Birational Geometry and Localisation of Categories

With Appendices by Jean-Louis Colliot-Thélène and Ofer Gabber

To Alexander Merkurjev, with warmest wishes for his 60th birthday

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Received: September 25, 2014
Revised: October 7, 2014

Abstract. We explore connections between places of function fields over a base field F and birational morphisms between smooth F-varieties. This is done by considering various categories of fractions involving function fields or varieties as objects, and constructing functors between these categories. The main result is that in the localised category $S_b^{-1}\mathbf{Sm}(F)$, where $\mathbf{Sm}(F)$ denotes the usual category of smooth varieties over F and S_b is the set of birational morphisms, the set of morphisms between two objects X and Y, with Y proper, is the set of R-equivalence classes $Y(F(X))/R$.

2010 Mathematics Subject Classification: 14E05, 18F99
Keywords and Phrases: Localisation, birational geometry, places, R-equivalence.

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[1]The first author acknowledges the support of Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005 and the second author that of an NSERC Grant. Both authors acknowledge the support of CEFIPRA project 2501-1.

Introduction

Let Φ be a functor from the category of smooth proper varieties over a field F to the category of sets. We say that Φ is *birational* if it transforms birational morphisms into isomorphisms. In characteristic 0, examples of such functors are obtained by choosing a function field K/F and defining $\Phi_K(X) = X(K)/R$, the set of R-equivalence classes of K-rational points [5, Prop. 10]. One of the main results of this paper is that *any birational functor* Φ *is canonically a direct limit of functors of the form* Φ_K.

This follows from Theorem 1 below via the complement to Yoneda's lemma ([SGA4, Exp. I, Prop. 3.4 p. 19] or [28, Ch. III, §, Th. 1 p. 76]). Here is the philosophy which led to this result and others presented here:

Birational geometry over a field F is the study of function fields over F, viewed as generic points of algebraic varieties[2], or alternately the study of algebraic F-varieties "up to proper closed subsets". In this context, two ideas seem related:

- places between function fields;
- rational maps.

The main motivation of this paper has been to understand the precise relationship between them. We have done this by defining two rather different "birational categories" and comparing them.

The first idea gives the category **place** (objects: function fields; morphisms: F-places), that we like to call the *coarse birational category*. For the second idea, one has to be a little careful: the naïve attempt at taking as objects smooth varieties and as morphisms rational maps does not work because, as was pointed out to us by Hélène Esnault, one cannot compose rational maps in general. On the other hand, one can certainly start from the category **Sm** of smooth F-varieties and *localise* it (in the sense of Gabriel-Zisman [12]) with respect to the set S_b of birational morphisms. We like to call the resulting category $S_b^{-1}\mathbf{Sm}$ the *fine birational category*. By hindsight, the problem mentioned by Esnault can be understood as a problem of calculus of fractions of S_b in **Sm**.

In spite of the lack of calculus of fractions, the category $S_b^{-1}\mathbf{Sm}$ was studied in [21] and we were able to show that, under resolution of singularities, the natural functor $S_b^{-1}\mathbf{Sm}^{\mathrm{prop}} \to S_b^{-1}\mathbf{Sm}$ is an equivalence of categories, where $\mathbf{Sm}^{\mathrm{prop}}$ denotes the full subcategory of smooth proper varieties (*loc. cit.* , Prop. 8.5).

What was not done in [21] was the computation of Hom sets in $S_b^{-1}\mathbf{Sm}$. This is the first main result of this paper:

Theorem 1 (*cf.* Th. 6.6.3 and Cor. 6.6.4). *Let X, Y be two smooth F-varieties, with Y proper. Then,*

a) In $S_b^{-1}\mathbf{Sm}$, we have an isomorphism

$$\mathrm{Hom}(X, Y) \simeq Y(F(X))/R$$

[2]By convention all varieties are irreducible here, although not necessarily geometrically irreducible.

where the right hand side is the set of R-equivalence classes in the sense of Manin.
b) The natural functor

$$S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \to S_b^{-1}\mathbf{Sm}$$

is fully faithful. Here $\mathbf{Sm}_*^{\mathrm{prop}}$ is the full subcategory of $\mathbf{Sm}$ with objects those smooth proper varieties whose function field has a cofinal set of smooth proper models (see Definition 4.2.1).

For the link with the result mentioned at the beginning of the introduction, note that $\mathbf{Sm}_*^{\mathrm{prop}} = \mathbf{Sm}^{\mathrm{prop}}$ in characteristic 0, and any birational functor on smooth proper varieties factors uniquely through $S_b^{-1}\mathbf{Sm}^{\mathrm{prop}}$, by the universal property of the latter category.
Theorem 1 implies that $X \mapsto X(F)/R$ is a birational invariant of smooth proper varieties in any characteristic (Cor. 6.6.6), a fact which seemed to be known previously only in characteristic 0 [5, Prop. 10]. It also implies that one can define a composition law on classes of R-equivalence (for smooth proper varieties), a fact which is not at all obvious a priori.
The second main result is a comparison between the coarse and fine birational categories. Let $\mathbf{dv}$ be the subcategory of $\mathbf{place}$ whose objects are separably generated function fields and morphisms are generated by field extensions and places associated to "good" discrete valuation rings (Definition 6.1.1).

Theorem 2 (*cf.* Th. 6.5.2 and 6.7.1). *a) There is an equivalence of categories*

$$\overline{\Psi} : (\mathbf{dv} \,/\, \mathrm{h}')^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}$$

where $\mathbf{dv} \,/\, \mathrm{h}'$ is the quotient category of $\mathbf{dv}$ by the equivalence relation generated by two elementary relations: homotopy of places (definition 6.4.1) and "having a common centre of codimension 2 on some smooth model".
b) If $\operatorname{char} F = 0$, the natural functor $\mathbf{dv} \,/ h' \to \mathbf{place} \,/\, \mathrm{h}''$ is an equivalence of categories, where h'' is generated by homotopy of places and "having a common centre on some smooth model".

(See §1.2 for the notion of an equivalence relation on a category.)
Put together, Theorems 1 and 2 provide an answer to a question of Merkurjev: given a smooth proper variety X/F, give a purely birational description of the set $X(F)/R$. This answer is rather clumsy because the equivalence relation h' is not easy to handle; we hope to come back to this issue later.
Let us introduce the set S_r of *stable birational morphisms*: by definition, a morphism $s : X \to Y$ is in S_r if it is dominant and the function field extension $F(X)/F(Y)$ is purely transcendental. We wondered about the nature of the localisation functor $S_b^{-1}\mathbf{Sm} \to S_r^{-1}\mathbf{Sm}$ for a long time, until the answer was given us by Colliot-Thélène through a wonderfully simple geometric argument (see Appendix A):

Theorem 3 (*cf.* Th. 1.7.2). *The functor $S_b^{-1}\mathbf{Sm} \to S_r^{-1}\mathbf{Sm}$ is an equivalence of categories.*

This shows a striking difference between birational functors and numerical birational invariants, many of which are not stably birationally invariant (for example, plurigenera).
Theorems 1 and 2 are substantial improvements of our results in the first version of this paper [22], which were proven only in characteristic 0: even in characteristic 0, Theorem 2 is new with respect to [22]. Their proofs are intertwined in a way we shall describe now.
The first point is to relate the coarse and fine birational categories, as there is no obvious comparison functor between them. There are two essentially different approaches to this question. In the first one:

- We introduce (Definition 2.2.1) an "incidence category" **SmP**, whose objects are smooth F-varieties and morphisms from X to Y are given by pairs (f,λ), where f is a morphism $X\to Y$, λ is a place $F(Y)\rightsquigarrow F(X)$ and f,λ are *compatible* in an obvious sense. This category maps to both $\mathbf{place}^{\mathrm{op}}$ and **Sm** by obvious forgetful functors. Replacing **Sm** by **SmP** turns out to have a strong rigidifying effect.
- We embed $\mathbf{place}^{\mathrm{op}}$ in the category of locally ringed spaces via the "Riemann-Zariski" variety attached to a function field.

In this way, we obtain a naturally commutative diagram

$$\begin{tikzcd}
 & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P} \arrow[dl, "\overline{\Phi}_1^*"'] \arrow[dr, "\overline{\Phi}_2^*"] & \\
\mathbf{place}_*^{\mathrm{op}} \arrow[dr, "\bar{\Sigma}"'] & & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \arrow[dl, "\bar{J}"] \\
 & S_b^{-1}\widehat{\mathbf{Sm}_*^{\mathrm{prop}}} &
\end{tikzcd}$$

where $\mathbf{place}_*$ denotes the full subcategory of **place** consisting of the function fields of varieties in $\mathbf{Sm}_*^{\mathrm{prop}}$ (compare Theorem 1). Then J is an *equivalence of categories*[3] and the induced functor

$$(*)\qquad \Psi_*:\mathbf{place}_*^{\mathrm{op}}\to S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$$

is *full and essentially surjective* (Theorems 4.2.3 and 4.2.4).
This is more or less where we were in the first version of this paper [22], except for the use of the categories $\mathbf{Sm}_*$ and $\mathbf{place}_*$ which allow us to state results in any characteristic; in [22], we also proved Theorem 1 when $\operatorname{char} F=0$, using resolution of singularities and a complicated categorical method.[4]
The second approach is to construct a functor $\mathbf{dv}^{\mathrm{op}}\to S_b^{-1}\mathbf{Sm}$ directly. Here the new and decisive input is the recent paper of Asok and Morel [1], and especially the results of its §6: they got the insight that, working with discrete

[3] So is $\overline{\Phi}_1^*$.

[4] Another way to prove Theorem 1 in characteristic 0, which was our initial method, is to define a composition law on R-equivalence classes by brute force (still using resolution of singularities) and to proceed as in the proof of Proposition 6.4.3.

valuations of rank 1, all the resolution that is needed is "in codimension 2". We implement their method in §6 of the present paper, which leads to a rather simple proof of Theorems 1 and 2 in any characteristic. Another key input is a recent uniformisation theorem of Knaf and Kuhlmann [23].

Let us now describe the contents in more detail. We start by setting up notation in Section 1, which ends with Theorem 3. In Section 2, we introduce the incidence category **SmP** sitting in the larger category **VarP**, the forgetful functors $\mathbf{VarP} \to \mathbf{Var}$ and $\mathbf{VarP} \to \mathbf{place}^{\mathrm{op}}$, and prove elementary results on these functors (see Lemmas 2.3.2 and 2.3.4). In Section 3, we endow the abstract Riemann variety with the structure of a locally ringed space, and prove that it is a cofiltered inverse limit of proper models, viewed as schemes (Theorem 3.2.8): this ought to be well-known but we couldn't find a reference. We apply these results to construct in §4 the functor (*), using calculus of fractions. In section 5, we study calculus of fractions in greater generality; in particular, we obtain a partial calculus of fractions in $S_b^{-1}\mathbf{Sm}_*$ in Proposition 5.4.1.

In §6, we introduce a notion of homotopy on **place** and the subcategory **dv**. We then relate our approach to the work of Asok-Morel [1] to prove Theorems 1 and 2. We make the link between the first and second approaches in Theorem 6.7.1 = Theorem 2 b).

Section 7 discusses variants of Kollár's notion of rational chain connectedness (which goes back to Chow under the name of linear connectedness), recalls classical theorems of Murre, Chow and van der Waerden, states new theorems of Gabber including the one proven in Appendix B, and draws some consequences in Theorem 7.3.1. Section 8 discusses some applications, among which we like to mention the existence of a "universal birational quotient" of the fundamental group of a smooth variety admitting a smooth compactfication (§8.4). We finish with a few open questions in §8.8.

This paper grew out of the preprint [20], where some of its results were initially proven. We decided that the best was to separate the present results, which have little to do with motives, from the rest of that work. Let us end with a word on the relationship between $S_b^{-1}\mathbf{Sm}$ and the $\mathbf{A}^1$-homotopy category of schemes $\mathcal{H}$ of Morel-Voevodsky [32]. One of the main results of Asok and Morel in [1] is a proof of the following conjecture of Morel in the proper case (*loc. cit.* Th. 2.4.3):

Conjecture 1 ([31, p. 386]). *If X is a smooth variety, the natural map*

$$X(F) \to \mathrm{Hom}_{\mathcal{H}}(\mathrm{Spec}\, F, X)$$

is surjective and identifies the right hand side with the quotient of the set $X(F)$ by the equivalence relation generated by

$$(x \sim y) \iff \exists h : \mathbf{A}^1 \to X \mid h(0) = x \text{ and } h(1) = y.$$

(Note that this "$\mathbf{A}^1$-equivalence" coincides with R-equivalence if X is proper.) Their result can then be enriched as follows:

Theorem 4 ([4]). *The Yoneda embedding of* $\mathbf{Sm}$ *into the category of simplicial presheaves of sets on* $\mathbf{Sm}$ *induces a fully faithful functor*

$$S_b^{-1}\mathbf{Sm} \longrightarrow S_b^{-1}\mathcal{H}$$

where $S_b^{-1}\mathcal{H}$ *is a suitable localisation of* $\mathcal{H}$ *with respect to birational morphisms.*

Acknowledgements. We would like to thank the many colleagues who gave us insights about this work, among whom Jean-Louis Colliot-Thélène, Hélène Esnault, Najmuddin Fakhruddin, Ofer Gabber, Hagen Knaf, Georges Maltsiniotis, Vikram Mehta, Bernard Teissier, Michael Temkin, I-Hsun Tsai and Michel Vaquié. Finally, we thank Colliot-Thélène and Gabber for kindly allowing us to include Appendices A and B in this paper.

Conventions. F is the base field. "Variety" means irreducible separated F-scheme of finite type. All morphisms are F-morphisms. If X is a variety, η_X denotes its generic point.

1. Preliminaries and notation

In this section, we collect some basic material that will be used in the paper. This allows us to fix our notation.

1.1. Localisation of categories and calculus of fractions. We refer to Gabriel-Zisman [12, Chapter I] for the necessary background. Recall [12, I.1] that if $\mathcal{C}$ is a small category and S is a collection of morphisms in $\mathcal{C}$, there is a category $\mathcal{C}[S^{-1}]$ and a functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ which is universal among functors from $\mathcal{C}$ which invert the elements of S. When S satisfies *calculus of fractions* [12, I.2] the category $\mathcal{C}[S^{-1}]$ is equivalent to another one, denoted $S^{-1}\mathcal{C}$ by Gabriel and Zisman, in which the Hom sets are more explicit.
If $\mathcal{C}$ is only essentially small, one can construct a category verifying the same 2-universal property by starting from an equivalent small category, provided S contains the identities. All categories considered in this paper are subcategories of $\mathbf{Var}(F)$ (varieties over our base field F) or $\mathbf{place}(F)$ (finitely generated extensions of F, morphisms given by places), hence are essentially small.
We shall encounter situations where calculus of fractions is satisfied, as well as others where it is not. We shall take the practice to abuse notation and write $S^{-1}\mathcal{C}$ rather than $\mathcal{C}[S^{-1}]$ even when calculus of fractions is not verified.

1.1.1. Notation. If $(\mathcal{C}, S)$ is as above, we write $\langle S\rangle$ for the *saturation* of S: it is the set of morphisms s in $\mathcal{C}$ which become invertible in $S^{-1}\mathcal{C}$. We have $S^{-1}\mathcal{C} = \langle S\rangle^{-1}\mathcal{C}$ and $\langle S\rangle$ is maximal for this property.

Note the following easy lemma:

1.1.2. Lemma. *Let* $T : \mathcal{C} \to \mathcal{D}$ *be a full and essentially surjective functor. Let* $S \in Ar(\mathcal{C})$ *be a set of morphisms. Then the induced functor* $\bar{T} : S^{-1}\mathcal{C} \to T(S)^{-1}\mathcal{D}$ *is full and essentially surjective.*

Proof. Essential surjectivity is obvious. Given two objects $X, Y \in S^{-1}\mathcal{C}$, a morphism from $\bar{T}(X)$ to $\bar{T}(Y)$ is given by a zig-zag of morphisms of $\mathcal{D}$. By the essential surjectivity of T, lift all vertices of this zig-zag, then lift its edges thanks to the fullness of T. □

1.2. EQUIVALENCE RELATIONS.

1.2.1. DEFINITION. Let $\mathcal{C}$ be a category. An *equivalence relation* on $\mathcal{C}$ consists, for all $X, Y \in \mathcal{C}$, of an equivalence relation $\sim_{X,Y} = \sim$ on $\mathcal{C}(X, Y)$ such that $f \sim g \Rightarrow fh \sim gh$ and $kf \sim kg$ whenever it makes sense.

In [28, p. 52], the above notion is called a 'congruence'. Given an equivalence relation $\sim$ on $\mathcal{C}$, we may form the *factor category* $\mathcal{C}/\sim$, with the same objects as $\mathcal{C}$ and such that $(\mathcal{C}/\sim)(X, Y) = \mathcal{C}(X, Y)/\sim$. This category and the projection functor $\mathcal{C} \to \mathcal{C}/\sim$ are universal for functors from $\mathcal{C}$ which equalise equivalent morphisms.

1.2.2. *Example.* Let $\mathcal{A}$ be an Ab-category (sets of morphisms are abelian groups and composition is bilinear). An *ideal* $\mathcal{I}$ in $\mathcal{A}$ is given by a subgroup $\mathcal{I}(X, Y) \subseteq \mathcal{A}(X, Y)$ for all $X, Y \in \mathcal{A}$ such that $\mathcal{I}\mathcal{A} \subseteq \mathcal{I}$ and $\mathcal{A}\mathcal{I} \subseteq \mathcal{I}$. Then the ideal $\mathcal{I}$ defines an equivalence relation on $\mathcal{A}$, compatible with the additive structure.

Let $\sim$ be an equivalence relation on the category $\mathcal{C}$. We have the collection $S_\sim = \{f \in \mathcal{C} \mid f \text{ is invertible in } \mathcal{C}/\sim\}$. The functor $\mathcal{C} \to \mathcal{C}/\sim$ factors into a functor $S_\sim^{-1}\mathcal{C} \to \mathcal{C}/\sim$. Conversely, let $S \subset \mathcal{C}$ be a set of morphisms. We have the equivalence relation $\sim_S$ on $\mathcal{C}$ such that $f \sim_S g$ if $f = g$ in $S^{-1}\mathcal{C}$, and the localisation functor $\mathcal{C} \to S^{-1}\mathcal{C}$ factors into $\mathcal{C}/\sim_S \to S^{-1}\mathcal{C}$. Neither of these two factorisations is an equivalence of categories in general; however, [15, Prop. 1.3.3] remarks that if $f \sim g$ implies $f = g$ in $S_\sim^{-1}\mathcal{C}$, then $S_\sim^{-1}\mathcal{C} \to \mathcal{C}/\sim$ is an isomorphism of categories.

1.2.3. *Exercise.* Let A be a commutative ring and $I \subseteq A$ an ideal.
a) Assume that the set of minimal primes of A that do not contain I is finite (*e.g.* that A is noetherian). Show that the following two conditions are equivalent:

(i) There exists a multiplicative subset S of A such that $A/I \simeq S^{-1}A$ (compatibly with the maps $A \to A/I$ and $A \to S^{-1}A$).
(ii) I is generated by an idempotent.

(Hint: show first that, without any hypothesis, (i) is equivalent to

(iii) For any $a \in I$, there exists $b \in I$ such that $ab = a$.)

b) Give a counterexample to (i) $\Rightarrow$ (ii) in the general case (hint: take $A = k^{\mathbf{N}}$, where k is a field).

1.3. PLACES, VALUATIONS AND CENTRES [40, Ch. VI], [2, Ch. 6]. Recall [2, Ch. 6, §2, Def. 3] that a *place* from a field K to a field L is a map $\lambda : K \cup \{\infty\} \to L \cup \{\infty\}$ such that $\lambda(1) = 1$ and λ preserves sum and product whenever they are defined. We shall usually denote places by screwdriver arrows:

$$\lambda : K \rightsquigarrow L.$$

Then $\mathcal{O}_\lambda = \lambda^{-1}(L)$ is a valuation ring of K and $\lambda_{|\mathcal{O}_\lambda}$ factors as

$$\mathcal{O}_\lambda \twoheadrightarrow \kappa(\lambda) \hookrightarrow L$$

where $\kappa(\lambda)$ is the residue field of $\mathcal{O}_\lambda$. Conversely, the data of a valuation ring $\mathcal{O}$ of K with residue field κ and of a field homomorphism $\kappa \to L$ uniquely defines a place from K to L (*loc. cit.* , Prop. 2). It is easily checked that the composition of two places is a place.

1.3.1. *Caution.* Unlike Zariski-Samuel [40] and other authors [39, 23], we compose places in the same order as extensions of fields: so if $K \overset{\lambda}{\rightsquigarrow} L \overset{\mu}{\rightsquigarrow} M$ are two successive places, their composite is written $\mu\lambda$ in this paper. We hope this will not create confusion.

If K and L are extensions of F, we say that λ is an F-*place* if $\lambda_{|F} = Id$ and then write $F(\lambda)$ rather than $\kappa(\lambda)$.

In this situation, let X be an integral F-scheme of finite type with function field K. A point $x \in X$ is a *centre* of a valuation ring $\mathcal{O} \subset K$ if $\mathcal{O}$ dominates the local ring $\mathcal{O}_{X,x}$. If $\mathcal{O}$ has a centre on X, we sometimes say that $\mathcal{O}$ is *finite* on X. As a special case of the valuative criterion of separatedness (*resp.* of the valuative criterion of properness), x is unique (*resp.* and exists) for all $\mathcal{O}$ if and only if X is separated (*resp.* proper) [16, Ch. 2, Th. 4.3 and 4.7].

On the other hand, if $\lambda : K \rightsquigarrow L$ is an F-place, then a point $x \in X(L)$ is a *centre* of λ if there is a map $\varphi : \operatorname{Spec} \mathcal{O}_\lambda \to X$ letting the diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_\lambda \arrow[dr, "\varphi"] & \operatorname{Spec} K \arrow[l] \arrow[d] \\
\operatorname{Spec} L \arrow[u, "\lambda^*"] \arrow[r, "x"] & X
\end{tikzcd}$$

commute. Note that the image of the closed point by φ is then a centre of the valuation ring $\mathcal{O}_\lambda$ and that φ uniquely determines x.

