured multiplication, for an arbitrary base-scheme X. For X smooth and of finite type over a Dedekind domain, $H\mathbb{Z}$ represents motivic cohomology defined as Bloch's higher Chow groups [Vo02]; this theory agrees with Voevodsky's motivic cohomology for smooth schemes of finite type over a perfect field. In addition, Spitzweck has extended theorem 4.1 to an arbitrary base-scheme.

Theorem 4.2 ([S13, theorem 11.3], [S14, corollary 6.6]). *Let p be a prime integer and let S be a scheme whose positive residue characteristics are all prime to p. Then the canonical maps $MGL_p/(\{x_i : i = 1, 2, \ldots\}) \to s_0MGL_p \to H\mathbb{Z}_{(p)}$ are isomorphisms in $\mathcal{SH}(S)$. In case $S = \operatorname{Spec} A$, A a Dedekind domain with all residue characteristics prime to p and with trivial class group, the inclusion $\mathbb{L}_{*p} \subset \pi_{2*,*}MGL_p(S)$ is an equality.*

We define a series of subsets of the set of generators $\{x_i \mid i = 1, 2 \ldots\}$,

$$\begin{aligned}
B_p^c &= \{x_i : i \neq p^k - 1, k \geq 1\},\\
B_p &= \{x_i : i = p^k - 1, k \geq 1\},\\
B\langle n\rangle_p^c &= \{x_i : i \neq p^k - 1, 1 \leq k \leq n\},\\
B\langle n\rangle_p &= \{x_i : i = p^k - 1, 1 \leq k \leq n\},\\
k\langle n\rangle_p &= \{x_{p^n-1}\}.
\end{aligned}$$

We also define

$$k\langle n\rangle_p^c = \{x_i : i \neq p^n - 1, \text{and } x_0 = p\} \subset \{p, x_i \mid i = 1, 2 \ldots\}.$$

Definition 4.3 (BP, $BP\langle n\rangle$ and $E(n)$). The Brown-Peterson spectrum BP is defined as

$$BP := MGL_p/(\{x_i \mid i \in B_p^c\}),$$

the truncated Brown-Peterson spectrum $BP\langle n\rangle$ is defined as

$$BP\langle n\rangle := MGL_p/(\{x_i \mid i \in B\langle n\rangle_p^c\})$$

and the Johnson-Wilson spectrum $E(n)$ is the localization

$$E(n) := BP\langle n\rangle[x_{p^n-1}^{-1}].$$

Definition 4.4 (Morava K-theories $k(n)$ and $K(n)$). *Effective Morava K-theory $k(n)$ is defined as*

$$k(n) := MGL_p/(\{x_i \mid i \in k\langle n\rangle_p^c\}) \cong BP\langle n\rangle/(x_{p-1}, \dots, x_{p^{n-1}-1}, p).$$

Define Morava K-theory $K(n)$ to be the localization

$$K(n) := k(n)[x_{p^n-1}^{-1}].$$

The spectra $BP, BP\langle n\rangle$, $E(n), k(n)$ and $K(n)$ are MGL_p-modules. BP and $E(n)$ are Landweber exact. We let $\mathcal{C}$ denote the category of MGL_p-modules.

Lemma 4.5. *The MGL_p-module spectra $BP, BP\langle n\rangle$ and $k(n)$ are effective. BP and $E(n)$ have the structure of oriented weak commutative ring T-spectra in $\mathcal{SH}(S)$.*

Proof. The effectivity of these theories follows from lemma 2.2 and the fact that homotopy colimits of effective spectra are effective. The ring structure for BP and $E(n)$ follows from the Landweber exactness (see [NSO09]). □

We first discuss the effective theories $BP, BP\langle n\rangle$ and $k(n)$.

Proposition 4.6. *Let p be a prime and S a scheme with all residue characteristics prime to p. Then in $\mathcal{SH}(S)$:*
1. The zeroth slices of both BP and $BP\langle n\rangle$ are isomorphic to p-local motivic Eilenberg-MacLane spectrum $H\mathbb{Z}_{(p)}$, and the zeroth slice of $k(n)$ is isomorphic to $H\mathbb{Z}/p$.
2. The quotient maps from MGL_p induce isomorphisms

$$s_0BP \simeq (s_0MGL)_p \simeq s_0BP\langle n\rangle,$$
$$s_0k(n) \simeq (s_0MGL)_p/p.$$

3. The respective quotient maps from BP, $BP\langle n\rangle$ and $k(n)$ induce isomorphisms

$$BP/(\{x_i : x_i \in B_p\}) \simeq s_0BP,$$
$$BP\langle n\rangle/(\{x_i : x_i \in B\langle n\rangle_p\}) \simeq s_0BP\langle n\rangle,$$
$$k(n)/(x_{p^n-1}) \simeq s_0k(n).$$

Proof. By theorem 4.1 (in case S is essentially smooth over a field) or theorem 4.2 (for general S), the classifying map $MGL \to H\mathbb{Z}$ for motivic cohomology induces isomorphisms

$$MGL_p/(\{x_i : i = 1, 2, \dots\}) \cong s_0MGL_p \cong H\mathbb{Z}_{(p)}$$

in $\mathcal{SH}(S) \otimes \mathbb{Z}_{(p)}$.

Now let $\mathcal{S} \subset \mathbb{N}$ be a subset and $\mathcal{S}^c$ its complement. By remark 1.5, we have an isomorphism

$$(MGL_p/(\{x_i : i \in \mathcal{S}^c\}))/(\{x_i : i \in \mathcal{S}\}) \cong MGL_p/(\{x_i : i \in \mathbb{N}\}).$$

Also, as x_i is a map $\Sigma^{2i,i} MGL_p \to MGL_p$, $i > 0$, the quotient map $MGL_p \to MGL_p/(\{x_i : i \in \mathcal{S}^c\})$ induces an isomorphism

$$s_0 MGL_p \to s_0[MGL_p/(\{x_i : i \in \mathcal{S}^c\})].$$

This gives us isomorphisms

$$\begin{aligned}(MGL_p/(\{x_i : i \in \mathcal{S}^c\}))/(\{x_i : i \in \mathcal{S}\}) &\cong \\ &\cong s_0[MGL_p/(\{x_i : i \in \mathcal{S}^c\})] \cong s_0 MGL_p,\end{aligned}$$

with the first isomorphism induced by the quotient map

$$MGL_p/(\{x_i : i \in \mathcal{S}^c\} \to (MGL_p/(\{x_i : i \in \mathcal{S}^c\}))/(\{x_i : i \in \mathcal{S}\}).$$

