Limit Mordell–Weil Groups and their p-Adic Closure

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Abstract. This is a twin article of [H14b], where we study the projective limit of the Mordell–Weil groups (called pro Λ-MW groups) of modular Jacobians of p-power level. We prove a control theorem of an ind-version of the K-rational Λ-MW group for a number field K. In addition, we study its p-adic closure in the group of $K_{\mathfrak{p}}$-valued points of the modular Jacobians for a $\mathfrak{p}$-adic completion $K_{\mathfrak{p}}$ for a prime $\mathfrak{p}|p$ of K. As a consequence, if $K_{\mathfrak{p}} = \mathbb{Q}_p$, we give an exact formula for the rank of the ordinary/co-ordinary part of the closure.

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1. Introduction

Consider a p-adic ordinary family of modular eigenforms of prime-to-p level N. This is an irreducible scheme $\mathrm{Spec}(\mathbb{I})$ which is finite torsion-free over the Iwasawa algebra $\mathbb{Z}_p[[T]]$, and whose points P of codimension one and not in the special fiber correspond to ordinary p-adic modular eigenforms f_P. Among those points, many corresponds to modular classical eigenforms of weight 2 and level Np^r (for variable r), and such points are Zariski dense in $\mathrm{Spec}(\mathbb{I})$. An old, well-known, and fundamental construction of Eichler–Shimura attaches to any modular cuspidal eigenform f of weight 2 an abelian variety A_f defined over $\mathbb{Q}$, of dimension the degree of the field generated by the coefficients of f over $\mathbb{Q}$. For these abelian varieties A_f, one can consider the Mordell–Weil group $A_f(\mathbb{Q})$ and more generally, $A_f(k)$ for k a fixed number field, which are finitely generated abelian groups. Let us set $\widehat{A}_f(k) = A_f(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We consider the following natural question: how does the Mordell–Weil group $\widehat{A}_f(k)$ varies as f varies among those cuspidal eigenforms of weight 2 in the family? We give a partial answer to this question in the form of control theorems (Theorems 1.1 and 6.6) for these Mordell–Weil groups. An analogous result is proved when the number field k is replaced by an l-adic field $k_{\mathfrak{l}}$, and also a consequence concerning the image of $A_f(k)$ in $A_f(k_{\mathfrak{l}})$.

Fix a prime p. This article concerns the p-slope 0 Hecke eigen cusp forms of level Np^r for $r > 0$ and $p \nmid N$, and for small primes $p = 2, 3$, they exists only when $N > 1$; thus, we may assume $Np^r \geq 4$. Then the open curve $Y_1(Np^r)$ (obtained from $X_1(Np^r)$ removing all cusps) gives the fine smooth moduli scheme classifying elliptic curves E with an embedding $\mu_{Np^r} \hookrightarrow E$. Anyway for simplicity, we assume that $p \geq 3$, although we indicate often any modification necessary for $p = 2$. A main difference in the case $p = 2$ is that we need to consider the level Np^r with $r \geq 2$, and whenever the principal ideal $(\gamma^{p^{r-1}} - 1)$ shows up in the statement for $p > 2$, we need to replace it by $(\gamma^{p^{r-2}} - 1)$ (assuming $r \geq 2$), as the maximal torsion-free subgroup of $\mathbb{Z}_2^\times$ is $1+2^2\mathbb{Z}_2$. We applied in [H86b] and [H14a] the techniques of $U(p)$-isomorphisms to p-divisible Barsotti–Tate groups of modular Jacobian varieties of all p-power level (with a fixed prime-to-p level N) in order to get coherent control under diamond operators. In this article, we apply the same techniques to Mordell–Weil groups of the Jacobians and see what we can say. We hope to study $U(p)$-isomorphisms of the Tate–Shafarevich groups of the Jacobians in a future article.

Let $X_r = X_1(Np^r)_{/\mathbb{Q}}$ be the compactified moduli of the classification problem of pairs (E, ϕ) of elliptic curves E and an embedding $\phi : \mu_{Np^r} \hookrightarrow E[Np^r]$ as finite flat group schemes. Since $\mathrm{Aut}(\mu_{p^r}) = (\mathbb{Z}/p^r\mathbb{Z})^\times$, $z \in \mathbb{Z}_p^\times$ acts on X_r via $\phi \mapsto \phi \circ \overline{z}$ for the image $\overline{z} \in (\mathbb{Z}/p^r\mathbb{Z})^\times$. We write X_s^r $(s > r)$ for the quotient curve $X_s/(1 + p^r\mathbb{Z}_p)$. The complex points $X_s^r(\mathbb{C})$ contains $\Gamma_s^r\backslash\mathfrak{H}$ as an open Riemann surface for $\Gamma_s^r = \Gamma_0(p^s) \cap \Gamma_1(Np^r)$. Write $J_{r/\mathbb{Q}}$ (resp. $J^r_{s/\mathbb{Q}}$) for the Jacobian of X_r (resp. X_s^r) whose origin is given by the infinity cusp ∞ of the modular curves. We regard J_r as the degree 0 component of the Picard scheme of X_r. For a number field k, we consider the group of k-rational points $J_r(k)$. The Hecke operator $U(p)$ and its dual $U^*(p)$ act on $J_r(k)$ and their p-adic limit $e = \lim_{n\to\infty} U(p)^{n!}$ and $e^* = \lim_{n\to\infty} U^*(p)^{n!}$ are well defined on the Barsotti–Tate group $J_r[p^\infty]$. For a general abelian variety over a number field k, we put $\widehat{X}(k) = X(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (though we give the definition of the sheaf $\widehat{X}$ in the following section for global and local field k and if k is local, $\widehat{X}$ may not be the tensor product as above).

By Picard functoriality, we have injective limits $J_\infty(k) = \varinjlim_r \widehat{J}_r(k)$ and $J_\infty[p^\infty](k) = \varinjlim_r J_r[p^\infty](k)$, on which e acts. Here $J_r[p^\infty]$ is the p-divisible Barsotti–Tate group of J_r over $\mathbb{Q}$). Write $\mathcal{G} = e(J_\infty[p^\infty])$, which is called the Λ-adic Barsotti–Tate group in [H14a] and whose integral property was scrutinized there. We define the p-adic completion of $J_\infty(k)$:

$$\check{J}_\infty(k) = \varprojlim_n J_\infty(k)/p^n J_\infty(k).$$

These groups we call ind (limit) MW-groups. Since projective limit and injective limit are left-exact, the functor $R \mapsto J_\infty(R)$ is a sheaf with values in abelian groups on the fppf site over $\mathbb{Q}$ (we call such a sheaf an fppf abelian sheaf).

Adding superscript or subscript "ord" (resp. "co-ord"), we indicate the image of e (resp. e^*). The compact cyclic group $\Gamma = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ acts on these modules by the diamond operators. In other words, we identify canonically $\mathrm{Gal}(X_r/X_0(Np^r))$ for modular curves X_r and $X_0(Np^r)$ with $(\mathbb{Z}/Np^r\mathbb{Z})^\times$, and the group Γ acts on J_r through its image in $\mathrm{Gal}(X_r/X_0(Np^r))$. We study control of $\check{J}_\infty(k)^{\mathrm{ord}}$ under diamond operators.

A compact or discrete $\mathbb{Z}_p$-module M is called an Iwasawa module if it has a continuous action of the multiplicative group $\Gamma = 1 + p\mathbb{Z}_p$ with a topological generator $\gamma = 1 + p$. If M is given by a projective or an injective limit of naturally defined compact $\mathbb{Z}_p[\Gamma/\Gamma^{p^r}]$-modules M_r, we say M has exact control if $M_r = M/(\gamma^{p^r} - 1)M$ in the case of a projective limit and $M_r = M[\gamma^{p^r} - 1] = \{x \in M | (\gamma^{p^r} - 1)x = 0\}$ in the case of an injective limit. If M is compact and $M/(\gamma - 1)M$ is finite (resp. of finite type over $\mathbb{Z}_p$), M is Λ-torsion (resp. of finite type over Λ), where $\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim_r \mathbb{Z}_p[\Gamma/\Gamma^{p^r}]$ (the Iwasawa algebra). When $p = 2$, we need to take $\Gamma = 1 + p^2\mathbb{Z}_2$ and $\gamma = 1 + 4 = 5 \in \Gamma$. In addition, we need to assume often $s > r > 1$ in place of $s > r > 0$ for odd primes.

The big ordinary Hecke algebra $\mathbf{h}$ (whose properties we recall at the end of this section) acts on $\check{J}_\infty^{\mathrm{ord}}$ and J_∞^{ord} as endomorphisms of functors. Let k be a number field or a finite extension of $\mathbb{Q}_l$ for a prime l. Write B_P for Shimura's abelian variety quotient of J_r in [Sh73] and A_P for his abelian subvariety $A_P \subset J_r$ [IAT, Theorem 7.14] associated to a Hecke eigenform f_P in an analytic family of slope 0 Hecke eigenforms $\{f_P | P \in \mathrm{Spec}(\mathbb{I})\}$ (for an irreducible component $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(\mathbf{h})$ for the big ordinary Hecke algebra $\mathbf{h}$). Here we assume that f_P has weight 2 and is a p-stabilized new form of level Np^r with $r = r(P) > 0$. Let $\mathrm{Spec}(\mathbb{T}) \subset \mathrm{Spec}(\mathbf{h})$ be the connected component containing $\mathrm{Spec}(\mathbb{I})$. For any $\mathbf{h}$-module, we write $M_\mathbb{T}$ (or $M^\mathbb{T}$) for the $\mathbb{T}$-eigen component $1_\mathbb{T} \cdot M = M \otimes_\mathbf{h} \mathbb{T}$ for the idempotent $1_\mathbb{T}$ of T in $\mathbf{h}$. Suppose that P is a principal ideal generated by $\alpha \in \mathbb{T}$ (regarding as $P \in \mathrm{Spec}(\mathbb{T})$). This principality assumption holds most of the cases (see Proposition 5.1). Then we may assume that $\alpha = \varprojlim_s \alpha_s$ (as an endomorphism of the fppf abelian sheaf J_∞) for $\alpha_s \in \mathrm{End}(J_s)$, $B_P = J_r/\alpha_r(J_r)$, and the abelian variety A_P is the connected component of $J_r[\alpha_r] = \mathrm{Ker}(\alpha_r)$. For a finite extension k of $\mathbb{Q}$ or $\mathbb{Q}_l$ (for a prime l), we show in Section 4 that the Pontryagin dual $\mathcal{G}_\mathbb{T}(k)^\vee$ is often a finite module and at worst is a torsion Λ-module of finite type.

In this paper as Proposition 6.4, we prove the following exact sequence:

$$\widehat{A}_P(k)^{\mathrm{ord},\mathbb{T}} \xrightarrow{\iota_\infty} \check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}} \xrightarrow{\alpha} \check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}}, \tag{1.1}$$

where $\mathrm{Ker}(\iota_\infty)$ is finite and $\mathrm{Coker}(\alpha)$ is a $\mathbb{Z}_p$-module of finite type with free rank less than or equal to $\dim_{\mathbb{Q}_p} \widehat{B}_r(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The main result Theorem 6.6 of this paper is basically the $\mathbb{Z}_p$-dual version of Proposition 6.4 for $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} := \mathrm{Hom}_{\mathbb{Z}_p}(\check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}}, \mathbb{Z}_p)$. Here is a shortened statement of our main theorem (Theorem 6.6 in the text):

THEOREM 1.1. *The sequence $\mathbb{Z}_p$-dual to the one in* (1.1)*:*

$$0 \to \mathrm{Coker}(\alpha)^*_{\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \to \widehat{A}_P(k)^*_{\mathrm{ord},\mathbb{T}} \to 0 \tag{1.2}$$

is exact up to finite error.

In Theorem 6.6, we give many control sequences similar to (1.2) for other incarnations of $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$.

These modules $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$ are modules over the big ordinary Hecke algebra $\mathbf{h}$. We cut down these modules to an irreducible component $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(\mathbf{h})$. In other words, we study the following $\mathbb{I}$-modules:

$$\check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{I}} := \check{J}_\infty(k)^{\mathrm{ord}} \otimes_{\mathbf{h}} \mathbb{I}.$$

We could ask diverse questions out of our control theorem. For example, when is $A_P(\kappa)$ dense in $A_P(\kappa_{\mathfrak{p}})$ for a prime $\mathfrak{p}|p$ of a number field κ? We can answer this question for almost all P if $\kappa_{\mathfrak{p}} = \mathbb{Q}_p$ and $\dim_{\mathbb{Q}} A_{P_0}(\kappa) \otimes_{\mathbb{Z}} \mathbb{Q} > 0$ for one sufficiently generic P_0 (see Corollary 7.2). In [H14b], we extend the control result to the projective limit $\varprojlim_r \widehat{J}_r(k)^{\mathrm{ord}}_{\mathbb{T}}$. In a forthcoming paper [H14c], we prove "almost" constancy of the Mordell–Weil rank of Shimura's abelian variety in a p-adic analytic family.
Our point is that we have a control theorem of the limit Mordell–Weil groups (under mild assumptions) which is possibly smaller than the Selmer groups studied more often. We hope to discuss the relation of our result to the limit Selmer group studied by Nekovár in [N06] in our future paper.
The control theorems for $\mathbf{h}$ proven for $p \geq 5$ in [H86a] and [H86b] and in [GME, Corollary 3.2.22] for general p assert that, for $p > 2$, the quotient $\mathbf{h}/(\gamma^{p^{r-1}} - 1)\mathbf{h}$ is canonically isomorphic to the Hecke algebra $\mathbf{h}_r$ ($r > 0$) in $\mathrm{End}_{\mathbb{Z}_p}(J_r[p^\infty]^{\mathrm{ord}})$ generated over $\mathbb{Z}_p$ by Hecke operators $T(n)$ (while for $p = 2$, $\mathbf{h}/(\gamma^{p^{r-2}} - 1)\mathbf{h} \cong \mathbf{h}_r$ for $r \geq 2$). By this control result, we showed that $\mathbf{h}$ is a free of finite rank over Λ (see [GK13] for the treatment for $p = 2$).

We recall succinctly how these control theorems were proven in [H86b] (and in [H86a]) for $p \geq 5$, as it gives a good introduction to the methods used in the present paper. The arguments in these papers work well for $p = 2, 3$ assuming that $Np^r \geq 4$ (see [GK13] for details in the case of $p = 2$). We have a well known commutative diagram of $U(p^{s-r})$-operators:

$$\begin{array}{ccc} J_{r,R} & \xrightarrow{\pi^*} & J^r_{s,R} \\ \downarrow u & \swarrow u' & \downarrow u'' \\ J_{r,R} & \xrightarrow{\pi^*} & J^r_{s,R}, \end{array} \tag{1.3}$$

where the middle u' is given by $U^s_r(p^{s-r})$ and u and u'' are $U(p^{s-r})$. These operators comes from the double coset $\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r} \end{pmatrix} \Gamma'$ for $\Gamma = \Gamma^r_s = \Gamma_0(p^s) \cap \Gamma_1(Np^r)$ and $\Gamma' = \Gamma^{r'}_{s'}$ for suitable $s \geq r, s' \geq r'$. Note that $U(p^n) = U(p)^n$. Then the above diagram implies

$$J_{r/\mathbb{Q}}[p^\infty]^{\mathrm{ord}} \cong J^r_{s/\mathbb{Q}}[p^\infty]^{\mathrm{ord}} \quad \text{and} \quad \widehat{J}_{r/\mathbb{Q}}(k)^{\mathrm{ord}} \cong \widehat{J}^r_{s/\mathbb{Q}}(k)^{\mathrm{ord}}. \tag{1.4}$$

The commutativity of the diagram (1.3) and the level lowering (1.4) are universally true even when we replace the fppf abelian sheaf J_r by any fppf sheaf with reasonable $U(p)$-action compatible with the modular tower $\cdots \to X_r \to \cdots \to X_1$.

For computational purpose, in [H86b], we identified $J(\mathbb{C})$ with a subgroup of $H^1(\Gamma, \mathbf{T})$ (for the Γ-module $\mathbf{T} := \mathbb{R}/\mathbb{Z}$ with trivial Γ-action). Since $\Gamma_s^r \triangleright \Gamma_1(Np^s)$, we may consider the finite cyclic quotient group $C := \frac{\Gamma_s^r}{\Gamma_1(Np^s)} = \Gamma^{p^{r-1}}/\Gamma^{p^{s-1}}$. By the inflation restriction sequence, we have the following commutative diagram with exact rows, writing $H^\bullet(?, \mathbf{T})$ as $H^\bullet(?)$:

$$\begin{tikzcd}
H^1(C) \arrow[r, hook] & H^1(\Gamma_s^r) \arrow[r] & H^1(\Gamma_1(Np^s))^{\gamma^{p^{r-1}}=1} \arrow[r] & H^2(C) \\
? \arrow[r] \arrow[u] & J_s^r(\mathbb{C}) \arrow[r] \arrow[u, "\cup"'] & J_s(\mathbb{C})[\gamma^{p^{r-1}} - 1] \arrow[r] \arrow[u, "\cup"] & ?. \arrow[u]
\end{tikzcd}$$

Since $H^2(C, \mathbf{T}) = 0$ and $U(p)^{s-r}(H^1(C, \mathbf{T})) = 0$, we have the control of Barsotti–Tate groups (see [H86b] and more recent [H14a, §4–5]):

$$J_s[p^\infty][\gamma^{p^{r-1}} - 1]^{\mathrm{ord}}_{/\mathbb{C}} \cong J_r[p^\infty]^{\mathrm{ord}}_{/\mathbb{C}}.$$

Out of this control by the Γ-action of the ordinary Barsotti–Tate groups $J_r[p^\infty]^{\mathrm{ord}}$, we proved the control of $\mathbf{h}$ (cited above) by the diamond operators.

A suitable power of $U(p)$-operator killing the kernel and cokernel of the restriction maps in (1) should be also universally true not just over $\mathbb{C}$ but over smaller rings. We will study almost the same diagram obtained by replacing $H^1(?, \mathbf{T})$ for $? = \Gamma_1(Np^s)$ and Γ_s^r by $H^1_{\mathrm{fppf}}(X_{/\mathbb{Q}}, \mathcal{O}_X^\times) = \mathrm{Pic}_{X/\mathbb{Q}}$ for $X = X_s$ and X_s^r. In an algebro-geometric way, we verify that an appropriate power of the $U(p)$-operator kills the corresponding kernel and cokernel. Technical points aside, this is a key to the proof of Theorem 1.1. This principle should hold for more general sheaves (under a Grothendieck topology) with $U(p)$-action compatible with the modular tower, and the author plans to present many other examples of such in his forthcoming papers.

We call a point $P \in \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ an arithmetic point of weight 2 if $P(\gamma^{p^j} - 1) = 0$ for some integer $j \geq 0$. Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism $\lambda : \mathbf{h} \to \overline{\mathbb{Q}}_p$ killing $\gamma^{p^{r-1}} - 1$ for sufficiently large $r > 0$ to a classical Hecke eigenform, we need to fix (once and for all) an embedding $\overline{\mathbb{Q}} \xrightarrow{i_p} \overline{\mathbb{Q}}_p$ of the algebraic closure $\overline{\mathbb{Q}}$ in $\mathbb{C}$ into a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

Contents

2. Sheaves associated to abelian varieties

Let k be a finite extension of $\mathbb{Q}$ or the l-adic field $\mathbb{Q}_l$. In this section, we set the notation used in the rest of the paper and present a general fact about an exact sequence of abelian varieties. Let $0 \to A \to B \to C \to 0$ be an exact sequence of algebraic groups proper over the field k. We assume that B and C are abelian varieties. However A can be an extension of an abelian variety by a finite (étale) group.

If k is a number field, let S be a finite set of places where all members of the above exact sequence have good reduction outside S; so, all archimedean places are included in S. Let $K = k^S$ (the maximal extension unramified outside S). If k is a finite extension of $\mathbb{Q}_l$, we put $K = \overline{k}$ (an algebraic closure of k). A general field extension of k is denoted by κ. We consider the étale topology, the smooth topology and the fppf topology on the small site over $\mathrm{Spec}(k)$. Here under the smooth topology, covering families are made of faithfully flat smooth morphisms.