In this paper, when X is separated we shall denote by $c_X(v) \in X$ the centre of a valuation v and by $c_X(\lambda) \in X(L)$ the centre of a place λ, and carefully distinguish between the two notions (one being a scheme-theoretic point and the other a rational point).

We have the following useful lemma from Vaquié [39, Prop. 2.4]; we reproduce its proof.

1.3.2. Lemma. *Let $X \in$ **Var**, $K = F(X)$, v a valuation on K with residue field κ and $\bar{v}$ a valuation on κ. Let $v' = \bar{v} \circ v$ denote the composite valuation.*
a) If v' is finite on X, so is v.
b) Assume that v is finite on X, and let $Z \subset X$ be the closure of its centre (so that $F(Z) \subseteq \kappa$). Then v' is finite on X if and only if [the restriction to $F(Z)$ of] $\bar{v}$ is finite on Z, and then $c(\bar{v}) \in Z$ equals $c(v') \in X$.

Proof. We may assume that $X = \operatorname{Spec} A$ is an affine variety. Denoting respectively by $V, V', \bar{V}$ and $\mathfrak{m}, \mathfrak{m}', \bar{\mathfrak{m}}$ the valuation rings associated to $v, v', \bar{v}$ and their maximal ideals, we have $(0) \subset \mathfrak{m} \subset \mathfrak{m}' \subset V' \subset V \subset K$ and $\bar{\mathfrak{m}} \subset \bar{V} = V'/\mathfrak{m} \subset \bar{K} = V/\mathfrak{m}$.
a) v' is finite on X if and only if $A \subset V'$, which implies $A \subset V$.
b) The centres of the valuations v and v' on X are defined by the prime ideals $\mathfrak{p} = A \cap \mathfrak{m}$ and $\mathfrak{p}' = A \cap \mathfrak{m}'$ of A, and the centre of the valuation $\bar{v}$ on $Z = \operatorname{Spec} \bar{A}$, with $\bar{A} = A/\mathfrak{p}$ is defined by the prime ideal $\bar{\mathfrak{p}} = \bar{A} \cap \bar{\mathfrak{m}}$ of $\bar{A}$. Then the claim is a consequence of the equality $\bar{\mathfrak{p}} = \mathfrak{p}'/\mathfrak{p}$. □

1.4. RATIONAL MAPS. Let X, Y be two F-schemes of finite type, with X integral and Y separated. Recall that a *rational map* from X to Y is a pair (U, f) where U is a dense open subset of X and $f : U \to Y$ is a morphism. Two rational maps (U, f) and (U', f') are equivalent if there exists a dense open subset U'' contained in U and U' such that $f_{|U''} = f'_{|U''}$. We denote by $\mathbf{Rat}(X, Y)$ the set of equivalence classes of rational maps, so that

$$\mathbf{Rat}(X, Y) = \varinjlim \operatorname{Map}_F(U, Y)$$

where the limit is taken over the open dense subsets of X. There is a largest open subset U of X on which a given rational map $f : X \dashrightarrow Y$ is defined [16, Ch. I, Ex. 4.2]. The (reduced) closed complement $X - U$ is called the *fundamental set* of f (notation: $\operatorname{Fund}(f)$). We say that f is *dominant* if $f(U)$ is dense in Y.
Similarly, let $f : X \to Y$ be a birational morphism. The complement of the largest open subset of X on which f is an isomorphism is called the *exceptional locus* of f and is denoted by $\operatorname{Exc}(f)$.
Note that the sets $\mathbf{Rat}(X, Y)$ only define a *precategory* (or diagram, or diagram scheme, or quiver) $\mathbf{Rat}(F)$, because rational maps cannot be composed in general. To clarify this, let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be two rational maps, where X, Y, Z are varieties. We say that f and g are *composable* if $f(\eta_X) \notin \operatorname{Fund}(g)$, where η_X is the generic point of X. Then there exists an open subset $U \subseteq X$ such that f is defined on U and $f(U) \cap \operatorname{Fund}(g) = \emptyset$, and $g \circ f$ makes sense as a rational map. This happens in two important cases:

- f is dominant;
- g is a morphism.

This composition law is associative wherever it makes sense. In particular, we do have the category $\mathbf{Rat}_{\mathrm{dom}}(F)$ with objects F-varieties and morphisms dominant rational maps. Similarly, the category $\mathbf{Var}(F)$ of 1.7 acts on $\mathbf{Rat}(F)$ on the left.

1.4.1. LEMMA ([21, Lemma 8.2]). *Let $f, g : X \to Y$ be two morphisms, with X integral and Y separated. Then $f = g$ if and only if $f(\eta_X) = g(\eta_X) =: y$ and f, g induce the same map $F(y) \to F(X)$ on the residue fields.* □

For X, Y as above, there is a well-defined map

$$\begin{aligned} \mathbf{Rat}(X,Y) &\to Y(F(X)) \\ (U,f) &\mapsto f_{|\eta_X} \end{aligned} \tag{1.1}$$

where η_X is the generic point of X.

1.4.2. Lemma. *The map* (1.1) *is bijective.*

Proof. Surjectivity is clear, and injectivity follows from Lemma 1.4.1. □

1.5. The graph trick. We shall often use this well-known and basic device, which allows us to replace a rational map by a morphism.
Let U, Y be two F-varieties. Let $j : U \to X$ be an open immersion (X a variety) and $g : U \to Y$ a morphism. Consider the graph $\Gamma_g \subset U \times Y$. By the first projection, $\Gamma_g \xrightarrow{\sim} U$. Let $\bar{\Gamma}_g$ be the closure of Γ_g in $X \times Y$, viewed as a reduced scheme. Then the rational map $g : X \dashrightarrow Y$ has been replaced by $g' : \bar{\Gamma}_g \to Y$ (second projection) through the birational map $p : \bar{\Gamma}_g \to X$ (first projection). Clearly, if Y is proper then p is proper.

1.6. Structure theorems on varieties. Here we collect two well-known results, for future reference.

1.6.1. Theorem (Nagata [34]). *Any variety X can be embedded into a proper variety $\bar{X}$. We shall sometimes call $\bar{X}$ a* compactification *of X.*

1.6.2. Theorem (Hironaka [17]). *If* $\operatorname{char} F = 0$,
a) For any variety X there exists a projective birational morphism $f : \tilde{X} \to X$ with $\tilde{X}$ smooth. (Such a morphism is sometimes called a modification.) Moreover, f may be chosen such that it is an isomorphism away from the inverse image of the singular locus of X. In particular, any smooth variety X may be embedded as an open subset of a smooth proper variety (projective if X is quasi-projective).
b) For any proper birational morphism $p : Y \to X$ between smooth varieties, there exists a proper birational morphism $\tilde{p} : \tilde{Y} \to X$ which factors through p and is a composition of blow-ups with smooth centres.

In some places we shall assume characteristic 0 in order to use resolution of singularities. We shall specify this by putting an asterisk to the statement of the corresponding result (so, the asterisk will mean that the characteristic 0 assumption is due to the use of Theorem 1.6.2).

1.7. Some multiplicative systems. Let $\mathbf{Var}(F) = \mathbf{Var}$ be the category of *F-varieties*: objects are F-varieties (*i.e.* integral separated F-schemes of finite type) and morphisms are all F-morphisms. We write $\mathbf{Sm}(F) = \mathbf{Sm}$ for its full subcategory consisting of smooth varieties. As in [21], the superscripts ${}^{\mathrm{qp}}, {}^{\mathrm{prop}}, {}^{\mathrm{proj}}$ respectively mean quasi-projective, proper and projective.
As in [21], we shall use various collections of morphisms of $\mathbf{Var}$ that are to be inverted:

- *Birational morphisms* S_b: $s \in S_b$ if s is dominant and induces an isomorphism of function fields.
- *Stably birational morphisms* S_r: $s \in S_r$ if s is dominant and induces a purely transcendental extension of function fields.

In addition, we shall use the following subsets of S_b:

- S_o: open immersions
- S_b^p: proper birational morphisms

and of S_r:

- S_r^p: proper stably birational morphisms
- S_h: the projections $pr_2 : X \times \mathbf{P}^1 \to X$.

We shall need the following lemma:

1.7.1. LEMMA. *a) In* **Var** *and* **Sm***, we have* $\langle S_b \rangle = \langle S_o \rangle$ *and* $\langle S_r \rangle = \langle S_b \cup S_h \rangle$ *(see Notation 1.1.1).*
b) We have $\langle S_r^p \rangle = \langle S_b^p \cup S_h \rangle$ *in* **Var***, *and also in* **Sm** *under resolution of singularities.*

Proof. a) The first equality is left to the reader. For the second one, given a morphism $s : Y \to X$ in S_r with $X, Y \in$ **Var** or **Sm**, it suffices to consider a commutative diagram

(1.2)
$$\begin{tikzcd}
& \tilde{Y} \arrow[dl, "t"'] \arrow[dr, "u"] & \\
Y \arrow[dr, "s"'] & & X \times (\mathbf{P}^1)^n \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

with $t, u \in S_o$, $\tilde{Y}$ a common open subset of Y and $X \times (\mathbf{P}^1)^n$.
b) For a morphism $s : Y \to X$ in S_r^p with $X, Y \in$ **Var**, we get a diagram (1.2), this time with $t, u \in S_b^p$ and $\tilde{Y}$ obtained by the graph trick. If $X, Y \in$ **Sm**, we use resolution to replace $\tilde{Y}$ by a smooth variety. □

Here is now the main result of this section.

1.7.2. THEOREM. *In* **Sm***, the sets* S_b *and* S_r *have the same saturation. *This is also true for* S_b^p *and* S_r^p *under resolution of singularities.*
In particular, the obvious functor $S_b^{-1}\mathbf{Sm} \to S_r^{-1}\mathbf{Sm}$ *is an equivalence of categories.*

Proof. Let us prove that S_h is contained in the saturation of S_b^p, hence in the saturation of S_b. Let Y be smooth variety, and let $f : Y \times \mathbf{P}^1 \to Y$ be the first projection. We have to show that f becomes invertible in $(S_b^p)^{-1}\mathbf{Sm}$. By Yoneda's lemma, it suffices to show that $F(f)$ is invertible for any (representable) functor $F : (S_b^p)^{-1}\mathbf{Sm}^{\mathrm{op}} \to \mathbf{Sets}$. This follows from taking the proof of Appendix A and "multiplying" it by Y.

To get Theorem 1.7.2, we now apply Lemma 1.7.1 a) and b). (Applying b) is where resolution of singularities is required.) □

1.7.3. *Remark.* Theorem 1.7.2 is also valid in **Var**, without resolution of singularities hypothesis (same proof). Recall however that the functor $S_b^{-1}\mathbf{Sm} \to S_b^{-1}\mathbf{Var}$ induced by the inclusion $\mathbf{Sm} \hookrightarrow \mathbf{Var}$ is far from being fully faithful [21, Rk. 8.11].

2. Places and morphisms

2.1. The category of places.

2.1.1. Definition. We denote by $\mathbf{place}(F) = \mathbf{place}$ the category with objects finitely generated extensions of F and morphisms F-places. We denote by $\mathbf{field}(F) = \mathbf{field}$ the subcategory of $\mathbf{place}(F)$ with the same objects, but in which morphisms are F-homomorphisms of fields. We shall sometimes call the latter *trivial places*.

2.1.2. *Remark.* If $\lambda : K \rightsquigarrow L$ is a morphism in **place**, then its residue field $F(\lambda)$ is finitely generated over F, as a subfield of the finitely generated field L. On the other hand, given a finitely generated extension K/F, there exist valuation rings of K/F with infinitely generated residue fields as soon as $\operatorname{trdeg}(K/F) > 1$, *cf.* [40, Ch. VI, §15, Ex. 4].

In this section, we relate the categories **place** and **Var**. We start with the main tool, which is the notion of compatibility between a place and a morphism.

2.2. A compatibility condition.

2.2.1. Definition. Let $X, Y \in \mathbf{Var}$, $f : X \dashrightarrow Y$ a rational map and $v : F(Y) \rightsquigarrow F(X)$ a place. We say that f and v are *compatible* if

- v is finite on Y (*i.e.* has a centre in Y).
- The corresponding diagram

$$\begin{array}{ccc} \eta_X & \xrightarrow{v^*} & \operatorname{Spec}\mathcal{O}_v \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & Y \end{array}$$

  commutes, where U is an open subset of X on which f is defined.

2.2.2. Proposition. *Let X, Y, v be as in Definition 2.2.1. Suppose that v is finite on Y, and let $y \in Y(F(X))$ be its centre. Then a rational map $f : X \dashrightarrow Y$ is compatible with v if and only if*

- $y = f(\eta_X)$ *and*

- *the diagram of fields*

$$\begin{tikzcd} F(v) \arrow[dr, "v"] & \\ F(y) \arrow[u] \arrow[r, "f^*"] & F(X) \end{tikzcd}$$

commutes.

In particular, there is at most one such f.

Proof. Suppose v and f compatible. Then $y = f(\eta_X)$ because $v^*(\eta_X)$ is the closed point of $\operatorname{Spec} \mathcal{O}_v$. The commutativity of the diagram then follows from the one in Definition 2.2.1. Conversely, if f verifies the two conditions, then it is obviously compatible with v. The last assertion follows from Lemma 1.4.1. □

2.2.3. Corollary. *a) Let $Y \in \mathbf{Var}$ and let $\mathcal{O}$ be a valuation ring of $F(Y)/F$ with residue field K and centre $y \in Y$. Assume that $F(y) \xrightarrow{\sim} K$. Then, for any rational map $f : X \dashrightarrow Y$ with X integral, such that $f(\eta_X) = y$, there exists a unique place $v : F(Y) \rightsquigarrow F(X)$ with valuation ring $\mathcal{O}$ which is compatible with f.*
b) If f is an immersion, the condition $F(y) \xrightarrow{\sim} K$ is also necessary for the existence of v.
c) In particular, let $f : X \dashrightarrow Y$ be a dominant rational map. Then f is compatible with the trivial place $F(Y) \hookrightarrow F(X)$, and this place is the only one with which f is compatible.

Proof. This follows immediately from Proposition 2.2.2. □

2.2.4. Proposition. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of varieties. Let $v : F(Y) \rightsquigarrow F(X)$ and $w : F(Z) \rightsquigarrow F(Y)$ be two places. Suppose that f and v are compatible and that g and w are compatible. Then $g \circ f$ and $v \circ w$ are compatible.*

Proof. We first show that $v \circ w$ is finite on Z. By definition, the diagram

$$\begin{tikzcd} \eta_Y \arrow[r, "w^*"] \arrow[d] & \operatorname{Spec} \mathcal{O}_w \arrow[d] \\ \operatorname{Spec} \mathcal{O}_v \arrow[r] & \operatorname{Spec} \mathcal{O}_{v \circ w} \end{tikzcd}$$

is cocartesian. Since the two compositions

$$\eta_Y \xrightarrow{w^*} \operatorname{Spec} \mathcal{O}_w \to Z$$

and

$$\eta_Y \to \operatorname{Spec} \mathcal{O}_v \to Y \xrightarrow{g} Z$$

coincide (by the compatibility of g and w), there is a unique induced (dominant) map $\operatorname{Spec}\mathcal{O}_{v\circ w} \to Z$. In the diagram

$$\begin{tikzcd}
\eta_X \arrow[r, "v^*"] \arrow[d] & \operatorname{Spec}\mathcal{O}_v \arrow[r] \arrow[d] & \operatorname{Spec}\mathcal{O}_{v\circ w} \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

the left square commutes by compatibility of f and v, and the right square commutes by construction. Therefore the big rectangle commutes, which means that $g \circ f$ and $v \circ w$ are compatible. □

2.3. The category VarP.

2.3.1. Definition. We denote by $\mathbf{VarP}(F) = \mathbf{VarP}$ the following category:

- Objects are F-varieties.
- Let $X, Y \in \mathbf{VarP}$. A morphism $\varphi \in \mathbf{VarP}(X,Y)$ is a pair (λ, f) with $f : X \to Y$ a morphism, $\lambda : F(Y) \rightsquigarrow F(X)$ a place and λ, f compatible.
- The composition of morphisms is given by Proposition 2.2.4.

If $\mathcal{C}$ is a full subcategory of $\mathbf{Var}$, we also denote by $\mathcal{C}\mathbf{P}(F) = \mathcal{C}\mathbf{P}$ the full subcategory of $\mathbf{VarP}$ whose objects are in $\mathcal{C}$.

We now want to do an elementary study of the two forgetful functors appearing in the diagram below:

$$\begin{tikzcd}
\mathbf{VarP} \arrow[r, "\Phi_1"] \arrow[d, "\Phi_2"'] & \mathbf{place}^{\mathrm{op}} \\
\mathbf{Var}\,.
\end{tikzcd} \tag{2.1}$$

Clearly, Φ_1 and Φ_2 are essentially surjective. Concerning Φ_2, we have the following partial result on its fullness:

2.3.2. Lemma. *Let $f : X \dashrightarrow Y$ be a rational map, with X integral and Y separated. Assume that $y = f(\eta_X)$ is a* regular *point (i.e. $A = \mathcal{O}_{Y,y}$ is regular). Then there is a place $v : F(Y) \rightsquigarrow F(X)$ compatible with f.*

Proof. By Corollary 2.2.3 a), it is sufficient to produce a valuation ring $\mathcal{O}$ containing A and with the same residue field as A.
The following construction is certainly classical. Let $\mathfrak{m}$ be the maximal ideal of A and let $(a_1, \dots, a_d)$ be a regular sequence generating $\mathfrak{m}$, with $d = \dim A = \operatorname{codim}_Y y$. For $0 \le i < j \le d+1$, let

$$A_{i,j} = (A/(a_j, \dots, a_d))_{\mathfrak{p}}$$

where $\mathfrak{p} = (a_{i+1}, \dots, a_{j-1})$ (for $i = 0$ we invert no a_k, and for $j = d+1$ we mod out no a_k). Then, for any (i,j), $A_{i,j}$ is a regular local ring of dimension

$j-i-1$. In particular, $F_i = A_{i,i+1}$ is the residue field of $A_{i,j}$ for any $j \geq i+1$. We have $A_{0,d+1} = A$ and there are obvious maps

$$\begin{aligned} A_{i,j} &\to A_{i+1,j} \quad \text{(injective)} \\ A_{i,j} &\to A_{i,j-1} \quad \text{(surjective).} \end{aligned}$$

Consider the discrete valuation v_i associated to the discrete valuation ring $A_{i,i+2}$: it defines a place, still denoted by v_i, from F_{i+1} to F_i. The composition of these places is a place v from $F_d = F(Y)$ to $F_0 = F(y)$, whose valuation ring dominates A and whose residue field is clearly $F(y)$. □

2.3.3. *Remark.* In Lemma 2.3.2, the assumption that y is a regular point is necessary. Indeed, take for f a closed immersion. By [2, Ch. 6, §1, Th. 2], there exists a valuation ring $\mathcal{O}$ of $F(Y)$ which dominates $\mathcal{O}_{Y,y}$ and whose residue field κ is an algebraic extension of $F(y) = F(X)$. However we cannot choose $\mathcal{O}$ such that $\kappa = F(y)$ in general. The same counterexamples as in [21, Remark 8.11] apply (singular curves, the point $(0,0,\dots,0)$ on the affine cone $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ over $\mathbf{R}$ for $n \geq 3$).

Now concerning Φ_1, we have:

2.3.4. Lemma. *Let X, Y be two varieties and $\lambda : F(Y) \rightsquigarrow F(X)$ a place. Assume that λ is finite on Y. Then there exists a unique rational map $f : X \dashrightarrow Y$ compatible with λ.*

Proof. Let y be the centre of $\mathcal{O}_\lambda$ on Y and $V = \operatorname{Spec} R$ an affine neighbourhood of y, so that $R \subset \mathcal{O}_\lambda$, and let S be the image of R in $F(\lambda)$. Choose a finitely generated F-subalgebra T of $F(X)$ containing S, with quotient field $F(X)$. Then $X' = \operatorname{Spec} T$ is an affine model of $F(X)/F$. The composition $X' \to \operatorname{Spec} S \to V \to Y$ is then compatible with v. Its restriction to a common open subset U of X and X' defines the desired map f. The uniqueness of f follows from Proposition 2.2.2. □

2.3.5. *Remark.* Let Z be a third variety and $\mu : F(Z) \rightsquigarrow F(Y)$ be another place, finite on Z; let $g : Y \dashrightarrow Z$ be the rational map compatible with μ. If f and g are composable, then $g \circ f$ is compatible with $\lambda \circ \mu$: this follows easily from Proposition 2.2.4. However it may well happen that f and g are not composable. For example, assume Y smooth. Given μ, hence g (that we suppose not to be a morphism), choose $y \in \operatorname{Fund}(g)$ and find a λ with centre y, for example by the method in the proof of Lemma 2.3.2. Then the rational map f corresponding to λ has image contained in $\operatorname{Fund}(g)$.

We conclude this section with a useful lemma which shows that places rigidify the situation very much.

2.3.6. Lemma. *a) Let Z, Z' be two models of a function field L, with Z' separated, and v a valuation of L with centres z, z' respectively on Z and Z'.*

Assume that there is a birational morphism $g : Z \to Z'$. Then $g(z) = z'$.
b) Consider a diagram

$$\begin{tikzcd} & Z \arrow[dd, "g"] \\ X \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & Z' \end{tikzcd}$$

with g a birational morphism. Let $K = F(X)$, $L = F(Z) = F(Z')$ and suppose given a place $v : L \rightsquigarrow K$ compatible both with f and f'. Then $f' = g \circ f$.