Taking $\mathcal{S} = B_p, B\langle n\rangle_p, \{x_{p^n-1}\}$ proves the result for BP, $BP\langle n\rangle$ and $k(n)$, respectively. $\square$

For motivic spectra $\mathcal{E} = BP,\ BP\langle n\rangle,\ k(n),\ E(n)$ and $K(n)$ defined in 4.3 and 4.4 let us denote the corresponding topological spectra by $\mathcal{E}^{top}$. The graded coefficient rings $\mathcal{E}^{top}_*$ of these topological spectra are

$$\mathcal{E}^{top}_* \simeq \left\{ \begin{array}{ll} \mathbb{Z}_p[v_1,\, v_2, \cdots] & \mathcal{E} = BP \\ \mathbb{Z}_p[v_1,\, v_2, \cdots, v_n] & \mathcal{E} = BP\langle n\rangle \\ \mathbb{Z}_p[v_1,\, v_2, \cdots, v_n,\, v_n^{-1}] & \mathcal{E} = E(n) \\ \mathbb{Z}/p[v_n] & \mathcal{E} = k(n) \\ \mathbb{Z}/p[v_n,\, v_n^{-1}] & \mathcal{E} = K(n) \end{array} \right\}$$

where $\deg v_n = 2(p^n - 1)$. The element v_n corresponds to the element $\bar{x}_n \in MGL^{2n,n}(k)$.

COROLLARY 4.7. *Let p be a prime integer and let S be a scheme whose positive residue characteristics are all prime to p. Then in $\mathcal{SH}(S)$, the slices of Brown-Peterson, Johnson-Wilson and Morava theories are given by*

$$s_i\mathcal{E} \simeq \left\{ \begin{array}{ll} \Sigma^i_T H_{\mathbb{Z}_p} \otimes \mathcal{E}^{top}_{2i} & \mathcal{E} = BP,\ BP\langle n\rangle \text{ and } E(n) \\ \Sigma^i_T H_{\mathbb{Z}/p} \otimes \mathcal{E}^{top}_{2i} & \mathcal{E} = k(n) \text{ and } K(n) \end{array} \right\}$$

where $\mathcal{E}^{top}_{2i}$ is degree $2i$ homogeneous component of coefficient ring of the corresponding topological theory.

Proof. The statement for BP and $BP\langle n\rangle$ follows from theorem 2.3, and remark 2.6. The case of $E(n)$ follows from corollary 2.4 and the cases of $k(n)$ and $K(n)$ follow from corollary 2.5. $\square$

THEOREM 4.8. *Let p be a prime integer and let S be a scheme whose positive residue characteristics are all prime to p. The slice spectral sequence for any of the spectra $\mathcal{E} = BP,\ BP\langle n\rangle,\ k(n),\ E(n)$ and $K(n)$ in $\mathcal{SH}(S)$ has the form*

$$\mathcal{E}^{p,q}_2(X,m) = H^{p-q}(X, \mathcal{Z}(m-q)) \otimes_{\mathbb{Z}} \mathcal{E}^{top}_{-2q} \Rightarrow \mathcal{E}^{p+q,m}(X),$$

where $\mathcal{Z} = \mathbb{Z}_p$ for $\mathcal{E} = BP$, $BP\langle n\rangle$ and $E(n)$, and $\mathcal{Z} = \mathbb{Z}/p$ for $\mathcal{E} = k(n)$ and $K(n)$. In case $S = \operatorname{Spec} k$ and k is perfect, these spectral sequences are all strongly convergent.

Proof. The form of the slice spectral sequence for $\mathcal{E}$ follows from corollary 4.7. The fact that the slice spectral sequences strongly converge for $S = \operatorname{Spec} k$, k perfect, follows from remark 3.4 and the fact that $MGL^{2r+s,r}(Y) = 0$ for all $Y \in \mathbf{Sm}/S$, $r \in \mathbb{Z}$ and $s \geq 1$. This in turn follows from the Hopkins-Morel-Hoyois spectral sequence

$$E_2^{p,q}(n) := H^{p-q}(Y, \mathbb{Z}(n-q)) \otimes \mathbb{L}_{-q} \Longrightarrow MGL^{p+q,n}(Y),$$

which is strongly convergent by [Hoy, theorem 8.12]. □

5. Modules for oriented theories

We will use the slice spectral sequence to compute the "geometric part" $\mathcal{E}^{2*,*}$ of a quotient spectrum $\mathcal{E} = MGL_p/(\{x_{i_j}\})$ in terms of algebraic cobordism, when working over a base field k of characteristic zero. As the quotient spectra are naturally MGL_p-modules but may not have a ring structure, we will need to extend the existing theory of oriented Borel-Moore homology and related structures to allow for modules over ring-based theories.

5.1. Oriented Borel-Moore homology. We first discuss the extension of oriented Borel-Moore homology. We use the notation and terminology of [LM09, §5]. Let $\mathbf{Sch}/k$ be the category of quasi-projective schemes over a field k and let $\mathbf{Sch}/k'$ denote the subcategory of projective morphisms in $\mathbf{Sch}/k$. Let $\mathbf{Ab}_*$ denote the category of graded abelian groups, $\mathbf{Ab}_{**}$ the category of bi-graded abelian groups.

Definition 5.1. Let A be an oriented Borel-Moore homology theory on $\mathbf{Sch}/k$ [LM09, definition 5.1.3]. An oriented A-module B is given by

(MD1) An additive functor $B_* : \mathbf{Sch}/k' \to \mathbf{Ab}_*$, $X \mapsto B_*(X)$.
(MD2) For each l. c. i. morphism $f : Y \to X$ in $\mathbf{Sch}/k$ of relative dimension d, a homomorphism of graded groups $f^* : B_*(X) \to B_{*+d}(Y)$.
(MD3) For each pair (X, Y) of objects in $\mathbf{Sch}/k$ a bilinear graded pairing

$$A_*(X) \otimes B_*(Y) \to B_*(X \times_k Y)$$
$$u \otimes v \mapsto u \times v$$

which is associative and unital with respect to the external products in the theory A.
These satisfy the conditions (BM1), (BM2), (PB) and (EH) of [LM09, definition 5.1.3]. In addition, these satisfy the following modification of (BM3).