We want to define p-adically completed sheaves $\widehat{X}$ for $X = A, B, C$ as above defined over these sites. The word "p-adically completed" does not always mean $\widehat{X}(R)$ is given by the projective limit $\varprojlim_n X(R)/p^n X(R)$, and the definition depends on the choice of k. For the moment, assume that k is a number field. In this case, for an extension X of abelian variety defined over k by a finite flat group scheme, we define $\widehat{X}(F) := X(F) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for an fppf extension F over k. We may regard its p-adic "completion" $0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$ as an exact sequence of fppf/smooth/étale abelian sheaves over k (or over any subring of k over which B and C extend to abelian schemes). Here the word "completion" means tensoring with $\mathbb{Z}_p$ over $\mathbb{Z}$. Indeed, for any ring R of finite type over k, $R \mapsto \widehat{C}(R) := C(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is an exact functor from the category of abelian fppf/smooth/étale sheaves into itself; therefore, the tensor construction gives $\widehat{C}(\kappa) = \varprojlim_n C(\kappa)/p^n C(\kappa)$ if κ is a field of finite type, since $C(\kappa)$ is an abelian group of finite type by a generalized Mordell-Weil theorem (e.g., [RTP, IV]). Let ϵ denote the dual number. Then we have a canonical identification $Lie(C)_{/\kappa} = \mathrm{Ker}(C(\kappa[\epsilon]) \to C(\kappa))$ (e.g. [EAI, §10.2.4]), and hence $Lie(C) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{Ker}(\widehat{C}(\kappa[\epsilon]) \to \widehat{C}(\kappa))$ is the p-adic completion of the κ-vector space $Lie(C)$ if κ is a finite extension of k. Since we find a complementary abelian subvariety C' of B such that C' is isogenous to C and $B = A + C'$ with finite $A \cap C'$, adding the primes dividing the order $|A \cap C'|$ to S, the intersection $A \cap C' \cong \mathrm{Ker}(C' \to C)$ extends to an étale finite group scheme outside S; so, $C'(K) \to C(K)$ is surjective. Thus we have an exact sequence of $\mathrm{Gal}(K/k)$-modules

$$0 \to A(K) \to B(K) \to C(K) \to 0.$$

Note that $\widehat{A}(K) = A(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p := \bigcup_F \widehat{A}(F)$ for F running over all finite extensions of k inside K. Then we have an exact sequence

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0. \tag{2.1}$$

Now assume that k is a finite extension of $\mathbb{Q}_l$. We put $K = \overline{k}$ (an algebraic closure of k). Suppose that F is a finite extension of k. Then $A(F) = O_F^{\dim A} \oplus \Delta_F$ for a finite group Δ_F and the l-adic integer ring O_F of F (see [M55] ot [T66]). Now suppose $l \neq p$. For an fppf extension $R_{/k}$, we define again $\widehat{A}(R) := A[p^\infty](R) = \varinjlim_n A[p^n]$ for $A[p^n] := \mathrm{Ker}(A(R) \xrightarrow{p^n} A(R))$. Then we have $\widehat{A}(F) = \varprojlim_n A(F)/p^n A(F) = \Delta_{F,p} := \Delta_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and we have $\widehat{A}(K) = \varinjlim_F \widehat{A}(F) = A[p^\infty](K)$, and $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are identical to the fppf/smooth/étale abelian sheaves $A[p^\infty]$, $B[p^\infty]$ and $C[p^\infty]$, where $X[p^\infty] := \varinjlim_n X[p^n]$ as an ind finite flat group scheme with $X[p^n] = \mathrm{Ker}(p^n : X \to X)$ for $X = A, B, C$. We again have the exact sequence (2.1) of $\mathrm{Gal}(\overline{k}/k)$-modules:

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0$$

and an exact sequence of fppf/smooth/étale abelian sheaves

$$0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$$

whose value at finite extension $\kappa/\mathbb{Q}_l$ coincides with the projective limit $\widehat{X}(\kappa) = \varprojlim_n X(\kappa)/p^n X(\kappa)$ for $X = A, B, C$.

Suppose $l = p$. For any module M, we define $M^{(p)}$ by the maximal prime-to-p torsion submodule of M. For $X = A, B, C$ and an fppf extension $R_{/k}$, the sheaf $R \mapsto X^{(p)}(R) = \varinjlim_{p \nmid N} X[N](R)$ is an fppf/smooth/étale abelian sheaf. Then we define the fppf/smooth/étale abelian sheaf $\widehat{X}$ by the sheaf quotient $X/X^{(p)}$. Since $X(F) = O_F^{\dim X} \oplus X[p^\infty](F) \oplus X^{(p)}(F)$ for a finite extension $F_{/k}$, on the étale site over k, $\widehat{X}$ is the sheaf associated to a presheaf $R \mapsto X(R)/X^{(p)}(R) = O_F^{\dim X} \oplus X[p^\infty](R)$. If X has semi-stable reduction over O_F, we have $\widehat{X}(F) = X^\circ(O_F) + X[p^\infty](F) \subset X(F)$ for the formal group X° of the identity connected component of the Néron model of X over O_F. Since X becomes semi-stable over a finite Galois extension F_0/k, in general $\widehat{X}(F) = H^0(\mathrm{Gal}(F_0F/F), X(F_0F))$ for any finite extension $F_{/K}$ (or more generally for each finite 'etale extension $F_{/k}$); so, $F \mapsto \widehat{X}(F)$ is a sheaf over the étale site over k. Thus by [ECH, II.1.5], the sheafication coincides over the étale site with the presheaf $F \mapsto \varprojlim_n X(F)/p^n X(F)$. Thus we conclude $\widehat{X}(F) = \varprojlim_n X(F)/p^n X(F)$ for any étale finite extensions $F_{/k}$. Moreover $\widehat{X}(K) = \bigcup_F \widehat{X}(F)$. Applying the snake lemma to the commutative diagram with exact rows (in the category of fppf/smooth/étale abelian sheaves):

$$\begin{array}{ccccc} A^{(p)} & \xrightarrow{\hookrightarrow} & B^{(p)} & \xrightarrow{\twoheadrightarrow} & C^{(p)} \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ A & \xrightarrow[\hookrightarrow]{} & B & \xrightarrow[\twoheadrightarrow]{} & C, \end{array}$$

the cokernel sequence gives rise to an exact sequence of fppf/smooth/étale abelian sheaves over k:

$$0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$$

and an exact sequence of $\mathrm{Gal}(\overline{k}/k)$-modules

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0.$$

In this way, we extended the étale sheaves $\widehat{A}, \widehat{B}, \widehat{C}$ defined on the étale site over $\mathrm{Spec}(k)$ to an abelian sheaves on the smooth, fppf and étale sites keeping the exact sequence $\widehat{A} \hookrightarrow \widehat{B} \twoheadrightarrow \widehat{C}$ intact. However note that our way of defining $\widehat{X}$ for $X = A, B, C$ depends on the base field $k = \mathbb{Q}, \mathbb{Q}_p, \mathbb{Q}_l$. In summary, we have, for fppf algebras $R_{/k}$:

$$\text{(S)} \qquad \widehat{X}(R) = \begin{cases} X(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \text{if } [k:\mathbb{Q}] < \infty, \\ X[p^\infty](R) & \text{if } [k:\mathbb{Q}_l] < \infty \ (l \neq p), \\ (X/X^{(p)})(R) \text{ as a sheaf quotient} & \text{if } [k:\mathbb{Q}_p] < \infty. \end{cases}$$

LEMMA 2.1. *Let the notation be as above (in particular, X is an extension of an abelian variety over k by a finite étale group scheme). If κ is either an integral domain or a field of finite type over k and either k is a number field or a local field with residual characteristic $l \neq p$, we have $\widehat{X}(\kappa) = \varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa)$. If κ is an étale extension of finite type over k and k is a p-adic field, we again have $\widehat{X}(\kappa) = \varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa)$.*

Proof. First suppose that k is a number field. If κ is a field extension of finite type over k, by [RTP, IV], $X(\kappa)$ is a $\mathbb{Z}$-module of finite type; so, we have $\widehat{X}(\kappa) = X(\kappa) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \varprojlim X(\kappa)/p^n X(\kappa)$. Here the first identity is just by the definition. More generally, if κ/k is a Krull domain of finite type over k, κ is a normal noetherian domain; and $\kappa = \bigcap_V V$ for discrete valuation ring V in $Q(\kappa)$ containing κ. By projectivity of the abelian variety, we have $X(V) = X(Q(\kappa)$ (by the valuative criterion of properness), which implies $X(\kappa) = \bigcap_V X(V) = X(Q(\kappa))$ (so, $\widehat{X}(\kappa) = \widehat{X}(Q(\kappa))$) for the quotient field $Q(\kappa)$ of κ. In particular, if κ is a smooth extension of finite type, an the result follows, Since the normalization $\widetilde{\kappa}$ of κ in $Q(\kappa)$ is a Krull domain, we find $\widehat{X}(\kappa) \subset \widehat{X}(\widetilde{\kappa}) = \widehat{X}(Q(\kappa))$; so, $\widehat{X}(\kappa)$ is an abelian group of finite type as long as κ is an integral domain of finite type over k (and hence is a reduced algbera of finite type over k).

If k is local of residual characteristic $l \neq p$, we have $\widehat{X} = X[p^\infty]$. If κ is an integral domain of finite type over k, then $\widehat{X}(\kappa)$ is a finite p-group, and the result is obvious.

The case where k is local of residual characteristic p is already dealt with before the lemma. $\square$

For a sheaf X under the topology ?, we write $H^\bullet_?(X)$ for the cohomology group $H^1_?(\mathrm{Spec}(\kappa), X)$ under the topology ?. If we have no subscript, $H^1(X)$ means the Galois cohomology $H^\bullet(\mathrm{Gal}(K/\kappa), X)$ for the $\mathrm{Gal}(K/\kappa)$-module X.

LEMMA 2.2. *Let X be an extension of an abelian variety over k by a finite étale group scheme of order prime to p. For any intermediate extension $K/\kappa/k$, We have a canonical injection*

$$\varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa) \hookrightarrow \varprojlim_n H^1(X[p^n]).$$

Similarly, for any fppf, smooth or étale extension κ/k of finite type which is an integral domain, we have an injection

$$\varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa) \hookrightarrow \varprojlim_n H^1_?(X[p^n])$$

for ? = fppf*,* sm *or* ét *according as κ/k is an fppf extension or a smooth extension.*

By Lemma 2.1, we have $\widehat{X}(\kappa) = \varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa)$ in the following cases:
(2.2)

$$\begin{cases} [k:\mathbb{Q}] < \infty \text{ and } \kappa \text{ is an integral domain of finite type over } k \\ [k:\mathbb{Q}_l] < \infty \text{ with } l \neq p \text{ and } \kappa \text{ is an integral domain of finite type over } k \\ [k:\mathbb{Q}_p] < \infty \text{ and } \kappa \text{ is a finite algebraic extension over } k. \end{cases}$$

Proof. We consider the sheaf exact sequence under the topology ? = fppf or sm or étale on $\mathrm{Spec}(\kappa)$

$$0 \to X[p^n] \to \widehat{X} \xrightarrow{p^n} \widehat{X}.$$

We want to show that the multiplication by p^n is surjective. If our cohomology theory is Galois cohomology (or equivalently ? = étale), we have an exact sequence

$$0 \to X[p^n](K) \to X(K) \xrightarrow{p^n} X(K) \to 0.$$

Since $\widehat{X}(K) = X(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, the desired exactness follows.
Let κ be an fppf extension of k. Then for each $x \in X(\kappa)$, we consider the Cartesian diagram

$$\begin{tikzcd} X_x \arrow[r] \arrow[d] & X \arrow[d, "p^n"] \\ \mathrm{Spec}(\kappa) \arrow[r, "x"'] & X. \end{tikzcd}$$

Then $X_x \cong X[p^n]$ as schemes over κ; so, $X_x = \mathrm{Spec}(R)$ for an étale finite extension R of κ, which is obviously smooth and also fppf extension of κ. Thus over the covering R/κ, x is the image of the point given by $\mathrm{Spec}(R) \hookrightarrow X$. Then by [ECH, II.2.5 (c)], $X \xrightarrow{p^n} X$ is an epimorphism of sheaves under étale, smooth

and also fppf topology. If k is a number field, we have $\widehat{X}(\kappa) = X(\kappa) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we get the exactness of $X[p^n] \hookrightarrow \widehat{X} \twoheadrightarrow \widehat{X}$ from the exactness of $X[p^n] \hookrightarrow X \twoheadrightarrow X$. If k is a finite extension of $\mathbb{Q}_l$ for $l \neq p$, we can argue as above replacing X by $\widehat{X} = X[p^\infty]$ and get the exactness of $X[p^n] \hookrightarrow \widehat{X} \twoheadrightarrow \widehat{X}$. Suppose that k is a finite extension of $\mathbb{Q}_p$. Then $\widehat{X} = X/X^{(p)}$ as a ?-sheaf. Take $x \in \widehat{X}(\kappa)$. Then by definition, we have an ?-extension R of κ such that x is the image of $y \in X(R)$. Then as above we can find a ?-extension R'/R such that $y = p^n y'$ for $y' \in X(R')$. Then for the image x' of $y' \in X(R')$ in $\widehat{X}(R')$, we have $p^n x' = x$. Thus again $\widehat{X} \xrightarrow{p^n} \widehat{X}$ is an epimorphism of sheaves under the topology ?.
Thus we can apply Kummer theory to the sheaf exact sequence

$$0 \to X[p^n] \hookrightarrow \widehat{X} \xrightarrow{p^n} \widehat{X} \to 0$$

with respect to the topology given by ?, we have an inclusion $\widehat{X}(\kappa)/p^n\widehat{X}(\kappa) \hookrightarrow H^1_?(X[p^n])$. Passing to the limit with respect to n, we have $\delta : \varprojlim_n X(\kappa)/p^n X(\kappa) \to \varprojlim_n H^1_?(X[p^n])$. Since taking projective limit is a left exact functor, δ is injective as desired. □

Taking instead an injective limit, we get

Lemma 2.3. *Let A be an abelian variety over k. For any intermediate extension $K/\kappa/k$, we have an exact sequence*

$$0 \to \widehat{A}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{T}_p \to H^1_?(A[p^\infty]) \to H^1_?(\widehat{A}) \to 0$$

for ? = fppf, sm *or* ét *according as κ/k is an fppf extension, a smooth extension or an étale extension. In particular, the Pontryagin dual of $H^1_?(\widehat{A})$ is a $\mathbb{Z}_p$-module of finite type; so, $H^1_?(\widehat{A})$ has the form $(\mathbb{Q}_p/\mathbb{Z}_p)^j \oplus \Delta$ for some $0 \leq j \in \mathbb{Z}$ and a finite p-group Δ.*

Proof. Since any smooth covering has finer étale covering, we have $H^\bullet_{\mathrm{sm}}(\widehat{A}) = H^\bullet_{\text{ét}}(\widehat{A})$ (cf. [ECH, III.3.4 (c)]). Since an étale covering is covered by a finer étale finite coverings, $H^q_{\text{ét}}(\widehat{A})$ and $H^q(A)$ for $q > 0$ is a torsion module. This torsionness is well known for Galois cohomology (as the Galois group is profinite; see [CNF, (1.6.1)]).
Pick any $x \in \widehat{A}(\kappa)$. We can find an étale finite extension κ'/κ such that $p^n y = x$ for some $y \in \widehat{A}(\kappa')$. Then y is unique modulo $\widehat{A}[p^n](\kappa')$. Therefore, the sheaf quotient $(\widehat{A}/A[p^\infty])(\kappa)$ is p-divisible and torsion-free; so, is a sheaf of $\mathbb{Q}_p$-vector spaces. In other words, $\widehat{A}/A[p^\infty]$ is isomorphic to the sheaf tensor product $\widehat{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Thus we have an exact sequence

$$0 \to A[p^\infty] \to \widehat{A} \to \widehat{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$

Since $H^1_?(\widehat{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is a $\mathbb{Q}_p$-vector space, the image in $H^1_?(\widehat{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ of the torsion module $H^1(\widehat{A})$ vanishes. Thus we have an exact sequence

$$0 \to \widehat{A}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{T}_p \to H^1_?(A[p^\infty]) \to H^1_?(\widehat{A}) \to 0.$$

Since $0 \to \widehat{A}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z} \to H^1_?(\widehat{A}[p]) \to H^1_?(\widehat{A})[p] \to 0$ is exact, by the finiteness of $H^1(\widehat{A}[p]) = H^1(A[p])$ (see [ADT, I.5]), the last assertion for Galois cohomology follows. Then using the comparison theorem (cf. [ECH, III.3.4 (c) and III.3.9]), we conclude the same for other topologies. $\square$

3. $U(p)$-isomorphisms

We recall the results in [H14b, §3] with detailed proofs for some results and a brief account for some others (as [H14b] is being written along with this paper). For $Z[U]$-modules X and Y, we call a $\mathbb{Z}[U]$-linear map $X \xrightarrow{f} Y$ a U-injection (resp. a U-surjection) if $\mathrm{Ker}(f)$ is killed by a power of U (resp. $\mathrm{Coker}(f)$ is killed by a power of U). If f is both U-injection and U-surjection, we call f is a U-isomorphism. Thus, f is a U-injection (resp. a U-surjection, a U-isomorphism) if after tensoring $\mathbb{Z}[U, U^{-1}]$, it becomes an injection (resp. a surjection, an isomorphism). In terms of U-isomorphisms, we describe briefly the facts we study in this article (and in later sections, we fill in more details in terms of the ordinary projector e).

Let N be a positive integer prime to p. We assume $Np^r \geq 4$ (without losing any generality as remarked in the introduction). We consider the (open) modular curve $Y_1(Np^r)_{/\mathbb{Q}}$ which classifies elliptic curves E with an embedding $\phi : \mu_{p^r} \hookrightarrow E[p^r] = \mathrm{Ker}(p^r : E \to E)$. Let $R_i = \mathbb{Z}_{(p)}[\mu_{p^i}]$, $K_i = \mathbb{Q}[\mu_{p^i}]$, $R_\infty = \bigcup_i R_i \subset \overline{\mathbb{Q}}$ and $K_\infty = \bigcup_i K_i \subset \overline{\mathbb{Q}}$. For a valuation subring or a subfield R of K_∞ over $\mathbb{Z}_{(p)}$ with quotient field K, we write $X_{r/R}$ for the normalization of the j-line $\mathbf{P}(j)_{/R}$ in the function field of $Y_1(Np^r)_{/K}$. The group $z \in (\mathbb{Z}/p^r\mathbb{Z})^\times$ acts on X_r by $\phi \mapsto \phi \circ z$, as $\mathrm{Aut}(\mu_{Np^r}) \cong (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Thus $\Gamma = 1 + p\mathbb{Z}_p = \gamma^{\mathbb{Z}_p}$ acts on X_r (and its Jacobian) through its image in $(\mathbb{Z}/Np^r\mathbb{Z})^\times$. Only in the following section, we need the result over a discrete valuation ring R. Hereafter, in most cases, we take $U = U(p)$ for the Hecke-Atkin operator $U(p)$ (though we take $U = U^*(p)$ sometimes for the dual $U^*(p)$ of $U(p)$).

Let $J_{r/R} = \mathrm{Pic}^0_{X_r/R}$ be the connected component of the Picard scheme. We state a result comparing $J_{r/R}$ and the Néron model of $J_{r/K}$ over R. Thus we assume that R is a valuation ring. By [AME, 13.5.6, 13.11.4], $X_{r/R}$ is regular; the reduction $X_r \otimes_R \mathbb{F}_p$ is a union of irreducible components, and the component containing the ∞ cusp has geometric multiplicity 1. Then by [NMD, Theorem 9.5.4], $J_{r/R}$ gives the identity connected component of the Néron model of the Jacobian of $X_{r/R}$. We write $X^s_{r/R}$ for the normalization of the j-line in the function field of the canonical $\mathbb{Q}$-curve associated to the modular curve of the congruence subgroup $\Gamma^r_s = \Gamma_1(Np^r) \cap \Gamma_0(p^s)$ (for $0 < r \leq s$). The open curve $Y^r_{s/\mathbb{Q}} = X^r_{s/\mathbb{Q}} - \{\text{cusps}\}$ classifies triples $(E, C, \phi : \mu_{Np^r} \hookrightarrow E)$ with a cyclic subgroup C of order p^s containing the image $\phi(\mu_{p^r})$.