Proof. a) Let $f : \operatorname{Spec} \mathcal{O}_v \to Z$ be the dominant map determined by z. Then $f' = g \circ f$ is a dominant map $\operatorname{Spec} \mathcal{O}_v \to Z'$. By the valuative criterion of separatedness, it must correspond to z'. b) This follows from a) and Proposition 2.2.2. $\square$

3. Places, valuations and the Riemann varieties

In this section, we give a second categorical relationship between the idea of places and that of algebraic varieties. This leads us to consider Zariski's "abstract Riemann surface of a field" as a locally ringed space. We start by giving the details of this theory, as we could not find it elaborated in the literature[5]. We remark however that the study of 'Riemann-Zariski spaces' has recently been revived by different authors independently (see [10], [36], [37], [39]).

3.1. Strict birational morphisms. It will be helpful to work here with the following notion of *strict birational morphisms*:

$$\underline{S}_b = \{s \in S_b \mid s \text{ induces an equality of function fields}\}$$

In fact, the difference between S_b and $\underline{S}_b$ is immaterial in view of the following

3.1.1. Lemma. *Any birational morphism of (separated) varieties is the composition of a strict birational morphism and an isomorphism.*

Proof. Let $s : X \to Y$ be a birational morphism. First assume X and Y affine, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Let $K = F(X)$ and $L = F(Y)$, so that K is the quotient field of A and L is the quotient field of B. Let $s^* : L \xrightarrow{\sim} K$ be the isomorphism induced by s. Then $A' \xrightarrow{\sim} A = s^*(A)$, hence s may be factored as $X \xrightarrow{s'} X' \xrightarrow{u} Y$ with $X' = \operatorname{Spec} A'$, where s' is strict birational and u is an isomorphism. In the general case, we may patch the above construction (which is canonical) over an affine open cover (U_i) of Y and an affine open cover of X refining $(s^{-1}(U_i))$. $\square$

[5] Except for a terse allusion in [17, 0.6, p. 146]: we thank Bernard Teissier for pointing out this reference.

3.2. The Riemann-Zariski variety as a locally ringed space.

3.2.1. Definition. We denote by $\mathcal{R}(F) = \mathcal{R}$ the full subcategory of the category of locally ringed spaces such that $(X, \mathcal{O}_X) \in \mathcal{R}$ if and only if $\mathcal{O}_X$ is a sheaf of local F-algebras.

(Here, we understand by "local ring" a commutative ring whose non-invertible elements form an ideal, but we don't require it to be Noetherian.)

3.2.2. Lemma. *Cofiltering inverse limits exist in $\mathcal{R}$. More precisely, if $(X_i, \mathcal{O}_{X_i})_{i\in I}$ is a cofiltering inverse system of objects of $\mathcal{R}$, its inverse limit is represented by $(X, \mathcal{O}_X)$ with $X = \varprojlim X_i$ and $\mathcal{O}_X = \varinjlim p_i^* \mathcal{O}_{X_i}$, where $p_i : X \to X_i$ is the natural projection.*

Sketch. Since a filtering direct limit of local rings for local homomorphisms is local, the object of the lemma belongs to $\mathcal{R}$ and we are left to show that it satisfies the universal property of inverse limits in $\mathcal{R}$. This is clear on the space level, while on the sheaf level it follows from the fact that inverse images of sheaves commute with direct limits. □

Recall from Zariski-Samuel [40, Ch. VI, §17] the *abstract Riemann surface* S_K of a function field K/F: as a set, it consists of all nontrivial valuations on K which are trivial on F. It is topologised by the following basis $\mathcal{E}$ of open sets: if R is a subring of K, finitely generated over F, $E(R) \in \mathcal{E}$ consists of all valuations v such that $\mathcal{O}_v \supseteq R$.
As has become common practice, we slightly modify this definition:

3.2.3. Definition. The *Riemann variety* Σ_K of K is the following ringed space:

- As a topological space, $\Sigma_K = S_K \cup \{\eta_K\}$ where η_K is the trivial valuation of K. (The topology is defined as for S_K.)
- The set of sections over $E(R)$ of the structural sheaf of Σ_K is the intersection $\bigcap_{v\in E(R)} \mathcal{O}_v$, *i.e.* the integral closure of R.

3.2.4. Lemma. *The stalk at $v \in \Sigma_K$ of the structure sheaf is $\mathcal{O}_v$. In particular, $\Sigma_K \in \mathcal{R}$.*

Proof. Let $x_1, \ldots, x_n \in \mathcal{O}_v$. The subring $F[x_1, \ldots, x_n]$ is finitely generated and contained in $\mathcal{O}_v$, thus $\mathcal{O}_v$ is the filtering direct limit of the R's such that $v \in E(R)$. □

Let R be a finitely generated F-subalgebra of K. We have a canonical morphism of locally ringed spaces $c_R : E(R) \to \operatorname{Spec} R$ defined as follows: on points we map $v \in E(R)$ to its centre $c_R(v)$ on $\operatorname{Spec} R$. On the sheaf level, the map is defined by the inclusions $\mathcal{O}_{X,c_X(v)} \subset \mathcal{O}_v$.
We now reformulate [40, p. 115 ff] in scheme-theoretic language. Let $X \in \mathbf{Var}$ be provided with a dominant morphism $\operatorname{Spec} K \to X$ such that the corresponding field homomorphism $F(X) \to K$ is an inclusion (as opposed to a monomorphism). We call such an X a *Zariski-Samuel model* of K; X is

a *model* of K if, moreover, $F(X) = K$. Note that Zariski-Samuel models of K form a cofiltering ordered set. Generalising $E(R)$, we may define $E(X) = \{v \in \Sigma_K \mid v \text{ is finite on } X\}$ for a Zariski-Samuel model of K; this is still an open subset of Σ_K, being the union of the $E(U_i)$, where (U_i) is some finite affine open cover of X. We still have a morphism of locally ringed spaces $c_X : E(X) \to X$ defined by glueing the affine ones. If X is proper, $E(X) = \Sigma_K$ by the valuative criterion of properness. Then:

3.2.5. Theorem (Zariski-Samuel). *The induced morphism of ringed spaces*

$$\Sigma_K \to \varprojlim X$$

where X runs through the proper Zariski-Samuel models of K, is an isomorphism in $\mathcal{R}$. The generic point η_K is dense in Σ_K.

Proof. Zariski and Samuel's theorem [40, th. VI.41 p. 122] says that the underlying morphism of topological spaces is a homeomorphism; thus, by Lemma 3.2.2, we only need to check that the structure sheaf of Σ_K is the direct limit of the pull-backs of those of the X. This amounts to showing that, for $v \in \Sigma_K$, $\mathcal{O}_v$ is the direct limit of the $\mathcal{O}_{X,c_X(v)}$.
We argue essentially as in [40, pp. 122–123] (or as in the proof of Lemma 3.2.4). Let $x \in \mathcal{O}_v$, and let X be the projective Zariski-Samuel model determined by $\{1, x\}$ as in *loc. cit.* , bottom p. 119, so that either $X \simeq \mathbf{P}^1_F$ or $X = \operatorname{Spec} F'$ where F' is a finite extension of F contained in K. In both cases, $c = c_X(v)$ actually belongs to $\operatorname{Spec} F[x]$ and $x \in \mathcal{O}_{X,c} \subset \mathcal{O}_v$.
Finally, η_K is contained in every basic open set, therefore is dense in Σ_K. □

3.2.6. Definition. Let $\mathcal{C}$ be a full subcategory of $\mathbf{Var}$. We denote by $\hat{\mathcal{C}}$ the full subcategory of $\mathcal{R}$ whose objects are cofiltered inverse limits of objects of $\mathcal{C}$ under morphisms of $\underline{S}_b$ (*cf.* §1.7). The natural inclusion $\mathcal{C} \subset \hat{\mathcal{C}}$ is denoted by J.

Note that, for any function field K/F, $\Sigma_K \in \widehat{\mathbf{Var}^{\text{prop}}}$ by Theorem 3.2.5. Also, for any $X \in \widehat{\mathbf{Var}}$, the function field $F(X)$ is well-defined.

3.2.7. Lemma. *Let $X \in \widehat{\mathbf{Var}}$ and $K = F(X)$.*
a) For a finitely generated F-algebra $R \subset K$, the set

$$E_X(R) = \{x \in X \mid R \subset O_{X,x}\}$$

is an open subset of X. These open subsets form a basis for the topology of X.
b) The generic point $\eta_K \in X$ is dense in X, and X is quasi-compact.

Proof. a) If X is a variety, then $E_X(R)$ is open, being the set of definition of the rational map $X \dashrightarrow \operatorname{Spec} R$ induced by the inclusion $R \subset K$. In general, let $(X, \mathcal{O}_X) = \varprojlim_\alpha (X_\alpha, \mathcal{O}_{X_\alpha})$ with the X_α varieties and let $p_\alpha : X \to X_\alpha$ be the projection. Since R is finitely generated, we have

$$E_X(R) = \bigcup_\alpha p_\alpha^{-1}(E_{X_\alpha}(R))$$

which is open in X.
Let $x \in X$: using Lemma 3.2.2, we can find an α and an affine open $U \subset X_\alpha$ such that $x \in p_\alpha^{-1}(U)$. Writing $U = \operatorname{Spec} R$, we see that $x \in E_X(R)$, thus the $E_X(R)$ form a basis of the topology of X.
In b), the density follows from a) since clearly $\eta_K \in E_X(R)$ for every R. The space X is a limit of spectral spaces under spectral maps, and hence quasi-compact. Alternately, X is compact in the constructible topology as compactness is preserved under inverse limits, and hence quasi-compact in the weaker Zariski topology. □

We are grateful to M. Temkin for pointing out an error in our earlier proof of quasi-compactness and providing the proof of b) above.

3.2.8. Theorem. *Let $X = \varprojlim X_\alpha$, $Y = \varprojlim Y_\beta$ be two objects of $\widehat{\mathbf{Var}}$. Then we have a canonical isomorphism*

$$\widehat{\mathbf{Var}}(X,Y) \simeq \varprojlim_\beta \varinjlim_\alpha \mathbf{Var}(X_\alpha, Y_\beta).$$

Proof. Suppose first that Y is constant. We then have an obvious map

$$\varinjlim_\alpha \mathbf{Var}(X_\alpha, Y) \to \widehat{\mathbf{Var}}(X,Y).$$

Injectivity follows from Lemma 1.4.1. For surjectivity, let $f : X \to Y$ be a morphism. Let $y = f(\eta_K)$. Since η_K is dense in X by Lemma 3.2.7 b), $f(X) \subseteq \overline{\{y\}}$. This reduces us to the case where f is *dominant*.
Let $x \in X$ and $y = f(x)$. Pick an affine open neighbourhood $\operatorname{Spec} R$ of y in Y. Then $R \subset \mathcal{O}_{X,x}$, hence $R \subset \mathcal{O}_{X_\alpha,x_\alpha}$ for some α, where $x_\alpha = p_\alpha(x)$, $p_\alpha : X \to X_\alpha$ being the canonical projection. This shows that the rational map $f_\alpha : X_\alpha \dashrightarrow Y$ induced by restricting f to the generic point is defined at x_α for α large enough.
Let U_α be the set of definition of f_α. We have just shown that X is the increasing union of the open sets $p_\alpha^{-1}(U_\alpha)$. Since X is quasi-compact, this implies that $X = p_\alpha^{-1}(U_\alpha)$ for some α, *i.e.* that f factors through X_α for this value of α.
In general we have

$$\widehat{\mathbf{Var}}(X,Y) \xrightarrow{\sim} \varprojlim_\beta \widehat{\mathbf{Var}}(X, Y_\beta)$$

by the universal property of inverse limits, which completes the proof. □

3.2.9. *Remark.* Let $\mathrm{pro}_{\underline{S}_b}-\mathbf{Var}$ be the full subcategory of the category of pro-objects of $\mathbf{Var}$ consisting of the (X_α) in which the transition maps $X_\alpha \to X_\beta$ are strict birational morphisms. Then Theorem 3.2.8 may be reinterpreted as saying that the functor

$$\varprojlim : \mathrm{pro}_{\underline{S}_b}-\mathbf{Var} \to \widehat{\mathbf{Var}}$$

is an *equivalence of categories.*

3.3. Riemann varieties and places. We are going to study two functors

$$\mathrm{Spec} : \mathbf{field}^{\mathrm{op}} \to \widehat{\mathbf{Var}}$$
$$\Sigma : \mathbf{place}^{\mathrm{op}} \to \widehat{\mathbf{Var}}$$

and a natural transformation $\eta : \mathrm{Spec} \Rightarrow \Sigma \circ \iota$, where ι is the embedding $\mathbf{field}^{\mathrm{op}} \hookrightarrow \mathbf{place}^{\mathrm{op}}$.
The first functor is simply $K \mapsto \mathrm{Spec}\, K$. The second one maps K to the Riemann variety Σ_K. Let $\lambda : K \rightsquigarrow L$ be an F-place. We define $\lambda^* : \Sigma_L \to \Sigma_K$ as follows: if $w \in \Sigma_L$, we may consider the associated place $\tilde{w} : L \rightsquigarrow F(w)$; then $\lambda^* w$ is the valuation underlying $\tilde{w} \circ \lambda$.
Let $E(R)$ be a basic open subset of Σ_K. Then

$$(\lambda^*)^{-1}(E(R)) = \begin{cases} \emptyset & \text{if } R \not\subseteq \mathcal{O}_\lambda \\ E(\lambda(R)) & \text{if } R \subseteq \mathcal{O}_\lambda. \end{cases}$$

Moreover, if $R \subseteq \mathcal{O}_\lambda$, then λ maps $\mathcal{O}_{\lambda^* w}$ to $\mathcal{O}_w$ for any valuation $w \in (\lambda^*)^{-1} E(R)$. This shows that λ^* is continuous and defines a morphism of locally ringed spaces. We leave it to the reader to check that $(\mu \circ \lambda)^* = \lambda^* \circ \mu^*$.
Note that we have for any K a morphism of ringed spaces

$$\eta_K : \mathrm{Spec}\, K \to \Sigma_K \tag{3.1}$$

with image the trivial valuation of Σ_K (which is its generic point). This defines the natural transformation η we alluded to.

3.3.1. Proposition. *The functors* Spec *and* Σ *are fully faithful; moreover, for any* K, L*, the map*

$$\widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K) \xrightarrow{\eta_L^*} \widehat{\mathbf{Var}}(\mathrm{Spec}\, L, \Sigma_K)$$

is bijective.

Proof. The case of Spec is obvious. For the rest, let $K, L \in \mathbf{place}(F)$ and consider the composition

$$\mathbf{place}(K, L) \xrightarrow{\Sigma} \widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K) \xrightarrow{\eta_L^*} \widehat{\mathbf{Var}}(\mathrm{Spec}\, L, \Sigma_K).$$

It suffices to show that η_L^* is injective and $\eta_L^* \circ \Sigma$ is bijective.
Let $\psi_1, \psi_2 \in \widehat{\mathbf{Var}}(\Sigma_L, \Sigma_K)$ be such that $\eta_L^* \psi_1 = \eta_L^* \psi_2$. Pick a proper model X of K; by Theorem 3.2.8, $c_X \circ \psi_1$ and $c_X \circ \psi_2$ factor through morphisms $f_1, f_2 : Y \to X$ for some model Y of L. By Lemma 1.4.1, $f_1 = f_2$, hence $c_X \circ \psi_1 = c_X \circ \psi_2$ and finally $\psi_1 = \psi_2$ by Theorem 3.2.5. Thus η_L^* is injective.
On the other hand, let $\varphi \in \widehat{\mathbf{Var}}(\mathrm{Spec}\, L, \Sigma_K)$ and $v = \varphi(\mathrm{Spec}\, L)$: then φ induces a homomorphism $\mathcal{O}_v \to L$, hence a place $\lambda : K \rightsquigarrow L$ and clearly $\varphi = \eta_L^* \circ \Sigma(\lambda)$. This is the only place mapping to φ. This shows that the composition $\eta_L^* \circ \Sigma$ is bijective, which concludes the proof. □

4. Two equivalences of categories

In this section, we compare the localised categories $S_r^{-1}\,\mathbf{place}$ and a suitable version of $S_b^{-1}\mathbf{Sm}^{\mathrm{prop}}$ by using the techniques of the previous section. First, we prove in Theorem 4.2.3 that a suitable version of the functor Φ_1 of (2.1) becomes an equivalence of categories after we invert birational morphisms. Next, we construct a full and essentially surjective functor

$$\mathbf{place}_*^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$$

in Corollary 4.2.4, where $\mathbf{Sm}_*^{\mathrm{prop}}$ is the full subcategory of $\mathbf{Sm}$ formed of smooth varieties having a cofinal system of smooth proper models, and $\mathbf{place}_* \subseteq \mathbf{place}$ is the full subcategory of their function fields.

4.1. The basic diagram. We start from the commutative diagram of functors

(4.1)
$$\begin{tikzcd} & \mathbf{VarP} \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] & \\ \mathbf{place}^{\mathrm{op}} \arrow[dr, "\Sigma"'] & & \mathbf{Var} \arrow[dl, "J"] \\ & \widehat{\mathbf{Var}} & \end{tikzcd}$$

where Φ_1, Φ_2 are the two forgetful functors of (2.1). Note that Σ takes values in $\widehat{\mathbf{Var}^{\mathrm{prop}}}$, so this diagram restricts to a similar diagram where $\mathbf{Var}$ is replaced by $\mathbf{Var}^{\mathrm{prop}}$.

We can extend the birational morphisms S_b to the categories appearing in this diagram:

4.1.1. Definition (*cf.* Theorem 3.2.8). Let $X, Y \in \widehat{\mathbf{Var}}$, with $X = \varprojlim X_\alpha$, $Y = \varprojlim Y_\beta$. A morphism $s : X \to Y$ is *birational* if, for each β, the projection $X \xrightarrow{s} Y \to Y_\beta$ factors through a birational map $s_{\alpha,\beta} : X_\alpha \to Y_\beta$ for some α (this does not depend on the choice of α). We denote by $S_b \subset Ar(\widehat{\mathbf{Var}})$ the collection of these morphisms.

In $\mathbf{Var}\,\mathbf{P}$, we write S_b for the set of morphisms of the form (u, f) where u is an isomorphism of function fields and f is a birational morphism. In $\mathbf{place}$, we take for S_b the set of isomorphisms.

4.2. Main results.

4.2.1. Definition. Let

- $\mathbf{place}_*$ be the full subcategory of $\mathbf{place}$ formed of function fields which have a cofinal system of smooth proper models.
- $\mathbf{Sm}_*^{\mathrm{prop}} \subseteq \mathbf{Sm}^{\mathrm{prop}}$ be the full subcategory of those X such that, for any $Y \in \mathbf{Var}^{\mathrm{prop}}$ birational to X, there exists $X' \in \mathbf{Sm}^{\mathrm{prop}}$ and a (proper) birational morphism $s : X' \to Y$.

Note that $\mathbf{Sm}_*^{\mathrm{prop}} = \mathbf{Sm}^{\mathrm{prop}}$ in characteristic 0 and that $X \in \mathbf{Sm}^{\mathrm{prop}} \Rightarrow X \in \mathbf{Sm}_*^{\mathrm{prop}}$ if $\dim X \le 2$ in any characteristic. On the other hand, it is not clear whether $\mathbf{Sm}_*^{\mathrm{prop}}$ is closed under products, or even under product with $\mathbf{P}^1$.

The following lemma is clear:

4.2.2. Lemma. *a) If $X, X' \in \mathbf{Sm}^{\mathrm{prop}}$ are birational, then $X \in \mathbf{Sm}_*^{\mathrm{prop}} \iff X' \in \mathbf{Sm}_*^{\mathrm{prop}}$.*
b) $K \in \mathbf{place}_* \iff K$ has a model in $\mathbf{Sm}_*^{\mathrm{prop}}$, and then any smooth proper model of K is in $\mathbf{Sm}_*^{\mathrm{prop}}$. □

If $X \in \mathbf{Sm}_*^{\mathrm{prop}}$, we have $F(X) \in \mathbf{place}_*$, hence with these definitions, (4.1) induces a commutative diagram of localised categories:

(4.2)
$$\begin{tikzcd}
& S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P} \arrow[dl, "\overline{\Phi}_1^*"'] \arrow[dr, "\overline{\Phi}_2^*"] & \\
\mathbf{place}_*^{\mathrm{op}} \arrow[dr, "\bar{\Sigma}"] & & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \arrow[dl, "\bar{J}"'] \\
& S_b^{-1}\widehat{\mathbf{Sm}_*^{\mathrm{prop}}} &
\end{tikzcd}$$

4.2.3. Theorem. *In (4.2), $\bar{J}$ and $\overline{\Phi}_1^*$ are equivalences of categories.*

Composing $\bar{\Sigma}$ with a quasi-inverse of $\bar{J}$, we get a functor

$$\Psi_* : \mathbf{place}_*^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}. \tag{4.3}$$

This functor is well-defined up to unique natural isomorphism, by the essential uniqueness of a quasi-inverse to $\bar{J}$.

4.2.4. Theorem. *a) The functor Ψ_* is full and essentially surjective.*
b) Let $K, L \in \mathbf{place}_*$ and $\lambda, \mu \in \mathbf{place}_*(K, L)$. Suppose that λ and μ have the same centre on some model $X \in \mathbf{Sm}_*^{\mathrm{prop}}$ of K. Then $\Psi_*(\lambda) = \Psi_*(\mu)$.
c) Let $S_r \subset \mathbf{place}_*$ denote the set of field extensions $K \hookrightarrow K(t)$ such that $K \in \mathbf{place}_*$ and $K(t) \in \mathbf{place}_*$. Then the composition $\mathbf{place}_*^{\mathrm{op}} \xrightarrow{\Psi_*} S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \to S_b^{-1}\mathbf{Sm}$ factors through a (full) functor, still denoted by Ψ_*:

$$\Psi_* : S_r^{-1}\mathbf{place}_*^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}.$$

The proofs of Theorems 4.2.3 and 4.2.4 go in several steps, which are given in the next subsections.