(MBM3) Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms in $\mathbf{Sch}/k$. If f and g are projective, then for $u' \in A_*(X')$, $v' \in B_*(Y')$, one has

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v').$$

If f and g are l. c. i. morphisms, then for $u \in A_*(X)$, $v \in B_*(Y)$, one has

$$(f \times g)^*(u \times v) = f_*(u) \times g_*(v).$$

Let $f : A \to A'$ be a morphism of Borel-Moore homology theories, let B be an oriented A-module, B' an oriented A'-module. A morphism $g : B \to B'$ over f is a collection of homomorphisms of graded abelian groups $g_X : B_*(X) \to B'_*(X)$, $X \in \mathbf{Sch}/k$ such that the g_X are compatible with projective push-forward, l. c. i. pull-back and external products.

We do not require the analog of the axiom (CD) of [LM09, definition 5.1.3]; this axiom plays a role only in the proof of universality of Ω_*, whereas the universality of Ω for A-modules follows formally from the universality for Ω among oriented Borel-Moore homology theories (see proposition 5.3 below).

EXAMPLE 5.2. Let N_* be a graded module for the Lazard ring $\mathbb{L}_*$ and let A_* be an oriented Borel-Moore homology theory. Define $A_*^N(X) := A_*(X) \otimes_{\mathbb{L}_*} N_*$. Then with push-forward $f_*^N := f_*^A \otimes \mathrm{id}_{N_*}$, pull-back $f_N^* := f_A^* \otimes \mathrm{id}_{N_*}$, and product $u \times (v \otimes n) := (u \times v) \otimes n$, for $u \in A_*(X)$, $v \in A_*(Y)$, $n \in N_*$, A_*^N becomes an oriented A-module. Sending N_* to A_*^N gives a functor from graded $\mathbb{L}_*$-modules to oriented A-modules.
In case k has characteristic zero, we note that, for $A_* = \Omega_*$, we have a canonical isomorphism $\theta_{N_*} : \Omega_*^{N_*}(k) \cong N_*$, as the classifying map $\mathbb{L}_* \to \Omega_*(k)$ is an isomorphism [LM09, theorem 1.2.7].

Just as for a Borel-Moore homology theory, one can define operations of $A_*(Y)$ on $B_*(Z)$ via a morphism $f : Z \to Y$, assuming that Y is in $\mathbf{Sm}/k$: for $a \in A_*(Y)$, $b \in B_*(Z)$, define $a \cap_f b \in B_*(Z)$ by

$$a \cap_f b := (f, \mathrm{id}_Z)^*(a \times b),$$

where $(f, \mathrm{id}_Z) : Z \to Y \times_k Z$ is the (transpose of) the graph embedding. As Y is smooth over k, (f, id_Z) is an l. c. i. morphism, so the pullback $(f, \mathrm{id}_Z)^*$ is defined. Similarly, $B_*(Y)$ is an $A_*(Y)$-module via

$$a \cup_Y b := \delta_Y^*(a \times b).$$

These products satisfy the analog of the properties listed in [LM09, §5.1.4, proposition 5.2.1].

PROPOSITION 5.3. *Let A be an oriented Borel-Moore homology theory on $\mathbf{Sch}/k$ and let B be an oriented A-module. Let $\vartheta_A : \Omega_* \to A_*$ be the classifying map. There is a unique morphism $\theta_{A/B} : \Omega_*^{B_*(k)} \to B_*$ over ϑ_A such that $\theta_{A/B}(k) : \Omega_*^{B_*(k)}(k) \to B_*(k)$ is the canonical isomorphism $\theta_{B_*(k)}$.*

Proof. For $X \in \mathbf{Sch}/k$, $b \in B_*(k)$ and $u \in \Omega_*(X)$, we define $\theta_{A/B}(u \otimes b) := \vartheta_A(u) \times b \in B_*(X \times_k k) = B_*(X)$. It is easy to check that this defines a morphism over ϑ_A. Uniqueness follows easily from the fact that the product structure in A and Ω is unital. □

5.2. Oriented duality theories. Next, we discuss a theory of modules for an oriented duality theory (H, A). We use the notation and definitions from [L08]. In particular, we have the category $\mathbf{SP}$ of smooth pairs over k, with objects (M, X), $M \in \mathbf{Sm}/k$, $X \subset M$ a closed subset, and where a morphism $f : (M, X) \to (N, Y)$ is a morphism $f : M \to N$ in $\mathbf{Sm}/k$ such that $f^{-1}(Y) \subset X$.

Definition 5.4. Let A be a bi-graded oriented ring cohomology theory, in the sense of [L08, definition 1.5, remark 1.6]. An oriented A-module B is a bi-graded cohomology theory on $\mathbf{SP}$, satisfying the analog of [L08, definition 1.5], that is: for each pair of smooth pairs (M, X), (N, Y) there is a bi-graded homomorphism

$$\times : A^{**}_X(M) \otimes B^{**}_Y(N) \to B^{**}_{X\times Y}(M \times_k N)$$

satisfying

(1) *associativity*: $(a \times b) \times c = a \times (b \times c)$ for $a \in A^{**}_X(M)$, $b \in A^{**}_Y(N)$, $c \in B^{**}_Z(P)$.
(2) *unit*: $1 \times a = a$.
(3) *Leibniz rule*: Given smooth pairs (M, X), (M, X'), (N, Y) with $X \subset X'$ we have

$$\partial_{M\times N,X'\times N,X\times N}(a \times b) = \partial_{M,X',X}(a) \times b$$

for $a \in A^{**}_{X'\setminus X}(M \setminus X)$, $b \in B^{**}_Y(N)$. For a triple (N, Y', Y) with $Y \subset Y' \subset N$, $a \in A^{m,*}_{X'}(M)$, $b \in B^{**}_{Y'\setminus Y}(N \setminus Y)$ we have

$$\partial_{M\times N,M\times Y',M\times Y}(a \times b) = (-1)^m a \times \partial_{N,Y',Y}(b).$$

We write $a \cup b \in B_{X\cap Y}(M)$ for $\delta^*_M(a \times b)$, $a \in A^{**}_X(M)$, $b \in B^{**}_Y(M)$.
In addition, we assume that the "Thom classes theory" [P09, lemma 3.7.2] arising from the orientation on A induces an orientation on B in the following sense: Let (M, X) be a smooth pair and let $p : E \to M$ be a rank r vector bundle on M. Then the cup product with the Thom class $th(E) \in A^{2r,r}_M(E)$

$$B^{**}_X(M) \xrightarrow{p^*} B^{**}_{p^{-1}(X)}(E) \xrightarrow{th(E)\cup(-)} B^{2r+*,r+*}_X(E)$$

is an isomorphism.