We denote $\mathrm{Pic}^0_{X^r_s/R}$ by $J^r_{s/R}$. Similarly, as above, $J^r_{s/R}$ is the connected component of the Néron model of $X^r_{s/K}$. Note that

$$\text{(3.1)}\quad \Gamma^r_s\backslash\Gamma^r_s \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r} \end{pmatrix} \Gamma_1(Np^r) = \left\{ \begin{pmatrix} 1 & a \\ 0 & p^{s-r} \end{pmatrix} \middle| a \bmod p^{s-r} \right\} = \Gamma_1(Np^r)\backslash\Gamma_1(Np^r) \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r} \end{pmatrix} \Gamma_1(Np^r).$$

Write $U^s_r(p^{s-r}) : J^s_{r/R} \to J_{r/R}$ for the Hecke operator of $\Gamma^s_r\alpha_{s-r}\Gamma_1(Np^r)$ for $\alpha_m = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\right)$. Strictly speaking, the Hecke operator induces a morphism of the generic fiber of the Jacobians and then extends to their connected components of the Néron models by the functoriality of the model (or equivalently by Picard functoriality). Then we have the following commutative diagram from the above identity, first over $\mathbb{C}$, then over K and by Picard functoriality over R:

$$\text{(3.2)}\qquad \begin{array}{ccc} J_{r/R} & \xrightarrow{\pi^*} & J^r_{s/R} \\ \downarrow u & \swarrow u' & \downarrow u'' \\ J_{r/R} & \xrightarrow{\pi^*} & J^r_{s/R}, \end{array}$$

where the middle u' is given by $U^s_r(p^{s-r})$ and u and u'' are $U(p^{s-r})$. Thus

(u1) $\pi^* : J_{r/R} \to J^r_{s/R}$ is a $U(p)$-isomorphism (for the projection $\pi : X^r_s \to X_r$).

Taking the dual $U^*(p)$ of $U(p)$ with respect to the Rosati involution associated to the canonical polarization on the Jacobians. We have a dual version of the above diagram for $s > r > 0$:

$$\text{(3.3)}\qquad \begin{array}{ccc} J_{r/R} & \xleftarrow{\pi_*} & J^r_{s/R} \\ \uparrow u^* & \nearrow u'^* & \uparrow u''^* \\ J_{r/R} & \xleftarrow{\pi_*} & J^r_{s/R}. \end{array}$$

Here the superscript "$*$" indicates the Rosati involution corresponding to the canonical divisor on the Jacobians, and $u^* = U^*(p)^{s-r}$ for the level $\Gamma_1(Np^r)$ and $u''^* = U^*(p)^{s-r}$ for Γ^r_s. Note that these morphisms come from the following double coset identity:

$$\text{(3.4)}\quad \Gamma^r_s\backslash\Gamma^r_s \begin{pmatrix} p^{s-r} & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(Np^r) = \left\{ \begin{pmatrix} p^{s-r} & a \\ 0 & 1 \end{pmatrix} \middle| a \bmod p^{s-r} \right\} = \Gamma_1(Np^r)\backslash\Gamma_1(Np^r) \begin{pmatrix} p^{s-r} & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(Np^r).$$

From this, we get

(u*1) $\pi_* : J^r_{s/R} \to J_{r/R}$ is a $U^*(p)$-isomorphism, where π_* is the dual of π^*.

In particular, if we take the ordinary and the co-ordinary projector $e = \lim_{n\to\infty} U(p)^{n!}$ and $e^* = \lim_{n\to\infty} U^*(p)^{n!}$ on $J[p^\infty]$ for $J = J_{r/R}, J_{s/R}, J^r_{s/R}$, noting $U(p^m) = U(p)^m$, we have

$$\pi^* : J^{\mathrm{ord}}_{r/R}[p^\infty] \cong J^{r,\mathrm{ord}}_{s/R}[p^\infty] \text{ and } \pi_* : J^{r,\text{co-ord}}_{s/R}[p^\infty] \cong J^{\text{co-ord}}_{r/R}[p^\infty]$$

where "ord" (resp. "co-ord") indicates the image of the projector e (resp. e^*). For simplicity, we write $\mathcal{G}_{r/R} := J^{\mathrm{ord}}_{r/R}[p^\infty]_{/R}$.

Suppose that we have morphisms of three noetherian schemes $X \xrightarrow{\pi} Y \xrightarrow{g} S$ with $f = g \circ \pi$. We look into

$$H^0_{\mathrm{fppf}}(T, R^1 f_* \mathbb{G}_m) = R^1 f_* O_X^\times(T) = \mathrm{Pic}_{X/S}(T)$$

for S-scheme T and the structure morphism $f : X \to S$ (see [NMD, Chapter 8]). Suppose that f and g have compatible sections $S \xrightarrow{s_g} Y$ and $S \xrightarrow{s_f} X$ so that $\pi \circ s_f = s_g$. Then we get (e.g., [NMD, Section 8.1])

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Ker}(s_f^1 : H^1_{\mathrm{fppf}}(X_T, O_X^\times) \to H^1_{\mathrm{fppf}}(T, O_T^\times))$$
$$\mathrm{Pic}_{Y/S}(T) = \mathrm{Ker}(s_g^1 : H^1_{\mathrm{fppf}}(Y_T, O_{Y_T}^\times) \to H^1_{\mathrm{fppf}}(T, O_T^\times))$$

for any S-scheme T, where $s_f^q : H^q_{\mathrm{fppf}}(X_T, O_{X_T}^\times) \to H^q_{\mathrm{fppf}}(T, O_T^\times)$ and $s_g^n : H^n_{\mathrm{fppf}}(Y_T, O_{Y_T}^\times) \to H^n_{\mathrm{fppf}}(T, O_T^\times)$ are morphisms induced by s_f and s_g, respectively. Here we wrote $X_T = X \times_S T$ and $Y_T = Y \times_S T$. We suppose that the functors $\mathrm{Pic}_{X/S}$ and $\mathrm{Pic}_{Y/S}$ are representable by smooth group schemes (for example, if X, Y are curves and $S = \mathrm{Spec}(k)$ for a field k; see [NMD, Theorem 8.2.3 and Proposition 8.4.2]). We then put $J_? = \mathrm{Pic}^0_{?/S}$ $(? = X, Y)$. Anyway we suppose hereafter also that X, Y, S are varieties (in the sense of [ALG, II.4]).

For an fppf covering $\mathcal{U} \to Y$ and a presheaf $P = P_Y$ on the fppf site over Y, we define via Čech cohomology theory an fppf presheaf $\mathcal{U} \mapsto \check{H}^q(\mathcal{U}, P)$ denoted by $\underline{\check{H}}^q(P_Y)$ (see [ECH, III.2.2 (b)]). The inclusion functor from the category of fppf sheaves over Y into the category of fppf presheaves over Y is left exact. The derived functor of this inclusion of an fppf sheaf $F = F_Y$ is denoted by $\underline{H}^\bullet(F_Y)$ (see [ECH, III.1.5 (c)]). Thus $\underline{H}^\bullet(\mathbb{G}_{m/Y})(\mathcal{U}) = H^\bullet_{\mathrm{fppf}}(\mathcal{U}, O_{\mathcal{U}}^\times)$ for a Y-scheme $\mathcal{U}$ as a presheaf (here $\mathcal{U}$ varies in the small fppf site over Y).

Assuming that f, g and π are all faithfully flat of finite presentation, we use the spectral sequence of Čech cohomology of the flat covering $\pi : X \twoheadrightarrow Y$ in the fppf site over Y [ECH, III.2.7]:

$$\check{H}^p(X_T/Y_T, \underline{H}^q(\mathbb{G}_{m/Y})) \Rightarrow H^n_{\mathrm{fppf}}(Y_T, O_{Y_T}^\times) \xrightarrow[\iota]{\sim} H^n(Y_T, O_{Y_T}^\times) \tag{3.5}$$

for each S-scheme T. Here $F \mapsto H^n_{\mathrm{fppf}}(Y_T, F)$ (resp. $F \mapsto H^n(Y_T, F)$) is the right derived functor of the global section functor: $F \mapsto F(Y_T)$ from the category of fppf sheaves (resp. Zariski sheaves) over Y_T to the category of abelian groups. The canonical isomorphism ι is the one given in [ECH, III.4.9]. By the sections $s_?$, we have a splitting $H^q(X_T, O_{X_T}^\times) = \mathrm{Ker}(s_f^q) \oplus H^q(T, O_T^\times)$ and $H^n(Y_T, O_{Y_T}^\times) = \mathrm{Ker}(s_g^n) \oplus H^n(T, O_T^\times)$. Write $\underline{H}^\bullet_{Y_T}$ for $\underline{H}^\bullet(\mathbb{G}_{m/Y_T})$ and $\check{H}^\bullet(\underline{H}^0_{Y_T})$ for $\check{H}^\bullet(Y_T/X_T, \underline{H}^0_{Y_T})$. Since

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Ker}(s^1_{f,T} : H^1(X_T, O_{X_T}^\times) \to H^1(T, O_T^\times))$$

for the morphism $f : X \to S$ with a section [NMD, Proposition 8.1.4], from this spectral sequence, we have the following commutative diagram with exact rows, writing $\check H^0(\frac{X_T}{Y_T},?)$ as $\check H^0(?)$ and $H^1(T,O_T^\times)$ as $H^1(O_T^\times)$:

(3.6)
$$\begin{array}{ccccc}
?_1 & \longrightarrow & \check H^1(\underline{H}^0_{Y_T}) & = & \check H^1(\underline{H}^0_{Y_T}) \\
\downarrow & & \downarrow & & \downarrow\cap \\
\mathrm{Pic}_T \oplus J_Y(T) & \xrightarrow{\hookrightarrow} & \mathrm{Pic}_T\oplus \mathrm{Pic}_{Y/S}(T) & \xrightarrow{\sim} & H^1(O_T^\times)\oplus \mathrm{Ker}(s^1_{g,T}) \\
c\downarrow & & b\downarrow & & a\downarrow \\
\mathrm{Pic}_T \oplus \check H^0(J_X(T)) & \xrightarrow{\hookrightarrow} & \check H^0(\mathrm{Pic}_Y(T)) & = & \check H^0(\underline{H}^1(\mathbb{G}_{m,Y})) \\
\downarrow & & \downarrow & & \downarrow \\
?_2 & \longrightarrow & \check H^2(\underline{H}^0_{Y_T}) & = & \check H^2(\underline{H}^0_{Y_T}),
\end{array}$$

where we have written $J_? = \mathrm{Pic}^0_{?/S}$ (the identity connected component of $\mathrm{Pic}_{?/S}$). Here the vertical exactness at the right two columns follows from the spectral sequence (3.5) (see [ECH, Appendix B]).

We now recall the definition of the Čech cohomology: for a general S-scheme T and Čech cochain $c_{i_0,\dots,i_q} \in H^0(X_T^{(q+1)}, O^\times_{X_T^{(q+1)}})$,

$$(3.7)\quad \check H^q(\frac{X_T}{Y_T}, \underline{H}^0(\mathbb{G}_{m/Y})) = \frac{\{(c_{i_0,\dots,i_q})|\prod_j (c_{i_0\dots\check i_j\dots i_{q+1}}\circ p_{i_0\dots\check i_j\dots i_{q+1}})^{(-1)^j}=1\}}{\{db_{i_0\dots i_q}=\prod_j(b_{i_0\dots\check i_j\dots i_q}\circ p_{i_0\dots\check i_j\dots i_q})^{(-1)^j}|b_{i_0\dots\check i_j\dots i_q}\in H^0(X_T^{(q)},O^\times_{X_T^{(q)}})\}}$$

where we agree to put $H^0(X_T^{(0)}, O^{(0)}_{X_T}) = 0$ as a convention,

$$X_T^{(q)} = \overbrace{X\times_Y X\times_Y\cdots\times_Y X}^{q}\times_S T,$$
$$O_{X_T^{(q)}} = \overbrace{O_X\times_{O_Y} O_X\times_{O_Y}\cdots\times_{O_Y} O_X}^{q}\times_{O_S} O_T,$$

the identity $\prod_j (c\circ p_{i_0\dots\check i_j\dots i_{q+1}})^{(-1)^j}=1$ takes place in $O_{X_T^{(q+2)}}$ and $p_{i_0\dots\check i_j\dots i_{q+1}}: X_T^{(q+2)}\to X_T^{(q+1)}$ is the projection to the product of X the j-th factor removed. Since $T\times_T T\cong T$ canonically, we have $X_T^{(q)}\cong \overbrace{X_T\times_T\cdots\times_T X_T}^{q}$ by transitivity of fiber product.

Take a correspondence $U\subset Y\times_S Y$ given by two finite flat projections $\pi_1,\pi_2: U\to Y$ of constant degree (i.e., $\pi_{j,*}\mathcal{O}_U$ is locally free of finite rank $\deg(\pi_j)$ over

$\mathcal{O}_Y$). Consider the pullback $U_X \subset X \times_S X$ given by the Cartesian diagram:

$$\begin{tikzcd} U_X = U \times_{Y \times_S Y} (X \times_S X) \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\ U \arrow[r, hook] & Y \times_S Y \end{tikzcd}$$

Let $\pi_{j,X} = \pi_j \times_S \pi : U_X \twoheadrightarrow X$ ($j = 1, 2$) be the projections.

We describe the correspondence action of U on $H^0(X, \mathcal{O}_X^\times)$ in down-to-earth terms. Consider $\alpha \in H^0(X, \mathcal{O}_X)$. Then we lift $\pi_{1,X}^* \alpha = \alpha \circ \pi_{1,X} \in H^0(U_X, \mathcal{O}_{U_X})$. Put $\alpha_U := \pi_{1,X}^* \alpha$. Note that $\pi_{2,X,*}\mathcal{O}_{U_X}$ is locally free of rank $d = \deg(\pi_2)$ over $\mathcal{O}_X$, the multiplication by α_U has its characteristic polynomial $P(T)$ of degree d with coefficients in $\mathcal{O}_X$. We define the norm $N_U(\alpha_U)$ to be the constant term $P(0)$. Since α is a global section, $N_U(\alpha_U)$ is a global section, as it is defined everywhere locally. If $\alpha \in H^0(X, \mathcal{O}_X^\times)$, $N_U(\alpha_U) \in H^0(X, \mathcal{O}_X^\times)$. Then define $U(\alpha) = N_U(\alpha_U)$, and in this way, U acts on $H^0(X, \mathcal{O}_X^\times)$.

For a degree q Čech cohomology class $[c] \in \check{H}^q(X_{/Y}, \underline{H}^0(\mathbb{G}_{m/Y}))$ of a Čech q-cocycle $c = (c_{i_0,\dots,i_q})$, $U([c])$ is given by the cohomology class of the Čech cocycle $U(c) = (U(c_{i_0,\dots,i_q}))$, where $U(c_{i_0,\dots,i_q})$ is the image of the global section $c_{i_0,\dots,i_q}$ under U. Indeed, $(\pi_{1,X}^* c_{i_0,\dots,i_q})$ plainly satisfies the cocycle condition, and $(N_U(\pi_{1,X}^* c_{i_0,\dots,i_q}))$ is again a Čech cocycle as N_U is a multiplicative homomorphism. By the same token, we see that this operation sends coboundaries to coboundaries and obtain the action of U on the cohomology group.

Lemma 3.1. *Let the notation and the assumption be as above. In particular, $\pi : X \to Y$ is a finite flat morphism of geometrically reduced proper schemes over $S = \mathrm{Spec}(k)$ for a field k. Suppose that X and U_X are proper schemes over a field k satisfying one of the following conditions:*

(1) *U_X is geometrically reduced, and for each geometrically connected component X° of X, its pull back to U_X by $\pi_{2,X}$ is also connected; i.e., $\pi^0(X) \xrightarrow[\sim]{\pi_{2,X}^*} \pi^0(U_X)$;*

(2) *$(f \circ \pi_{2,X})_* \mathcal{O}_{U_X} = f_* \mathcal{O}_X$.*

If $\pi_2 : U \to Y$ has constant degree $\deg(\pi_2)$, the action of U on $H^0(X, \mathcal{O}_X^\times)$ factors through the multiplication by $\deg(\pi_2) = \deg(\pi_{2,X})$.

Proof. By properness, under (1) or (2), $H^0(U_X, \mathcal{O}_{U_X}) \overset{\pi_{2,X,*}}{=} H^0(X, \mathcal{O}_X) (\overset{(1)}{=} k^{\pi^0(X)})$ for the set of connected components $\pi^0(X)$ of X. In particular, we see $\alpha_U \in H^0(U_X, \mathcal{O}_{U_X}) = H^0(X, \mathcal{O}_X)$, which tells us that $N_U(\alpha_U) = \alpha_U^{\deg(\pi_2)}$, and the result follows. □

Consider the iterated product $\pi_{i,X^{(q)}} = \pi_{i,X} \times_Y \cdots \times_Y \pi_{i,X} : U_X^{(q)} \to X^{(q)}$ ($i = 1, 2$). Here $U_X^{(q)} = \overbrace{U_X \times_Y U_X \times_Y \cdots \times_Y U_X}^{q}$. We plug in $U_X^{(j)}$ in the first

j slots of the fiber product (for $0 < j \le q$) and consider

$$U_X^{(j-1)} \times_Y X^{(q-j+1)} \xleftarrow{\pi_{1,j}^{(q)}} U_j := U_X^{(j)} \times_Y X^{(q-j)} \xrightarrow{\pi_{2,j}^{(q)}} U_X^{(j-1)} \times_Y X^{(q-j+1)}$$

which induces a correspondence U_j in $(U_X^{(j-1)} \times_Y X^{(q-j+1)}) \times_Y (U_X^{(j-1)} \times_Y X^{(q-j+1)})$. Here $\pi_{i,j}$ restricted to first $j-1$-factors U_X is the identity id_{U_X}; the last $q-j$ factors is the identity id_X and at the j-th factor, it is the projection π_i $(i = 1, 2)$. For example, if $q = 3$ and $i = 2$, we have

$$U_X \times_Y U_X \times_Y U_X \xrightarrow[\mathrm{id}_U \times \mathrm{id}_U \times \pi_2]{\pi_{2,3}^{(q)}} U_X \times_Y U_X \times_Y X$$
$$\xrightarrow[\mathrm{id}_U \times \pi_2 \times \mathrm{id}_X]{\pi_{2,2}^{(q)}} U_X \times_Y X \times_Y X \xrightarrow[\pi_2 \times \mathrm{id}_X \times \mathrm{id}_X]{\pi_{2,1}^{(q)}} X \times_Y \times_Y X.$$

Naturally $\pi_{2,X^{(q)}}$ factors through the following q consecutive coverings $U_q \xrightarrow{\rho_q} U_{q-1} \xrightarrow{\rho_{q-1}} \cdots \xrightarrow{\rho_1} X^{(q)}$ for $\rho_j = \pi_{2,j}^{(q)}$. Note that the norm map $N_{U_q} = N_{\pi_{2,X^{(q)}}} : \pi_{2,X^{(q)},*}\mathcal{O}_{U_q}^\times \to \mathcal{O}_{X^{(q)}}^\times$ factors through the corresponding norm maps:

$$N_{U_q} = N_q \circ N_{q-1} \circ \cdots \circ N_1,$$

where N_j is the norm map with respect to $U_j \to U_{j-1}$. The last norm is essentially the product of N_U and the identity of $X^{(q-1)}$ corresponding to $U \times_Y X^{(q-1)} \twoheadrightarrow X^{(q)}$. In particular, $\rho_{1,*}(\mathcal{O}_{U_1}) = \pi_{2,X,*}(\mathcal{O}_{U_X}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{X^{(q-1)}}$ and

$$(f \circ \rho_1)_*(\mathcal{O}_{U_1}) = (f \circ \pi_{2,X})_*(\mathcal{O}_{U_X}) \otimes_{\mathcal{O}_Y} \overbrace{f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \cdots \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X}^{q-1}.$$

Thus if the assumption (2) in Lemma 3.1 is satisfied, the corresponding assumption for U_1 is satisfied. The assumption (1) implies (2) which is really necessary for the proof of Lemma 3.1. Applying the argument proving Lemma 3.1 to the correspondence U_1 and the last factor N_1 of the norm, we get

COROLLARY 3.2. *Let the notation and the assumption be as in* Lemma 3.1. *Then the action of $U^{(q)}$ on $H^0(X, \mathcal{O}_{X^{(q)}}^\times)$ factors through the multiplication by* $\deg(\pi_2) = \deg(\pi_{2,X})$.