4.3. Proof of Theorem 4.2.3: the case of $\bar{J}$. We apply Proposition 5.10 b) of [21]. To lighten notation we drop the functor J. We have to check Conditions (b1), (b2) and (b3) of *loc. cit.* , namely:

(b1) *Given two maps $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathbf{Sm}_*^{\mathrm{prop}}$ and a map $s : Z = \varprojlim Z_\alpha \to X$ in $S_b \subset \widehat{\mathbf{Sm}_*^{\mathrm{prop}}}$, $fs = gs \Rightarrow f = g$.* This is clear by Lemma 1.4.1, since by Theorem 3.2.8 s factors through some Z_α, with $Z_\alpha \to X$ birational.

(b2) *For any* $X = \varprojlim X_\alpha \in \widehat{\mathbf{Sm}^{\mathrm{prop}}_*}$, *there exists a birational morphism* $s : X \to X'$ *with* $X' \in \mathbf{Sm}^{\mathrm{prop}}_*$. It suffices to take $X' = X_\alpha$ for some α.

(b3) *Given a diagram*

$$\begin{array}{ccc} X_1 & & \\ \big\uparrow{\scriptstyle s_1} & & \\ X = \varprojlim X_\alpha & \xrightarrow{f} & Y \end{array}$$

with $X \in \widehat{\mathbf{Sm}^{\mathrm{prop}}_*}$, $X_1, Y \in \mathbf{Sm}^{\mathrm{prop}}_*$ *and* $s_1 \in S_b$, *there exists* $s_2 : X \to X_2$ *in* S_b, *with* $X_2 \in \mathbf{Sm}^{\mathrm{prop}}_*$, *covering both* s_1 *and* f. Again, it suffices to take $X_2 = X_\alpha$ for α large enough (use Theorem 3.2.8).

4.4. Calculus of fractions.

4.4.1. Proposition. *The category* $\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$ *admits a calculus of right fractions with respect to* S_b^p. *In particular, in* $(S_b^p)^{-1}\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$, *any morphism may be written in the form* fp^{-1} *with* $p \in S_b^p$. *The latter also holds in* $(S_b^p)^{-1}\mathbf{Sm}^{\mathrm{prop}}_*$.

Proof. Consider a diagram

$$\begin{array}{ccc} & & Y' \\ & & \big\downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array} \tag{4.4}$$

in $\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$, with $s \in S_b^p$. Let $\lambda : F(Y) \rightsquigarrow F(X)$ be the place compatible with u which is implicit in the statement. By Proposition 2.2.2, λ has centre $z = u(\eta_X)$ on Y. Since s is proper, λ therefore has also a centre z' on Y'. By Lemma 2.3.6 a), $s(z') = z$. By Lemma 2.3.4, there exists a unique rational map $\varphi : X \dashrightarrow Y'$ compatible with λ, and $s \circ \varphi = u$ by Lemma 2.3.6 b). By the graph trick, we get a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ {\scriptstyle s'}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array} \tag{4.5}$$

in which $X' \subset X \times_Y Y'$ is the closure of the graph of φ, $s' \in S_b^p$ and u' is compatible with λ. Since $X \in \mathbf{Sm}^{\mathrm{prop}}_*$, we may birationally dominate X' by an $X'' \in \mathbf{Sm}^{\mathrm{prop}}_*$ by Lemma 4.2.2, hence replace X' by X'' in the diagram.
Since Φ_1^* is full by Lemma 2.3.2, the same construction works in $\mathbf{Sm}^{\mathrm{prop}}_*$, hence the structure of morphisms in $(S_b^p)^{-1}\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$ and $(S_b^p)^{-1}\mathbf{Sm}^{\mathrm{prop}}_*$.
Let now

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Y'$$

be a diagram in $\mathbf{Sm}^{\mathrm{prop}}_*\mathbf{P}$ with $s \in S_b^p$, such that $sf = sg$. By Corollary 2.2.3 c), the place underlying s is the identity. Hence the two places underlying f and g must be equal. But then $f = g$ by Proposition 2.2.2. □

4.4.2. Proposition. *a) Consider a diagram in* $\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$

(4.6)
$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "f"] & \\ X & & Y \\ & Z' \arrow[ul, "p'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

where $p,p' \in S_b^p$. *Let* $K = F(Z) = F(Z') = F(X)$, $L = F(Y)$ *and suppose given a place* $\lambda : L \rightsquigarrow K$ *compatible both with* f *and* f'. *Then* $(\lambda, fp^{-1}) = (\lambda, f'p'^{-1})$ *in* $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$.
b) Consider a diagram (4.6) *in* $\mathbf{Sm}_*^{\mathrm{prop}}$. *Then* $fp^{-1} = f'p'^{-1}$ *in* $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$ *if* (f,p) *and* (f',p') *define the same rational map from* X *to* Y.

Proof. a) By the graph trick, complete the diagram as follows:

(4.7)
$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "f"] & \\ X & Z'' \arrow[u, "p_1"] \arrow[d, "p'_1"'] & Y \\ & Z' \arrow[ul, "p'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

with $p_1, p'_1 \in S_b^p$ and $Z'' \in \mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$. Since $X \in \mathbf{Sm}_*^{\mathrm{prop}}$, we may take Z'' in $\mathbf{Sm}_*^{\mathrm{prop}}$.Then we have
$$pp_1 = p'p'_1, \quad fp_1 = f'p'_1$$
(the latter by Lemma 2.3.6 b)), hence the claim.
b) If (f,p) and (f',p') define the same rational map, then arguing as in a) we get a diagram (4.7) in $\mathbf{Sm}_*^{\mathrm{prop}}$, hence $fp^{-1} = f'p'^{-1}$ in $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$. □

4.5. The morphism associated to a rational map. Let $X, Y \in \mathbf{Sm}_*^{\mathrm{prop}}$, and let $\varphi : Y \dashrightarrow X$ be a rational map. By the graph trick, we may find $p : Y' \to Y$ proper birational and a morphism $f : Y' \to X$ such that φ is represented by (f,p); since $Y \in \mathbf{Sm}_*^{\mathrm{prop}}$, we may choose Y' in $\mathbf{Sm}_*^{\mathrm{prop}}$. Then $fp^{-1} \in (S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$ does not depend on the choice of Y' by Proposition 4.4.2 b): we simply write it φ.

4.6. Proof of Theorem 4.2.4. Let $K, L \in \mathbf{place}_*$ and $\lambda \in \mathbf{place}_*(K,L)$. Put $X = \Psi_*(K), Y = \Psi_*(L)$, so that X (resp. Y) is a smooth proper model of K (resp. L) in $\mathbf{Sm}_*$ (see 4.2.1). Since X is proper, λ is finite on X and by Lemma 2.3.4 there exists a unique rational map $\varphi : Y \dashrightarrow X$ compatible with λ, that we view as a morphism in $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$ by §4.5.

4.6.1. Lemma. *With the above notation, we have* $\Psi_*(\lambda) = \varphi$.

Proof. Consider the morphisms $(\lambda, f) \in \mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}(Y', X)$ and $(1_L, s) \in \mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}(Y', Y)$. In (4.2) Φ_1^* sends the first morphism to λ and the second one to 1_L, while Φ_2^* sends the first morphism to f and the second one to s. The conclusion now follows from the commutativity of (4.2) and the construction of Ψ_*. $\square$

We can now prove Theorem 4.2.4:
a) The essential surjectivity of Ψ_* is tautological. Let now $X = \Psi_*(K), Y = \Psi_*(L)$ for some $K, L \in \mathbf{place}_*$ and let $\varphi \in (S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}(X, Y)$. By Proposition 4.4.1, we may write $\varphi = fs^{-1}$ where f, s are morphisms in $\mathbf{Sm}_*^{\mathrm{prop}}$ and $s \in S_b^p$. Let $\tilde{\varphi} : X \dashrightarrow Y$ be the corresponding rational map. By Lemma 2.3.2, f is compatible with some place λ and by Corollary 2.2.3 c), s is compatible with the corresponding isomorphism ι of function fields. Then $\tilde{\varphi}$ is compatible with $\iota^{-1}\lambda$, and $\Psi_*(\iota^{-1}\lambda) = \varphi$ by Lemma 4.6.1. This proves the fullness of Ψ_*. (One could also use Lemma 1.1.2.)
b) By Lemma 4.6.1, $\Psi_*(\lambda)$ and $\Psi_*(\mu)$ are given by the respective rational maps $f, g : \Psi_*(L) \dashrightarrow \Psi_*(K)$ compatible with λ, μ. By the definition of $\mathbf{Sm}_*^{\mathrm{prop}}$, we can find a model $X' \in \mathbf{Sm}_*^{\mathrm{prop}}$ of K and two birational morphisms $s : X' \to X$, $t : X' \to \Psi_*(K)$. The hypothesis and Lemma 2.3.4 imply that $st^{-1}f = st^{-1}g$, hence $f = g$ in $S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$.
c) The said composition sends morphisms in S_r to morphisms in S_r, hence induces a functor

$$S_r^{-1}\,\mathbf{place}_*^{\mathrm{op}} \to S_r^{-1}\mathbf{Sm}.$$

But $S_b^{-1}\mathbf{Sm} \xrightarrow{\sim} S_r^{-1}\mathbf{Sm}$ by Theorem 1.7.2.

4.7. Proof of Theorem 4.2.3: the case of $\overline{\Phi}_1^*$. Essential surjectivity is obvious by definition of $\mathbf{place}_*$. Let $X, Y \in \mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$, and $K = \Phi_1^*(X), L = \Phi_1^*(Y)$. By Lemma 2.3.4, a place $\lambda : L \rightsquigarrow K$ is compatible with a (unique) rational map $\varphi : X \dashrightarrow Y$. Since $X \in \mathbf{Sm}_*^{\mathrm{prop}}$, we may write $\varphi = fs^{-1}$ with $f : X' \to Y$ for $X' \in \mathbf{Sm}_*^{\mathrm{prop}}$, and $s : X' \to X$ is a birational morphism. This shows the fullness of $\overline{\Phi}_1^*$.
We now prove the faithfulness of $\overline{\Phi}_1^*$. Let $(\lambda_1, \psi_1), (\lambda_2, \psi_2)$ be two morphisms from X to Y in $(S_b^p)^{-1}\mathbf{Sm}_*^{\mathrm{prop}}\mathbf{P}$ having the same image under $\overline{\Phi}_1^*$. By Proposition 4.4.1, we may write $\psi_i = f_i p_i^{-1}$ with f_i, p_i morphisms and $p_i \in S_b$. As they have the same image, it means that the places λ_1 and λ_2 from $F(Y)$ to $F(X)$ are equal. By Lemma 2.3.4, (f_1, p_1) and (f_2, p_2) define the same rational map $\varphi : X \dashrightarrow Y$. Therefore $\psi_1 = \psi_2$ by Proposition 4.4.2 b), and $(\lambda_1, \psi_1) = (\lambda_2, \psi_2)$.

4.8. Dominant rational maps. Recall from Subsection 1.4 the category $\mathbf{Rat}_{\mathrm{dom}}$ of dominant rational maps between F-varieties. Writing $\mathbf{Var}_{\mathrm{dom}}$ for the category of F-varieties and dominant maps, we have inclusions of categories

$$\mathbf{Var} \supset \mathbf{Var}_{\mathrm{dom}} \overset{\rho}{\hookrightarrow} \mathbf{Rat}_{\mathrm{dom}}\,. \tag{4.8}$$

Recall [16, Ch. I, Th. 4.4] that there is an anti-equivalence of categories

$$\begin{aligned}\mathbf{Rat}_{\mathrm{dom}} &\xrightarrow{\sim} \mathbf{field}^{\mathrm{op}}\\ X &\mapsto F(X).\end{aligned}\tag{4.9}$$

Actually this follows easily from Lemma 1.4.2. We want to revisit this theorem from the current point of view. For simplicity, we restrict to smooth varieties and separably generated extensions of F. Recall:

4.8.1. Lemma. *A function field K/F has a smooth model if and only if it is separably generated.*

Proof. Necessity: let p be the exponential characteristic of F. If X is a smooth model of K/F, then $X\otimes_F F^{1/p}$ is smooth over $F^{1/p}$ and irreducible, hence $K\otimes_F F^{1/p}$ is still a field. The conclusion then follows from Mac Lane's separability criterion [27, Chapter 8, §4]
Sufficiency: if K/F is separably generated, pick a separable transcendence basis $\{x_1,\dots,x_n\}$. Writing $F(x_1,\dots,x_n) = F(\mathbf{A}^n)$, we can find an affine model of finite type X of K/F with a dominant generically finite morphism $f : X \to \mathbf{A}^n$. By generic flatness [EGA4, 11.1.1], there is an open subset $U \subseteq \mathbf{A}^n$ such that $f^{-1}(U) \to U$ is flat. On the other hand, since $K/F(x_1,\dots,x_n)$ is separable, there is another open subset $V \subseteq \mathbf{A}^n$ such that $\Omega^1_{f^{-1}(V)/V} = 0$. Then $f^{-1}(U\cap V)$ is flat and unramified, hence étale, over $U\cap V$, hence is smooth over F since $U\cap V$ is smooth [EGA4, 17.3.3]. □

Instead of (4.1) and (4.2), consider now the commutative diagrams of functors

$$\begin{tikzcd} & \mathbf{Sm}_{\mathrm{dom}}\mathbf{P} \arrow[dl, "\Phi_{1,\mathrm{dom}}"'] \arrow[dr, "\Phi_{2,\mathrm{dom}}"] & \\ \mathbf{field}_s^{\mathrm{op}} \arrow[dr, "\mathrm{Spec}"] & & \mathbf{Sm}_{\mathrm{dom}} \arrow[dl, "J_{\mathrm{dom}}"'] \\ & \widehat{\mathbf{Sm}_{\mathrm{dom}}} & \end{tikzcd}\tag{4.10}$$

$$\begin{tikzcd} & S_b^{-1}\mathbf{Sm}_{\mathrm{dom}}\mathbf{P} \arrow[dl, "\overline{\Phi}_{1,\mathrm{dom}}"'] \arrow[dr, "\overline{\Phi}_{2,\mathrm{dom}}"] & \\ \mathbf{field}_s^{\mathrm{op}} \arrow[dr, "\overline{\mathrm{Spec}}"] & & S_b^{-1}\mathbf{Sm}_{\mathrm{dom}}. \arrow[dl, "\bar{J}_{\mathrm{dom}}"'] \\ & S_b^{-1}\widehat{\mathbf{Sm}_{\mathrm{dom}}} & \end{tikzcd}$$

Here, $\mathbf{field}_s \subseteq \mathbf{field}$ is the full subcategory of separably generated extensions, $\mathbf{Sm}_{\mathrm{dom}}\mathbf{P}$ is the subcategory of $\mathbf{VarP}$ given by varieties in $\mathbf{Sm}$ and morphisms are pairs (λ, f) where f is dominant (so that λ is an inclusion of function fields) and $\Phi_{1,\mathrm{dom}}$, $\Phi_{2,\mathrm{dom}}$ are the two forgetful functors of (2.1), restricted to $\mathbf{Sm}_{\mathrm{dom}}\mathbf{P}$. Similarly, J_{dom} is the analogue of J for $\mathbf{Sm}_{\mathrm{dom}}$. We extend the birational morphisms S_b as in Definition 4.1.1.

4.8.2. Theorem. *In the top diagram of (4.10), $\Phi_{2,\mathrm{dom}}$ is an isomorphism of categories. In the bottom diagram, all functors are equivalences of categories.*

Proof. The first claim follows from Corollary 2.2.3 c). In the right diagram, the proofs for $\bar{J}_{\mathrm{dom}}$ and $\overline{\Phi}_{1,\mathrm{dom}}$ are exactly parallel to those of Theorems 4.2.3 and 4.2.4 with a much simpler proof for the latter. As $\overline{\Phi}_{2,\mathrm{dom}}$ is an isomorphism of categories, the 4th functor $\overline{\mathrm{Spec}}$ is an equivalence of categories as well. □

In Theorem 4.8.2, we could replace $\mathbf{Sm}_{\mathrm{dom}}$ by $\mathbf{Var}_{\mathrm{dom}}$ or $\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}}$ (proper varieties) and $\mathbf{field}_s$ by $\mathbf{field}$ (same proofs).[6] Since $\Phi_{2,\mathrm{dom}}$ is an isomorphism of categories in both cases, we directly get a naturally commutative diagram of categories and functors

$$\begin{tikzcd}
 & S_b^{-1}\mathbf{Sm}_{\mathrm{dom}} \arrow[r, "\sim"] \arrow[d] & \mathbf{field}_s^{\mathrm{op}} \arrow[d] \\
S_b^{-1}\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \arrow[r, "\sim"] & S_b^{-1}\mathbf{Var}_{\mathrm{dom}} \arrow[r, "\sim"] & \mathbf{field}^{\mathrm{op}}.
\end{tikzcd} \tag{4.11}$$

where the horizontal ones are equivalences.

To make the link with (4.9), note that the functor ρ of (4.8) sends a birational morphism to an isomorphism. Hence ρ induces functors

$$S_b^{-1}\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \to S_b^{-1}\mathbf{Var}_{\mathrm{dom}} \to \mathbf{Rat}_{\mathrm{dom}} \tag{4.12}$$

whose composition with the second equivalence of (4.11) is (4.9).

4.8.3. Proposition. *Let $S = S_o, S_b$ or S_b^p.*
a) S admits a calculus of right fractions within $\mathbf{Var}_{\mathrm{dom}}$.
b) The functors in (4.12) are equivalences of categories.

Proof. a) For any pair (u, s) of morphisms as in Diagram (4.4), with $s \in S$ and u dominant, the pull-back of s by u exists and is in S. Moreover, if $sf = sg$ with f and g dominant and $s \in S$, then $f = g$.
b) This follows from (4.11) and (4.9). □

Taking a quasi-inverse of (4.11), we now get an equivalence of categories

$$\Psi_{\mathrm{dom}} : \mathbf{field}_s^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}_{\mathrm{dom}} \tag{4.13}$$

which will be used in Section 6.

4.8.4. *Remark.* The functor $(S_b^p)^{-1}\mathbf{Var}_{\mathrm{dom}} \to \mathbf{field}^{\mathrm{op}}$ is not full (hence is not an equivalence of categories). For example, let X be a proper variety and Y an affine open subset of X, and let K be their common function field. Then the identity map $K \to K$ is not in the image of the above functor. Indeed,

[6] We could also replace dominant morphisms by flat morphisms, as in [19].

if it were, then by calculus of fractions it would be represented by a map of the form fs^{-1} where $s : X' \to X$ is proper birational. But then X' would be proper and $f : X' \to Y$ should be constant, a contradiction.
It can be shown that the localisation functor

$$(S_b^p)^{-1}\,\mathbf{Var}_{\mathrm{dom}} \to S_b^{-1}\,\mathbf{Var}_{\mathrm{dom}}$$

has a (fully faithful) right adjoint given by

$$(S_b^p)^{-1}\,\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \to (S_b^p)^{-1}\,\mathbf{Var}_{\mathrm{dom}}$$

via the equivalence $(S_b^p)^{-1}\,\mathbf{Var}_{\mathrm{dom}}^{\mathrm{prop}} \xrightarrow{\sim} S_b^{-1}\,\mathbf{Var}_{\mathrm{dom}}$ given by Proposition 4.8.3 b). The proof is similar to that of Theorem 5.3.1 (ii) below.

4.9. Recapitulation. We constructed a full and essentially surjective functor (Theorem 4.2.4)

$$\Psi_* : S_r^{-1}\,\mathbf{place}_*^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}$$

and an equivalence of categories (4.13)

$$\Psi_{\mathrm{dom}} = \bar{J}_{\mathrm{dom}}^{-1} \circ \overline{\mathrm{Spec}} : \mathbf{field}_s^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}_{\mathrm{dom}}.$$

Consider the natural functor

$$\theta : S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \to S_b^{-1}\mathbf{Sm}. \tag{4.14}$$

In characteristic zero, θ is an equivalence of categories by [21, Prop. 8.5], noting that in this case $\mathbf{Sm}_*^{\mathrm{prop}} = \mathbf{Sm}^{\mathrm{prop}}$ by Hironaka. Let ι be the inclusion $\mathbf{field}_s^{\mathrm{op}} \hookrightarrow \mathbf{place}_*^{\mathrm{op}}$. Then the natural transformation $\eta : \mathrm{Spec} \Rightarrow \Sigma$ of (3.1) provides the following naturally commutative diagram

$$\begin{tikzcd}
\mathbf{field}_s^{\mathrm{op}} \arrow[r, "\Psi_{\mathrm{dom}}", "\sim"'] \arrow[d, "\iota"] & S_b^{-1}\mathbf{Sm}_{\mathrm{dom}} \arrow[r] & S_b^{-1}\mathbf{Sm} \\
S_r^{-1}\,\mathbf{place}_*^{\mathrm{op}} \arrow[r, "\Psi_*"] & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}. \arrow[ur, "\theta", "\sim"'] &
\end{tikzcd} \tag{4.15}$$

(Note that η induces a natural isomorphism $\bar{\eta} : \overline{\mathrm{Spec}} \overset{\sim}{\Rightarrow} \bar{\Sigma}$.)
We can replace $^{\mathrm{prop}}$ by $^{\mathrm{proj}}$ in all this story.
In characteristic p, we don't know if $\mathbf{field}_s \subset \mathbf{place}_*$: to get an analogue of (4.15) we would have to take the intersection of these categories. We shall do this in Section 6 in an enhanced way, using a new idea (Lemma 6.3.4 a)). As a byproduct, we shall get the full faithfulness of θ in any characteristic (Corollary 6.6.4)

5. Other classes of varieties

In this section we prove that, given a full subcategory $\mathcal{C}$ of $\mathbf{Var}$ satisfying certain hypotheses, the functor

$$S_b^{-1}\mathcal{C}\mathbf{P} \to \mathbf{place}^{\mathrm{op}}$$

induced by the functor Φ_1 of Diagram (4.1) is fully faithful.