We call an orientation on A that induces an orientation on B as above an orientation on (A, B), or just an orientation on B.
Given an orientation ω on A, one has 1st Chern classes in A for line bundles, where for $L \to M$ a line bundle over $M \in \mathbf{Sm}/k$ with zero section $s : M \to L$, one defines $c_1(L) \in A^{2,1}(X)$ as $s^*(th(L))$.
Let $\mathbf{SP}'$ be the category with the same objects (M, X) as in $\mathbf{SP}$, where a morphism $f : (M, X) \to (N, Y)$ is a projective morphism $f : M \to N$ such that $f(X) \subset Y$. One proceeds just as in [L08] to show that the orientation on B gives rise to an integration on B. To describe this more precisely, we first need to extend the notion of an *integration with support* [L08, definition 1.8] to the setting of bi-graded A-modules.

The discussion in [L08] is carried out in the setting of an ungraded cohomology theory; we modify this by introducing a bi-grading on the cohomology theory A as well as on the A-module B as above. An *integration with supports* for the pair (A, B) is defined by modifying the axioms of [L08, definition 1.8] as follows. We first discuss the modifications for A. The bi-grading is incorporated in that the pushforward map F_* associated to a morphisms $F : (M, X) \to (N, Y)$ in $\mathbf{SP'}$ has the form $F_* : A^{**}_X(M) \to A^{*-2d,*-d}_Y(N)$, where $d = \dim_k M - \dim_k N$. With this refinement, the remaining parts of definition 1.8 for A remain the same. For the module B, one requires as above that one has for each morphism $F : (M, X) \to (N, Y)$ in $\mathbf{SP'}$ a pushforward map $F_* : B^{**}_X(M) \to B^{*-2d,*-d}_Y(N)$. In addition, one modifies the multiplicative structure $f^*(-)\cup$ and $\cup$ for A in definition 1.8(2) of *loc. cit.* to bi-graded products

$$f^*(-)\cup : A^{**}_Z(M) \otimes B^{**}_Y(N) \to B^{**}_{Y\cap f^{-1}(Z)}(N)$$

and

$$\cup : A^{**}_Z(M) \otimes B^{**}_X(M) \to B_{X\cap Z}(M),$$

and, with these changes, we require that B satisfies the conditions of definition 1.8(2) of *loc. cit.* We call such a structure an *integration with supports on* (A, B).

Given an integration with supports on (A, B) and an orientation ω on (A, B) we say (as in [L08, definition 1.11]) that the integration with supports is subjected to ω if for each smooth pair (M, X) and each line bundle $p : L \to M$ with zero section $s : M \to L$, the compositions

$$A^{**}_X(M) \xrightarrow{s_*} A^{*-2,*-1}_{p^{-1}(X)}(L) \xrightarrow{s^*} A^{*-2,*-1}_X(M),$$

$$B^{**}_X(M) \xrightarrow{s_*} B^{*-2,*-1}_{p^{-1}(X)}(L) \xrightarrow{s^*} B^{*-2,*-1}_X(M)$$

are given by respective cup product with $c_1(L)$.

We have the analog of [L08, theorem 1.12] in the setting of oriented modules.

Theorem 5.5. *Let A be a bi-graded ring cohomology theory with orientation ω and let B be an oriented A-module with orientation induced by ω. Then there is a unique integration with supports on (A, B) subjected to the orientation ω.*

The proof is exactly the same way as the proof of theorem 1.12 of *loc. cit.* We now extend the notion of an oriented duality theory to the setting of modules.

Definition 5.6. Let (H, A) be an oriented duality theory, in the sense of [L08, definition 3.1]. An oriented (H, A)-module is a pair (J, B), where

(D1) $J : \mathbf{Sch}/k' \to \mathbf{Ab}_{**}$ is a functor.

(D2) B is an oriented A-module.

(D3) For each open immersion $j : U \to X$ there is a pullback map $j^* : J_{**}(X) \to J_{**}(U)$.

(D4) i. For each smooth pair (M, X) and each morphism $f : Y \to M$ in $\mathbf{Sch}/k$, there is a bi-graded cap product map

$$f^*(-)\cap : A_X(M) \otimes H(Y) \to H(f^{-1}(X)).$$

ii. For $X, Y \in \mathbf{Sch}/k$, there is a bi-graded external product

$$\times : H_{**}(X) \otimes J_{**}(Y) \to J_{**}(X \times Y).$$

(D5) For each smooth pair (M, X), there is a graded isomorphism

$$\beta_{M,X} : J_{**}(X) \to B_X^{2d-*,d-*}(M); \quad d = \dim_k M.$$

(D6) For each $X \in \mathbf{Sch}/k$ and each closed subset $Y \subset X$, there is a map

$$\partial_{X,Y} : J_{*+1,*}(X \setminus Y) \to J_{**}(Y).$$

These satisfy the evident analogs of properties (A1)-(A4) of [L08, definition 3.1], where we make the following changes: Let $d = \dim_k M$, $e = \dim_k N$. One replaces H with J_{**} throughout (except in (A3)(ii)), and

- in (A1) one replaces $A_Y(N)$, $A_X(M)$ with $B_Y^{2d-*,d-*}(N)$, $B_X^{2d-*,d-*}(M)$,
- in (A2) one replaces $A_Y(N)$, $A_X(M)$ with $B_Y^{2e-*,e-*}(N)$, $B_X^{2d-*,d-*}(M)$,
- in (A3)(i) one replaces $A_Y(M)$ with $B_Y^{2d-*,d-*}(M)$ and $A_{Y\cap f^{-1}(X)}(N)$ with $B_{Y\cap f^{-1}(X)}^{2e-*,e-*}(N)$,
- in (A3)(ii) one replaces $A_Y(M)$ with $B_Y^{2e-*,e-*}(N)$ and $A_{X\times Y}(M\times N)$ with $B_{X\times Y}^{2(d+e)-*,d+e-*}(M\times N)$, $H(X)$ with $H_{**}(X)$, $H(Y)$ with $J_{**}(Y)$ and $H(X \times Y)$ with $J_{**}(X \times Y)$,
- in (A4) one replaces $A_{X\setminus Y}(M \setminus Y)$ with $B_{X\setminus Y}^{2d-*,d-*}(M \setminus Y)$.