Here is a main result of this section:

PROPOSITION 3.3. *Suppose that $S = \mathrm{Spec}(k)$ for a field k. Let $\pi : X \to Y$ be a finite flat covering of (constant) degree d of geometrically reduced proper varieties over k, and let $Y \xleftarrow{\pi_1} U \xrightarrow{\pi_2} Y$ be two finite flat coverings (of constant degree) identifying the correspondence U with a closed subscheme $U \overset{\pi_1 \times \pi_2}{\hookrightarrow} Y \times_S Y$. Write $\pi_{j,X} : U_X = U \times_Y X \to X$ be the base-change to X. Suppose one of the conditions* (1) *and* (2) *of* Lemma 3.1 *for (X, U). Then*

1. *The correspondence $U \subset Y \times_S Y$ sends $\check{H}^q(\underline{H}_Y^0)$ into* $\deg(\pi_2)(\check{H}^q(\underline{H}_Y^0))$ *for all $q > 0$.*
2. *If d is a p-power and* $\deg(\pi_2)$ *is divisible by p, $\check{H}^q(\underline{H}_Y^0)$ for $q > 0$ is killed by U^M if $p^M \ge d$.*

(3) *The cohomology $\check{H}^q(\underline{H}^0_Y)$ with $q > 0$ is killed by d.*

Proof. The first assertion follows from Corollary 3.2. Indeed, by (3.7), $U^{(q)}$ acts on each Čech q-cocycle, through an action factoring through the multiplication by $\deg(\pi_{2,X}) = \deg(\pi_2)$ by Corollary 3.2.
Now we regard $X \xrightarrow{\pi} Y$ as a correspondence of Y (with multiplicity d) by the projection $\pi_1 = \pi_2 = \pi : X \to Y$. Then $[X](c) = dc$ for $c \in \check{H}^q(X/Y, \underline{H}^0(\mathbb{G}_{m/Y}))$. On the other hand, by the definition of the correspondence action, $[X]$ factors through $\check{H}^q(X/X, \underline{H}^0(\mathbb{G}_{m/Y})) = 0$, and hence $dx = 0$. This shows that if X/Y is a covering of degree d, $\check{H}^q(X/Y, \underline{H}^0(\mathbb{G}_{m/Y}))$ is killed by d proving (3), and the assertion (2) follows from the first (1). □

We apply the above proposition to $(U, X, Y) = (U(p), X_s, X_s^r)$ with U given by $U(p) \subset X_s^r \times X_s^r$ over $\mathbb{Q}$. Indeed, $U := U(p) \subset X_s^r \times X_s^r$ corresponds to $X(\Gamma)$ given by $\Gamma = \Gamma_1(Np^r) \cap \Gamma_0(p^{s+1})$ and U_X is given by $X(\Gamma')$ for $\Gamma' = \Gamma_1(Np^s) \cap \Gamma_0(p^{s+1})$ both geometrically irreducible curves. In this case π_1 is induced by $z \mapsto \frac{z}{p}$ on the upper complex plane and π_2 is the natural projection of degree p. In this case, $\deg(X_s/X_s^r) = p^{s-r}$ and $\deg(\pi_2) = p$.

Assume that a finite group G acts on $X_{/Y}$ faithfully. Then we have a natural morphism $\phi : X \times G \to X \times_Y X$ given by $\phi(x, \sigma) = (x, \sigma(x))$. In other words, we have a commutative diagram

$$\begin{tikzcd} X \times G \arrow[r, "(x,\sigma)\mapsto\sigma(x)"] \arrow[d, "(x,\sigma)\mapsto x"'] & X \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd}$$

which induces $\phi : X \times G \to X \times_Y X$ by the universality of the fiber product. Suppose that ϕ is surjective; for example, if Y is a geometric quotient of X by G; see [GME, §1.8.3]). Under this map, for any fppf abelian sheaf F, we have a natural map $\check{H}^0(X/Y, F) \to H^0(G, F(X))$ sending a Čech 0-cocycle $c \in H^0(X, F) = F(X)$ (with $p_1^* c = p_2^* c$) to $c \in H^0(G, F(X))$. Obviously, by the surjectivity of ϕ, the map $\check{H}^0(X/Y, F) \to H^0(G, F(X))$ is an isomorphism (e.g., [ECH, Example III.2.6, page 100]). Thus we get

Lemma 3.4. *Let the notation be as above, and suppose that ϕ is surjective. For any scheme T fppf over S, we have a canonical isomorphism: $\check{H}^0(X_T/Y_T, F) \cong H^0(G, F(X_T))$.*

We now assume $S = \mathrm{Spec}(k)$ for a field k and that X and Y are proper reduced connected curves. Then we have from the diagram (3.6) with the exact middle two columns and exact horizontal rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 \\
\uparrow & & \deg\uparrow\text{onto} & & \deg\uparrow\text{onto} & & \uparrow \\
\check{H}^1(\underline{H}^0_Y) & \longrightarrow & \mathrm{Pic}_{Y/S}(T) & \xrightarrow{b} & \check{H}^0(\frac{X_T}{Y_T}, \mathrm{Pic}_{Y/S}(T)) & \longrightarrow & \check{H}^2(\underline{H}^0_Y) \\
\uparrow & & \cup\uparrow & & \uparrow\cup & & \uparrow \\
?_1 & \longrightarrow & J_Y(T) & \xrightarrow[c]{} & \check{H}^0(\frac{X_T}{Y_T}, J_X(T)) & \longrightarrow & ?_2,
\end{array}$$

Thus we have $?_j = \check{H}^j(\underline{H}^0_Y)$ $(j = 1, 2)$.
By Proposition 3.3, if $q > 0$ and X/Y is of degree p-power and $p|\deg(\pi_2)$, $\check{H}^q(\underline{H}^0_Y)$ is a p-group, killed by U^M for $M \gg 0$. Taking $(X, Y, U)_{/S}$ to be $(X_{s/\mathbb{Q}}, X^r_{s/\mathbb{Q}}, U(p))_{/\mathbb{Q}}$ for $s > r \geq 1$ for p odd and $s > r \geq 2$ for $p = 2$, we get for the projection $\pi : X_s \to X^r_s$

Corollary 3.5. *Let F be a number field or a finite extension of $\mathbb{Q}_l$ (for a prime l not necessarily equal to p). Then we have*

(u) $\pi^* : J^r_{s/\mathbb{Q}}(F) \to \check{H}^0(X_s/X^r_s, J_{s/\mathbb{Q}}(F)) \overset{(*)}{=} J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-1}} - 1]$ *is a $U(p)$-isomorphism,*

where $J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-1}} - 1] = \mathrm{Ker}(\gamma^{p^{r-1}} - 1 : J_s(F) \to J_s(F))$.

From these, we got the following facts as [H14b, Lemma 3.7]

Lemma 3.6. *We have morphisms*

$$\iota^r_s : J_{s/\mathbb{Q}}[\gamma^{p^{r-1}} - 1] \to J^r_{s/\mathbb{Q}} \quad \text{and} \quad \iota^{r,*}_s : J^r_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}}/(\gamma^{p^{r-1}} - 1)(J_{s/\mathbb{Q}})$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc}
J^r_{s/\mathbb{Q}} & \xrightarrow{\pi^*} & J_{s/\mathbb{Q}}[\gamma^{p^{r-1}} - 1] \\
\downarrow u & \swarrow \iota^r_s & \downarrow u'' \\
J^r_{s/\mathbb{Q}} & \xrightarrow{\pi^*} & J_{s/\mathbb{Q}}[\gamma^{p^{r-1}} - 1],
\end{array} \tag{3.8}$$

and

$$\begin{array}{ccc}
J^r_{s/\mathbb{Q}} & \xleftarrow{\pi_*} & J_{s/\mathbb{Q}}/(\gamma^{p^{r-1}} - 1)(J_{s/\mathbb{Q}}) \\
\uparrow u^* & \nearrow \iota^{r,*}_s & \uparrow u''^* \\
J^r_{s/\mathbb{Q}} & \xleftarrow{\pi^*} & J_{s/\mathbb{Q}}/(\gamma^{p^{r-1}} - 1)(J_{s/\mathbb{Q}}),
\end{array} \tag{3.9}$$

where u and u'' are $U(p^{s-r}) = U(p)^{s-r}$ and u^* and u''^* are $U^*(p^{s-r}) = U^*(p)^{s-r}$. In particular, for an fppf extension $T_{/\mathbb{Q}}$, the evaluated map at T: $(J_{s/\mathbb{Q}}/(\gamma^{p^{r-1}} - 1)(J_{s/\mathbb{Q}}))(T) \xrightarrow{\pi_*} J^r_s(T)$ (resp. $J^r_s(T) \xrightarrow{\pi^*} J_s[\gamma^{p^{r-1}} - 1](T)$) is a $U^*(p)$-isomorphism (resp. $U(p)$-isomorphism).

Remark 3.7. Note here that the natural morphism:

$$\frac{J_s(T)}{(\gamma^{p^{r-1}} - 1)(J_s(T))} \to (J_s/(\gamma^{p^{r-1}} - 1)(J_s))(T)$$

may have non-trivial kernel and cokernel which may not be killed by a power of $U^*(p)$. In other words, the left-hand-side is an fppf presheaf (of T) and the right-hand-side is its sheafication. On the other hand, $T \mapsto J_s[\gamma^{p^{r-1}} - 1](T)$ is already an fppf abelian sheaf; so, $J_r(T) \xrightarrow{\pi^*} J_s[\gamma^{p^{r-1}} - 1](T)$ is a $U(p)$-isomorphism without ambiguity by the above Lemma 3.6 and Corollary 3.5 combined. Also, as remarked in the introduction, we need to replace $\gamma^{p^{r-1}} - 1$ in the above statement by $\gamma^{p^{r-2}} - 1$ if $p = 2$.

4. Structure of Λ-BT groups over number fields and local fields

Let $\mathcal{G}_{/R_\infty} := \varinjlim_r J_r[p^\infty]^{\mathrm{ord}}_{/R_\infty}$, which is a Λ-BT group in the sense of [H14a, Sections 3 and 5] with a canonical $\mathbf{h}$-action. Here for an abelian variety $A_{/R}$, $A[p^n] = \mathrm{Ker}(A \xrightarrow{p^n} A)$ and $A[p^\infty]_{/R} = \varinjlim_n A[p^n]$ (the p-divisible Barsotti–Tate group of A over R). For an $\mathbf{h}$-algebra A, we put $\mathcal{G}_A = \mathcal{G} \otimes_{\mathbf{h}} A$. Pick a reduced local ring $\mathbb{T}$ of $\mathbf{h}$ and write $a(l^m)$ for the image in $\mathbb{T}$ of $U(l^m)$ or $T(l^m)$ for a prime l according as $l|Np$ or $l \nmid Np$ and $\mathfrak{m}_\mathbb{T}$ for the maximal ideal of $\mathbb{T}$. Since $\mathcal{G}_\mathbb{T}$ is a Λ-BT group in the sense of [H14a, Theorem 5.4, Remark 5.5], we have the connected-étale exact sequence over $\mathbb{Z}_p[\mu_{p^\infty}]$:

$$0 \to \mathcal{G}^\circ_\mathbb{T} \to \mathcal{G}_\mathbb{T} \to \mathcal{G}^{\mathrm{\acute{e}t}}_\mathbb{T} \to 0,$$

where $\mathcal{G}^\circ_\mathbb{T}$ is the connected component of the flat group $\mathcal{G}_\mathbb{T}$ and $\mathcal{G}^{\mathrm{\acute{e}t}}_\mathbb{T}$ is the quotient of $\mathcal{G}_\mathbb{T}$ by $\mathcal{G}^\circ_\mathbb{T}$. The étale group $\mathcal{G}_{\mathbb{T}/\mathbb{Q}}$ over $\mathbb{Q}$ is a Λ-BT group over $\mathbb{Q}$ (in the sense of [H14a, §4]) on which $\mathbb{Z}_p^\times$ act by diamond operators. The entire group $\mathcal{G}_\mathbb{T}$ extends to a Λ-BT group over $\mathbb{Z}_p[\mu_{p^\infty}]$ (see [H14a, Remark 5.5]). The $\overline{\mathbb{Q}}_p$-points of this sequence descent to $\mathbb{Q}_p$ giving an exact sequence:

$$0 \to \mathcal{G}^\circ_\mathbb{T}(\overline{\mathbb{Q}}_p) \to \mathcal{G}_\mathbb{T}(\overline{\mathbb{Q}}_p) \to \mathcal{G}^{\mathrm{\acute{e}t}}_\mathbb{T}(\overline{\mathbb{Q}}_p) \to 0$$

with $\mathcal{G}^{\mathrm{\acute{e}t}}_\mathbb{T}(\overline{\mathbb{Q}}_p) = H_0(I_p, \mathcal{G}_\mathbb{T}(\overline{\mathbb{Q}}_p))$ for the inertia group $I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.
We know that $\mathcal{G}^\circ_\mathbb{T}$ and $\mathcal{G}^{\mathrm{\acute{e}t}}_\mathbb{T}$ are well controlled, and the Pontryagin dual modules of $\mathcal{G}^\circ_\mathbb{T}(\overline{\mathbb{Q}})$ and $\mathcal{G}^{\mathrm{\acute{e}t}}_\mathbb{T}(\overline{\mathbb{Q}})$ are Λ-free modules of (equal) finite rank (see [H86b, §9] or [H14a, Sections 4–5]). Here we equip these Λ-divisible modules with the discrete topology. Take a field k as a base field. Pick a $\mathbb{T}$-ideal $\mathfrak{a}$. Write $\mathcal{G}_\mathbb{T}[\mathfrak{a}]$ for the kernel of $\mathfrak{a}$:

$$\mathcal{G}_\mathbb{T}[\mathfrak{a}](R) = \{x \in \mathcal{G}_\mathbb{T}(R) | ax = 0 \ \forall a \in \mathfrak{a}\},$$

where R is an fppf extension of k. Write $a(p)$ for the image of $U(p)$ in $\mathbb{T}$.
For the moment, assume that k is a finite extension k of $\mathbb{Q}_p$ with p-adic integer ring W. If the residual degree of k is f and $a(p)^f \not\equiv 1 \mod \mathfrak{m}_\mathbb{T}$ for the maximal ideal $\mathfrak{m}_\mathbb{T}$ of $\mathbb{T}$, we have

$$\mathcal{G}_\mathbb{T}[\mathfrak{m}_\mathbb{T}]^{\mathrm{\acute{e}t}}(k) = 0,$$

since the action of $Frob_p$ on $\mathcal{G}_\mathbb{T}[\mathfrak{m}_\mathbb{T}]^{\mathrm{\acute{e}t}}(\overline{\mathbb{Q}}_p)$ is given by multiplication by $a(p)$. On the other hand, the action of $\mathrm{Gal}(\overline{k}/k)$ on $e \cdot J_\infty[p^\infty]^\circ(\overline{k}) \otimes_{\mathbf{h}} \mathbb{T}$ factors through $\mathrm{Gal}(k[\mu_{p^\infty}]/k) \hookrightarrow \mathbb{Z}_p^\times \to \Lambda^\times$, where the factor $\Gamma = 1 + p\mathbb{Z}_p$ of $\mathbb{Z}_p^\times = \Gamma \times \mu_{p-1}$ is

embedded into $\Lambda = \mathbb{Z}_p[[\Gamma]]$ by natural inclusion and $\zeta \in \mu_{p-1}$ is sent to ζ^a for some $0 \le a =: a(\mathbb{T}) = a_k(\mathbb{T}) < p-1$. Thus if $a(\mathbb{T}) \neq 0$, we have $\mathcal{G}^\circ_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = 0$. We have a natural projection $\pi = \pi^r_s : \mathcal{G}_s := J_s[p^\infty]^{\mathrm{ord}}_{/\mathbb{Q}} \to \mathcal{G}_r$ for $s > r$ (see [H13a, Section 4] where π^r_s is written as N^s_r). This induces a projective system of Tate modules $\{T\mathcal{G}_{s,\mathbb{T}} := T\mathcal{G}_s \otimes_{\mathbf{h}} \mathbb{T}\}_s$ and $\{T\mathcal{G}^?_{s,\mathbb{T}}\}$ for $? = \circ, \text{ét}$. We put $T\mathcal{G}^?_{\mathbb{T}} = \varprojlim_s T\mathcal{G}^?_{s,\mathbb{T}}(\overline{\mathbb{Q}})$ for ? = nothing, $\circ$ or ét. They are Λ-free modules with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Write $\rho_{\mathbb{T}}$ for the Galois representation realized on $T\mathcal{G}_{\mathbb{T}}$, and put $\rho_P = \rho_{\mathbb{T}} \mod P$ acting on $T\mathcal{G}_{\mathbb{T}}/PT\mathcal{G}_{\mathbb{T}}$ for $P \in \mathrm{Spec}(\mathbb{T})$. In particular, we simply write $\overline{\rho} = \overline{\rho}_{\mathbb{T}} = \rho_{\mathfrak{m}_{\mathbb{T}}}$ for the maximal ideal $\mathfrak{m}_{\mathbb{T}}$ of $\mathbb{T}$.
If $\mathbb{T}$ is a Gorenstein ring, then for the Tate modules $T\mathcal{G}_{\mathbb{T}}$, $T\mathcal{G}^\circ_{\mathbb{T}}$ and $T\mathcal{G}^{\text{ét}}_{\mathbb{T}}$ as above, we have

$$T\mathcal{G}_{\mathbb{T}} \cong \mathbb{T}^2 \text{ and } T\mathcal{G}^\circ_{\mathbb{T}} \cong \mathbb{T} \cong T\mathcal{G}^{\text{ét}}_{\mathbb{T}}$$

as $\mathbb{T}$-modules (e.g., [H13a, Section 4]), and if $\overline{\rho}_{\mathbb{T}}(I_p)$ contains a non-trivial unipotent element for the inertia group I_p in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/k)$, again we have $\mathcal{G}^{\text{ét}}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = 0$. Thus we get

Lemma 4.1. *Let $k_{/\mathbb{Q}_p}$ in $\overline{\mathbb{Q}}_p$ be a finite extension and $\mathbb{T}$ be a reduced local ring of $\mathbf{h}$. Assume that k has residual degree f and one of the following two conditions:*

(1) $a_k(\mathbb{T}) \neq 0$ *and* $a(p)^f \not\equiv 1 \mod \mathfrak{m}_{\mathbb{T}}$,
(2) $\mathbb{T}$ *is a Gorenstein ring, and $\overline{\rho}_{\mathbb{T}}(I_p)$ has non-trivial unipotent element for the inertia group I_p of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/k)$.

Then we have $\mathcal{G}_{\mathbb{T}}(k) = 0$.