5.1. The $_*$ construction. We generalise Definition 4.2.1 as follows:

5.1.1. Definition. Let $\mathcal{C}$ be a full subcategory of **Var**. We write $\mathcal{C}_*$ for the full subcategory of $\mathcal{C}$ with the following objects: $X \in \mathcal{C}_*$ if and only if, for any $Y \in \mathbf{Var}^{\mathrm{prop}}$ birational to X, there exists $X' \in \mathcal{C}$ and a proper birational morphism $s : X' \to Y$.

5.1.2. Lemma. *a) $\mathcal{C}_*$ is closed under birational equivalence.*
*b) We have $\mathcal{C}_* = \mathcal{C}$ for the following categories: $\mathbf{Var}, \mathbf{Norm}$ *and $\mathbf{Sm}, \mathbf{Sm}^{\mathrm{qp}}$ if $\operatorname{char} F = 0$.*
c) We have $\mathcal{C}_* \cap \mathcal{C}^{\mathrm{prop}} = (\mathcal{C}^{\mathrm{prop}})_*$, where $\mathcal{C}^{\mathrm{prop}} := \mathbf{Var}^{\mathrm{prop}} \cap \mathcal{C}$.

Proof. a) is tautological. b) is trivial for **Var**, is true for **Norm** because normalisation is finite and birational in **Var**, and follows from Hironaka's resolution for **Sm**. Finally, c) is trivial. □

5.1.3. Lemma. *Suppose $\mathcal{C}$ verifies the following condition: given a diagram*

$$\begin{array}{ccc} X' & \xrightarrow{j} & \tilde{X} \\ {\scriptstyle p}\downarrow & & \\ X & & \end{array}$$

with $X, \tilde{X} \in \mathcal{C}_*$, $p \in S_b^p$, $j \in S_o$ and $\tilde{X}$ proper, we have $X' \in \mathcal{C}$. (This holds in the following special cases: $\mathcal{C} \subseteq \mathbf{Var}^{\mathrm{prop}}$, or $\mathcal{C}$ stable under open immersions.)
a) Let $X \in \mathcal{C}_*$. Then the following holds: for any $s : Y \to X$ with $Y \in \mathbf{Var}$ and $s \in S_b^p$, there exists $t : X' \to Y$ with $X' \in \mathcal{C}_*$ and $t \in S_b^p$.
b) Let $X, Y \in \mathcal{C}_*$ with Y proper, and let $\gamma : X \dashrightarrow Y$ be a rational map. Then there exists $X' \in \mathcal{C}_*$, $s : X' \to X$ in S_b^p and a morphism $f : X' \to Y$ such that $\gamma = fs^{-1}$.

Proof. a) By Nagata's theorem, choose a compactification $\bar{Y}$ of Y. By hypothesis, there exists $\bar{X}' \in \mathcal{C}$ and a proper birational morphism $t' : \bar{X}' \to \bar{Y}$. If $X' = t'^{-1}(Y)$, then $t : X' \to Y$ is a proper birational morphism. The hypothesis on $\mathcal{C}$ then implies that $X' \in \mathcal{C}$, hence $X' \in \mathcal{C}_*$ by Lemma 5.1.2 a).
b) Apply a) to the graph of γ, which is proper over X. □

5.2. Calculus of fractions.

5.2.1. Proposition. *Under the condition of Lemma 5.1.3, Propositions 4.4.1 and 4.4.2 remain valid for $\mathcal{C}_*\mathbf{P}$. In particular, any morphism in $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$ or $(S_b^p)^{-1}\mathcal{C}_*$ is of the form fp^{-1}, with $f \in \mathcal{C}_*\mathbf{P}$ or $\mathcal{C}_*$ and $p \in S_b^p$.*

Proof. Indeed, the only fact that is used in the proofs of Propositions 4.4.1 and 4.4.2 is the conclusion of Lemma 5.1.3 a). □

To go further, we need:

5.2.2. Proposition. *In $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$, S_o admits a calculus of left fractions. In particular (cf. Proposition 5.2.1), any morphism in $S_b^{-1}\mathcal{C}_*\mathbf{P}$ may be written as $j^{-1}fq^{-1}$, with $j \in S_o$ and $q \in S_b^p$.*

Proof. a) Consider a diagram in $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "\varphi"'] & X' \\ Y & \end{tikzcd}$$

with $j \in S_o$. By Proposition 5.2.1, we may write $\varphi = fp^{-1}$ with $p \in S_b^p$ and f a morphism of $\mathcal{C}_*\mathbf{P}$ (f, p originate from some common $\bar{X}$). We may embed Y as an open subset of a proper $\bar{Y}$. This gives us a rational map $X' \dashrightarrow \bar{Y}$. Using the graph trick, we may "resolve" this rational map into a morphism $g : \tilde{X}' \to \bar{Y}$, with $\tilde{X}' \in \mathbf{Var}$ provided with a proper birational morphism $q : \tilde{X}' \to X'$. Since $Y \in \mathcal{C}_*$, we may assume $\bar{X}' \in \mathcal{C}_*$. Let $\psi = gq^{-1} \in (S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$. Then the diagram in $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "\varphi"'] & X' \arrow[d, "\psi"] \\ Y \arrow[r, "j_1"] & \bar{Y} \end{tikzcd}$$

commutes because the following bigger diagram commutes in $\mathcal{C}_*\mathbf{P}$:

$$\begin{tikzcd} \tilde{X} \arrow[dr, "p"] \arrow[ddr, "f"'] & & \tilde{X}'' \arrow[ll, "r"'] \arrow[rr, "r'"] & & \tilde{X}' \arrow[dl, "q"'] \arrow[ddl, "g"] \\ & X \arrow[rr, "j"] & & X' & \\ & Y \arrow[rr, "j_1"] & & \bar{Y} & \end{tikzcd}$$

thanks to Lemma 2.3.6, for suitable $\tilde{X}'' \in \mathcal{C}_*$ and $r, r' \in S_b^p$.
b) Consider a diagram

$$X' \xrightarrow{j} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$, where $j \in S_o$ and $fj = gj$. By Proposition 5.2.1, we may write $f = \tilde{f}p^{-1}$ and $g = \tilde{g}p^{-1}$, where $\tilde{f}, \tilde{g}$ are morphisms in $\mathcal{C}_*\mathbf{P}$ and $p : \tilde{X} \to X$ is in S_b^p. Let U be a common open subset to X' and $\tilde{X}$: then the equality $fj = gj$ implies that the restrictions of $\tilde{f}$ and $\tilde{g}$ to U coincide as morphisms of $(S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$. Hence the places underlying $\tilde{f}$ and $\tilde{g}$ are equal, which implies that $\tilde{f} = \tilde{g}$ (Proposition 2.2.2), and thus $f = g$. □

5.2.3. *Remark.* S_o does not admit a calculus of *right* fractions, even in $(S_b^p)^{-1}\,\mathbf{VarP}$. Indeed, consider a diagram in $(S_b^p)^{-1}\,\mathbf{VarP}$

$$\begin{tikzcd} & Y' \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $j \in S_o$ and, for simplicity, f comes from $\mathbf{VarP}$. Suppose that we can complete this diagram into a commutative diagram in $(S_b^p)^{-1}\,\mathbf{VarP}$

$$\begin{tikzcd} \tilde{X}' \arrow[d, "p"'] \arrow[dr, "g"] & \\ X' \arrow[r, "\varphi"] \arrow[d, "j'"'] & Y' \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $p \in S_b^p$ and g comes from $\mathbf{VarP}$. By Proposition 2.2.2 the localisation functor $\mathbf{VarP} \to (S_b^p)^{-1}\,\mathbf{VarP}$ is faithful, so the diagram (without φ) must already commute in $\mathbf{VarP}$. If $f(X) \cap Y' = \emptyset$, this is impossible.

5.3. GENERALISING THEOREM 4.2.3.

5.3.1. THEOREM. *Let $\mathcal{C}$ be a full subcategory of* $\mathbf{Var}$. *In diagram* (4.1),
a) J induces an equivalence of categories $S_b^{-1}\mathcal{C} \to S_b^{-1}\hat{\mathcal{C}}$.
b) Suppose that $\mathcal{C}$ verifies the condition of Lemma 5.1.3. Consider the string of functors

$$(S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}}\mathbf{P} \xrightarrow{S} (S_b^p)^{-1}\mathcal{C}_*\mathbf{P} \xrightarrow{T} S_b^{-1}\mathcal{C}_*\mathbf{P} \xrightarrow{\overline{\Phi_1^*}} \mathbf{place}^{\mathrm{op}}.$$

where S and T are the obvious ones and $\overline{\Phi_1^*}$ is induced by Φ_1. Then

(i) *S is fully faithful and T is faithful.*
(ii) *For any $X \in (S_b^p)^{-1}\mathcal{C}_*P$ and $Y \in (S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}}\mathbf{P}$, the map*

$$T : \mathrm{Hom}(X, S(Y)) \to \mathrm{Hom}(T(X), TS(Y)) \tag{5.1}$$

is an isomorphism.
(iii) *TS is an equivalence of categories.*
(iv) *$\overline{\Phi_1^*}$ is fully faithful.*

Proof. a) It is exactly the same proof as for the case of $\bar{J}$ in Theorem 4.2.3.
b) In 4 steps:
A) We run through the proof of Theorem 4.2.3 given in §4.7 for $\overline{\Phi_1^*}$ in the case $\mathcal{C} = \mathbf{Sm}^{\mathrm{prop}}$. In view of Proposition 5.2.1, the proof of faithfulness for $\overline{\Phi_1^*}T$ goes through verbatim. The proof of fullness for $\overline{\Phi_1^*}TS$ also goes through (note that in *loc. cit.* , we need Y to be proper in order for λ to be finite on it). It follows that S is fully faithful and T is faithful.

B) By A), (5.1) is injective. Let $\varphi \in \operatorname{Hom}(T(X), TS(Y))$. By Proposition 5.2.2, $\varphi = j^{-1}fp^{-1}$ with $j \in S_o$ and $p \in S_b^p$. Since Y is proper, j is necessarily an isomorphism, which shows the surjectivity of (5.1). This proves (ii).
C) It follows from A) and B) that TS is fully faithful. Essential surjectivity follows from Lemma 5.1.2 a) and c) plus Nagata's theorem. This proves (iii).
D) We come to the proof of (iv). Since $\overline{\Phi_1^*}TS$ is faithful (see A)) and TS is an equivalence, $\overline{\Phi_1^*}$ is faithful. To show that it is full, let $X, Y \in \mathcal{C}_*\mathbf{P}$ and $\lambda : F(Y) \rightsquigarrow F(X)$ a place. Let $Y \to \bar{Y}$ be a compactification of Y. By Definition 5.1.1, we may choose $\bar{Y}' \xrightarrow{s} \bar{Y}$ with $s \in S_b^p$ and $\bar{Y}' \in \mathcal{C}_*^{\mathrm{prop}}$. Then λ is finite over $\bar{Y}'$. By Lemma 2.3.4, there is a rational map $f : X \dashrightarrow \bar{Y}'$ compatible with λ. Applying Lemma 5.1.3 b) to the rational maps $X \dashrightarrow \bar{Y}'$ and $Y \dashrightarrow \bar{Y}'$, we find a diagram in $\mathcal{C}_*$

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "t"] & \bar{Y}' & Y' \arrow[l, "t'"'] \arrow[d, "s'"] \\ X & & Y \end{tikzcd}$$

with $t, s' \in S_b^p$ (and $t' \in S_b$). Then $\varphi = s't'^{-1}ft^{-1} : X \to Y$ is such that $\overline{\Phi_1^*}(\varphi) = \lambda$. □

5.3.2. Corollary. *The localisation functor T has a right adjoint, given explicitly by $(TS)^{-1} \circ S$.* □

Consider now the commutative diagram of functors:
(5.2)

$$\begin{tikzcd} (S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}}\mathbf{P} \arrow[r, "S"] \arrow[d] & (S_b^p)^{-1}\mathcal{C}_*\mathbf{P} \arrow[r, "T"] \arrow[d] & S_b^{-1}\mathcal{C}_*\mathbf{P} \arrow[r, "\overline{\Phi_1^*}"] \arrow[d] & \mathbf{place}^{\mathrm{op}} \arrow[d, equal] \\ (S_b^p)^{-1}\,\mathbf{Var}^{\mathrm{prop}}\,\mathbf{P} \arrow[r, "S"] & (S_b^p)^{-1}\,\mathbf{VarP} \arrow[r, "T"] & S_b^{-1}\,\mathbf{VarP} \arrow[r, "\overline{\Phi_1^*}"] & \mathbf{place}^{\mathrm{op}}. \end{tikzcd}$$

5.3.3. Corollary. *All vertical functors in (5.2) are fully faithful.*

Proof. For the first and third vertical functors, this is a byproduct of Theorem 5.3.1. The middle one is faithful by the faithfulness of T and $\overline{\Phi_1^*}$ in Theorem 5.3.1. For fullness, let $X, Y \in (S_b^p)^{-1}\mathcal{C}_*\mathbf{P}$ and $\varphi : X \to Y$ be a morphism in $(S_b^p)^{-1}\,\mathbf{VarP}$. By Proposition 5.2.1, we may write $\varphi = fp^{-1}$, with $p : \tilde{X} \to X$ proper birational. By Lemma 5.1.3 a), we may find $p' : \tilde{X}' \to \tilde{X}$ proper birational with $\tilde{X}' \in \mathcal{C}_*$, and replace fp^{-1} by $fp'(pp')^{-1}$. □

5.3.4. *Remarks.* 1) Take $\mathcal{C} = \mathbf{Var}$ in Theorem 5.3.1 and let $X, Y \in (S_b^p)^{-1}\,\mathbf{VarP}$. Then the image of $\operatorname{Hom}(X, Y)$ in $\operatorname{Hom}(\overline{\Phi_1^*}T(Y), \overline{\Phi_1^*}T(X))$ via $\overline{\Phi_1^*}T$ is contained in the set of places which are finite on Y. If X and Y are proper, then the image is all of $\operatorname{Hom}(\overline{\Phi_1^*}T(Y), \overline{\Phi_1^*}T(X))$. On the other hand, if X is proper and Y is affine, then for any map $\varphi = fp^{-1} : X \to Y$, the

source X' of p is proper hence $f(X')$ is a closed point of Y, so that the image is contained in the set of places from $F(Y)$ to $F(X)$ whose centre on Y is a closed point (and one sees easily that this inclusion is an equality). In general, the description of this image seems to depend heavily on the geometric nature of X and Y.
2) For "usual" subcategories $\mathcal{C} \subseteq \mathbf{Var}$, the functors Φ_1^*, $\Phi_1^* T$ and $\Phi_1^* TS$ of Theorem 5.3.1 b) are essentially surjective (hence so are those in Corollary 5.3.3): this is true for $\mathcal{C} = \mathbf{Var}$ or $\mathbf{Norm}$ (any function field has a normal proper model), and for $\mathcal{C} = \mathbf{Sm}$ in characteristic 0. For $\mathcal{C} = \mathbf{Sm}$ in positive characteristic, the essential image of these functors is the category $\mathbf{place}_*^{\mathrm{op}}$ of Definition 4.2.1.

5.4. LOCALISING $\mathcal{C}_*$. In Theorem 5.3.1, we generalised Theorem 4.2.3 which was used to construct the functor Ψ_* of (4.3). A striking upshot is Corollary 5.3.3. What happens if we study $S_b^{-1}\mathcal{C}_*$ instead of $S_b^{-1}\mathcal{C}_*\mathbf{P}$?
This was done previously in [21, §8], by completely different methods. The two main points were:

- In characteristic 0, we have the following equivalences of categories:

$$S_b^{-1}\mathbf{Sm}^{\mathrm{proj}} \simeq S_b^{-1}\mathbf{Sm}^{\mathrm{prop}} \simeq S_b^{-1}\mathbf{Sm}^{\mathrm{qp}} \simeq S_b^{-1}\mathbf{Sm} \tag{5.3}$$

  induced by the obvious inclusion functors [21, Prop. 8.5].
- Working with varieties that are not smooth or at least regular leads to pathologies: for example, the functor $S_b^{-1}\mathbf{Sm} \to S_b^{-1}\mathbf{Var}$ is neither full nor faithful [21, Rk. 8.11]. This contrasts starkly with Corollary 5.3.3. The issue is closely related to the regularity condition appearing in Lemma 2.3.2; it is dodged in [21, Prop. 8.6] by restricting to those morphisms that send smooth locus into smooth locus.

Using the methods of [21], one can show that the functor

$$(S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}} = S_b^{-1}\mathcal{C}_*^{\mathrm{prop}} \to S_b^{-1}\mathcal{C}_* \tag{5.4}$$

is an equivalence of categories for any $\mathcal{C} \subseteq \mathbf{Var}$ satisfying the condition of Lemma 5.1.3. For this, one should use [21, Th. 5.14] under a form similar to that given in [21, Prop. 5.10]. One can then deduce from Corollary 5.3.2 that the localisation functor

$$(S_b^p)^{-1}\mathcal{C}_* \xrightarrow{T} S_b^{-1}\mathcal{C}_*$$

has a right adjoint given (up to the equivalence (5.4)) by $(S_b^p)^{-1}\mathcal{C}_*^{\mathrm{prop}} \xrightarrow{S} (S_b^p)^{-1}\mathcal{C}_*$ (in particular, S is fully faithful): indeed, the unit and counit of the adjunction in Corollary 5.3.2 map by the essentially surjective forgetful functors

$$S_b^{-1}\mathcal{C}_*\mathbf{P} \to S_b^{-1}\mathcal{C}_*, \quad \text{etc.} \tag{5.5}$$

to natural transformations which keep enjoying the identities of an adjunction. Note however that (5.5) is not full unless $\mathcal{C} \subseteq \mathbf{Sm}$ (see Lemma 1.1.2 and Lemma 2.2.2 for this case).
For $\mathcal{C} = \mathbf{Sm}$ or $\mathbf{Sm}^{\mathrm{qp}}$, the equivalence (5.4) extends a version of (5.3) to positive characteristic. We won't give a detailed proof however, because it would be

tedious and we shall obtain a better result later (Corollary 6.6.4) by a different method.

The proofs given in [21] do not use any calculus of fractions. In fact, S_b^p *does not admit any calculus of fractions within* **Var**, contrary to the case of **VarP** (*cf.* Proposition 4.4.1). This is shown by the same examples as in Remark 2.3.3. If we restrict to $\mathbf{Sm}_*$, we can use Proposition 4.4.1 and Lemma 2.3.2 to prove a helpful part of calculus of fractions:

5.4.1. Proposition. *a) Let $s : Y \to X$ be in S_b^p, with X smooth. Then s is an envelope [9]: for any extension K/F, the map $Y(K) \to X(K)$ is surjective.*
b) The multiplicative set S_b^p verifies the second axiom of calculus of right fractions within $\mathbf{Sm}_*$.
c) Any morphism in $S_b^{-1}\mathbf{Sm}_*$ may be represented as $j^{-1}fp^{-1}$, where $j \in S_o$ and $p \in S_b^p$.

Proof. a) Base-changing to K, it suffices to deal with $K = F$. Let $x \in X(F)$. By lemma 2.3.2, there is a place λ of $F(X)$ with centre x and residue field F. The valuative criterion for properness implies that λ has a centre y on Y; then $s(y) = x$ by Lemma 2.3.6 and $F(y) \subseteq F(\lambda) = F$.
b) We consider a diagram (4.4) in $\mathbf{Sm}_*$, with $s \in S_b^p$. By a), $z = u(\eta_X)$ has a preimage $z' \in Y'$ with same residue field. Let $Z = \overline{\{z\}}$ and $Z' = \overline{\{z'\}}$: the map $Z' \to Z$ is birational. Since the map $\bar{u} : X \to Z$ factoring u is dominant, we get by Theorem 4.8.3 b) a commutative diagram like (4.5), with s' proper birational. By Lemma 5.1.3 a), we may then replace X' by an object of $\mathbf{Sm}_*$.
c) As that of Proposition 5.2.2. □

5.4.2. *Remark.* On the other hand, S_b^p is far from verifying the third axiom of calculus of right fractions within $\mathbf{Sm}_*$. Indeed, let $s : Y \to X$ be a proper birational morphism that contracts some closed subvariety $i : Z \subset Y$ to a point. Then, given any two morphisms $f, g : Y' \rightrightarrows Z$, we have $sif = sig$. But if $ift = igt$ for some $t \in S_b^p$, then $if = ig$ (hence $f = g$) since t is dominant.

6. Homotopy of places and R-equivalence

In this section, we do several things. In Subsection 6.1 we prove elementary results on divisorial valuations with separably generated residue fields. In Subsection 6.2 we introduce a subcategory **dv** of **place**, where morphisms are generated by field inclusions and places given by discrete valuation rings. We relate it in Subsection 6.3 with a construction of Asok-Morel [1] to define a functor

$$\Psi : S_r^{-1}\,\mathbf{dv} \to S_b^{-1}\mathbf{Sm}$$

extending the functor Ψ_{dom} of (4.13). This functor is compatible with the functor Ψ_* of Theorem 4.2.4. We then show in Proposition 6.4.3 that the localisation **place** $\to S_r^{-1}$ **place** is also a quotient by a certain equivalence relation h; although remarkable, this fact is elementary.

Next, we reformulate a result of Asok-Morel to enlarge the equivalence relation h to another, h′, so that the functor Ψ factors through an equivalence of categories

$$\mathbf{dv} / \mathrm{h}' \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}.$$

Finally, we use another result of Asok-Morel to compute some Hom sets in $S_b^{-1}\mathbf{Sm}$ as R-equivalence classes: in the first version of this paper, we had proven this only in characteristic 0 by much more complicated arguments.

6.1. Good dvr's.

6.1.1. Definition. A discrete valuation ring (dvr) R containing F is *good* if its quotient field K and its residue field E are finitely and separably generated over F, with $\operatorname{trdeg}(E/F) = \operatorname{trdeg}(K/F) - 1$.