Remark 5.7. Let (H, A) be an oriented duality theory on $\mathbf{Sch}/k$, for k a field admitting resolution of singularities. By [L08, proposition 4.2] there is a unique natural transformation

$$\vartheta_H : \Omega_* \to H_{2*,*}$$

of functors $\mathbf{Sch}/k' \to \mathbf{Ab}_*$ compatible with all the structures available for $H_{2*,*}$ and, after restriction to $\mathbf{Sm}/k$ is just the classifying map $\Omega^* \to A^{2*,*}$ for the oriented cohomology theory $X \mapsto A^{2*,*}(X)$. We refer the reader to [L08, §4] for a complete description of the properties satisfied by ϑ_H.

Via ϑ_H and the ring homomorphism $\rho_\Omega : \mathbb{L}_* \to \Omega_*(k)$ classifying the formal group law for Ω_*, we have the ring homomorphism $\rho_H : \mathbb{L}_* \to H_{2*,*}(k)$. If (J, B) is an oriented (H, A)-module, then via the $H_{2*,*}(k)$-module structure on $J_{2*,*}(k)$, ρ_H makes $J_{2*,*}(k)$ a $\mathbb{L}_*$-module. We write J_* for the $\mathbb{L}_*$-module $J_{2*,*}(k)$.

Proposition 5.8. *Let k be a field admitting resolution of singularities. Let (H, A) be an oriented duality theory and (J, B) an oriented (H, A)-module. There is a unique natural transformation $\vartheta_{H/J} : \Omega_*^{J_*} \to J_{2*,*}$ from $\mathbf{Sch}/k' \to \mathbf{Ab}_*$ satisfying*

(1) *$\vartheta_{H/J}$ is compatible with pullback maps j^* for $j : U \to X$ an open immersion in $\mathbf{Sch}/k$.*
(2) *$\vartheta_{H/J}$ is compatible with fundamental classes.*
(3) *$\vartheta_{H/J}$ is compatible with external products.*

(4) *$\vartheta_{H/J}$ is compatible with the action of 1st Chern class operators.*
(5) *Identifying $\Omega_*^{J_*}(k)$ with $J_{2*,*}(k)$ via the product map $\Omega_*(k) \otimes_{\mathbb{L}_*} J_{2*,*}(k) \to J_{2*,*}(k)$, $\vartheta_{H/J}(k) : \Omega_*^{J_*}(k) \to J_{2*,*}$ is the identity map.*

Proof. For $X \in \mathbf{Sch}/k$, we define $\vartheta_{H/J}(X)$ by

$$\vartheta_{H/J}(u \otimes j) = \vartheta_H(u) \times j \in J_{2*,*}(X \times_k \operatorname{Spec} k) = J_{2*,*}(X),$$

for $u \otimes j \in \Omega_*^{J_*}(X) := \Omega_*(X) \otimes_{\mathbb{L}_*} J_{2*,*}(k)$. The properties (1)-(5) follow directly from the construction. As $\Omega_*(X)$ is generated by push-forwards of fundamental classes, the properties (2), (3) and (5) determine $\vartheta_{H/J}$ uniquely. □

Remark 5.9. Let k, (H, A) and (J, B) be as in proposition 5.8. Suppose that $J_* := J_{2*,*}$ has external products $\times_J$ and there is a unit element $1_J \in J_0(k)$ for these external products. Suppose further that these are compatible with the external products $H_*(X) \otimes J_*(Y) \to J_*(X \times_k Y)$, in the sense that

$$(h \times 1_J) \times_J b = h \times b \in J_*(X \times_k Y)$$

for $h \in H_*(X)$, $b \in J_*(Y)$, and that $1_H \times 1_J = 1_J$. Then $\vartheta_{H/J}$ is compatible with external products and is unital. This follows directly from our assumptions and the identity

$$\vartheta_{H/J}((u \otimes h) \times (u' \otimes j')) = \vartheta_H(u) \times \vartheta_{H/J}(u' \otimes (h \times j)).$$

5.3. MODULES FOR ORIENTED RING SPECTRA. We now discuss the oriented duality theory and oriented Borel-Moore homology associated to a module spectrum for an oriented weak commutative ring T-spectrum.

Let ph be the two-sided ideal of phantom maps in $\mathcal{SH}(S)$, where a phantom map is a map $f : \mathcal{E} \to \mathcal{F}$ such that $f \circ g = 0$ for each compact object $\mathcal{A}$ in $\mathcal{SH}(S)$ and each morphism $g : \mathcal{A} \to \mathcal{E}$. Let $\mathcal{E}$ be a weak commutative ring T-spectrum, that is, there are maps $\mu : \mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$, $\eta : \mathbb{S}_S \to \mathcal{E}$ in $\mathcal{SH}(S)$ that satisfy the axioms for a monoid in $\mathcal{SH}(S)/\mathrm{ph}$. An $\mathcal{E}$-module is similarly an object $\mathcal{N} \in \mathcal{SH}(S)$ together with a multiplication map $\rho : \mathcal{E} \wedge \mathcal{N} \to \mathcal{E}$ in $\mathcal{SH}(S)$ that makes $\mathcal{N}$ into a unital $\mathcal{E}$-module in $\mathcal{SH}(S)/\mathrm{ph}$ (see for example [NSO09, §8], where a weak commutative ring T-spectrum is referred to as a T-spectrum $\mathcal{E}$ with a quasi-multiplication $\mu : \mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$).

Suppose that $(\mathcal{E}, c)$ is an oriented weak commutative ring T-spectrum in $\mathcal{SH}(k)$, k a field admitting resolution of singularities. We have constructed in [L08, theorem 3.4] a bi-graded oriented duality theory $(\mathcal{E}'_{**}, \mathcal{E}^{**})$ by defining $\mathcal{E}'_{a,b}(X) := \mathcal{E}_X^{2m-a,m-b}(M)$, where $M \in \mathbf{Sm}/k$ is a chosen smooth quasi-projective scheme containing X as a closed subscheme and $m = \dim_k M$. Let $\mathcal{N}$ be an $\mathcal{E}$-module. For $E \to M$ a rank r vector bundle on $M \in \mathbf{Sm}/k$ and $X \subset M$ a closed subscheme, the Thom classes for $\mathcal{E}$ give rise to a Thom isomorphism $\mathcal{N}_X^{**}(M) \to \mathcal{N}_X^{2r+*,r+*}(E)$.