Proof. Let V be the Λ-dual of $T\mathcal{G}_{\mathbb{T}}$, which is also the Pontryagin dual of $\mathcal{G}_{\mathbb{T}}$. Then we have $H_0(k, V/\mathfrak{m}_{\mathbb{T}}V) \cong \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = H^0(k, \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}])$. By the assumption (1) or (2), we have the vanishing $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = 0$. Look into the following exact sequence of sheaves

$$0 \to \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}] \to \mathcal{G}_{\mathbb{T}} \xrightarrow{\varphi} \bigoplus_{\alpha \in I} \mathcal{G}_{\mathbb{T}}$$

with $\varphi(x) = (\alpha x)_\alpha$ for a finite set $I = \{\alpha\}_\alpha$ of generators of $\mathfrak{m}_{\mathbb{T}}$. Taking the $\mathrm{Gal}(\overline{k}/k)$-invariant, we get another exact sequence

$$0 \to \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) \to \mathcal{G}_{\mathbb{T}}(k) \xrightarrow{\varphi_k} \bigoplus_{\alpha \in I} \mathcal{G}_{\mathbb{T}}(k).$$

Since $\mathrm{Ker}(\varphi_k) = \mathcal{G}_{\mathbb{T}}(k)[\mathfrak{m}_{\mathbb{T}}]$, we conclude $(\mathcal{G}_{\mathbb{T}}(k))[\mathfrak{m}_{\mathbb{T}}] = \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](k) = 0$. Taking the Pontryagin dual module written as $M^\vee$ for a compact or discrete module M, we have, setting $V = \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}})^\vee$,

$$H_0(k, V)/\mathfrak{m}_{\mathbb{T}}H_0(k, V) \cong (\mathcal{G}_{\mathbb{T}}(k))^\vee/\mathfrak{m}_{\mathbb{T}}(\mathcal{G}_{\mathbb{T}}(k))^\vee = (\mathcal{G}_{\mathbb{T}}(k)[\mathfrak{m}_{\mathbb{T}}])^\vee = 0,$$

which implies $\mathcal{G}_{\mathbb{T}}(k)^\vee = H_0(k, V) = 0$ by Nakayama's lemma, and hence $\mathcal{G}_{\mathbb{T}}(k) = 0$. This proves the assertion under (1) or (2). $\square$

In the $l \neq p$ case, we remark the following fact:

LEMMA 4.2. *Let $k_{/\mathbb{Q}_l}$ in $\overline{\mathbb{Q}}_l$ be a finite extension for a prime $l \neq p$ and $\mathbb{T}$ be a reduced local ring of $\mathbf{h}$. If the semi-simplification of $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/k)$ does not contain the identity representation, then $\mathcal{G}_{\mathbb{T}}(k) = 0$. In general, $\mathcal{G}_{\mathbb{T}}(k)^\vee$ is always a torsion Λ-module of finite type.*

Proof. If the semi-simplification of $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/k)$ does not contain the identity representation, we have $H^0(k, \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]) = 0$; so, $H_0(k, V/\mathfrak{m}_{\mathbb{T}}V) = 0$ for $V = \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}})^\vee$. Writing $\mathfrak{m}_{\mathbb{T}} = (\alpha_i)_{i\in I}$ for $\alpha_i \in \mathbb{T}$ with a finite index set I, we have an exact sequence:

$$0 \to \mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}](\overline{\mathbb{Q}}) \to \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}}) \xrightarrow{x \mapsto (\alpha_i x)_i} \bigoplus_{i\in I} \mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}}).$$

Taking the Pontryagin dual we have another exact sequence of Galois modules:

$$0 \leftarrow V/\mathfrak{m}_{\mathbb{T}}V \leftarrow V \xleftarrow{x \mapsto (\alpha_i x)_i} \prod_{i\in I} V.$$

Since Galois homology functor is right exact, the above exact sequence implies

$$H_0(k, V) \otimes_{\mathbb{T}} \mathbb{T}/\mathfrak{m}_{\mathbb{T}} = H_0(k, V/\mathfrak{m}_{\mathbb{T}}V) = 0.$$

Then by Nakayama's lemma, we get $H_0(k, V) = 0$, which implies $\mathcal{G}_{\mathbb{T}}(k) = 0$.
Let f be the residual degree of k as before. Consider the Hecke polynomial $H_{f,l}(X) = X^2 - A(l^f)X + l^f\langle l\rangle^f$, where $A(l^f)$ is determined by the following recurrence relation: $A(l) = a(l)$ and $A(l^m) = a(l^m) - l\langle l\rangle a(l^{m-1})$ for $m \geq 2$. If $l \nmid Np$, $\mathcal{G}_{\mathbb{T}}$ is unramified over k. By the Eichler–Shimura congruence relation (e.g. [GME, Theorem 4.2.1]), if $l \nmid Np$, for the l-Frobenius element $\phi \in \mathrm{Gal}(\overline{k}/k)$, the linear operator $H_{f,l}(\phi)$ annihilates $\mathcal{G}_{\mathbb{T}}$. Thus if $H_{f,l}(X)$ mod $\mathfrak{m}_{\mathbb{T}}$ is not divisible by $X-1$, $\mathcal{G}_{\mathbb{T}}[\mathfrak{m}_{\mathbb{T}}]$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/k)$ does not contain the identity representation.
For an arithmetic prime P, $H_{f,l}(X) \mod P$ does not have a factor $X-1$. Thus after the localization at P of the Pontryagin dual $(\mathcal{G}_{\mathbb{T}}(k)^\vee)_P$ is killed by $H_{f,l}(\phi)$ and $\phi - 1$, and hence $\mathcal{G}_{\mathbb{T}}(k)^\vee$ is a torsion Λ-module.
Now assume that $l|N$. By the solution of the local Langlands conjecture (see [C86] and [AAG]), after replacing k by its finite extension, the Galois module $\mathcal{G}_{\mathbb{T}}[P]$ for an arithmetic point P becomes unramified unless ρ_P is Steinberg at l (i.e., is multiplicative type at l). Suppose that we have a non-Steinberg P. Then characteristic polynomial $H(X)$ of ϕ modulo P is prime to $X-1$ (as $H(X) \mod P$ has Weil numbers of weight f as its roots). Then by the same argument, we conclude the torsion property.
Suppose that all arithmetic point of $\mathrm{Spec}(\mathbb{T})$ is Steinberg at l (this often happens; see a remark below Conjecture 3.4 of [H11, §3]). Write ρ_P for the 2-dimensional Galois representation realized on $(\mathcal{G}_{\mathbb{T}}(\overline{\mathbb{Q}}_l)^\vee) \otimes_{\mathbb{T}} \kappa(P)$. Again by Langlands-Carayol, $\rho_P(I_l)$ for the inertia group $I_l \subset \mathrm{Gal}(\overline{\mathbb{Q}}_l/k)$ contains a non-trivial unipotent element. Thus ρ_P does not have a quotient on which I_l acts trivially. This shows again the Λ-torsion property. □

Let $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T})$ be an irreducible component. Without assuming the Gorenstein condition, we have $(T\mathcal{G}_{\mathbb{I}})_P \cong \mathbb{I}_P^2$ for almost all height one primes $P \in \mathrm{Spec}(\Lambda)$; so, we have $\rho_{\mathbb{I}}$ with values in $GL_2(\mathbb{I}_P)$ for most of P. We call $\mathbb{I}$ a CM component if $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \Psi$ for a Galois character $\Psi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}_P^\times$ (for an imaginary quadratic field M). If $\mathbb{I}$ is not a CM component, again for almost all P, by [Z14], $\rho_{\mathbb{T}}(I_p)$ contains an unipotent element conjugate to $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ with non-zero-divisor $u \in \mathbb{T}_P^\times$. In this case, we have $H_0(\mathrm{Gal}(\overline{k}/k), T\mathcal{G}_{\mathbb{I}})_P = 0$; so, $\mathcal{G}_{\mathbb{I}}(k)$ is a co-torsion Λ-module.

Lemma 4.3. *Let $k_{/\mathbb{Q}_p}$ in $\overline{\mathbb{Q}}_p$ be a finite extension with residual degree f and $\mathbb{T}$ be a reduced local ring of $\mathbf{h}$. Then the Pontryagin dual $\mathcal{G}_{\mathbb{T}}(k)^\vee$ of $\mathcal{G}_{\mathbb{T}}(k)$ is a torsion Λ–module of finite type.*

Proof. We may suppose either $a(p)^f \equiv 1 \mod \mathfrak{m}_{\mathbb{T}}$ or $a_k(\mathbb{T}) = 0$, as otherwise $\mathcal{G}_{\mathbb{T}}(k) = 0$ by Lemma 4.1. Replacing $\mathbb{T}$ by its irreducible component $\mathbb{I}$, we only need to prove torsion-ness for $\mathcal{G}_{\mathbb{I}}(k)^\vee$. Write V for the Λ-torsion free quotient of $T\mathcal{G}_{\mathbb{I}}$. Then for any $P \in \mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}}_p)$, we have $V_P = (T\mathcal{G}_{\mathbb{I}})_P$ (as the reflexive closure in [BCM, Chapter 7] of $\mathbb{I}$ is Λ-free).

If $\mathbb{I}$ is not a CM component (i.e., $\rho_{\mathbb{I}}$ is not an induced representation from the Galois group over an imaginary quadratic field), the assertion follows from the same argument proving Lemma 4.1 replacing $\mathfrak{m}_{\mathbb{T}}$ by $P\mathbb{T}_P$ and $\mathbb{T}$ by $\mathbb{T}_P$. Indeed, taking an arithmetic point P of weight 2. Then by [Z14], we have $u \in \mathbb{T}_P^\times$. Then $H^0(k, V_P/PV_P)$ is a submodule of $H^0(I, V_P/PV_P)$ (for the inertia group I at p) killed by $a(p)^f - 1$. Since P is an arithmetic point of weight 1, we may choose P so that $a(p) \mod P$ is a Weil number of weight 1 (indeed, we only need to assume that the Neben character of f_P is non-trivial at p; see [MFM, Theorem 4.6.17]), and hence $a(p)^f \not\equiv 1 \mod P$. Thus $H^0(k, V_P/PV_P) = 0$. This implies $\mathcal{G}_{\mathbb{I}}(k)[P]$ is a finite module; so, $\mathcal{G}_{\mathbb{I}}(k)^\vee$ is a torsion Λ-module.

Now assume that $\mathbb{I}$ is a CM component with $\rho_{\mathbb{I}} = \mathrm{Ind}_M^{\mathbb{Q}} \Psi$. Define $\Psi^c(\sigma) = \Psi(c\sigma c^{-1})$ for a complex conjugation c. In the imaginary quadratic field M, p splits into a product of two primes $\mathfrak{p}\overline{\mathfrak{p}}$ as $\rho_{\mathbb{I}}$ is ordinary. For any arithmetic point $\mathfrak{P} \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ $\Psi_{\mathfrak{P}} := \Psi \mod \mathfrak{P}$ ramifies at $\mathfrak{p}$ and its restriction to the inertia group at $\mathfrak{p}$ has infinite order, and Ψ^c is unramified at $\mathfrak{p}$ with infinite order $\Psi^c(Frob_{\mathfrak{p}})$ (from an explicit description of Ψ; cf, [H13a, §3]). Then we have $V_{\mathfrak{P}} = V \otimes_{\mathbb{I}} \mathbb{I}_{\mathfrak{P}} \cong \mathbb{I}_{\mathfrak{P}}^2$. Thus replacing k by the composite $kM_{\mathfrak{p}}$, we have $V_{\mathfrak{P}} \cong \Psi \oplus \Psi^c$ over $\mathrm{Gal}(\overline{\mathbb{Q}}_p/k)$. Since Ψ^c is unramified at $\mathfrak{p}$ and $\Psi^c_{\mathfrak{P}}(Frob_{\mathfrak{p}})$ has infinite order. This shows that $H^0(k, V_{\mathfrak{P}}/\mathfrak{P}V_{\mathfrak{P}}) = 0$, and again we find that $\mathcal{G}_{\mathbb{I}}^\vee(k)$ is a torsion $\mathbb{I}$-module and hence a torsion Λ-module. □

Corollary 4.4. *If k is a number field or a finite extension of $\mathbb{Q}_l$, the localization of $\mathcal{G}_{\mathbb{T}}(k)^\vee$ at an arithmetic prime of weight 2 vanishes.*

Proof. We only need to prove this for a finite extension k of $\mathbb{Q}_l$. Write W for the integer ring of k. Replacing k by its finite extension, we may assume that A_P has semi-stable reduction over W for an arithmetic prime at P. If A_P has good reduction and $l \neq p$, the l-Frobenius acts on T_pA_P by a Weil number of weight ≥ 1, and then $A_P[p^\infty](k)$ is finite; so, $\mathcal{G}_{\mathbb{T}}[P](k)$ is finite. If $l = p$, by [Z14], the inertia image in $\mathrm{Aut}(T_pA_P)$ contains a non-trivial unipotent element, and hence again $A_P[p^\infty](k)$ is finite, and the result follows. If A_P has multiplicative reduction, $A_P[p^\infty](k)$ is finite by a theorem of Tate–Mumford as the Tate period of A_P is non-trivial. This shows that $\mathcal{G}_{\mathbb{T}}[P](k)$ is finite, and hence the result follows. □

5. Abelian factors of modular Jacobians

Let $h_r(\mathbb{Z})$ be the subalgebra generated by $T(n)$ (including $U(l)$ for $l|Np$) of $\mathrm{End}(J_{r/\mathbb{Q}})$. Then $h_r(\mathbb{Z}_p) = h_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is canonically isomorphic to the $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(J_r[p^\infty])$ generated by $T(n)$ (including $U(l)$ for $l|Np$). Then $\mathbf{h}_r = h_r(\mathbb{Z}_p)^{\mathrm{ord}}$ by the control theorems in [H86a] and [H86b].
As before, let k be a finite extension of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$ or a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$. Let A_r be a abelian subvariety of J_r defined over k. Write A_s ($s > r$) for the image of A_r in J_s under the morphism $\pi^* : J_r \to J_s$ given by Picard functoriality from the projection $\pi : X_s \to X_r$. If A_r is Shimura's abelian subvariety attached to a Hecke eigenform f, we sometimes write $A_{f,s}$ for A_s to indicate this fact. Hereafter we assume

(A) We have a coherent sequence $\alpha_s \in \mathrm{End}(J_{s/\mathbb{Q}})$ (for all $s \geq r$) having the limit $\alpha = \varprojlim_s \alpha_s \in \mathrm{End}(J_{\infty/\mathbb{Q}})$ such that
  - (a) A_s is the connected component of $J_s[\alpha_s]$ with $J_s = A_s + \alpha_s(J_s)$ so that the inclusion: $A_s[p^\infty] \cong J_s[\alpha_s][p^\infty]$ is a $U(p)$-isomorphism,
  - (b) the restriction $\alpha_s|_{\alpha_s(J_s)} \in \mathrm{End}(\alpha_s(J_s))$ is a self-isogeny.

Here for $s' > s$, coherency of α_s means the following commutative diagram:

$$\begin{array}{ccc} J_s & \xrightarrow{\pi^*} & J_{s'} \\ {\scriptstyle\alpha_s}\downarrow & & \downarrow{\scriptstyle\alpha_{s'}} \\ J_s & \xrightarrow[\pi^*]{} & J_{s'} \end{array}$$

The Rosati involution $h \mapsto h^*$ and $T(n) \mapsto T^*(n)$ (with respect to the canonical divisor on J_r) brings $h_r(\mathbb{Z})$ to $h_r^*(\mathbb{Z}) \subset \mathrm{End}(J_{r/\mathbb{Q}})$. Define A_s^* to be the identity connected component of $J_s[\alpha^*]$. The condition (A) is equivalent to

(B) The abelian quotient map $J_s \twoheadrightarrow B_s = \mathrm{Coker}(\alpha_s)$ dual to $A_s^* \subset J_s$ induces an $U(p)$-isomorphism of Tate modules: $T_p(J_s/\alpha_s(J_s)) \to T_pB_s$ and α_s induces an automorphism of the $\mathbb{Q}_p$-vector space $T_p\alpha_s(J_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Again if A_r is Shimura's abelian subvariety of J_r associated to a Hecke eigenform f, we sometimes write $B_{f,s}$ for B_s as above. The condition (A) (and hence (B)) is a mild condition. Here are sufficient conditions for (α, A_s, B_s) to satisfy (A) (and (B)):

PROPOSITION 5.1. *Let* $\mathrm{Spec}(\mathbb{T})$ *be a connected component of* $\mathrm{Spec}(\mathbf{h})$ *and* $\mathrm{Spec}(\mathbb{I})$ *be an irreducible component of* $\mathrm{Spec}(\mathbb{T})$*. Then the condition* (A) *holds for the following choices of* (α, A_s, B_s)*:*

(P1) *Fix $r > 0$. Then $\alpha_s = \alpha$ for a factor $\alpha|\gamma^{p^{r-1}} - 1$ in Λ, $A_s = J_s[\alpha]^\circ$ (the identity connected component) and $B_s = \mathrm{Pic}^0_{A_s/\mathbb{Q}}$ for all $s \geq r$.*

(P2) *Suppose that an eigen cusp form $f = f_P$ new at each prime $l|N$ belongs to $\mathrm{Spec}(\mathbb{T})$ and that $\mathbb{T} = \mathbb{I}$ is regular (or more generally a unique factorization domain). Then writing the level of f_P as Np^r, the algebra homomorphism $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ given by $f|T(l) = \lambda(T(l))f$ gives rise to the prime ideal $P = \mathrm{Ker}(\lambda)$. Since P is of height 1, it is principal generated by $\varpi \in \mathbb{T}$. This ϖ has its image $\varpi_s \in \mathbb{T}_s = \mathbb{T} \otimes_\Lambda \Lambda_s$ for $\Lambda_s = \Lambda/(\gamma^{p^{s-1}} - 1)$. Since $\mathbf{h}_s = \mathbf{h} \otimes_\Lambda \Lambda_s = \mathbb{T}_s \oplus X_s$ as an algebra direct sum, $\mathrm{End}(J_{s/\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \supset h_s(\mathbb{Z}_p) = \mathbb{T}_s \oplus Y_s$ with Y_s projecting down onto X_s. Then, we can approximate $a_s = \varpi_s \oplus 1_s \in h_s(\mathbb{Z}_p)$ for the identity 1_s of Y_s by $\alpha_s \in h_s(\mathbb{Z})$ so that $\alpha_s h_s(\mathbb{Z}_p) = a_s h_s(\mathbb{Z}_p)$ (hereafter we call α_s "sufficiently close" to a_s if $\alpha_s h_s(\mathbb{Z}_p) = a_s h_s(\mathbb{Z}_p)$). For this choice of α_s, $A_s := A_{f,s}$ and $B_s := B_{f,s}$.*

(P3) *More generally than (P2), we pick a general connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$. Pick a (classical) Hecke eigenform $f = f_P$ (of weight 2) for $P \in \mathrm{Spec}(\mathbb{T})$. Assume that $\mathbf{h}_s$ (for every $s \geq r$) is reduced and $P = (\varpi)$ for $\varpi \in \mathbb{T}$, and write ϖ_s for the image of ϖ in $h_s(\mathbb{Z}_p)$. Take the complementary direct summand Y_s of $\mathbb{T}_s$ in $h_s(\mathbb{Z}_p)$ and approximate $a_s := \varpi_s \oplus 1_s$ in $h_s(\mathbb{Z}_p)$ to get α_s sufficiently close to a_s. Then for this choice of α_s, $A_s := A_{f,s}$ and $B_s := B_{f,s}$.*

(P4) *Suppose that $\mathbb{T}/(\varpi)$ for a non-zero divisor $\varpi \in \mathbb{T}$ is a reduced algebra of characteristic 0 factoring through $\mathbf{h}_r := \mathbf{h}/(\gamma^{p^{r-1}} - 1)\mathbf{h}$ for some $r > 0$. Assume that $\mathbb{T}_s$ is reduced for every $s \geq r$, and write ϖ_s for the image of ϖ in $\mathbb{T}_s$. Then approximating $a_s = \varpi_s \oplus 1_s$ by $\alpha_s \in h_s(\mathbb{Z})$ sufficiently closely for each $s \geq r$, we define A_s to be the connected component of $J_s[\alpha_s]$ and B_s to be its dual quotient.*

Proof. We first prove (P4). Since α_s is sufficiently close to a_s, we have the identity $\alpha_s h_s(\mathbb{Z}_p) = a_s h_s(\mathbb{Z}_p)$ of ideals. By reducedness of $\mathbb{T}_s$, we have an algebra product decomposition: $h_s(\mathbb{Q}_p) := h_s(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \alpha_s(\mathbb{T}_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \times Z_s$ for the complementary $\mathbb{Q}_p$-subalgebra Z_s, which is given by $(\mathbb{T}_s/(\varpi_s)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Write the idempotent of Z_s as $\epsilon_s \in Z_s$. Then $\epsilon_s + a_s$ is invertible in $h_s(\mathbb{Q}_p)$. For some positive integer M_s, $\beta_s := U(p)^{M_s}\epsilon_s \in h_s(\mathbb{Z}) \subset \mathrm{End}(J_s)$. Then by $\epsilon_s + a_s \in h_s(\mathbb{Q}_p)^\times$, the connected component A_s of $J_s[\alpha_s]$ is given by $\beta_s(J_s)$, $J_s = \beta_s(J_s) + \alpha_s(J_s) = A_s + \alpha(J_s)$, and the inclusion map $A_s \hookrightarrow J_s[\alpha_s]$ is an $U(p)$-isomorphism. Since α_s is invertible in $\alpha_s(h_s(\mathbb{Q}_p))$, α_s induces a self-isogeny of $\alpha(J_s)$. Thus the triple satisfies (A). Since $\varpi_{s'} h_{s'}(\mathbb{Z}_p)$ surjects down to $\varpi_s h_s(\mathbb{Z}_p)$ for all $s' \geq s$, we can adjust α_s inductively to have a projective system $\{\alpha_s \in \mathrm{End}(J_s)\}_{s \geq r}$. Thus $\alpha = \varprojlim_s \alpha_s \in \mathrm{End}(J_\infty)$ does the job. This proves (P4). The assertions (P2) and (P3) are direct consequences of (P4).
As for (P1), since $\alpha|(\gamma^{p^{r-1}} - 1)|(\gamma^{p^{s-1}} - 1)$ in the unique factorization domain Λ, factoring $\gamma^{p^s} - 1 = \alpha_s\beta_s$, the ideals (α_s) and (β_s) are co-prime in the unique-factorization domain Λ. From this, we have $J_s = \beta_s(J_s) + \alpha(J_s) =$

$A_s + \alpha(J_s)$, and $\alpha|_{\alpha(J_s)}$ is a self isogeny of $\alpha(J_s)$ as $\alpha|_{\alpha(J_s)}$ is a non-zero-divisor in $\mathrm{End}(\alpha(J_s))$. $\square$

Remark 5.2. (i) Under (P2), all arithmetic points P of weight 2 in $\mathrm{Spec}(\mathbb{I})$ satisfies (A).
(ii) For a given weight 2 Hecke eigenform f, for density 1 primes $\mathfrak{p}$ of $\mathbb{Q}(f)$, f is ordinary at $\mathfrak{p}$ (i.e., $a(p,f) \not\equiv 0 \mod \mathfrak{p}$; see [H13b, §7]). Except for finitely many primes $\mathfrak{p}$ as above, f belongs to a connected component $\mathbb{T}$ which is regular (see [F02, §3.1]); so, (P2) is satisfied for such $\mathbb{T}$.
(iii) If N is square-free (as assumed for simplicity in the introduction), $\mathbf{h}_s$ is reduced [H13a, Corollary 1.3]; so, if an arithmetic prime $P \in \mathrm{Spec}(\mathbf{h}_r)$ is principal, α_s as in (P3) satisfies (A).