6.1.2. Lemma. *A dvr R containing F is good if and only if there exist a smooth F-variety X and a smooth divisor $D \subset X$ such that $R \simeq \mathcal{O}_{X,D}$.*

Proof. Sufficiency is clear by Lemma 4.8.1. Let us show necessity. The condition on the transcendence degrees means that R is *divisorial* = a "prime divisor" in the terminology of [40]. By *loc. cit.* , Ch. VI, Th. 31, there exists then a model X of K/F such that $R = \mathcal{O}_{X,x}$ for some point x of codimension 1. (In particular, granting the finite generation of K, that of E is automatic.) Furthermore, the separable generation of E yields a short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{R/F} \otimes_R E \to \Omega^1_{E/F} \to 0$$

where $\mathfrak{m}$ is the maximal ideal of R (see Exercise 8.1 (a) of [16, Ch. II]). Therefore $\dim_E \Omega^1_{R/F} \otimes_R E = \operatorname{trdeg}(K/F) = \dim_K \Omega^1_{R/F} \otimes_R K$, thus $\Omega^1_{R/F}$ is free of rank $\operatorname{trdeg}(K/F)$ and x is a smooth point of X. Shrinking X around x, we may assume that it is smooth; if $D = \overline{\{x\}}$, it is generically smooth by Lemma 4.8.1, hence we may assume D is smooth up to shrinking X further. □

6.1.3. Lemma. *Let R be a good dvr containing F, with quotient field K and residue field E, and let K_0/F be a subextension of K/F. Then $R \cap K_0$ is either K_0 or a good dvr.*

Proof. By Mac Lane's criterion, K_0 is separably generated, and the same applies to the residue field $E_0 \subseteq E$ of $R \cap K_0$ if the latter is a dvr. □

6.2. The category dv.

6.2.1. Definition. Let K/F and L/F be two separably generated extensions. We denote by $\mathbf{dv}(K, L)$ the set of morphisms in $\mathbf{place}(K, L)$ of the form

$$K \rightsquigarrow K_1 \rightsquigarrow \ldots \rightsquigarrow K_n \hookrightarrow L \tag{6.1}$$

where for each i, the place $K_i \rightsquigarrow K_{i+1}$ corresponds to a good dvr with quotient field K_i and residue field K_{i+1}. (Compare [40, Ch. VI, §3].)

6.2.2. Lemma. *In* $\mathbf{dv}(K,L)$*, the decomposition of a morphism in the form* (6.1) *is unique. The collection of the* $\mathbf{dv}(K,L)$*'s defines a subcategory* $\mathbf{dv} \subset \mathbf{place}$*, with objects the separably generated function fields.*

Proof. Uniqueness follows from [40, p. 10]. To show that $Ar(\mathbf{dv})$ is closed under composition, we immediately reduce to the case of a composition

$$K \overset{i}{\hookrightarrow} L \overset{\lambda}{\rightsquigarrow} L_1 \tag{6.2}$$

where (L, L_1) correspond to a good dvr R. Then the claim follows from applying Lemma 6.1.3 to the commutative diagram in **place**

$$\begin{tikzcd} L \arrow[r, "\lambda"] & L_1 \\ K \arrow[u, "i"] \arrow[r, "\lambda_1"] & K_1 \arrow[u, "i_1"'] \end{tikzcd} \tag{6.3}$$

where K_1 is the residue field of $R \cap K$ if this is a dvr, and $K_1 = K$ otherwise (and then λ_1 is a trivial place). □

We shall need the following variant of a theorem of Knaf and Kuhlmann [23, Th. 1.1] (compare *loc. cit.* , pp. 834/835):

6.2.3. Theorem. *Let* $\lambda : K \rightsquigarrow L$ *be a morphism in* $\mathbf{dv}$*. Then* λ *is finite over a smooth model of* K*. Moreover, let* $K' \subseteq K$ *be a subextension of* K*, and let* Z *be a model of* K' *on which* $\lambda_{|K'}$ *has a centre* z*. Then there is a smooth model* X *of* K *on which* λ *has a centre of codimension* n*, the rank of* λ*, and a morphism* $X \to Z$ *inducing the extension* K/K'*.*

Proof. This actually follows from [23, Th. 1.1][7]: let U be an open affine neighbourhood of z and let $E := \{y_1, \dots, y_r\}$ be a set of generators of the F-algebra $\mathcal{O}_Z(U)$ (ring of sections). Then by [23, Th. 1.1], there exists a model X_0 of K/F such that:

- λ is centred at a smooth point x of X_0,
- $\dim \mathcal{O}_{X_0,x} = n = \dim \mathcal{O}_\lambda$,
- E is contained in the maximal ideal of $\mathcal{O}_{X_0,x}$.

Hence $\mathcal{O}_Z(U) \subseteq \mathcal{O}_{X_0}(X)$ for some open affine neighbourhood X of x, which yields a morphism $X \to U$ that maps x to z. □

6.3. Relationship with the work of Asok and Morel. In [1, §6], Asok and Morel prove closely related results: let us translate them in the present setting.

Let us write $\mathcal{C}^\vee$ for the category of presheaves of sets on a category $\mathcal{C}$. In [1], the authors denote the category $(S_r^{-1}\mathbf{Sm})^\vee$ by $Shv_F^{h\mathbf{A}^1}$. Similarly, they write $\mathcal{F}_F^r - \mathbf{Set}$ for the category consisting of objects of $(\mathbf{field}_s^{\mathrm{op}})^\vee$ provided with

[7] We thank Hagen Knaf for his help in this proof.

"specialisation maps" for good dvrs. In [1, Th. 6.1.7], they construct a full embedding

$$Shv_F^{h\mathbf{A}^1} \to \mathcal{F}_F^r - \mathbf{Set} \tag{6.4}$$

(evaluate presheaves on function fields), and show that its essential image consists of those functors $\mathcal{S} \in \mathcal{F}_F^r - \mathbf{Set}$ satisfying a list of axioms (A1) – (A4) (ibid., Defn. 6.1.6).

The proof of Lemma 6.2.2 above shows that Conditions (A1) and (A2) mean that $\mathcal{S}$ defines a functor $\mathbf{dv}^{\mathrm{op}} \to \mathbf{Set}$, and Condition (A4) means that $\mathcal{S}$ factors through $S_r^{-1}\,\mathbf{dv}^{\mathrm{op}}$. In other words, they essentially[8] construct a functor

$$(S_r^{-1}\mathbf{Sm})^\vee \to (S_r^{-1}\,\mathbf{dv}^{\mathrm{op}})^\vee.$$

We now check that this functor is induced by a functor

$$\Psi : S_r^{-1}\,\mathbf{dv}^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}. \tag{6.5}$$

For this, we need a lemma:

6.3.1. Lemma. *Let $\mathbf{Sm}^{\mathrm{ess}}$ be the category of irreducible separated smooth F-schemes essentially of finite type. Then the full embedding $\mathbf{Sm} \hookrightarrow \mathbf{Sm}^{\mathrm{ess}}$ induces an equivalence of categories*

$$S_b^{-1}\mathbf{Sm} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}^{\mathrm{ess}}.$$

Proof. We use again the techniques of [21], to which we refer the reader: actually the first part of the proof of [21, Prop. 8.4] works with a minimal change. Namely, with notation as in *loc. cit.* , there are 3 conditions (b1) – (b3) to check:

(b1) *Given $f, g : X \to Y$ in $\mathbf{Sm}$ and $s : Z \to X$ in $\mathbf{Sm}^{\mathrm{ess}}$ with $s \in S_b$, $fs = gs \Rightarrow f = g$*: this follows from Lemma 1.4.1 (birational morphisms are dominant).

(b2) follows from the fact that any essentially smooth scheme may be embedded in a smooth scheme of finite type by an "essentially open immersion".

(b3) We are given $i : X \to \bar{X}$ and $j : X \to Y$ where $X \in \mathbf{Sm}^{ess}$, $\bar{X}, Y \in \mathbf{Sm}$ and $i \in S_b$; we must factor i and j through $X \xrightarrow{s} U$ with U in $\mathbf{Sm}$ and s, $U \to \bar{X}$ in S_b. We take for U the smooth locus of the closure of the diagonal image of X in $\bar{X} \times Y$.

□

To define Ψ, it is now sufficient to construct it as a functor $\Psi : S_r^{-1}\,\mathbf{dv}^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}^{\mathrm{ess}}$. We first construct Ψ on $\mathbf{dv}^{\mathrm{op}}$ by extending the functor Ψ_{dom} of (4.13) from $\mathbf{field}_s^{\mathrm{op}}$ to $\mathbf{dv}^{\mathrm{op}}$. For this, we repeat the construction given on [1, p. 2041]: if $K \in \mathbf{dv}$ and $\mathcal{O}$ is a good dvr with quotient field K and residue field E, then the morphism $\operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}$ is an isomorphism in $S_b^{-1}\mathbf{Sm}^{\mathrm{ess}}$, hence the quotient map $\mathcal{O} \to E$ induces a morphism $\operatorname{Spec} E \to \operatorname{Spec} K$.

[8]Essentially because Condition (A1) of [1, §6] only requires a commutation of diagrams coming from (6.3) when the ramification index is 1.

By Lemma 6.2.2, any morphism in $\mathbf{dv}$ has a unique expression in the form (6.1), which extends the definition of Ψ to all morphisms. To show that Ψ is a functor, it now suffices to check that it converts any diagram (6.3) into a commutative diagram, which is obvious by going through its construction. Finally, Ψ factors through $S_r^{-1}\mathbf{dv}^{\mathrm{op}}$ thanks to Theorem 1.7.2. It is now clear that the dual of Ψ gives back the Asok-Morel functor (6.4).
As in §4.5, we associate to a rational map f between smooth varieties a morphism in $S_b^{-1}\mathbf{Sm}$, still denoted by f. We need the following analogue of Lemma 4.6.1:

6.3.2. Proposition. *Let $\lambda : K \rightsquigarrow L$ be a morphism in $\mathbf{dv}$. Then, for any smooth model X of K on which λ is finite, we have $\Psi(\lambda) = st^{-1}f$, where $f : \Psi(L) \dashrightarrow X$ is the corresponding rational map and $s : U \hookrightarrow \Psi(K)$, $t : U \hookrightarrow X$ are open immersions of a common open subset U.*

Proof. We proceed by induction on the length n of a chain (6.1): If $n = 0$ the claim is trivial and if $n = 1$ it is true by construction. If $n > 1$, break λ as

$$K \overset{\lambda_1}{\rightsquigarrow} K_{n-1} \overset{\lambda_2}{\rightsquigarrow} K_n \hookrightarrow L$$

where λ_1 has rank $n-1$ and λ_2 has rank 1. We now apply Lemma 1.3.2: since λ is finite on X, so is λ_1, and if we write Z for the closure of $c_X(\lambda_1)$, then $z = c_X(\lambda) = c_Z(\lambda_2)$. If $n = 0$ the claim is trivial and if $n = 1$ it is true by construction. If $n > 1$, Theorem 6.2.3 provides us with $\pi : X_{n-1} \to Z$, X_{n-1} smooth with function field K_{n-1} on which λ_2 has a centre of codimension 1. Then we have a diagram

$$\begin{tikzcd}
\Psi(K) & \Psi(K_{n-1}) & \Psi(K_n) \\
U \arrow[u, "s"] \arrow[d, "t"'] & U_{n-1} \arrow[u, "s_{n-1}"] \arrow[d, "t_{n-1}"'] & U_n \arrow[u, "s_n"] \arrow[d, "t_n"'] \\
X & X_{n-1} \arrow[d, "\pi"] & X_n \arrow[l, "g"'] \\
 & Z \arrow[ul, "i"] &
\end{tikzcd}$$

where i is the closed immersion $Z \hookrightarrow X$, $s, s_{n-1}, s_n, t, t_{n-1}$ are open immersions and g is the closed immersion of a smooth divisor obtained by applying Lemma 6.1.2 after possibliy shrinking X_{n-1}. Thus (gt_n, s_n) represents the rational map given by the centre of λ_2 on X_{n-1}. The rational map corresponding to λ_1 is represented by (f_{n-1}, s_{n-1}) with

$$f_{n-1} = i\pi t_{n-1}$$

and the one corresponding to $\lambda_2\lambda_1$ is represented by (f_n, s_n) with

$$f_n = i\pi g t_n$$

because this is compatible with $\lambda_2\lambda_1$ by Proposition 2.2.4 (also use the uniqueness in Lemma 2.3.4).
By induction and definition, we have

$$\Psi(\lambda_1) = st^{-1}f_{n-1}s_{n-1}^{-1}, \quad \Psi(\lambda_2) = s_{n-1}t_{n-1}^{-1}gt_ns_n^{-1}$$

so we have to show that

$$st^{-1}f_{n-1}s_{n-1}^{-1}s_{n-1}t_{n-1}^{-1}gt_ns_n^{-1} = st^{-1}f_ns_n^{-1}$$

or

$$f_{n-1}t_{n-1}^{-1}gt_n = f_n = i\pi gt_n$$

which is true because $f_{n-1} = i\pi t_{n-1}$. This concludes the proof. □

6.3.3. *Remark.* In this proof, there is no codimension condition on $c_X(\lambda)$. So Theorem 6.2.3 is used twice in a weak form: once, implicitly, to ensure the existence of X. Then a second time, to deal with Z. But here λ_2 is a discrete valuation of rank 1, so this special case can perhaps already be obtained by examining the proof of [40, Th. 31] (which may have been a source of inspiration for [23].)

6.3.4. Lemma. *a) Let* $\mathbf{dv}_*$ *be the full subcategory of* $\mathbf{dv}$ *whose objects are in* $\mathbf{place}_*$. *Then the diagram of functors*

$$\begin{tikzcd}
S_r^{-1}\,\mathbf{place}_*^{\mathrm{op}} \arrow[r, "\Psi_*"] & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \arrow[dd] \\
S_r^{-1}\,\mathbf{dv}_*^{\mathrm{op}} \arrow[u] \arrow[d] & \\
S_r^{-1}\,\mathbf{dv}^{\mathrm{op}} \arrow[r, "\Psi"] & S_b^{-1}\mathbf{Sm}
\end{tikzcd}$$

is naturally commutative.
b) Let $K, L \in \mathbf{dv}$ *and* $\lambda, \mu \in \mathbf{dv}(K, L)$ *with the same residue field* $K' \subseteq L$. *Suppose that* λ *and* μ *have a common centre on some smooth model of* K. *Then* $\Psi(\lambda) = \Psi(\mu)$.

Proof. a) Same argument as in §4.9, using the natural transformation Spec $\Rightarrow \Sigma$ of (3.1). b) follows from Proposition 6.3.2 (compare proof of Theorem 4.2.4 b) in §4.6). □

6.4. Homotopy of places.

6.4.1. Definition. Let $K, L \in \mathbf{place}$. Two places $\lambda_0, \lambda_1 : K \rightsquigarrow L$ are *elementarily homotopic* if there exists a place $\mu : K \rightsquigarrow L(t)$ such that $s_i \circ \mu = \lambda_i, i = 0, 1$, where $s_i : L(t) \rightsquigarrow L$ denotes the place corresponding to specialisation at i.

The property of two places being elementarily homotopic is preserved under composition on the right. Indeed if λ_0 and λ_1 are elementarily homotopic and if $\mu : M \rightsquigarrow K$ is another place, then obviously so are $\lambda_0 \circ \mu$ and $\lambda_1 \circ \mu$. If on the other hand $\tau : L \rightsquigarrow M$ is another place, then $\tau \circ \lambda_0$ and $\tau \circ \lambda_1$ are not in general elementarily homotopic (we are indebted to Gabber for pointing this out), as one can see for example from the uniqueness of factorisation of places [40, p. 10].

Consider the equivalence relation h generated by elementary homotopy (*cf.* Definition 1.2.1). So h is the coarsest equivalence relation on morphisms in **place** which is compatible with left and right composition and such that two elementarily homotopic places are equivalent with respect to h.

6.4.2. Definition (*cf.* Def. 1.2.1). We denote by **place** /h the factor category of **place** by the homotopy relation h.

Thus the objects of **place** /h are function fields, while the set of morphisms consists of equivalence classes of homotopic places between the function fields. There is an obvious full surjective functor

$$\Pi : \mathbf{place} \to \mathbf{place}/\mathrm{h}.$$

The following proposition provides a more elementary description of S_r^{-1} **place** and of the localisation functor.

6.4.3. Proposition. *There is a unique isomorphism of categories*

$$\mathbf{place}/\mathrm{h} \to S_r^{-1}\,\mathbf{place}$$

which makes the diagram of categories and functors

$$\begin{tikzcd} & \mathbf{place} \arrow[dl, "\Pi"'] \arrow[dr, "S_r^{-1}"] & \\ \mathbf{place}/\mathrm{h} \arrow[rr, "\sim"] & & S_r^{-1}\,\mathbf{place} \end{tikzcd}$$

commutative. In particular, the localisation functor S_r^{-1} is full and its fibres are the equivalence classes for h*. These results remain true when restricted to the subcategory* **dv**.

Proof. [9] We first note that any two homotopic places become equal in S_r^{-1} **place**. Clearly it suffices to prove this when they are elementarily homotopic. But then s_0 and s_1 are left inverses of the natural inclusion $i : L \to L(t)$, which becomes an isomorphism in S_r^{-1} **place**. Thus s_0 and s_1 become equal in S_r^{-1} **place**. So the localisation functor **place** $\to S_r^{-1}$ **place** canonically factors through Π into a functor **place** /h $\longrightarrow$ S_r^{-1} **place**.

On the other hand we claim that, with the above notation, $i \circ s_0 : L(t) \rightsquigarrow L(t)$ is homotopic to $1_{L(t)}$ in **place**. Indeed they are elementarily homotopic via the trivial place $L(t) \rightsquigarrow L(t, s)$ that is the identity on L and maps t to st. Hence

[9] See also [15, Remark 1.3.4] for a closely related statement.

the projection functor Π factors as $S_r^{-1}\,\mathbf{place} \to \mathbf{place}\,/\mathrm{h}$, and it is plain that this functor is inverse to the previous one.
The claim concerning $\mathbf{dv}$ is clear since the above proof only used good dvr's. □

6.5. ANOTHER EQUIVALENCE OF CATEGORIES. In this subsection, we study the "fibres" of the functor Ψ of (6.5) in the light of the last condition of [1, §6], (A3). Using Proposition 6.4.3, we may view Ψ as a functor

$$\Psi : (\mathbf{dv}\,/\,\mathrm{h})^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}.$$

Condition (A3) of [1, §6] for a functor $\mathcal{S} \in \mathcal{F}_F^r - \mathbf{Set}$ requires that for any $X \in \mathbf{Sm}$ with function field K, for any $z \in X^{(2)}$ (with separably generated residue field) and for any $y_1, y_2 \in X^{(1)}$ both specialising to z, the compositions

$$\mathcal{S}(K) \to \mathcal{S}(F(y_i)) \to \mathcal{S}(z), \quad i = 1, 2$$

are equal. We can interpret this condition in the present context by introducing the equivalence relation h_{AM} in $\mathbf{dv}$ generated by h and the following relation $\equiv$:

> *Given $K, L \in \mathbf{dv}$ and two places $\lambda_1, \lambda_2 : K \rightsquigarrow L$ of the form*
>
> (6.6) $$\begin{gathered} K \xrightarrow{\mu_1} K_1 \xrightarrow{\nu_1} L \\ K \xrightarrow{\mu_2} K_2 \xrightarrow{\nu_2} L \end{gathered}$$
>
> *where $\mu_1, \nu_1, \mu_2, \nu_2$ stem from good dvr's, $\lambda_1 \equiv \lambda_2$ if λ_1 and λ_2 have a common centre with residue field L on some smooth model of K.*

By Yoneda's lemma, [1, Th. 6.1.7] then yields an equivalence of categories

$$(\mathbf{dv}\,/\,\mathrm{h}_{AM})^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}. \tag{6.7}$$

Here we implicitly used Lemma 6.3.4 b) and Theorem 6.2.3 to see that the functor $(\mathbf{dv}\,/\,\mathrm{h})^{\mathrm{op}} \to S_b^{-1}\mathbf{Sm}$ factors through h_{AM}, as well as the following lemma:

6.5.1. LEMMA. *Let $\psi : \mathcal{C} \to \mathcal{D}$ be a functor such that the induced functor $\psi^* : \mathcal{D}^\vee \to \mathcal{C}^\vee$ is an equivalence of categories. Then ψ is fully faithful, hence an equivalence of categories if it is essentially surjective.*

(Note that the essential surjectivity of (6.7) is obvious.)