Using these Thom isomorphisms, the arguments used to construct the oriented duality theory $(\mathcal{E}'_{**}, \mathcal{E}^{**})$ go through without change to give $\mathcal{N}^{**}$ the structure of an oriented $\mathcal{E}^{**}$-module, and to define an oriented $(\mathcal{E}'_{**}, \mathcal{E}^{**})$-module $(\mathcal{N}'_{**}, \mathcal{N}^{**})$, with canonical isomorphisms $\mathcal{N}'_{a,b}(X) \cong \mathcal{N}_X^{2m-a,m-b}(M)$,

$m = \dim_k M$, and where the cap products are induced by the $\mathcal{E}$-modules structure on $\mathcal{N}$.

5.4. Geometrically Landweber exact modules.

Definition 5.10. Let $(\mathcal{E}, c)$ be a weak oriented ring T-spectrum and let $\mathcal{N}$ be an $\mathcal{E}$-module. The *geometric part* of $\mathcal{E}^{**}$ is the $(2*, *)$-part $\mathcal{E}^* := \mathcal{E}^{2*,*}$ of $\mathcal{E}^{**}$, the geometric part of $\mathcal{N}$ is the $\mathcal{E}^*$-module $\mathcal{N}^{2*,*}$, and the geometric part of $\mathcal{N}'$ is similarly given by $X \mapsto \mathcal{N}'_*(X) := \mathcal{N}'_{2*,*}(X)$. This gives us the $\mathbb{Z}$-graded oriented duality theory $(\mathcal{E}'_*, \mathcal{E}^*)$ and the oriented $(\mathcal{E}'_*, \mathcal{E}^*)$-module $(\mathcal{N}'_*, \mathcal{N}^*)$.

Let $(\mathcal{E}, c)$ be a weak oriented ring T-spectrum and let $\mathcal{N}$ be an $\mathcal{E}$-module. By proposition 5.8, we have a canonical natural transformation

$$\vartheta_{\mathcal{E}'/\mathcal{N}'} : \Omega_*^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$$

satisfying the compatibilities listed in that proposition.
We extend the definition of a geometrically Landweber exact weak commutative ring T -spectrum (see [L15, definition 3.7]) to the case of an $\mathcal{E}$-module:

Definition 5.11. Let $(\mathcal{E}, c)$ be a weak oriented ring T-spectrum and let $\mathcal{N}$ be an $\mathcal{E}$-module. We say that $\mathcal{N}$ is *geometrically Landweber exact* if for each point $\eta \in X \in \mathbf{Sm}/k$

i. The structure map $p_\eta : \eta \to \operatorname{Spec} k$ induces an isomorphism $p_\eta^* : \mathcal{N}^{2*,*}(k) \to \mathcal{N}^{2*,*}(\eta)$.
ii. The product map $\cup_\eta : \mathcal{E}^{1,1}(\eta) \otimes \mathcal{N}^{2*,*}(\eta) \to \mathcal{N}^{2*+1,*+1}(\eta)$ induces a surjection $k(\eta)^\times \otimes \mathcal{N}^{2*,*}(\eta) \to \mathcal{N}^{2*+1,*+1}(\eta)$.

Here we use the canonical natural transformation $t_{\mathcal{E}} : \mathbb{G}_m \to \mathcal{E}^{1,1}(-)$ defined in [L15, remark 1.5] to define the map $k(\eta)^\times \to \mathcal{E}^{1,1}(\eta)$ needed in (ii).

The following result generalizes [L15, theorem 6.2] from oriented weak commutative ring T-spectra to modules:

Theorem 5.12. *Let k be a field of characteristic zero, $\mathcal{N}$ an MGL-module in $\mathcal{SH}(k)$, $(\mathcal{N}'_{**}, \mathcal{N}^{**})$ the associated oriented (MGL'_{**}, MGL^{**})-module, and $\mathcal{N}'_*$ the geometric part of $\mathcal{N}'$. Suppose that $\mathcal{N}$ is geometrically Landweber exact. Then the classifying map*

$$\vartheta_{MGL'_*/\mathcal{N}'_*} : \Omega_*^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$$

is an isomorphism.

Remark 5.13. Let k be a field of characteristic zero, let $(\mathcal{E}, c)$ be an oriented weak commutative ring T-spectrum in $\mathcal{SH}(S)$, and let $\mathcal{N}$ be an $\mathcal{E}$-module. Via the classifying map $\varphi_{\mathcal{E},c} : MGL \to \mathcal{E}$, $\mathcal{N}$ becomes an MGL-module. In addition, the classifying map $\vartheta_{\mathcal{E}'} : \Omega_* \to \mathcal{E}'_*$ is induced from $\varphi_{\mathcal{E},c}$ and the classifying map $\vartheta_{MGL'_*/\mathcal{N}'_*}$ factors through the classifying map $\vartheta_{\mathcal{E}'_*/\mathcal{N}'_*} : \mathcal{E}'^{\mathcal{N}'_*(k)}_* \to \mathcal{N}'_*$ as

$$\vartheta_{MGL'_*/\mathcal{N}'_*} = \vartheta_{\mathcal{E}'_*/\mathcal{N}'_*} \circ (\varphi_{\mathcal{E},c} \otimes \mathrm{id}_{\mathcal{N}'_*(k)}).$$

Thus, theorem 5.12 applies to $\mathcal{E}$-modules for arbitrary $(\mathcal{E}, c)$. Moreover, if $(\mathcal{E}, c)$ is geometrically Landweber exact in the sense of [L15, definition 3.7], the map $\bar{\vartheta}_{\mathcal{E}'_*} : \Omega_*^{\mathcal{E}'_*(k)} \to \mathcal{E}'_*$ is an isomorphism ([L15, theorem 6.2]) hence the map $\vartheta_{\mathcal{E}'_*/\mathcal{N}'_*}$ is an isomorphism as well.

Proof of theorem 5.12. The proof of theorem 5.12 is essentially the same as the proof of [L15, theorem 6.2]. Indeed, just as in *loc. cit.*, one constructs a commutative diagram (see [L09, (6.4)])