If $A_r = A_{f,r}$ is Shimura's abelian subvariety associated to a primitive form f as in [IAT, Theorem 7.14], its dual quotient $J_r \twoheadrightarrow B_r = B_{f,r}$ is also associated to f in the sense of [Sh73]. However, if A_r is not associated to a new form, the dual quotient may not be associated to the Hecke eigen form f. To clarify this point, we introduce an involution of J_s. We fix a generator ζ of the $\mathbb{Z}_p$-module $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}(\overline{\mathbb{Q}})$; so, ζ is a coherent sequence of generators ζ_{p^n} of $\mu_{p^n}(\overline{\mathbb{Q}})$ (i.e., $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n > 0$). We also fix a generator ζ_N of $\mu_N(\overline{\mathbb{Q}})$, and put $\zeta_{Np^r} := \zeta_N \zeta_{p^r}$. Identify the étale group scheme $\mathbb{Z}/Np^n\mathbb{Z}_{/\mathbb{Q}[\zeta_N,\zeta_{p^n}]}$ with μ_{Np^n} by sending $m \in \mathbb{Z}$ to $\zeta_{Np^n}^m$. Then for a couple $(E, \phi_{Np^r} : \mu_{Np^r} \hookrightarrow E)_{/K}$ for a $\mathbb{Q}[\mu_{Np^r}]$-algebra K, let $\phi^* : E[Np^r] \twoheadrightarrow \mathbb{Z}/Np^r\mathbb{Z}$ be the Cartier dual of ϕ_{Np^r}. Then ϕ^* induces $E[Np^r]/\mathrm{Im}(\phi_{Np^r}) \cong \mathbb{Z}/Np^r\mathbb{Z}$. Define $i : \mathbb{Z}/p^r\mathbb{Z} \cong (E/\mathrm{Im}(\phi_{Np^r}))[Np^r]$ by the inverse of ϕ^*. Then we define $\varphi_{Np^r} : \mu_{Np^r} \hookrightarrow E/\mathrm{Im}(\phi_{Np^r})$ by $\varphi_{Np^r} : \mu_{Np^r} \cong \mathbb{Z}/Np^r\mathbb{Z} \xrightarrow{i} (E/\mathrm{Im}(\phi_{Np^r}])[p^r] \subset E/\mathrm{Im}(\phi_{Np^r})$. This induces an involution w_r of X_r defined over $\mathbb{Q}[\mu_{Np^r}]$, which in turn induces an automorphism w_r of $J_{r/\mathbb{Q}[\zeta_{Np^r}]}$.
Let $P \in \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ be an arithmetic point of weight 2. Then we have a p-stabilized Hecke eigenform form f_P associated to P; i.e., $f_P|T(n) = P(T(n))f_P$ for all n. Suppose $f = f_P$ and write $A_{f,r} = A_P$. Then $f_P^* = w_r(f_P)$ is the dual common eigenform of $T^*(n)$. If f_P is new at every prime $l|Np$, f_P^* is a constant multiple of the complex conjugate f_P^c of f_P (but otherwise, it could be different). Then the abelian quotient associated to f_P^* is the dual abelian variety of A_P. Thus if f_P^* is not constant multiple of f_P^c, $B_{f,r}$ is not assocaited to f_P^* (see a remark at the end of [H14b, §6] for more details of this fact).
Pick an automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{Np^r})/\mathbb{Q})$ with $\zeta_{Np^r}^\sigma = \zeta_{Np^r}^z$ for $z \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Since w_r^σ is defined with respect to $\zeta_{Np^r}^\sigma = \zeta_{Np^r}^z$, we find $w_r^\sigma = \langle z \rangle \circ w_r$. By this formula, if $x \in A_P(\overline{\mathbb{Q}})$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\zeta^\sigma = \zeta^z$ for $z \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times = \varprojlim_s (\mathbb{Z}/Np^s\mathbb{Z})^\times$, we have $w_r(x)^\sigma = \langle z \rangle (w_r(x^\sigma))$. Thus $w_r^\sigma = \langle z \rangle \circ w_r = w_r \circ \langle z^{-1} \rangle$ (see [MW86, page 237] and [MW84, 2.5.6]). Let $\pi_{s,r,*} : J_s \to J_r$ for $s > r$ be the morphism induced by the covering map $X_s \twoheadrightarrow X_r$ through Albanese functoriality. Then we define $\pi_s^r = w_r \circ \pi_{s,r,*} \circ w_s$. Then $(\pi_s^r)^\sigma = w_r \langle z^{-1} \rangle \pi_{s,r,*} \langle z \rangle w_s = \pi_s^r$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{Np^s})/\mathbb{Q})$; thus, π_s^r is well defined over $\mathbb{Q}$, and satisfies $T(n) \circ \pi_s^r = \pi_s^r \circ T(n)$ for all n prime to

Np and $U(q) \circ \pi_s^r = \pi_s^r \circ U(q)$ for all $q|Np$ (as $w_? \circ h \circ w_? = h^*$ for $h \in h_?(\mathbb{Z})$ $(? = s, r)$ by [MFM, Section 4.6]. Since $w_r^2 = 1$, $\{J_s, \pi_s^r\}_{s>r}$ form a Hecke equivariant projective system of abelian varieties defined over $\mathbb{Q}$. We then define as described in (S) just above Lemma 2.1 an fppf abelian sheaf $\widehat{X}$ for any abelian variety quotient or subvariety X of $J_{s/k}$ over the fppf site over $k = \mathbb{Q}$ and $\mathbb{Q}_l$ (note here the definition of $\widehat{X}$ depends on k).

In general, for A_s in (A), we have $A_s^* = w_s(A_s) \subset J_s$ because $T(n) \circ w_s = w_s \circ T^*(n)$ for all n (see [MFM, Theorem 4.5.5]). Thus (B_s, π_s^r) in (B) gives rise to a natural projective system of abelian variety quotients of J_s.

6. Structure of ind-Λ-MW groups over number fields and local field

We return to the setting of Section 2; so, K/k is the infinite Galois extension defined there. In this section, unless otherwise mentioned, we often let κ denote an intermediate finite extension of k inside K (although the results in this section are valid for κ satisfying (2.2) unless otherwise mentioned).

We assume (A) in Section 5 for (α_s, A_s, B_s). By (A), the inclusion $A_s[p^\infty] \hookrightarrow J_s[\alpha_s][p^\infty]$ is a $U(p)$-isomorphism; so, we have the identity of the ordinary parts: $\widehat{A}_s^{\mathrm{ord}} = \widehat{J}_s^{\mathrm{ord}}[\alpha_s]$. From the exact sequence

$$0 \to J_s[\alpha_s] \to J_s \xrightarrow{\alpha_s} J_s \to B_s \to 0,$$

we get the following exact sequence of sheaves:

$$0 \to \widehat{A}_s^{\mathrm{ord}} \to \widehat{J}_s^{\mathrm{ord}} \xrightarrow{\alpha_s} \widehat{J}_s^{\mathrm{ord}} \to \widehat{B}_s^{\mathrm{ord}} \to 0. \tag{6.1}$$

This is because tensoring $\mathbb{Z}_p$ (or taking the p-primary part $X/X^{(p)}$ as in (S)) is an exact functor. Since taking injective limit is an exact functor, writing $X_\infty^{\mathrm{ord}} = \varinjlim_s \widehat{X}_s^{\mathrm{ord}}$, we get the following exact sequence of sheaves:

$$0 \to A_\infty^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \xrightarrow{\alpha} J_\infty^{\mathrm{ord}} \to B_\infty^{\mathrm{ord}} \to 0. \tag{6.2}$$

First, we shall describe A_∞^{ord} and B_∞^{ord} in terms of $\widehat{A}_r$ and $\widehat{B}_r$. The Picard functoriality induces a morphism $\pi_{r,s}^* : J_r \to J_s$. This gives a Hecke equivariant inductive system $\{J_s, \pi_{r,s}^*\}_{s>r}$ of abelian varieties defined over $\mathbb{Q}$. Since the two morphisms $J_r \to J_s^r$ and $J_s^r \to J_s[\gamma^{p^{r-1}} - 1]$ (Picard functoriality) are $U(p)$-isomorphisms of fppf abelian sheaves by (u1) and Corollary 3.5 (see also Remark 3.7), we get the following two isomorphisms of fppf abelian sheaves:

$$A_r[p^\infty]^{\mathrm{ord}} \xrightarrow[\pi_{r,s}^*]{\sim} A_s[p^\infty]^{\mathrm{ord}} \text{ and } \widehat{A}_r^{\mathrm{ord}} \xrightarrow[\pi_{r,s}^*]{\sim} \widehat{A}_s^{\mathrm{ord}}, \tag{6.3}$$

since $\widehat{A}_s^{\mathrm{ord}}$ is the isomorphic image of $\widehat{A}_r^{\mathrm{ord}} \subset \widehat{J}_r$ in $\widehat{J}_s[\gamma^{p^{r-1}} - 1]$. Since $w_r \circ T(n) = T^*(n) \circ w_r$ (by [MFM, Theorem 4.5.5]), twisting Cartier duality pairing $[\cdot, \cdot] : J_r[p^r] \times J_r[p^r] \to \mu_{p^r}$ coming from the canonical polarization, we get a perfect pairing $(\cdot, \cdot) : J_r[p^r] \times J_r[p^r] \to \mu_{p^r}$ with $(x|T(n), y) = (x, y|T(n))$

(e.g., [H14a, Section 4]). By this w-twisted Cartier duality applied to the first identity of (6.3), we have

$$B_s[p^\infty]^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} B_r[p^\infty]^{\mathrm{ord}}. \tag{6.4}$$

Thus, by Kummer sequence, we have the following commutative diagram

$$\begin{array}{ccc} \widehat{B}_s^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} = (B_s(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z})^{\mathrm{ord}} & \xrightarrow[\hookrightarrow]{} & H^1(B_s[p^m]^{\mathrm{ord}}) \\ {\scriptstyle \pi_s^r}\downarrow & & \wr\downarrow{\scriptstyle (6.4)} \\ \widehat{B}_r^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} = (B_r(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z})^{\mathrm{ord}} & \xrightarrow[\hookrightarrow]{} & H^1(B_r[p^m]^{\mathrm{ord}}) \end{array}$$

This shows

$$\widehat{B}_s^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z} \cong \widehat{B}_r^{\mathrm{ord}}(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z}.$$

Passing to the limit, we get

$$\widehat{B}_s^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} \widehat{B}_r^{\mathrm{ord}} \text{ and } (B_s\otimes_{\mathbb{Z}}\mathbb{T}_p)^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} (B_r\otimes_{\mathbb{Z}}\mathbb{T}_p)^{\mathrm{ord}} \tag{6.5}$$

as fppf abelian sheaves. As long as κ is either a field extension of finite type of a number field or a finite extension of $\mathbb{Q}_l$ ($l \neq p$) or a finite algebraic extension of $\mathbb{Q}_p$, the projective limit of $B_?(\kappa)\otimes\mathbb{Z}/p^m\mathbb{Z}$ (with respect to m) is equal to $\widehat{B}_?$ (by Lemma 2.1). In short, we get

Lemma 6.1. *Assume κ to be given either by a field extension of finite type of k if k is a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_l$ ($l \neq p$) or by a finite algebraic extension of k if $[k:\mathbb{Q}_p] < \infty$. Then we have the following isomorphism*

$$\widehat{A}_r(\kappa)^{\mathrm{ord}} \xrightarrow[\pi_{s,r}^*]{\sim} \widehat{A}_s(\kappa)^{\mathrm{ord}} \quad \text{and} \quad \widehat{B}_s(\kappa)^{\mathrm{ord}} \xrightarrow[\pi_s^r]{\sim} \widehat{B}_r(\kappa)^{\mathrm{ord}}$$

for all $s > r$ including $s = \infty$.

By computation, we get $\pi_s^r \circ \pi_{r,s}^* = p^{s-r}U(p^{s-r})$. To see this, as Hecke operators, $\pi_{r,s}^* = [\Gamma_s^r]$, $\pi_{r,s,*} = [\Gamma_r]$. Thus we have

$$\begin{aligned} \pi_s^r \circ \pi_{r,s}^* &= [\Gamma_s^r]\cdot w_s\cdot[\Gamma_r]\cdot w_r = [\Gamma_s]\cdot[w_s w_r]\cdot[\Gamma_r] \\ &= [\Gamma_s^r:\Gamma_s][\Gamma_s^r \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r}\end{pmatrix} \Gamma_r] = p^{s-r}U(p^{s-r}). \end{aligned} \tag{6.6}$$

Then we have the commutative diagram of fppf abelian sheaves for $s' > s$

$$\begin{array}{ccc} \widehat{A}_{s'}^{\mathrm{ord}} & \xleftarrow[\pi_{r,s'}^*]{\sim} & \widehat{A}_r^{\mathrm{ord}} \\ {\scriptstyle \pi_{s'}^s}\downarrow & & \downarrow{\scriptstyle p^{s'-s}U(p)^{s'-s}} \\ \widehat{A}_s^{\mathrm{ord}} & \xleftarrow[\pi_{r,s}^*]{\sim} & \widehat{A}_r^{\mathrm{ord}}. \end{array} \tag{6.7}$$

Note that A_s and B_s are mutually (w-twisted) dual as abelian varieties (see Section 5), and the w-twisted duality is compatible with Hecke operators. Thus $B_s[p^n]$ is the w-twisted Cartier dual of $A_s[p^n]$. The w-twisted Cartier duality pairing in [H14a, Section 4] satisfies $(x|X, y) = (x, y|X)$ for $X = T(n), U(q)$, and π^s_r and $\pi^*_{r,s}$ are adjoint each other under this duality. Then we have the dual commutative diagram of fppf abelian sheaves:

$$\begin{array}{ccc} \widehat{B}^{\mathrm{ord}}_{s'} & \xrightarrow[\pi^r_{s'}]{\sim} & \widehat{B}^{\mathrm{ord}}_{r} \\ {\scriptstyle \pi^*_{s,s'}}\uparrow & & \uparrow{\scriptstyle p^{s'-s}U(p)^{s'-s}} \\ \widehat{B}^{\mathrm{ord}}_{s} & \xrightarrow[\pi^r_{s}]{\sim} & \widehat{B}^{\mathrm{ord}}_{r}. \end{array} \tag{6.8}$$

By (6.7) and (6.8), we have the following four exact sequences of fppf abelian sheaves:

$$\begin{aligned} 0 \to & A_s[p^{s-r}]^{\mathrm{ord}} \to A_s[p^\infty]^{\mathrm{ord}} \xrightarrow{\pi^r_s} A_r[p^\infty]^{\mathrm{ord}} \to 0, \\ 0 \to & B_r[p^{s-r}]^{\mathrm{ord}} \to B_r[p^\infty]^{\mathrm{ord}} \xrightarrow{\pi^*_{r,s}} B_s[p^\infty]^{\mathrm{ord}} \to 0 \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} 0 \to & A_s[p^{s-r}]^{\mathrm{ord}} \to \widehat{A}^{\mathrm{ord}}_s \xrightarrow{\pi^r_s} \widehat{A}^{\mathrm{ord}}_r \to 0, \\ 0 \to & B_r[p^{s-r}]^{\mathrm{ord}} \to \widehat{B}^{\mathrm{ord}}_r \xrightarrow{\pi^*_{r,s}} \widehat{B}^{\mathrm{ord}}_s \to 0. \end{aligned} \tag{6.10}$$

LEMMA 6.2. *Let the notation and assumtions be as in* Lemma 6.1. *Then we have a canonical isomorphism*

$$\varinjlim_{s,\pi^*_{r,s}} \widehat{B}^{\mathrm{ord}}_s(\kappa) \cong \varinjlim_{s,p^{s-r}U(p)^{s-r}} \widehat{B}^{\mathrm{ord}}_r(\kappa) \cong \widehat{B}_r(\kappa)^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proof. Identifying the left and the right column of (6.8), we have the cohomology exact sequence of the second exact sequence of (6.10):

$$0 \to B_r[p^{s-r}]^{\mathrm{ord}}(\kappa) \xrightarrow{\pi^s_r} \widehat{B}^{\mathrm{ord}}_r(\kappa) \xrightarrow{\pi^*_{r,s}} \widehat{B}^{\mathrm{ord}}_s(\kappa) \to H^1_?(B_r[p^{s-r}]^{\mathrm{ord}}). \tag{6.11}$$

Passing to the inductive limit of $\{B_r[p^{s-r}]^{\mathrm{ord}}, p^{s-r}U(p)^{s-r}\}_s$, $\{\widehat{B}_r(\kappa)^{\mathrm{ord}}, p^{s-r}U(p)^{s-r}\}_s$ and $\{\widehat{B}_s, \pi^*_{r,s}\}_s$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \varinjlim_s \widehat{B}^{\mathrm{ord}}_r(\kappa) & \to & \varinjlim_s \widehat{B}^{\mathrm{ord}}_s(\kappa) & \to & \varinjlim_s H^1_?(B_r[p^{s-r}]^{\mathrm{ord}}) \\ \| & & \| & & \wr\downarrow \\ \varinjlim_s \widehat{B}^{\mathrm{ord}}_r(\kappa) & \to & \varinjlim_s \widehat{B}^{\mathrm{ord}}_s(\kappa) & \to & H^1_?(\varinjlim_s B_r[p^{s-r}]^{\mathrm{ord}}). \end{array} \tag{6.12}$$

Here the last isomorphism comes from the commutativity of injective limit and cohomology.