Proof. By [SGA4, I.5.3], ψ^* has a left adjoint $\psi_!$ which commutes naturally with ψ via the Yoneda embeddings $y_{\mathcal{C}}, y_{\mathcal{D}}$. Since ψ^* is an equivalence of categories, so is $\psi_!$; the conclusion then follows from the full faithfulness of $y_{\mathcal{C}}$ and $y_{\mathcal{D}}$. □

We now slightly refine the equivalence (6.7):

6.5.2. Theorem. *a) The functor* Ψ *induces an equivalence of categories:*

$$\overline{\Psi} : (\mathbf{dv} \,/\, \mathrm{h}')^{\mathrm{op}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}$$

where h' *is the equivalence relation generated by* h *and the relation* (6.6) *restricted to the tuples* $(\mu_1, \nu_1, \mu_2, \nu_2)$ *such that* ν_2 *is of the form* $s_0 : L(t) \rightsquigarrow L$ *(specialisation at* 0*). In particular,* Ψ *is full.*
b) Any morphism of $\mathbf{dv} \,/\, \mathrm{h}'$ *may be written in the form* $\iota^{-1} f$ *for* f *a morphism of the form* (6.2) *and* ι *a rational extension of function fields.*

Proof. a) Let us show that $\mathrm{h}' = \mathrm{h}_{AM}$. Starting from K, λ_1 and λ_2 as above, we get a smooth model X of K and $z, y_1, y_2 \in X$ with z of codimension 2, such that μ_i is specialisation to y_i and ν_i is specialisation from y_i to z. Shrinking, we may assume that the closures Z, Y_1, Y_2 of z, y_1, y_2 are smooth. Let $X' = \mathrm{Bl}_Z(X)$ be the blow-up of X at Z and let Y_1', Y_2' be the proper transforms of Y_1 and Y_2 in X'. The exceptional divisor P is a projective line over Z and $Z_i = P \cap Y_i'$ maps isomorphically to Z for $i = 1, 2$. We then get new places

$$\begin{aligned} \lambda_1' &: K \xrightarrow{\mu'} M \xrightarrow{\nu_1'} L \\ \lambda_2' &: K \xrightarrow{\mu'} M \xrightarrow{\nu_2'} L \end{aligned} \tag{6.8}$$

where $M = F(P)$, $L = F(Z)$ and $\lambda_i' \equiv \lambda_i$.
In $\mathbf{dv} \,/\, \mathrm{h} \simeq S_r^{-1}\,\mathbf{dv}$, the morphisms ν_1' and ν_2' are inverse to the rational extension $L \hookrightarrow L(t) \simeq M$, hence are equal, which concludes the proof that $\mathrm{h}' = \mathrm{h}_{AM}$.
The fullness of Ψ now follows from the obvious fullness of $\mathbf{dv} \to \mathbf{dv} \,/\, \mathrm{h}'$.
The argument in the proof of a) shows in particular that any composition $\nu \circ \mu$ of two good dvr's is equal in $\mathbf{dv} \,/\, \mathrm{h}'$ to such a composition in which ν is inverse to a purely transcendental extension of function fields: b) follows from this by induction on the number of dvr's appearing in a decomposition (6.1). □

6.5.3. *Remarks.* 1) Via $\overline{\Psi}$, Theorem 6.5.2 yields a structural result for morphisms in $S_b^{-1}\mathbf{Sm}$, closely related to Proposition 5.4.1 c) but weaker. See however Theorem 6.6.3 below.
2) We don't know any example of an object in $\mathcal{F}_F^r - \mathbf{Set}$ which verifies (A1), (A2) and (A4) but not (A3): it would be interesting to exhibit one.

6.6. R-equivalence. Recall the following definition of Manin:

6.6.1. Definition. a) Two rational points x_0, x_1 of a (separated) F-scheme X of finite type are *directly* R*-equivalent* if there is a rational map $f : \mathbf{P}^1 \dashrightarrow X$ defined at 0 and 1 and such that $f(0) = x_0$, $f(1) = x_1$.
b) R-equivalence on $X(F)$ is the equivalence relation generated by direct R-equivalence.

Recall that, for any X, Y, we have an isomorphism

$$(X \times Y)(F)/R \xrightarrow{\sim} X(F)/R \times Y(F)/R. \tag{6.9}$$

The proof is easy.

If X is proper, any rational map as in Definition 6.6.1 a) extends to a morphism; the notion of R-equivalence is therefore the same as Asok-Morel's notion of $\mathbf{A}^1$-equivalence in [1]. Another of their results is then, in the above language:

6.6.2. Theorem ([1, Th. 6.2.1]). *Let X be a proper F-scheme. Then the rule*

$$Y \mapsto X(F(Y))/R$$

defines a presheaf of sets $\Upsilon(X) \in (S_b^{-1}\mathbf{Sm})^\vee$.

Note that $X \mapsto \Upsilon(X)$ is obviously functorial.
The main point is that R-equivalence classes on X specialise well with respect to good discrete valuations. Such a result was originally indicated by Kollár [25, p. 1] for smooth proper schemes over a discrete valuation ring R, and proven by Madore [29, Prop. 3.1] for projective schemes over R. Asok and Morel's proof uses Lipman's resolution of 2-dimensional schemes as well as a strong factorisation result of Lichtenbaum; as hinted by Colliot-Thélène, it actually suffices to use the more elementary results of Šafarevič [35, Lect. 4, Theorem p. 33].
Let X be proper and smooth. Its generic point $\eta_X \in X(F(X))$ defines by Yoneda's lemma a morphism of presheaves

$$\eta(X) : y(X) \to \Upsilon(X) \tag{6.10}$$

where $y(X) \in (S_b^{-1}\mathbf{Sm})^\vee$ is the presheaf of sets represented by X; $\eta : X \mapsto \eta(X)$ is clearly a morphism of functors.

6.6.3. Theorem. *η is an isomorphism of functors. Explicitly: for $Y \in \mathbf{Sm}$, $\eta(X)$ induces an isomorphism*

$$S_b^{-1}\mathbf{Sm}(Y, X) \xrightarrow{\sim} X(F(Y))/R. \tag{6.11}$$

Proof. Since $K \mapsto X(K)$ is a functor on $\mathbf{dv}^{\mathrm{op}}$ (compare [1, Lemma 6.2.3]), we have a commutative diagram for any $Y \in \mathbf{Sm}$:

$$\begin{tikzcd} \mathbf{dv}^{\mathrm{op}}(F(Y), F(X)) \arrow[r, "\tilde{\eta}"] \arrow[d, "\Psi"'] & X(F(Y)) \arrow[d, "\pi"] \arrow[dl, "\varepsilon"'] \\ S_b^{-1}\mathbf{Sm}(Y, X) \arrow[r, "\eta"] & X(F(Y))/R. \end{tikzcd} \tag{6.12}$$

Here $\tilde{\eta}$ is obtained from η_X by Yoneda's lemma in the same way as (6.10), Ψ is (obtained from) the functor of (6.5), π is the natural projection and ε associates to a rational map its class in $S_b^{-1}\mathbf{Sm}(Y, X)$ (see comment just before Proposition 6.3.2). Here the commutativity of the top triangle follows from Proposition 6.3.2. The surjectivity of π shows the surjectivity of η. Note further that Ψ is surjective by Theorem 6.5.2 a). This shows that ε is also surjective.
To conclude, it suffices to show that ε factors through π, thus yielding an inverse to η. If $x_0, x_1 \in X(F(Y))$ are directly R-equivalent, up to shrinking Y

we have a representing commutative diagram

$$\begin{tikzcd} Y \arrow[r, shift left, "x_0"] \arrow[r, shift right, "x_1"'] \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & X \\ \mathbf{P}^1_Y \arrow[ur, "h"'] \end{tikzcd}$$

with s_0, s_1 the inclusions of 0 and 1. But if we view $X(F(Y))$ and $S_b^{-1}\mathbf{Sm}(Y,X)$ as functors of $F(Y) \in \mathbf{dv}^{\mathrm{op}}$ (the second one via Ψ), then ε is checked to be a natural transformation: indeed, this is easy in the case of an inclusion of function fields and follows from the properness of X in the case of a good dvr. Hence we get $\varepsilon(x_0) = \varepsilon(x_1)$ since $S_b^{-1}\mathbf{Sm}(Y,X) \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}(\mathbf{P}^1_Y, X)$ by Theorem 1.7.2. □

6.6.4. Corollary. *The functor $\theta : S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \to S_b^{-1}\mathbf{Sm}$ of (4.14) is fully faithful.*

Proof. For $X, Y \in \mathbf{Sm}_*^{\mathrm{prop}}$, we have a commutative diagram similar to (6.12) replacing $\mathbf{dv}$ by $\mathbf{place}_*$ and $\mathbf{Sm}$ by $\mathbf{Sm}_*^{\mathrm{prop}}$. The map η_* corresponding to η is obtained from (6.12) by composition, while the map $\tilde\eta_*$ corresponding to $\tilde\eta$ exists because $K \mapsto X(K)$ is a functor on $\mathbf{place}^{\mathrm{op}}$ by the valuative criterion of properness. Further, the map corresponding to ε is well-defined thanks to Proposition 4.4.2 b) and the top triangle commutes thanks to Lemma 4.6.1. The natural map from this diagram to (6.12) yields a commutative diagram thanks to Lemma 6.3.4. Moreover, the map corresponding to Ψ is surjective thanks to Theorem 4.2.4 a). The same reasoning as above then shows that η_* is bijective: we just have to replace "up to shrinking Y" by "up to replacing Y by a birationally equivalent smooth projective variety", using the graph trick and the definition of $\mathbf{Sm}_*^{\mathrm{prop}}$. The graph trick can also be used to reduce the verification that ε is natural to the case where the rational maps involved are in fact morphisms. Hence the conclusion. □

6.6.5. *Remark.* One could replace $\mathbf{Sm}_*^{\mathrm{prop}}$ by $\mathbf{Sm}_*^{\mathrm{proj}}$ in Corollary 6.6.4, thus getting a full embedding $S_b^{-1}\mathbf{Sm}_*^{\mathrm{proj}} \hookrightarrow S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}}$.

The following corollary generalises [5, Prop. 10] to any characteristic:

6.6.6. Corollary. *Let $s : Y \dashrightarrow X$ be a rational map, with $X, Y \in \mathbf{Sm}^{\mathrm{prop}}$, Then s induces an map $s_* : Y(K)/R \to X(K)/R$ for any $K \in \mathbf{dv}$. Moreover, s_* is a bijection for any $K \in \mathbf{dv}$ if and only if the morphism $\tilde s$ associated to s in $S_b^{-1}\mathbf{Sm}$ (see comment just before Proposition 6.3.2) is an isomorphism.*
In particular, $s_* : Y(K)/R \xrightarrow{\sim} X(K)/R$ for any $K \in \mathbf{dv}$ when s is dominant and the field extension $F(Y)/F(X)$ is rational.

Proof. The morphism $\tilde s$ induces a morphism $S_b^{-1}(U,Y) \to S_b^{-1}(U,X)$ for any $U \in \mathbf{Sm}$, hence the first claim follows from Theorem 6.6.3. "If" is obvious, and "only if" follows from Yoneda's lemma. Finally, Theorem 1.7.2 implies that $\tilde s$ is an isomorphism under the last hypothesis on s, hence the conclusion. □

See Theorem 7.3.1 for a further generalisation.

6.7. CORONIDIS LOCO. Let us go back to the diagram in Lemma 6.3.4 a). Let h'_* be the equivalence relation on $\mathbf{dv}_*$ defined exactly as h' on $\mathbf{dv}$ (using objects of $\mathbf{dv}_*$ instead of objects of $\mathbf{dv}$). On the other hand, let h'' be the equivalence relation on $\mathbf{place}_*$ generated by h and

> *For $\lambda, \mu : K \rightsquigarrow L$, $\lambda \sim \mu$ if λ and μ have a common centre on some model $X \in \mathbf{Sm}_*^{\mathrm{prop}}$ of K.*

Clearly, the restriction of h'' to $\mathbf{dv}_*$ is coarser than h'; hence, using Theorem 4.2.4 b) and Proposition 6.4.3, we get an induced naturally commutative diagram:

$$\begin{tikzcd}
(\mathbf{place}_* / \mathrm{h}'')^{\mathrm{op}} \arrow[r, "\overline{\Psi}_*"] & S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \arrow[dd, "\theta"] \\
(\mathbf{dv}_* / \mathrm{h}'_*)^{\mathrm{op}} \arrow[u, "a"] \arrow[d, "b"] & \\
(\mathbf{dv} / \mathrm{h}')^{\mathrm{op}} \arrow[r, "\overline{\Psi}"] & S_b^{-1}\mathbf{Sm}.
\end{tikzcd}$$

In this diagram, $\overline{\Psi}_*$ is full and essentially surjective by Theorem 4.2.4 a), $\overline{\Psi}$ is an equivalence of categories by Theorem 6.5.2 a) and θ is fully faithful by Corollary 6.6.4. Moreover, a is full by Lemma 2.3.4 and the proof of Lemma 2.3.2, and essentially surjective by defnition. All this implies:

6.7.1. *THEOREM. *If* $\operatorname{char} k = 0$*, all functors in the above diagram are equivalences of categories.*

Proof. If $\operatorname{char} k = 0$, $\mathbf{dv}_* = \mathbf{dv}$ hence b is the identity functor. In view of the above remarks, the diagram then shows that a is faithful, hence an equivalence of categories. It follows that $\overline{\Psi}_*$ is also an equivalence of categories. Finally θ is essentially surjective, which completes the proof. □

As an application, we get a generalisation of the specialisation theorem to arbitrary places (already obtained in [20, Cor. 7.1.2]):

6.7.2. *COROLLARY. *Suppose* $\operatorname{char} F = 0$*. Let* $X \in \mathbf{Var}^{\mathrm{prop}}$*,* $K, L \in \mathbf{place}_*$ *and* $\lambda : K \rightsquigarrow L$ *be a place. Then* λ *induces a map*

$$\lambda_* : X(K)/R \to X(L)/R.$$

If $\mu : L \rightsquigarrow M$ *is another place, with* $M \in \mathbf{place}_*$*, then* $(\mu\lambda)_* = \mu_*\lambda_*$*.*

Proof. By Theorem 6.6.2, $K \mapsto X(K)/R$ defines a presheaf on $(\mathbf{dv} / h')^{\mathrm{op}}$, which extends to a presheaf on $(\mathbf{place}_* / \mathrm{h}'')^{\mathrm{op}}$ by Theorem 6.7.1. □

7. Linear connectedness of exceptional loci

7.1. Linear connectedness. We have the following definition of Chow [3]:

7.1.1. Definition. A (separated) F-scheme X of finite type is *linearly connected* if any two points of X (over a universal domain) may be joined by a chain of rational curves.

Linear connectedness is closely related to the notion of rational chain-connectedness of Kollár et al., for which we refer to [7, p. 99, Def. 4.21]. In fact:

7.1.2. Proposition. *The following conditions are equivalent:*

(i) *X is linearly connected.*
(ii) *For any algebraically closed extension K/F, $X(K)/R$ is reduced to a point.*
If X is a proper F-variety, these conditions are equivalent to:
(iii) *X is rationally chain-connected.*

Proof. (ii) $\Rightarrow$ (i) is obvious by definition (take for K a universal domain). For the converse, let $x_0, x_1 \in X(K)$. Then x_0 and x_1 are defined over some finitely generated subextension E/F. By assumption, there exists a universal domain $\Omega \supset E$ such that x_0 and x_1 are R-equivalent in $X(\Omega)$. Then the algebraic closure $\bar{E}$ of E embeds into Ω and K. If x_0 and x_1 are R-equivalent in $X(\bar{E})$, so are they in $X(K)$; this reduces us to the case where $K \subseteq \Omega$.
Let $\gamma_1, \dots, \gamma_n : \mathbf{P}^1_\Omega \dashrightarrow X_\Omega$ be a chain of rational curves linking x_0 and x_1 over Ω. Pick a finitely generated extension L of K over which all the γ_i are defined. We may write $L = K(U)$ for some K-variety U. Then the γ_i define rational maps $\tilde{\gamma}_i : U \times \mathbf{P}^1 \dashrightarrow X$. Since each γ_i is defined at 0 and 1 with $\gamma_i(1) = \gamma_{i+1}(0)$, we may if needed shrink U so that the domains of definition of all the $\tilde{\gamma}_i$ contain $U \times \{0\}$ and $U \times \{1\}$. Moreover, these restrictions coincide in the same style as above, since they do at the generic point of U. Pick a rational point $u \in U(K)$: then the fibres of the $\tilde{\gamma}_i$ at u are rational curves defined over K that link x_0 to x_1.
A rationally chain connected F-scheme is a proper variety by definition; then (i) $\iff$ (iii) if F is uncountable by [7, p. 100, Remark 4.22 (2)]. On the other hand, the property of linear connectedness is clearly invariant under algebraically closed extensions, and the same holds for rational chain-connectedness by [7, p. 100, Remark 4.22 (3)]. Thus (i) $\iff$ (iii) holds in general. □

We shall discuss the well-known relationship with rationally connected varieties in §8.5.
Proposition 7.1.2 suggests the following definition:

7.1.3. Definition. A separated F-scheme X of finite type is *strongly linearly connected* if $X(K)/R = *$ for any separable extension K/F.

7.2. Theorems of Murre, Chow, van der Waerden and Gabber. We start with the following not so well-known but nevertheless basic theorem of Murre [33], which was later rediscovered by Chow and van der Waerden [3, 38].

7.2.1. Theorem (Murre, Chow, van der Waerden). *Let $f : X \to Y$ be a projective birational morphism of F-varieties and $y \in Y$ be a smooth rational point. Then the fibre $f^{-1}(y)$ is linearly connected. In particular, by Proposition 7.1.2, $f^{-1}(y)(K)/R$ is reduced to a point for any algebraically closed extension K/F.*

For the sake of completeness, we give the general statement of Chow, which does not require a base field:

7.2.2. Theorem (Chow). *Let A be a regular local ring and $f : X \to \operatorname{Spec} A$ be a projective birational morphism. Let s be the closed point of $\operatorname{Spec} A$ and F its residue field. Then the special fibre $f^{-1}(s)$ is linearly connected (over F).*

Gabber has recently refined these theorems:

7.2.3. Theorem (Gabber). *Let A, X, f, s, F be as in Theorem 7.2.2, but assume only that f is proper. Let X_{reg} be the regular locus of X and $f^{-1}(s)^{\mathrm{reg}} = f^{-1}(s) \cap X_{\mathrm{reg}}$, which is known to be open in $f^{-1}(s)$. Then, for any extension K/F, any two points of $f^{-1}(s)^{\mathrm{reg}}(K)$ become R-equivalent in $f^{-1}(s)(K)$.*
In particular, if X is regular, then $f^{-1}(s)$ is strongly linearly connected.

See Appendix B for a proof of Theorem 7.2.3.

7.2.4. Theorem (Gabber [11]). *If F is a field, X is a regular irreducible F-scheme of finite type and K/F a field extension, then the map*

$$\varprojlim X'(K)/R \to X(K)/R$$

has a section, which is contravariant in X and covariant in K. The limit is over the proper birational $X' \to X$.

7.3. Applications. The following theorem extends part of Corollary 6.6.6 to a relative setting:

7.3.1. Theorem. *a) Let $s : Y \to X$ be in S_b^p, with X, Y regular. Then the induced map $Y(K)/R \to X(K)/R$ is bijective for any field extension K/F. If K is algebraically closed, the hypothesis "Y regular" is not necessary.*
b) Let $f : Y \dashrightarrow Z$ be a rational map with Y regular and Z proper. Then there is an induced map $f_* : Y(K)/R \to Z(K)/R$, which depends functorially on K/F.

Proof. a) As in the proof of Proposition 5.4.1 a), it suffices to deal with $K = F$. By this proposition, we have to show injectivity.
We assume that $s \in S_b^p$. Let $y_0, y_1 \in Y(F)$. Suppose that $s(y_0)$ and $s(y_1)$ are R-equivalent. We want to show that y_0 and y_1 are then R-equivalent. By definition, $s(y_0)$ and $s(y_1)$ are connected by a chain of direct R-equivalences. Applying Proposition 5.4.1 a), the intermediate rational points lift to $Y(F)$. This reduces us to the case where $s(y_0)$ and $s(y_1)$ are directly R-equivalent.

Let $\gamma : \mathbf{P}^1 \dashrightarrow X$ be a rational map defined at 0 and 1 such that $\gamma(i) = s(y_i)$. Applying Proposition 5.4.1 a) with $K = F(t)$, we get that γ lifts to a rational map $\tilde{\gamma} : \mathbf{P}^1 \dashrightarrow Y$. Since s is proper, $\tilde{\gamma}$ is still defined at 0 and 1. Let $y_i' = \tilde{\gamma}(i) \in Y(F)$: then $y_i, y_i' \in s^{-1}(s(y_i))$. If F is algebraically closed, they are R-equivalent by Theorem 7.2.1, thus y_0 and y_1 are R-equivalent. If F is arbitrary but Y is regular, then we appeal to Theorem 7.2.3.
b) By the usual graph trick, as Z is proper, we can resolve f to get a morphism

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "p"'] \arrow[dr, "\tilde{f}"] & \\ Y & & Z \end{tikzcd}$$

such that p is a proper birational morphism. By Theorem 7.2.4, the map $p_* : \tilde{Y}(K)/R \to Y(K)/R$ has a section which is "natural" in p (*i.e.* when we take a finer p, the two sections are compatible). The statement follows. □

7.3.2. *Remark.* Concerning Theorem 7.2.3, Fakhruddin pointed out that $f^{-1}(s)$ is in general not strongly linearly connected, while Gabber pointed out that $f^{-1}(s)^{\mathrm{reg}}(F)$ may be empty even if X is normal, when F is not algebraically closed. Here is Gabber's example: in dimension 2, blow-up the maximal ideal of A and then a non-rational point of the special fiber, then contract the proper transform of the special fiber. Gabber also gave examples covering Fakhruddin's remark: suppose $\dim A = 2$ and start from $X_0 =$ the blow-up of $\operatorname{Spec} A$ at s. Using [8], one can "pinch" X_0 so as to convert a non-rational closed point of the special fibre into a rational point. The special fibre of the resulting $X \to \operatorname{Spec} A$ is then a singular quotient of $\mathbf{P}^1_F$, with two R-equivalence classes. He also gave a normal example [11].

8. Examples, applications and open questions

In this section, we put together some concrete applications of the above results and list some open questions.

8.1. Composition of R-equivalence classes. As a by-product of Theorem 6.6.3, one gets for three smooth proper varieties X, Y, Z over a field of characteristic 0 a composition law

$$Y(F(X))/R \times Z(F(Y))/R \to Z(F(X))/R \tag{8.1}$$

which is by no means obvious. As a corollary, we have:

8.1.1. Corollary. *Let X be a smooth proper variety with function field K. Then $X(K)/R$ has a structure of a monoid with η_X as the identity element.* □

8.2. R-equivalence and birational functors. Here is a more concrete reformulation of part of Theorem 6.6.3 and Corollary 6.6.4:

8.2.1. Corollary. *Let*

$$P : \mathbf{Sm} \to \mathcal{A}$$

be a functor to some category $\mathcal{A}$. Suppose that P is a birational functor. Then if X, Y are two smooth varieties with X proper, any class $x \in X(F(Y))/R$ induces a morphism $x_* : P(Y) \to P(X)$. This assignment is compatible with the composition of R-equivalence classes from (8.1).
In particular, for two morphisms $f, g : Y \to X$, $P(f) = P(g)$ as soon as $f(\eta_Y)$ and $g(\eta_Y)$ are R-equivalent.
The same statement holds for a birational functor $P : \mathbf{Sm}_*^{\text{prop}} \to \mathcal{A}$, with $X, Y \in \mathbf{Sm}_*^{\text{prop}}$.