(5.1)

$$\begin{tikzcd}
\oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathcal{N}'_{*-d+1} \arrow[r, "\mathrm{Div}_{\mathcal{N}}"] & \Omega_*^{\mathcal{N}'_*(1)}(X) \arrow[r, "i_*"] \arrow[d, "\bar{\vartheta}^{(1)}"'] & \Omega_*^{\mathcal{N}'_*}(X) \arrow[r, "j^*"] \arrow[d, "\bar{\vartheta}(X)"'] & \oplus_{\eta\in X_{(d)}} \Omega_*^{\mathcal{N}'_*}(\eta) \arrow[r] \arrow[d, "\bar{\vartheta}"'] & 0 \\
\oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathcal{N}'_{*-d+1} \arrow[r, "div_{\mathcal{N}}"'] & \mathcal{N}'^{(1)}_{2*,*}(X) \arrow[r, "i_*"'] & \mathcal{N}'_{2*,*}(X) \arrow[r, "j^*"'] & \oplus_{\eta\in X_{(d)}} \mathcal{N}'_{2*,*}(\eta) \arrow[r] & 0
\end{tikzcd}$$

where we write $\mathcal{N}'_*$ for $\mathcal{N}'_*(k)$, d is the maximum of $\dim_k X_i$ as X_i runs over the irreducible components of X, and $\mathcal{N}'^{(1)}_{2*,*}(X)$ is the colimit of $\mathcal{N}'_{2*,*}(W)$, as W runs over closed subschemes of X containing no dimension d generic point of X. A similarly defined colimit of the $\Omega_*^{\mathcal{N}'_*}(W)$ gives us $\Omega_*^{\mathcal{N}'_*(1)}(X)$. The maps $\bar{\vartheta}^{(1)}$, $\bar{\vartheta}(X)$ and $\bar{\vartheta}$ are all induced by the classifying map $\vartheta_{MGL'_*/\mathcal{N}'_*}$. The top row is a complex and the bottom row is exact; this latter fact follows from the surjectivity assumption in definition 5.11(ii). The map $\bar{\vartheta}$ is an isomorphism by part (i) of definition 5.11 and $\bar{\vartheta}^{(1)}$ is an isomorphism by induction on d. To show that $\bar{\vartheta}(X)$ is an isomorphism, it suffices to show that the identity map on $\oplus_\eta \mathcal{N}'_{*-d+1} \otimes k(\eta)^\times$ extends diagram (5.1) to a commutative diagram.
To see this, we note that the map $div_{\mathcal{N}}$ is defined by composing the boundary map

$$\partial : \oplus_{\eta\in X_{(d)}} \mathcal{N}'_{2*+1,*}(\eta) \to \mathcal{N}'^{(1)}_{2*,*}(X)$$

with the sum of the product maps $MGL'_{2d-1,d-1}(\eta)\otimes\mathcal{N}'_{*-d+1}(k) \to \mathcal{N}'_{2*+1,*}(\eta)$ and the canonical map $t_{MGL}(\eta) : k(\eta)^\times \to MGL^{1,1}(\eta) = MGL'_{2d-1,d-1}(\eta)$ (see [L09, remark 1.5]). For MGL', we have the similarly defined map

$$div_{MGL} : \oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathbb{L}_{*-d+1} \to MGL'^{(1)}_{2*,*}(X),$$

after replacing $MGL'_{*-d+1}(k)$ with $\mathbb{L}_{*-d+1}$ via the classifying map $\mathbb{L}_* \to MGL'_*(k)$. We have as well the commutative diagram (see [L09, (5.4)])

$$\begin{tikzcd}
\oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathbb{L}_{*-d+1} \arrow[r, "\mathrm{Div}"] \arrow[d, equal] & \Omega_*^{(1)}(X) \arrow[d, "\vartheta^{(1)}_{MGL}"] \\
\oplus_{\eta\in X_{(d)}} k(\eta)^\times \otimes \mathbb{L}_{*-d+1} \arrow[r, "div_{MGL}"'] & MGL'^{(1)}_{2*,*}(X),
\end{tikzcd}$$

which after applying $-\otimes_{\mathbb{L}_*}\mathcal{N}'_*$ gives us the commutative diagram

(5.2)
$$\begin{tikzcd}
\oplus_{\eta\in X_{(d)}} k(\eta)^\times\otimes\mathcal{N}_{*-d+1} \arrow[r, "\mathrm{Div}_{\mathcal{N}}"] \arrow[d, equal] & \Omega_*^{\mathcal{N}'_*(1)}(X) \arrow[d, "\vartheta^{(1)}_{MGL}\otimes\mathrm{id}"] \\
\oplus_{\eta\in X_{(d)}} k(\eta)^\times\otimes\mathcal{N}_{*-d+1} \arrow[r, "\overline{div}_{MGL}"'] & MGL'^{(1)}_{2*,*}(X)\otimes_{\mathbb{L}_*}\mathcal{N}'_*.
\end{tikzcd}$$

The Leibniz rule for ∂ gives us the commutative diagram

(5.3)
$$\begin{tikzcd}
\oplus_{\eta\in X_{(d)}} k(\eta)^\times\otimes\mathcal{N}_{*-d+1} \arrow[r, "\overline{div}_{MGL}"] \arrow[d, equal] & MGL'^{(1)}_{2*,*}(X)\otimes_{\mathbb{L}_*}\mathcal{N}'_* \arrow[d, "\cup"] \\
\oplus_{\eta\in X_{(d)}} k(\eta)^\times\otimes\mathcal{N}'_{*-d+1} \arrow[r, "div_{\mathcal{N}}"'] & \mathcal{N}'^{(1)}_{2*,*}(X);
\end{tikzcd}$$

combining diagrams (5.2) and (5.3) yields the desired commutativity. □

6. Applications to quotients of MGL

We return to our discussion of quotients of MGL_p and their localizations. We select a system of polynomial generators for the Lazard ring, $\mathbb{L}_*\cong\mathbb{Z}[x_1,x_2,\ldots]$, $\deg x_i=i$. Let $\mathcal{S}\subset\mathbb{N}$, $\mathcal{S}^c$ its complement and let $\mathbb{Z}[\mathcal{S}^c]$ denote the graded polynomial ring on the x_i, $i\in\mathcal{S}^c$, $\deg x_i=i$. Let $\mathcal{S}_0\subset\mathbb{Z}[\mathcal{S}^c]$ be a collection of homogeneous elements, $\mathcal{S}_0=\{z_j\in\mathbb{Z}[\mathcal{S}^c]_{e_j}\}$, and let $\mathbb{Z}[\mathcal{S}^c][\mathcal{S}_0^{-1}]$ denote the localization of $\mathbb{Z}[\mathcal{S}^c]$ with respect to $\mathcal{S}_0$.
We consider a quotient spectrum $MGL_p/(\mathcal{S}):=MGL_p/(\{x_i\mid i\in\mathcal{S}\})$ or an integral version $MGL/(\mathcal{S}):=MGL/(\{x_i\mid i\in\mathcal{S}\})$. We consider as well the localizations

$$MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]:=MGL_p/(\mathcal{S})[\{z_j^{-1}\mid z_j\in\mathcal{S}_0\}],$$
$$MGL/(\mathcal{S})[\mathcal{S}_0^{-1}]:=MGL/(\mathcal{S})[\{z_j^{-1}\mid z_j\in\mathcal{S}_0\}].$$

and the mod p version

$$MGL/(\mathcal{S},p)[\mathcal{S}_0^{-1}]:=MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]/p.$$