For a free $\mathbb{Z}_p$-module F of finite rank, we suppose to have a commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{p^n} & F \\ \|\downarrow & & \downarrow{\scriptstyle p^{-n}} \\ F & \xrightarrow[\hookrightarrow]{} & p^{-n}F. \end{array}$$

Thus we have $\varinjlim_{n,x\mapsto p^n x} F = \varinjlim_{n,x\mapsto p^{-n}x} p^{-n}F \cong F\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. If T is a torsion $\mathbb{Z}_p$-module with $p^B T = 0$ for $B \gg 0$, we have $\varinjlim_{n,x\mapsto p^n x} T = 0$. Thus for general $M = F\oplus T$, we have $\varinjlim_{n,x\mapsto p^n x} M \cong M\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Applying this consideration to $M = \widehat{B}_r(\kappa)$, we get

$$\varinjlim_{s,x\mapsto p^s U(p)^s x} \widehat{B}_r(\kappa) \cong \widehat{B}_r(\kappa)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

Similarly, $\varinjlim_{n,x\mapsto p^n U(p)^n x} B_r[p^n](\kappa) = \varinjlim_{n,x\mapsto p^n U(p)^n x} B_r[p^n](K) = 0$. Thus from the above diagram (6.12), we conclude the lemma. $\square$

Consider the composite morphism $\varpi_s : A_s \hookrightarrow J_s \twoheadrightarrow B_s$ of fppf abelian sheaves. Since $B_s = J_s/\alpha_s(J_s)$ and $J_s = A_s + \alpha_s(J_s)$ with finite intersection $J_s = A_s\times_{J_s}\alpha_s(J_s)$, we have a commutative diagram with exact rows in the category of fppf abelian sheaves:

$$(6.13)\qquad \begin{tikzcd} \alpha(J_s) \arrow[r, "\hookrightarrow"] & J_s \arrow[r, "\twoheadrightarrow", "\rho_s"'] & B_s \\ 0\to\alpha(J_s)\times_{J_s}A_s \arrow[u, "\cup"] \arrow[r, "\hookrightarrow"'] & A_s \arrow[u, "\cup"] \arrow[r, "\varpi_s", "\twoheadrightarrow"'] & B_s. \arrow[u, "\|"'] \end{tikzcd}$$

We have this diagram over $R_s := \mathbb{Z}_{(p)}[\mu_{p^s}]$ (not just over $\mathbb{Q}$) by taking the connected components of the Néron models of J_s, A_s and B_s. The intersection $\alpha(J_s)\times_{J_s}A_s = \mathrm{Ker}(\varpi_s)$ is an étale finite group scheme over $\mathbb{Q}$. These abelian varieties are known to have semi-stable reduction over R_s by the good reduction theorem of Carayol–Langlands. If the character $\mathbb{Z}_p^\times \ni z\mapsto\langle z\rangle\in\mathrm{End}(A_s)^\times$ is non-trivial, we may replace J_s by its complement $J_s^{(0)}$ of the image of J_s^0 in J_s. Under this circumstance, $\alpha(J_s)\times_{J_s}A_s = \mathrm{Ker}(\varpi_s)$ is a finite flat group scheme over R_s. Since A_s and B_s has good reduction over R_r, $\mathrm{Ker}(\varpi_s)$ is a finite flat group scheme defined over R_r. We consider the exact sequence

$$0\to\mathrm{Ker}(\varpi_s)\to A_s\xrightarrow{\varpi_s} B_s\to 0.$$

which is an exact sequence of fppf abelian sheaves over R_r (and smooth abelian sheaves over $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]$). From this, writing C_s for the p-primary part of $\mathrm{Ker}(\varpi_s)$, we have an exact sequence of fppf abelian sheaves over R_r (and smooth abelian sheaves over $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]$):

$$0\to C_s\to\widehat{A}_s\to\widehat{B}_s\to 0.$$

We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
A_s[p^{s-r}]^{\mathrm{ord}} \arrow[r, "\sim"] \arrow[d] & A_s[p^{s-r}]^{\mathrm{ord}} \arrow[r, two heads] \arrow[d] & 0 \arrow[d] \\
C_s^{\mathrm{ord}} \arrow[r, hook] \arrow[d] & \widehat{A}_s^{\mathrm{ord}} \arrow[r, two heads] \arrow[d] & \widehat{B}_s^{\mathrm{ord}} \arrow[d, "\wr"'] \\
C_r^{\mathrm{ord}} \arrow[r, hook] & \widehat{A}_r^{\mathrm{ord}} \arrow[r, two heads] & \widehat{B}_r^{\mathrm{ord}}.
\end{tikzcd}$$

By the snake lemma applied to the right two exact columns of the above diagram, we get the following exact sequence:

$$0 \to A_r[p^{s-r}]^{\mathrm{ord}} \to C_s^{\mathrm{ord}} \to C_r^{\mathrm{ord}} \to 0 \tag{6.14}$$

with $C_s^{\mathrm{ord}} \hookrightarrow A_s[p^\infty]^{\mathrm{ord}} \underset{\sim}{\xleftarrow{\pi_{r,s}^*}} A_r[p^\infty]^{\mathrm{ord}}$.

PROPOSITION 6.3. *We have the following exact sequence under the ?-topology over k, where ? = sm, étale, nothing and fppf:*

$$0 \to \widehat{A}_r^{\mathrm{ord}} \to J_\infty^{\mathrm{ord}} \xrightarrow{\alpha} J_\infty^{\mathrm{ord}} \xrightarrow{\rho_\infty} \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0 \tag{6.15}$$

with $\widehat{A}_r^{\mathrm{ord}}/\widehat{A}_r^{\mathrm{ord}}[p^\infty] \cong \widehat{B}_r^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. By (6.13), C_s^{ord} is equal to $\widehat{A}_s^{\mathrm{ord}} \cap \widehat{\alpha}(\widehat{J}_s^{\mathrm{ord}})$. Since A_s is the connected component of $J_s[\alpha]$ with $U(p)$-isomorphism $A_s \hookrightarrow J_s[\alpha_s]$, we have $C_s^{\mathrm{ord}} = \alpha(\widehat{J}_s^{\mathrm{ord}})[\alpha]$. Since α is an isogeny on $\alpha(J_s)$, we have an exact sequence of sheaves indexed by s under ?-topology

$$0 \to C_s^{\mathrm{ord}} \to \alpha(\widehat{J}_s^{\mathrm{ord}}) \xrightarrow{\alpha_s} \alpha(\widehat{J}_s^{\mathrm{ord}}) \to 0.$$

Passing to the inductive limit of these exact sequences (and noting $\varinjlim_s C_s^{\mathrm{ord}} = A_r[p^\infty]^{\mathrm{ord}}$ by (6.14)), we get another exact sequences:

$$0 \to \widehat{A}_r^{\mathrm{ord}}[p^\infty] \to \alpha(J_\infty^{\mathrm{ord}}) \xrightarrow{\alpha} \alpha(J_\infty^{\mathrm{ord}}) \to 0.$$

Therefore by (6.14), we get the following exact sequences (indexed by s) of sheaves under ?-topology:

$$0 \to C_s^{\mathrm{ord}} \to (\widehat{A}_s^{\mathrm{ord}} \times \alpha(\widehat{J}_s^{\mathrm{ord}})) \to \widehat{J}_s^{\mathrm{ord}} \to 0. \tag{6.16}$$

Passing again to the inductive limit of these exact sequences (and noting $\widehat{A}_r^{\mathrm{ord}} \cong \widehat{A}_s^{\mathrm{ord}}$ by $\pi_{r,s}^*$ and $\varinjlim_s C_s^{\mathrm{ord}} = A_r[p^\infty]^{\mathrm{ord}}$), we get the top and the bottom exact sequences of the following commutative diagram:

$$\begin{tikzcd}
\widehat{A}_r^{\mathrm{ord}}[p^\infty] \arrow[r, hook] \arrow[d, "0"'] & (\widehat{A}_r^{\mathrm{ord}} \times \alpha(J_\infty^{\mathrm{ord}})) \arrow[r, two heads] \arrow[d, "\alpha"'] & J_\infty^{\mathrm{ord}} \arrow[d, "\alpha"'] \\
\widehat{A}_r^{\mathrm{ord}}[p^\infty] \arrow[r, hook] & (\widehat{A}_r^{\mathrm{ord}} \times \alpha(J_\infty^{\mathrm{ord}})) \arrow[r, two heads] & J_\infty^{\mathrm{ord}}.
\end{tikzcd}$$

Applying the snake lemma (noting that the connection map is the zero map), we get

$$\mathrm{Coker}(J^{\mathrm{ord}}_\infty \xrightarrow{\alpha} J^{\mathrm{ord}}_\infty) = \widehat{A}^{\mathrm{ord}}_r/\widehat{A}^{\mathrm{ord}}_r[p^\infty].$$

Thus we have the following exact sequence of sheaves:

$$0 \to \widehat{A}^{\mathrm{ord}}_r \to J^{\mathrm{ord}}_\infty \to J^{\mathrm{ord}}_\infty \to \widehat{A}^{\mathrm{ord}}_r/\widehat{A}^{\mathrm{ord}}_r[p^\infty] \to 0. \tag{6.17}$$

There is another way to see (6.17). Passing to the inductive limit of the exact sequences of sheaves

$$0 \to \widehat{A}^{\mathrm{ord}}_s \to \widehat{J}^{\mathrm{ord}}_s \xrightarrow{\alpha_s} \widehat{J}^{\mathrm{ord}}_s \xrightarrow{\rho_s} \widehat{B}^{\mathrm{ord}}_s \to 0,$$

we get the following exact sequence of sheaves:

$$0 \to \widehat{A}^{\mathrm{ord}}_r \to J^{\mathrm{ord}}_\infty \xrightarrow{\alpha} J^{\mathrm{ord}}_\infty \xrightarrow{\rho_\infty} \varinjlim_{s, x\mapsto p^{s-r}U(p)^{s-r}} \widehat{B}^{\mathrm{ord}}_r \to 0$$

as $\widehat{A}^{\mathrm{ord}}_r \cong \widehat{A}^{\mathrm{ord}}_s$ by $\pi^*_{r,s}$. This combined with (6.17) and Lemma 6.2 proves the exact sequence in (6.15). By (6.16), we have $\widehat{A}^{\mathrm{ord}}_s \cap \alpha(\widehat{J}^{\mathrm{ord}}_s) \cong C^{\mathrm{ord}}_s$; thus $\mathrm{Ker}(\widehat{A}^{\mathrm{ord}}_s \to \widehat{B}^{\mathrm{ord}}_s) \cong C^{\mathrm{ord}}_s$ with $\varinjlim_s C^{\mathrm{ord}}_s = A_r[p^\infty]^{\mathrm{ord}}$, passing to the inductive limit we again get the identity of sheaves:

$$\varinjlim_{s, x\mapsto p^{s-r}U(p)^{s-r}} \widehat{B}^{\mathrm{ord}}_r \cong \widehat{A}^{\mathrm{ord}}_r/\widehat{A}_r[p^\infty]^{\mathrm{ord}} \cong \widehat{B}^{\mathrm{ord}}_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

This finishes the proof. □

We have two exact sequences of sheaves:

$$\begin{aligned} &0 \to \widehat{A}^{\mathrm{ord}}_r \to J^{\mathrm{ord}}_\infty \xrightarrow{\alpha} \alpha(J^{\mathrm{ord}}_\infty) \to 0, \\ &0 \to \alpha(J^{\mathrm{ord}}_\infty) \to J^{\mathrm{ord}}_\infty \xrightarrow{\rho_\infty} \widehat{B}^{\mathrm{ord}}_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0. \end{aligned} \tag{6.18}$$

These leave us to study the two error terms

$$E_1(\kappa) := \alpha(J^{\mathrm{ord}}_\infty)(\kappa)/\alpha(J^{\mathrm{ord}}_\infty(\kappa)) \quad \text{and} \quad E_2(\kappa) := \widehat{B}^{\mathrm{ord}}_r(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\rho_\infty(J^{\mathrm{ord}}_\infty(\kappa)).$$

Let $E^s_1(\kappa) := \alpha(J^{\mathrm{ord}}_s)(\kappa)/\alpha(J^{\mathrm{ord}}_s(\kappa))$ and $E^s_2(\kappa) := \widehat{B}^{\mathrm{ord}}_s(\kappa)/\rho_s(\widehat{J}^{\mathrm{ord}}_s(\kappa)) = \mathrm{Coker}(\rho_s)$ for $\rho_s : J^{\mathrm{ord}}_s(\kappa) \to \widehat{B}^{\mathrm{ord}}_s(\kappa)$. Note that

$$\begin{aligned} &E^s_1(\kappa)(\hookrightarrow H^1_?(\widehat{A}^{\mathrm{ord}}_r) = H^1_?(A^{\mathrm{ord}}_r) \otimes_{\mathbb{Z}} \mathbb{Z}_p) \\ &\qquad\qquad \text{and} \quad E^s_2(\kappa) = B^{\mathrm{ord}}_s(\kappa)/\rho_s(\widehat{J}^{\mathrm{ord}}_s(\kappa))(\hookrightarrow H^1_?(\alpha(\widehat{J}^{\mathrm{ord}}_s))[\alpha]) \end{aligned}$$

are p-torsion finite modules as long as s is finite. Note that $\alpha|_{\alpha(J_s)}$ is a self isogeny; so,

$$0 \to \alpha(J_s)[\alpha]^{\mathrm{ord}} \to \alpha(\widehat{J}^{\mathrm{ord}}_s) \xrightarrow{\alpha_s} \alpha(\widehat{J}^{\mathrm{ord}}_s) \to 0$$

is an exact sequence of sheaves. Since $\alpha(J_s)[\alpha]^{\mathrm{ord}} = C^{\mathrm{ord}}_s$, we have another exact sequence:

$$0 \to \alpha(\widehat{J}^{\mathrm{ord}}_s)(\kappa)/\alpha(\alpha(\widehat{J}^{\mathrm{ord}}_s)(\kappa)) \to H^1_?(\widehat{C}^{\mathrm{ord}}_s) \to H^1_?(\alpha(\widehat{J}^{\mathrm{ord}}_s))[\alpha] \to 0.$$

We have the following commutative diagram with exact rows and exact columns:

$$\begin{tikzcd}
E_1^s(\kappa) \arrow[r, hook] & H^1_?(\widehat{A}_r^{\mathrm{ord}}) \arrow[r] & H^1_?(\widehat{J}_s^{\mathrm{ord}}) \\
\frac{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa))} \arrow[u, "\text{onto}"] \arrow[r, hook] & H^1_?(C_s^{\mathrm{ord}}) \arrow[u] \arrow[r, two heads] & H^1_?(\alpha(\widehat{J}_s^{\mathrm{ord}}))[\alpha] \arrow[u] \\
\frac{\widehat{J}_s^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)} \arrow[u, "\overline{\alpha}_s"] \arrow[r, hook, "\rho_s"] & \widehat{B}_s^{\mathrm{ord}}(\kappa) \arrow[u] \arrow[r, two heads] & E_2^s(\kappa). \arrow[u, "\cup"']
\end{tikzcd}$$

The left column is exact by definition. The middle column is the part of the long exact sequence attached to the short one $C_s^{\mathrm{ord}} \hookrightarrow \widehat{A}_s^{\mathrm{ord}} \twoheadrightarrow \widehat{B}_s^{\mathrm{ord}}$, and the right column is the same for $\alpha(\widehat{J}_s^{\mathrm{ord}}) \hookrightarrow \widehat{J}_s^{\mathrm{ord}} \twoheadrightarrow \widehat{B}_s^{\mathrm{ord}}$. Note $\varinjlim_s C_s^{\mathrm{ord}} = A_r[p^\infty]^{\mathrm{ord}}$. Passing to the limit, we have the limit commutative diagram with exact rows and exact columns:

$$\begin{tikzcd}
E_1(\kappa) \arrow[r, hook] & H^1_?(\widehat{A}_r^{\mathrm{ord}}) \arrow[r] & H^1_?(J_\infty^{\mathrm{ord}}) \\
\frac{\alpha(J_\infty^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(J_\infty^{\mathrm{ord}})(\kappa))} \arrow[u, "\text{onto}"] \arrow[r, hook] & H^1_?(A_r[p^\infty]^{\mathrm{ord}}) \arrow[u, "\pi_B"] \arrow[r, two heads] & H^1_?(\alpha(J_\infty^{\mathrm{ord}}))[\alpha] \arrow[u] \\
\frac{J_\infty^{\mathrm{ord}}(\kappa)}{\alpha(J_\infty^{\mathrm{ord}})(\kappa)} \arrow[u, "\overline{\alpha}"] \arrow[r, hook, "\rho_\infty"] & \widehat{B}_r^{\mathrm{ord}}(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \arrow[u, "\delta_B"] \arrow[r, two heads] & E_2(\kappa). \arrow[u, "\cup"']
\end{tikzcd} \tag{6.19}$$

We have seen, $\widehat{A}_r^{\mathrm{ord}}/A_r[p^\infty]^{\mathrm{ord}} \cong \widehat{B}_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as sheaves of $\mathbb{Q}_p$-vector space; so, $H^1_?(\widehat{A}_r^{\mathrm{ord}}/A_r[p^\infty]^{\mathrm{ord}})$ is a $\mathbb{Q}_p$-vector space. On the other hand, $H^1_?(\widehat{A}_r^{\mathrm{ord}})$ is a p-torsion module (e.g., Lemma 2.2). Therefore the natural map $H^1_?(\widehat{A}_r^{\mathrm{ord}}) \to H^1_?(\widehat{A}^{\mathrm{ord}}/A_r[p^\infty]^{\mathrm{ord}})$ is the zero map. Thus by long exact sequence attached to $0 \to A_r[p^\infty]^{\mathrm{ord}} \to \widehat{A}_r^{\mathrm{ord}} \to \widehat{A}_r^{\mathrm{ord}}/A[p^\infty] \to 0$, the morphism π_B is onto. Since $A_r(\kappa) \otimes_{\mathbb{Z}} \mathbb{T}_p = B_r(\kappa) \otimes_{\mathbb{Z}} \mathbb{T}_p$, the map δ_B factors through the Kummer map $A_r(\kappa) \otimes_{\mathbb{Z}} \mathbb{T}_p \hookrightarrow H^1(A_r[p^\infty]^{\mathrm{ord}})$. Thus

$$\mathrm{Ker}(\delta_B) = \mathrm{Im}(\widehat{A}_r(\kappa) \to \widehat{B}_r(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cong \mathrm{Ker}(\overline{\alpha}),$$

where the last identity follows from the snake lemma applied to the above diagram.

Consider the following exact sequence:

$$\begin{aligned} E_1(\kappa)[p^n] = \mathrm{Tor}_1^{\mathbb{Z}}(E_1(\kappa), \mathbb{Z}/p^n\mathbb{Z}) &\xrightarrow{i_n} \alpha(J_\infty^{\mathrm{ord}}(\kappa)) \otimes \mathbb{Z}/p^n\mathbb{Z} \\ &\to \alpha(J_\infty^{\mathrm{ord}})(\kappa) \otimes \mathbb{Z}/p^n\mathbb{Z} \to E_1(\kappa) \otimes \mathbb{Z}/p^n\mathbb{Z} \to 0, \end{aligned}$$

which produces the following commutative diagram with exact rows for $n > m$:

$$\begin{tikzcd}
E_1(\kappa)[p^n] \arrow[r, "i_n"] \arrow[d] & \frac{\alpha(J_\infty^{\mathrm{ord}}(\kappa))}{p^n\alpha(J_\infty^{\mathrm{ord}}(\kappa))} \arrow[r, "j_n"] \arrow[d] & \frac{\alpha(J_\infty^{\mathrm{ord}})(\kappa)}{p^n\alpha(J_\infty^{\mathrm{ord}})(\kappa)} \arrow[r, two heads] \arrow[d] & E_1(\kappa) \otimes \mathbb{Z}/p^n\mathbb{Z} \arrow[d] \\
E_1(\kappa)[p^m] \arrow[r, "i_m"] & \frac{\alpha(J_\infty^{\mathrm{ord}}(\kappa))}{p^m\alpha(J_\infty^{\mathrm{ord}}(\kappa))} \arrow[r, "j_m"] & \frac{\alpha(J_\infty^{\mathrm{ord}})(\kappa)}{p^m\alpha(J_\infty^{\mathrm{ord}})(\kappa)} \arrow[r, two heads] & E_1(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z}
\end{tikzcd}$$

This in turn produces two commutative diagrams with exact rows:

$$\begin{tikzcd}
E_1(\kappa)[p^n] \arrow[r, "i_n"] \arrow[d] & \frac{\alpha(J_\infty^{\mathrm{ord}}(\kappa))}{p^n\alpha(J_\infty^{\mathrm{ord}}(\kappa))} \arrow[r] \arrow[d] & \mathrm{Coker}(i_n) = \mathrm{Im}(j_n) \to 0 \arrow[d] \\
E_1(\kappa)[p^m] \arrow[r, "i_m"] & \frac{\alpha(J_\infty^{\mathrm{ord}}(\kappa))}{p^m\alpha(J_\infty^{\mathrm{ord}}(\kappa))} \arrow[r] & \mathrm{Coker}(i_n) = \mathrm{Im}(j_m) \to 0
\end{tikzcd} \tag{6.20}$$

and

$$\begin{tikzcd}
0 \to \mathrm{Ker}(i_n) \arrow[r] \arrow[d] & E_1(\kappa)[p^n] \arrow[r, "i_n"] \arrow[d] & \mathrm{Im}(i_n) \to 0 \arrow[d] \\
0 \to \mathrm{Ker}(i_m) \arrow[r] & E_1(\kappa)[p^m] \arrow[r, "i_m"] & \mathrm{Im}(i_m) \to 0.
\end{tikzcd} \tag{6.21}$$

Since the diagram of (6.21) is made of finite modules (as $E_1(\kappa) \subset H^1(\widehat{A}_r^{\mathrm{ord}})$; Lemma 2.3), projective limit is an exact functor (from the category of compact modules), and passing to the limit, we get

$$\varprojlim_n \mathrm{Im}(i_n) = \mathrm{Im}(i_\infty : \varprojlim_n E_1(\kappa)[p^n] \to \varprojlim_n \frac{\alpha(J_\infty^{\mathrm{ord}}(\kappa))}{p^n\alpha(J_\infty^{\mathrm{ord}}(\kappa))}).$$

By the snake lemma (cf. [BCM, I.1.4.2 (2)]) applied to (6.20), $\mathrm{Im}(j_n) \to \mathrm{Im}(j_m)$ is a surjection for all $n > m$. Thus the projective system of the following exact sequences:

$$\{0 \to \mathrm{Im}(j_n) \to \frac{\alpha(J_\infty^{\mathrm{ord}})(\kappa)}{p^n\alpha(J_\infty^{\mathrm{ord}})(\kappa)} \to E_1(\kappa) \otimes \mathbb{Z}/p^n\mathbb{Z} \to 0\}_n$$

satisfies the Mittag–Leffler condition. Passing to the projective limit, we get the exact sequence

$$0 \to \alpha(\check{J}_\infty^{\mathrm{ord}}(\kappa)) = \mathrm{Im}(j_\infty) \to \alpha(\check{J}_\infty^{\mathrm{ord}})(\kappa) \to \varprojlim_n E_1(\kappa) \otimes \mathbb{Z}/p^n\mathbb{Z} \to 0.$$

Since $E_1(\kappa) = (\mathbb{Q}_p/\mathbb{Z}_p)^R \oplus \Delta \hookrightarrow H^1_?(\widehat{A}_r)^{\mathrm{ord}}$ for a finite group Δ and an integer $R \geq 0$ (by Lemma 2.3), $\varprojlim_n E_1(\kappa) \otimes \mathbb{Z}/p^n\mathbb{Z}$ is a finite group isomorphic to the torsion subgroup Δ of $E_1(\kappa)$. Thus

$$\check{J}_\infty^{\mathrm{ord}}(\kappa) \xrightarrow{\alpha} \alpha(\check{J}_\infty^{\mathrm{ord}})(\kappa) \text{ has finite cokernel } \Delta, \tag{6.22}$$

and Δ is isomorphic to the maximal torsion submodule of $E_1(\kappa)^\vee$.