Theorem 6.6.3 further says that R-equivalence is "universal" among birational functors.

8.3. Algebraic groups and R-equivalence. As a special case of Corollary 8.1.1, we consider a connected algebraic group G defined over F. Recall that for any extension K/F, the set $G(K)/R$ is in fact a group. Let $\bar{G}$ denote a smooth compactification of G over F (we assume that there is one). It is known (P. Gille, [13]) that the natural map $G(F)/R \to \bar{G}(F)/R$ is an isomorphism if F has characteristic zero and G is reductive.
Let K denote the function field $F(G)$. By the above corollary, there is a composition law $\circ$ on $\bar{G}(K)/R$. On the other hand, the multiplication morphism

$$m : G \times G \to G$$

considered as a rational map on $\bar{G} \times \bar{G}$ induces a product map (Theorem 7.3.1)

$$\bar{G}(K)/R \times \bar{G}(K)/R \to \bar{G}(K)/R$$

which we denote by $(g, h) \mapsto g \cdot h$; this is clearly compatible with the corresponding product map on $G(K)/R$ obtained using the multiplication homomorphism on G. Thus we have two composition laws on $\bar{G}(K)/R$.
The following lemma is a formal consequence of Yoneda's lemma:

8.3.1. Lemma. *Let $g_1, g_2, h \in \bar{G}(K)/R$. Then we have $(g_1 \cdot g_2) \circ h = (g_1 \circ h) \cdot (g_2 \circ h)$.* □

In particular, let us take $G = SL_{1,A}$, where A is a central simple algebra over F. It is then known that $G(K)/R \simeq SK_1(A_K)$ for any function field K. If $\operatorname{char} F = 0$, we may use Gille's theorem and find that, for $K = F(G)$, $SK_1(A_K)$ admits a second composition law with unit element the generic element, which is distributive on the right with respect to the multiplication law. However, it is not distributive on the left in general:
Note that the natural map $\operatorname{Hom}(\operatorname{Spec} F, \bar{G}) = \bar{G}(F)/R \to \bar{G}(K)/R = \operatorname{Hom}(\bar{G}, \bar{G})$ is split injective, a retraction being induced by the unit section $\operatorname{Spec} F \to G \to \bar{G}$. Now let $g \in G(F)$; for any $\varphi \in G(K) = \mathbf{Rat}(G, G)$, we

clearly have $[g] \circ [\varphi] = [g]$. In particular, $[g] \circ ([\varphi] \cdot [\varphi']) \neq ([g] \circ [\varphi]) \cdot ([g] \circ [\varphi'])$ unless $[g] = 1$. (This argument works for any group object in a category with finite products.)

8.4. Kan extensions and Π_1. Let as before $\mathbf{Sm}_*$ denote the full subcategory of $\mathbf{Sm}$ given by those smooth varieties which admit a cofinal system of smooth proper compactifications: then the functor θ of Corollary 6.6.4 induces an equivalence of categories $S_b^{-1}\mathbf{Sm}_*^{\mathrm{prop}} \xrightarrow{\sim} S_b^{-1}\mathbf{Sm}_*$. Suppose we are given a functor $F : \mathbf{Sm}_* \to \mathcal{C}$ whose restriction to $\mathbf{Sm}_*^{\mathrm{prop}}$ is birational. We then get an induced functor $\bar{F} : S_b^{-1}\mathbf{Sm}_* \to \mathcal{C}$ plus a natural transformation

$$\rho_X : F(X) \to \bar{F}(X)$$

for any $X \in \mathbf{Sm}_*$.
To construct $\bar{F}$, we set

$$\bar{F}(X) = \varprojlim_{\bar{X}} F(\bar{X})$$

where the limit is on the category of open immersions $j : X \hookrightarrow \bar{X}$ with $\bar{X} \in \mathbf{Sm}_*^{\mathrm{prop}}$: this is an inverse limit of isomorphisms, hence makes sense without any hypothesis on $\mathcal{C}$ and may be computed by taking any representative $\bar{X}$. To construct ρ_X, an open immersion $j : X \hookrightarrow \bar{X}$ as above yields a map $F(X) \xrightarrow{F(j)} F(\bar{X}) \simeq \bar{F}(X)$, and one checks that this does not depend on the choice of j. This is an instance of a *left Kan extension* [28, Ch. X, §3], compare [21, §3] and [18, lemme 6.5].
We may apply this to $F = \Pi_1$, the fundamental groupoid[10] (here $\mathcal{C}$ is the category of groupoids): the required property is [SGA1, Exp. X, Cor. 3.4]. As an extra feature, we get that the universal transformation ρ is an epimorphism, because $\Pi_1(U) \twoheadrightarrow \Pi_1(X)$ if $U \hookrightarrow X$ is an open immersion of smooth schemes. Thus, $\Pi_1(X)$ has a "universal birational quotient" which is natural in X.
As another application, we get that for $X \in \mathbf{Sm}_*^{\mathrm{prop}}$, the "section map" (subject to a famous conjecture of Grothendieck when X is a curve)

$$X(F) \to \operatorname{Hom}_{\Pi_1(\operatorname{Spec} F)}(\Pi_1(\operatorname{Spec} F), \Pi_1(X)) \tag{8.2}$$

factors through R-equivalence. On the other hand, if X is projective and Y is a smooth hyperplane section, then $\Pi_1(Y) \xrightarrow{\sim} \Pi_1(X)$ as long as $\dim X > 2$ by [SGA2, Exp. XII, Cor. 3.5]; so there are more morphisms to invert if one wishes to study (8.2) for $\dim X > 1$ by the present methods.

8.5. Strongly linearly connected smooth proper varieties. One natural question that arises is the following: characterise morphisms $f : X \to Y$ between smooth proper varieties which become invertible in the category $S_b^{-1}\mathbf{Sm}$. Here we shall study this question only in the simplest case, where $Y = \operatorname{Spec} F$.

8.5.1. Theorem. *a) Let X be a smooth proper variety over F. Consider the following conditions:*

[10] Rather than fundamental group, to avoid the choices of base points.

(1) *$p : X \to \operatorname{Spec} F$ is an isomorphism in $S_b^{-1}\mathbf{Sm}$.*
(2) *p is an isomorphism in $S_r^{-1}\mathbf{Sm}$.*
(3) *For any separable extension E/F, $X(E)/R$ has one element (i.e. X is strongly linearly connected according to Definition 7.1.3).*
(4) *Same, for E/F of finite type.*
(5) *$X(F) \neq \emptyset$ and $X(K)/R$ has one element for $K = F(X)$.*
(6) *$X(F) \neq \emptyset$ and, given $x_0 \in X(F)$, there exists a chain of rational curves $(f_i : \mathbf{P}^1_K \to X_K)_{i=1}^n$ such that $f_1(0) = \eta_X$, $f_{i+1}(0) = f_i(1)$ and $f_n(1) = x_0$. Here $K = F(X)$ and η_X is the generic point of X.*
(7) *Same as (6), but with $n = 1$.*

Then $(1) \iff (2) \iff (3) \iff (4) \iff (5) \iff (6) \Leftarrow (7)$.
b) If $\operatorname{char} F = 0$, *$X$ satisfies Conditions* $(1) - (6)$ *and is projective, it is rationally connected.*

Proof. a) (1) $\Rightarrow$ (2) is trivial and the converse follows from Theorem 1.7.2. Thanks to Theorem 6.6.3, (2) $\iff$ (4) is an easy consequence of the Yoneda lemma. The implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Leftarrow$ (7) are trivial and (4) $\Rightarrow$ (3) is easy by a direct limit argument. To see (6) $\Rightarrow$ (1), note that by Theorem 6.6.3 (6) implies that $1_X = x_0 \circ p$ in $S_b^{-1}\mathbf{Sm}(X, X)$, hence p is an isomorphism.
b) This follows from Proposition 7.1.2 plus the famous theorem of Kollár-Miyaoka-Mori [24, Th. 3.10], [7, p. 107, Cor. 4.28]. □

8.5.2. *Remark.* The example of an anisotropic conic shows that, in (5), the assumption $X(F) \neq \emptyset$ does not follow from the next one.

8.5.3. *Question.* In the situation of Theorem 8.5.1 b), does X verify condition (7)? We give a partial result in this direction in Proposition 8.6.2 below. (The reader may consult the first version of this paper for a non-conclusive attempt to answer this question in general.)

8.6. Retract-rational varieties. Recall that, following Saltman, X (smooth but not necessarily proper) is *retract-rational* if it contains an open subset U such that U is a retract of an open subset of $\mathbf{A}^n$. When F is infinite, this includes the case where there exists Y such that $X \times Y$ is rational, as in [5, Ex. A. pp. 222/223].
We have a similar notion for function fields:

8.6.1. Definition. A function field K/F is *retract-rational* if there exists an integer $n \geq 0$ and two places $\lambda : K \rightsquigarrow F(t_1, \dots, t_n)$, $\mu : F(t_1, \dots, t_n) \rightsquigarrow K$ such that $\mu\lambda = 1_K$.

Note that this forces λ to be a trivial place (*i.e.* an inclusion of fields). Using Lemma 2.3.2, we easily see that X is retract-rational if and only if $F(X)$ is retract-rational.

8.6.2. Proposition. *If X is a retract-rational smooth variety, then $X \xrightarrow{\sim} \operatorname{Spec} F$ in $S_b^{-1}\mathbf{Sm}$. If moreover X is proper and F is infinite, then X verifies Condition* (7) *of Theorem 8.5.1 for a Zariski dense set of points x_0.*

Proof. The first statement is obvious by Yoneda's lemma. Let us prove the second: by hypothesis, there exist open subsets $U \subseteq X$ and $V \subseteq \mathbf{A}^n$ and morphisms $f : U \to V$ and $g : V \to U$ such that $gf = 1_U$. This already shows that $U(F)$ is Zariski-dense in X. Let now $x_0 \in U(F)$, and let $K = F(X)$. Consider the straight line $\gamma : \mathbf{A}^1_K \to \mathbf{A}^n_K$ such that $\gamma(0) = f(x_0)$ and $\gamma(1) = f(\eta_X)$. Then $g \circ \gamma$ links x_0 to η_X, as desired. □

8.6.3. Corollary. *We have the following implications for a smooth proper variety X over a field F of characteristic* 0*: retract-rational ⇒ strongly linearly connected ⇒ rationally connected.*

Proof. The first implication follows from Theorem 8.5.1 and Proposition 8.6.2; the second implication follows from the theorem of Kollár-Miyaoka-Mori already quoted. □

8.6.4. *Remark.* In characteristic 0, if X is a smooth compactification of a torus, then it verifies Conditions (1) – (6) of Theorem 8.5.1 if and only if it is retract-rational, by [6, Prop. 7.4] (*i.e.* the first implication in the previous corollary is an equivalence for such X). This may also be true by replacing "torus" by "connected reductive group": at least it is so in many special cases, see [14, Th. 7.2 and Cor. 5.10].

8.7. S_r-local objects. Recall:

8.7.1. Definition. Let $\mathcal{C}$ be a category and S a family of morphisms of $\mathcal{C}$. An object $X \in \mathcal{C}$ is *local* relatively to S or S-*local* (*left closed* in the terminology of [12, Ch. 1, Def. 4.1 p. 19]) if, for any $s : Y \to Z$ in S, the map

$$\mathcal{C}(Z, X) \xrightarrow{s^*} \mathcal{C}(Y, X)$$

is bijective.

In this rather disappointing subsection, we show that there are not enough of these objects. They are the exact opposite of rationally connected varieties.

8.7.2. Definition. A proper F-variety X is *nonrational* if it does not carry any nonconstant rational curve (over the algebraic closure of F), or equivalently if the map

$$X(\bar{F}) \to X(\bar{F}(t))$$

is bijective.

8.7.3. Lemma. *a) Nonrationality is stable by product and by passing to closed subvarieties.*
b) Curves of genus > 0 and torsors under abelian varieties are nonrational.
c) Nonrational smooth projective varieties are minimal in the sense that their canonical bundle is nef.

Proof. a) and b) are obvious; c) follows from the Miyaoka-Mori theorem ([30], see also [26, Th. 1.13] or [7, Th. 3.6]). □

On the other hand, an anisotropic conic is not a nonrational variety. This is also true for some minimal models in dimension 2, even when F is algebraically closed.

Smooth nonrational varieties are the local objects of **Sm** with respect to S_r in the sense of Definition 8.7.1:

8.7.4. LEMMA. *a) A proper variety X is nonrational if and only if, for any morphism $f : Y \to Z$ between smooth varieties such that $f \in S_r$, the map*

$$f^* : Map(Z, X) \to Map(Y, X)$$

is bijective.

b) A smooth proper nonrational variety X is stably minimal in the following sense: any morphism in S_r with source X is an isomorphism.

Proof. a) Necessity is clear (take $f : \mathbf{P}^1 \to \operatorname{Spec} F$). For sufficiency, f^* is clearly injective since f is dominant, and we have to show surjectivity. We may assume F algebraically closed. Let U be a common open subset to Y and $Z \times \mathbf{P}^n$ for suitable n. Let $\psi : Y \to X$. By [26, Cor. 1.5] or [7, Cor. 1.44], $\psi_{|U}$ extends to a morphism φ on $Z \times \mathbf{P}^n$. But for any closed point $z \in Z$, $\varphi(\{z\} \times \mathbf{P}^1)$ is a point, where $\mathbf{P}^1$ is any line of $\mathbf{P}^n$. Therefore $\varphi(\{z\} \times \mathbf{P}^n)$ is a point, which implies that φ factors through the first projection.
b) immediately follows from a). □

8.7.5. LEMMA. *If X is nonrational, it remains nonrational over any extension K/F.*

Proof. It is a variant of the previous one: we may assume that F is algebraically closed and that K/F is finitely generated. Let $f : \mathbf{P}^1_K \to X_K$. Spread f to a U-morphism $\tilde{f} : U \times \mathbf{P}^1 \to U \times X$ and compose with the second projection. Any closed point $u \in U$ defines a map $f_u : \mathbf{P}^1 \to X$, which is constant, hence $p_2 \circ \tilde{f}$ factors through the first projection, which implies that f is constant. □

8.8. OPEN QUESTIONS. We finish by listing a few problems that are not answered in this paper.

(1) Compute Hom sets in $S_b^{-1}\mathbf{Var}$. In [21, Rk. 8.11], it is shown that the functor $S_b^{-1}\mathbf{Sm} \to S_b^{-1}\mathbf{Var}$ is neither full nor faithful and that the Hom sets are in fact completely different.
(2) Compute Hom sets in $(S_b^p)^{-1}\mathbf{Sm}$.
(3) Let $d_{\leq n}\mathbf{Sm}$ be the full subcategory of **Sm** consisting of smooth varieties of dimension $\leq n$. Is the induced functor $S_b^{-1}d_{\leq n}\mathbf{Sm} \to S_b^{-1}\mathbf{Sm}$ fully faithful?
(4) Give a categorical interpretation of rationally connected varieties.
(5) Finally one should develop additional functoriality: products and internal Homs, change of base field.

Appendix A. Invariance birationnelle et invariance homotopique

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14 septembre 2006.

Soit k un corps. Soit F un foncteur contravariant de la catégorie des k-schémas vers la catégorie des ensembles. Si sur les morphismes k-birationnels de surfaces projectives, lisses et géométriquement connexes ce foncteur induit des bijections, alors l'application $F(k) \to F(\mathbf{P}^1_k)$ est une bijection.

Démonstration. Toutes les variétés considérées sont des k-variétés. On écrit $F(k)$ pour $F(\mathrm{Spec}(k))$. Soit W l'éclaté de $\mathbf{P}^1 \times \mathbf{P}^1$ en un k-point M. Les transformés propres des deux génératrices L_1 et L_2 passant par M sont deux courbes exceptionnelles de première espèce $E_1 \simeq \mathbf{P}^1$ et $E_2 \simeq \mathbf{P}^1$ qui ne se rencontrent pas. On peut donc les contracter simultanément, la surface que l'on obtient est le plan projectif $\mathbf{P}^2$. Notons M_1 et M_2 les k-points de $\mathbf{P}^2$ sur lesquels les courbes E_1 et E_2 se contractent.
On réalise facilement cette construction de manière concrète. Dans $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ avec coordonnées multihomogènes $(u, v; w, z; X, Y, T)$ on prend pour W la surface définie par l'idéal $(uT - vX, wT - zY)$, et on considère les deux projections $W \to \mathbf{P}^1 \times \mathbf{P}^1$ et $W \to \mathbf{P}^2$.
On a un diagramme commutatif de morphismes

$$\begin{array}{ccc} E_1 & \longrightarrow & W \\ \wr\downarrow & & \downarrow \\ L_1 & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1. \end{array}$$

Le composé de l'inclusion $L_1 \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ et d'une des deux projections $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ est un isomorphisme. Par fonctorialité, la restriction $F(\mathbf{P}^1 \times \mathbf{P}^1) \to F(L_1)$ est donc surjective. Par fonctorialité, le diagramme ci-dessus implique alors que la restriction $F(W) \to F(E_1)$ est surjective.
Considérons maintenant la projection $W \to \mathbf{P}^2$. On a ici le diagramme commutatif de morphismes

$$\begin{array}{ccc} E_1 & \longrightarrow & W \\ \downarrow & & \downarrow \\ M_1 & \longrightarrow & \mathbf{P}^2. \end{array}$$

Par l'hypothèse d'invariance birationnelle, on a la bijection $F(\mathbf{P}^2) \xrightarrow{\sim} F(W)$. Donc la flèche composée $F(\mathbf{P}^2) \to F(W) \to F(E_1)$ est surjective. Mais par le diagramme commutatif ci-dessus la flèche composée se factorise aussi comme $F(\mathbf{P}^2) \to F(M_1) \to F(E_1)$. Ainsi $F(M_1) \to F(E_1)$, c'est-à-dire $F(k) \to F(\mathbf{P}^1)$, est surjectif. L'injectivité de $F(k) \to F(\mathbf{P}^1)$ résulte de la fonctorialité et de la considération d'un k-point sur $\mathbf{P}^1$.

Appendix B. A letter from O. Gabber

June 12, 2007

Dear Kahn,

I discuss a proof of

B.0.1. Theorem. *Let A be a regular local ring with residue field k, $X' \to X = Spec(A)$ a proper birational morphism, X'_{reg} the regular locus of X', X'_s the special fiber of X', $X'_{reg,s} = X'_s \cap X'_{reg}$, which is known to be open in X'_s, F a field extension of k, then any two points of $X'_{reg,s}(F)$ are R-equivalent in $X'_s(F)$.*

The proof I tried to sketch by joining centers of divisorial valuations has a gap in the imperfect residue field case. It is easier to adapt the proof by deformation of local arcs.
(1) If $Y' \to Y$ is proper surjective map between separated k-schemes of finite type whose fibers are projective spaces then for every $F/k, Y'(F)/R \to Y(F)/R$ is bijective. In particular the theorem holds if X' is obtained from X by a sequence of blow-ups with regular centers.
(2) If A is a regular local ring of dimension > 1 with maximal ideal $\mathfrak{m}$, U an open non empty in Spec(A), then there is $f \in \mathfrak{m} - \mathfrak{m}^2$ s.t. the generic point of $V(f)$ is in U.
This is because U omits only a finite number of height 1 primes and there are infinitely many possibilities for $V(f)$, e.g. $V(x - y^i)$ where x, y is a part of a regular system of parameters.
Inductively we get that there is $P \in U$ s.t. A/P is regular 1-dimensional.
(3) If A is a regular local ring and P, P' different prime ideals with A/P and A/P' regular one dimensional, then there is a prime ideal $Q \subset P \cap P'$ with A/Q regular 2-dimensional.
Indeed let $x_1, \dots, x_n$ be a minimal system of generators of P; their images in A/P' generate a principal ideal; we may assume this ideal is generated by the image of x_1, and then we can substract some multiples of x_1 from $x_2, \dots, x_n$ so that the images of $x_2, \dots, x_n$ are 0; take $Q = (x_2, \dots, x_n)$.
To prove the theorem we may assume F is a finitely generated extension of k, so F is a finite extension of a purely transcendental extension k' of k. We replace A by the local ring at the generic point of the special fiber of an affine space over A that has residue field k'. So we reduce to F/k finite. Let x, y be F-points of X'_s centered at closed points a, b at which X' is regular. Let U be dense open of X above which $X' \to X$ is an isomorphism. Let $X'(a), X'(b)$ be the local schemes (Spec of the local rings at a and b). There are regular one dimensional closed subschemes

$$C \subset X'(a), C' \subset X'(b)$$

whose generic points map to U.
By EGA 0_{III} 10.3 there are finite flat $D \to C$, $D' \to C'$ which are $Spec(F)$ over the closed points of C, C'. Then D, D' are Spec's of DVRs essentially

of finite type over A (localization of finite type A-algebras). We form the pushout of $D \leftarrow Spec(F) \to D'$, which is Spec of a fibered product ring, which by some algebraic exercise is still an A-algebra essentially of finite type. The pushout can be embedded as a closed subscheme in Spec of a local ring of an affine space over A and then by (3) in some Y a 2-dimensional local regular A-scheme essentially of finite type. Now D, D' are subschemes of Y. We have a rational map $Y \to X'$ defined on the inverse image of U and in particular at the generic points of D and D'. By e.g. Theorem 26.1 in Lipman's paper on rational singularities (Publ. IHES 36) there is $Y' \to Y$ obtained as a succession of blow-ups at closed points s.t. the rational map gives a morphism $Y' \to X'$. Then x, y are images of F-points of Y' (closed points of the proper transforms of D, D'),and by (1) any two F-points of the special fiber of $Y' \to Y$ are R-equivalent.

Sincerely,

Ofer Gabber

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