Proposition 6.1. *Let p be a prime, and let $S=\operatorname{Spec}k$, k a perfect field with exponential characteristic prime to p. Let $\mathcal{S}$ be a subset of $\mathbb{N}$ and $\mathcal{S}_0$ a set of homogeneous elements of $\mathbb{Z}[\mathcal{S}^c]$. Then the spectra $MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]$ and $MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}]$ are geometrically Landweber exact. In case $\operatorname{char}k=0$, $MGL/(\mathcal{S})[\mathcal{S}_0^{-1}]$ is geometrically Landweber exact.*

Proof. We discuss the cases $MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]$ and $MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}]$; the case of $MGL/(\mathcal{S})[\mathcal{S}_0^{-1}]$ is exactly the same.
Let A be a finitely generated abelian group and let η be a point in some $X\in\mathbf{Sm}/k$. Then the motivic cohomology $H^*(\eta,A(*))$ satisfies

$$H^{2r}(\eta,A(r))=H^{2r+1}(\eta,A(r+1))=0$$

for $r \neq 0$,

$$H^0(\eta, A(0)) = A, \quad H^1(\eta, A(1)) = k(\eta)^\times \otimes_{\mathbb{Z}} A.$$

We consider the slice spectral sequences

$$E_2^{p,q}(n) := H^{p-q}(\eta, \mathbb{Z}(n-q)) \otimes \mathbb{Z}[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{-q} \Rightarrow (MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{p+q,n}(\eta)$$

and

$$E_2^{p,q}(n) := H^{p-q}(\eta, \mathbb{Z}/p(n-q)) \otimes \mathbb{Z}[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{-q} \Rightarrow (MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{p+q,n}(\eta)$$

given by proposition 3.3. As in the proof of theorem 4.8, $MGL_p^{2n+a,n}(\eta) = 0$ for $a > 0$ and $n \in \mathbb{Z}$, and thus by remark 3.4, the convergence hypotheses in proposition 3.3 are satisfied. Thus, these spectral sequences are strongly convergent. As discussed in the proof of [L15, proposition 3.8], the only non-zero E_2 term contributing to $(MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{2n,n}(\eta)$ or to $(MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{2n,n}(\eta)$ is $E_2^{n,n}(n)$, the only non-zero E_2 term contributing to $(MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{2n-1,n}(\eta)$ or contributing to $(MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{2n-1,n}(\eta)$ is $E_2^{n,n-1}(n)$, and all differentials entering or leaving these terms are zero.
This gives us isomorphisms

$$\begin{aligned}
&(MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{2n,n}(\eta) \cong \mathbb{Z}_{(p)}[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{-n}\\
&(MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{2n,n}(\eta) \cong \mathbb{Z}/(p)[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{-n}\\
&(MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}])^{2n-1,n}(\eta) \cong \mathbb{Z}_{(p)}[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{1-n} \otimes k(\eta)^\times\\
&(MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}])^{2n-1,n}(\eta) \cong \mathbb{Z}/(p)[\mathcal{S}^c][\mathcal{S}_0^{-1}]_{1-n} \otimes k(\eta)^\times
\end{aligned}$$

from which it easily follows that $MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]$ and $MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}]$ are geometrically Landweber exact. □

COROLLARY 6.2. *Let $S = \operatorname{Spec} k$, k a field of characteristic zero. Fix a prime p and let $\mathcal{N} = MGL/(\mathcal{S})[\mathcal{S}_0^{-1}]$, $MGL_p/(\mathcal{S})[\mathcal{S}_0^{-1}]$ or $MGL_p/(\mathcal{S},p)[\mathcal{S}_0^{-1}]$, let $(\mathcal{N}', \mathcal{N})$ be the associated (MGL', MGL)-module and $\mathcal{N}'_*$ the geometric part of $\mathcal{N}'_{**}$. Then the classifying map*

$$\vartheta_{\mathcal{N}'_*(k)} : \Omega_*^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$$

is an isomorphism of Ω_*-modules.

This follows directly from theorem 5.12 and proposition 6.1. As an immediate consequence, we have

COROLLARY 6.3. *Let $S = \operatorname{Spec} k$, k a field of characteristic zero. Fix a prime p and let $\mathcal{N} = BP$, $BP\langle n\rangle$, $E(n)$, $k(n)$ or $K(n)$, let $(\mathcal{N}', \mathcal{N})$ be the associated (MGL', MGL)-module and $\mathcal{N}'_*$ the geometric part of $\mathcal{N}'_{**}$. Then the classifying map*

$$\vartheta_{\mathcal{N}'_*(k)} : \Omega_*^{\mathcal{N}'_*(k)} \to \mathcal{N}'_*$$

is an isomorphism of Ω_*-modules. In case $\mathcal{N} = BP$ or $E(n)$, $\vartheta_{\mathcal{N}'_*(k)}$ is compatible with external products.

Remark 6.4. Suppose that the theory with supports $\mathcal{N}^{2*,*}$ has products and a unit, compatible with its $MGL^{2*,*}$-module structure. Then by remark 5.9, the classifying map $\vartheta_{\mathcal{N}'_*(k)}$ is also compatible with products.
In the case of a quotient $\mathcal{E}$ of MGL or MGL_p by a subset $\{x_i : i \in I\}$ of the set of polynomial generators, the vanishing of $MGL^{2r+s,r}(k)$ for $s > 0$ shows that $\mathcal{E}^{2*,*}(k) = MGL^{2*,*}(k)/(\{x_i : i \in I\})$, which has the evident ring structure induced by the natural $MGL^{2*,*}(k)$-module structure. Thus, the rational theory $\Omega_*^{\mathcal{E}_*(k)}$ has a canonical structure of an oriented Borel-Moore homology theory on $\mathbf{Sch}/k$; the same holds for $\mathcal{E}$ a localization of this type of quotient. The fact that the classifying homomorphism $\vartheta_{\mathcal{E}} : \Omega_*^{\mathcal{E}'_*(k)} \to \mathcal{E}'_*$ is an isomorphism induces on $\mathcal{E}'_*$ the structure of an oriented Borel-Moore homology theory on $\mathbf{Sch}/k$; it appears to be unknown if this arises from a multiplicative structure on the spectrum level.

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