Consider the "big" ordinary Hecke algebra $\mathbf{h}$ given by $\varprojlim_s \mathbf{h}_s$ as in the introduction. For a Λ-algebra homomorphism $\mathbf{h} \to R$ and an $\mathbf{h}$-module M, we put $M_R = M \otimes_{\mathbf{h}} R$. Take a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$

such that α in (A) restricted to $\mathrm{Spec}(\mathbb{T})$ is a non-unit; so, $\widehat{A}_s^{\mathrm{ord}}(K)_{\mathbb{T}} \neq 0$. Note that $M_{\mathbb{T}}$ is a direct summand of M; so, the above diagrams and exactness are valid after tensoring $\mathbb{T}$ over $\mathbf{h}$ (attaching subscript $\mathbb{T}$). Note that $\alpha(J_\infty)^{\mathrm{ord}}[p^\infty](\kappa) \subset J_\infty^{\mathrm{ord}}[p^\infty](\kappa) = \mathcal{G}(\kappa)$.
Since $\mathrm{Im}(\rho_\infty)_{\mathbb{T}}$ is a direct summand in $J_\infty(\kappa)_{\mathbb{T}}$ and $\alpha(J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}})[p^n] \cong \mathrm{Tor}_1^{\mathbb{Z}_p}(\alpha(J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}}), \mathbb{Z}/p^n\mathbb{Z})$, we have the following exact sequences:

(6.23)
$$\alpha(J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}})[p^n] \to \frac{\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}}{p^n\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}} \to \frac{J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}}}{p^n J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}}} \xrightarrow{\alpha} \frac{\alpha(J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}})}{p^n\alpha(J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}})} \to 0$$
$$0 \to \alpha(J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}}) \otimes \mathbb{Z}/p^n\mathbb{Z} \to J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to \mathrm{Im}(\rho_\infty)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to 0.$$

The module $\alpha(J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}})[p^n]$ is killed by the annihilator $\mathfrak{a}$ of $\mathcal{G}_{\mathbb{T}}(\kappa)^\vee$ in Λ which is prime to $\gamma^{p^r} - 1$ (note that $\gamma^{p^r} - 1$ kills $\widehat{A}_r^{\mathrm{ord}}(\kappa)$). Thus the image of $\alpha(J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}})[p^n]$ in $\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is killed by $\mathfrak{A} = \mathfrak{a} + (\gamma^{p^{r-1}} - 1) \subset \Lambda$. Since $\Lambda/\mathfrak{A}$ is a finite ring and $\mathcal{G}_{\mathbb{T}}^\vee$ is a Λ-module of finite type, we get

$$|\,\mathrm{Ker}(\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z})| < B \tag{6.24}$$

for a constant $B > 0$ independent of n.
Applying the snake lemma to the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
p^n\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} & \xrightarrow{\hookrightarrow} & \widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} & \xrightarrow{\twoheadrightarrow} & \widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
p^n J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} & \xrightarrow[\hookrightarrow]{} & J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} & \xrightarrow[\twoheadrightarrow]{} & J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z},
\end{array}$$

we have an isomorphism, for $\mathcal{F}_n := p^n J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \cap \widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}$,

$$\mathcal{F}_n/p^n\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \cong \mathrm{Ker}(\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \otimes \mathbb{Z}/p^n\mathbb{Z})$$

whose right-hand-side is finite with bounded order independent of n by (6.24).
Consider the two filters on $\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}$:

$$\mathcal{F} := \{\mathcal{F}_n = (p^n J_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \cap \widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}})\}_n \text{ and } \{p^n\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}\}_n$$

with $\mathcal{F}_n \supset p^n\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}$. On the free quotient $\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}/\widehat{A}_r^{\mathrm{ord}}[p^\infty](\kappa)_{\mathbb{T}}$, the two filters induce the same p-adic topology. Writing $\widetilde{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}$ for the completion of $\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}$ with respect to $\mathcal{F}$, therefore we find
(6.25)
the natural surjective morphism: $\widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \twoheadrightarrow \widetilde{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}}$ has finite kernel.

This shows that the following sequence is exact by [CRT, Theorem 8.1 (ii)]:

$$0 \to \widetilde{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \to \check{J}_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \xrightarrow{\alpha} \alpha(\check{J}_\infty^{\mathrm{ord}}(\kappa)_{\mathbb{T}}) \to 0. \tag{6.26}$$

By this sequence combined with finiteness of $\mathrm{Ker}(\widetilde{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}} \to \widehat{A}_r^{\mathrm{ord}}(\kappa)_{\mathbb{T}})$, we get

Proposition 6.4. *Take a connected component* $\mathrm{Spec}(\mathbb{T})$ *of* $\mathrm{Spec}(\mathbf{h})$ *with* $\widehat{A}^{\mathrm{ord}}_{s,\mathbb{T}} \neq 0$*. Then we have the following exact sequence:*

$$0 \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \check{J}^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \xrightarrow{\alpha} \check{J}^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}},$$

where $\mathrm{Coker}(\alpha)$ *is a* $\mathbb{Z}_p$*-module of finite type with* $\dim_{\mathbb{Q}_p}\mathrm{Coker}(\alpha)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \le \dim_{\mathbb{Q}_p}\widehat{B}_r(\kappa)_{\mathbb{T}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$*. Moreover we have a natural surjection:* $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$ *with finite kernel. If* $\mathcal{G}_{\mathbb{T}}(\kappa) = 0$*, then* $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \cong \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$

We will see that the torsion submodule of $\mathrm{Coker}(\alpha)$ is isomorphic to the maximal p-torsion submodule of $E_1(\kappa)^\vee$.

Proof. The second sequence of (6.18) evaluated at κ produces the following exact sequence:

$$0 \to \alpha(J^{\mathrm{ord}}_\infty)(\kappa)_{\mathbb{T}} \to J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \xrightarrow{\rho_\infty} \widehat{B}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

By the exact sequence of the bottom row in the diagram (6.19), the image $\mathrm{Im}(\rho_\infty)$ is embedded into $(\widehat{B}_r\otimes\mathbb{Q}_p)(\kappa)_{\mathbb{T}}$, and thus $\mathrm{Im}(\rho_\infty)\cong\mathbb{Q}_p^i\oplus\mathbb{Z}_p^j$ with $i+j\le\dim\widehat{B}_r(\kappa)_{\mathbb{T}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Thus we get the following exact sequences indexed by n:

$$\begin{aligned}0 = \mathrm{Im}(\rho_\infty)[p^n] &\cong \mathrm{Tor}_1^{\mathbb{Z}_p}(\mathrm{Im}(\rho_\infty),\mathbb{Z}/p^n\mathbb{Z}) \to \alpha(J^{\mathrm{ord}}_\infty)(\kappa)_{\mathbb{T}}\otimes_{\mathbb{Z}_p}\mathbb{Z}/p^n\mathbb{Z}\\ &\to J^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}}\otimes_{\mathbb{Z}_p}\mathbb{Z}/p^n\mathbb{Z} \to \mathrm{Im}(\rho_\infty)_{\mathbb{T}}\otimes_{\mathbb{Z}_p}\mathbb{Z}/p^n\mathbb{Z}\cong(\mathbb{Z}/p^n\mathbb{Z})^j\to 0.\end{aligned}$$

Since these sequences satisfy the Mittag–Leffler condition, passing to the limit, we get another exact sequence:

$$0 \to \alpha(\check{J}^{\mathrm{ord}}_\infty)(\kappa)_{\mathbb{T}} \to \check{J}^{\mathrm{ord}}_\infty(\kappa)_{\mathbb{T}} \xrightarrow{\rho_\infty} \mathbb{Z}_p^j \to 0.$$

Then the assertion follows from (6.22).
We can check the last assertion by scrutinizing our computation, but here is a short cut. Since $\mathrm{Ker}(\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}})$ is a submodule of $A[p^\infty]^{\mathrm{ord}}(\kappa) \subset \mathcal{G}_{\mathbb{T}}(\kappa) = 0$. Thus the morphism has to be an isomorphism. □

Lemma 6.5. *Let* κ *be as in* Lemma 6.1*. Then the maximal torsion submodule of* $\check{J}_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}}$ *is equal to* $\mathcal{G}_{\mathbb{T}}(\kappa)$ *if* $\mathcal{G}_{\mathbb{T}}(\kappa)$ *is finite. Otherwise, it is killed by* p^B *for some* $0 < B \in \mathbb{Z}$*.*

Proof. By definition, the maximal torsion submodule of $\widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}} = (J_s(\kappa)\otimes_{\mathbb{Z}}\mathbb{Z}_p)^{\mathrm{ord}}_{\mathbb{T}}$ for finite s is given by $\mathcal{G}_s(\kappa)_{\mathbb{T}} := J_s[p^\infty](\kappa)^{\mathrm{ord}}_{\mathbb{T}}$. For $s=\infty$, the maximal torsion submodule of $J_\infty(\kappa)_{\mathbb{T}} = \varinjlim_s \widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}}$ is given by $\mathcal{G}(\kappa)_{\mathbb{T}}$. Thus we have an exact sequence for finite s:

$$0 \to \mathcal{G}_s(\kappa)_{\mathbb{T}} \to \widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \to F_s \to 0$$

for the maximal $\mathbb{Z}_p$-free quotient $F_s := \widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}}/\mathcal{G}_s(\kappa)_{\mathbb{T}}$. This is a split exact sequence as the right term $\widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}}/\mathcal{G}_s(\kappa)_{\mathbb{T}}$ is $\mathbb{Z}_p$-free. By taking p-adic completion: $M \mapsto \check{M} = \varprojlim_n M/p^nM$, we get a split exact sequence for finite s:

$$0 \to \check{\mathcal{G}}_s(\kappa)_{\mathbb{T}} \to \widehat{J}_s(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \to F_s \to 0.$$

This shows $\check{\mathcal{G}}_s(\kappa)_{\mathbb{T}} = \mathcal{G}_s(\kappa)_{\mathbb{T}}$ for finite s, and $\check{\mathcal{G}}_s(\kappa)_{\mathbb{T}}$ is a finite module if κ is as in Lemma 6.1. Since F_s is $\mathbb{Z}_p$-flat for all $s \geq r$, $F = \varinjlim_s F_s$ is a $\mathbb{Z}_p$-flat module. For $s = \infty$, we have the limit exact sequence (noting $\mathcal{G}(\kappa)_{\mathbb{T}} = \mathcal{G}_\infty(\kappa)_{\mathbb{T}}$)

$$0 \to \mathcal{G}(\kappa)_{\mathbb{T}} \to J_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \to F \to 0,$$

and $F = J_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}}/\mathcal{G}(\kappa)_{\mathbb{T}}$. By $\mathbb{Z}_p$-flatness of F, after tensoring $\mathbb{Z}/p^n\mathbb{Z}$ over $\mathbb{Z}_p$, we still have an exact sequence (cf. [BCM, I.2.5]) indexed by $0 < n \in \mathbb{Z}$:

$$0 \to \mathcal{G}(\kappa)_{\mathbb{T}}/p^n\mathcal{G}(\kappa)_{\mathbb{T}} \to J_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z} \to F/p^nF \to 0,$$

which obviously satisfies the Mittag–Leffler condition (with respect to n). Passing to the projective limit with respect to n, we get the limit exact sequence:

$$0 \to \check{\mathcal{G}}(\kappa)_{\mathbb{T}} \to \check{J}_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \to \check{F} \to 0,$$

Since F is $\mathbb{Z}_p$-flat, $\check{F}$ is torsion-free (and hence $\mathbb{Z}_p$-flat by [BCM, I.2.4]). Indeed, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Tor}_{\mathbb{Z}_p}(F/pF, \mathbb{Z}/p^n\mathbb{Z}) \arrow[r, hook] \arrow[d, "\wr"'] & F/p^nF \arrow[r, "x \mapsto px"] \arrow[d, "\|"'] & F/p^nF \arrow[r, two heads] \arrow[d, "\|"'] & F/pF \arrow[d, "\|"] \\
F/pF \arrow[r, hook] & F/p^nF \arrow[r, "x \mapsto px"] & F/p^nF \arrow[r, two heads] & F/pF.
\end{tikzcd}$$

Regard this as a projective system of exact sequences indexed by $0 < n \in \mathbb{Z}$. Then the transition maps of F/pF at the extreme right end is the identity and at the extreme left end is multiplication by p (i.e., the zero map). Passing to the limit, from left exactness of projective limit, we get an exact sequence

$$0 = \varprojlim_{x \mapsto px} F/pF \to \check{F} \xrightarrow{f \mapsto pf} \check{F},$$

and hence $\check{F}$ is p-torsion-free.

If $\mathcal{G}(\kappa)$ is killed by p^B for some $0 < B \in \mathbb{Z}$, we still have $\check{\mathcal{G}}(\kappa)_{\mathbb{T}} = \mathcal{G}(\kappa)_{\mathbb{T}}$. Otherwise, for some $0 < j \in \mathbb{Z}$, $\mathcal{G}(\kappa)_{\mathbb{T}}$ fits into the following split exact sequence by Lemmas in Section 4,

$$0 \to (\mathbb{Q}_p/\mathbb{Z}_p)^j \to \mathcal{G}(\kappa)_{\mathbb{T}} \to \mathcal{G}(\kappa)^{tor}_{\mathbb{T}} \to 0$$

for $\mathcal{G}(\kappa)^{tor}_{\mathbb{T}}$ killed by p^B for some $0 < B \in \mathbb{Z}$. Thus $\check{\mathcal{G}}(\kappa)_{\mathbb{T}} = \mathcal{G}(\kappa)^{tor}_{\mathbb{T}}$, which is the maximal torsion submodule of $\check{J}_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}}$. □

We put $M^* = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ for a $\mathbb{Z}_p$-module M and

$$\check{X}_{s,\mathbb{T}}(k)^*_{\mathrm{ord}} := \mathrm{Hom}_{\mathbb{Z}_p}(\check{X}_s(k)^{\mathrm{ord}}_{\mathbb{T}}, \mathbb{Z}_p) \quad \text{and} \quad \widehat{X}_{s,\mathbb{T}}(k)^*_{\mathrm{ord}} := \mathrm{Hom}_{\mathbb{Z}_p}(\widehat{X}_s(k)^{\mathrm{ord}}_{\mathbb{T}}, \mathbb{Z}_p)$$

with $s = r, r+1, \dots, \infty$ for $X = J, A, B$. The algebra $\mathbf{h}$ acts on $\check{J}_\infty$ naturally. As before, we write for an $\mathbf{h}$-algebra R, $\check{J}_\infty(k)^{\mathrm{ord}}_R = \check{J}_\infty(k)^{\mathrm{ord}} \otimes_{\mathbf{h}} R$ and $\check{J}_\infty(k)^*_{\mathrm{ord},\, R} = \check{J}_\infty(k)^*_{\mathrm{ord}} \otimes_{\mathbf{h}} R$.

Assume the condition (A) in Section 5 for (α, A_s, B_s). Take a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ in which the image of α is non-unit. Replacing α by $1_{\mathbb{T}}\alpha$ for the idempotent $1_{\mathbb{T}}$ of $\mathbb{T}$, we may assume that $\alpha \in \mathbb{T}$ as in the setting of

(P4) in Proposition 5.1. Recall $\mathcal{G}_{\mathbb{T}}(k)^{\vee}_{tor}$ is the maximal $\mathbb{Z}_p$-torsion submodule of $\mathcal{G}_{\mathbb{T}}(k)^{\vee}$. We now state the principal result of this paper:

THEOREM 6.6. *Let k be either a number field or a finite extension of $\mathbb{Q}_l$ for a prime l. Then we get*

(1) *Consider the following sequence $\mathbb{Z}_p$-dual to the one in* Proposition 6.4*:*

$$0 \to \mathrm{Coker}(\alpha)^*_{\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \xrightarrow{\alpha^*} \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \xrightarrow{\iota^*_\infty} \widehat{A}_r(k)^*_{\mathrm{ord},\mathbb{T}} \to 0.$$

Then

(a) *If $\mathcal{G}_{\mathbb{T}}(k) = 0$, the sequence is exact except that* $\mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$ *is finite;*

(b) *If $\mathcal{G}_{\mathbb{T}}(k)^{\vee}_{tor} = 0$, the sequence is exact except that* $\mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$ *and* $\mathrm{Coker}(\iota^*_\infty)$ *are both finite;*

(c) *If $\mathcal{G}_{\mathbb{T}}(k)^{\vee}_{tor} \neq 0$, the sequence is exact up to finite error.*

(d) *The module $\mathcal{G}_{\mathbb{T}}(k)^{\vee}_{tor}$ is killed by p^B for some finite $0 \le B \in \mathbb{Z}$, and the cokernel* $\mathrm{Coker}(\iota^*_\infty)$ *is finite and is killed by p^B. In particular, after localizing the sequence by any prime divisor $P \in \mathrm{Spec}(\Lambda)$, the sequence is exact.*

(2) *After tensoring $\mathbb{Q}_p$ with the sequence* (1)*, the following sequence*

$$0 \to \mathrm{Coker}(\alpha)^*_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \widehat{A}_r(k)^*_{\mathrm{ord},\,\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$$

is an exact sequence of p-adic $\mathbb{Q}_p$-Banach spaces (with respect to the Banach norm having the image of $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$ in $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as its closed unit ball).

(3) *The compact module $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$ is a Λ-module of finite type, and $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a $\Lambda[\frac{1}{p}]$-module of finite type.*

Proof. We prove the exactness of the sequence (1). Since $\widehat{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}} \to \widetilde{A}^{\mathrm{ord}}_r(\kappa)_{\mathbb{T}}$ has finite kernel and is an isomorphism if $\mathcal{G}_{\mathbb{T}}(k) = 0$ by Proposition 6.4, we only need to prove the various exactness of (1). By Proposition 6.4, the following sequence is exact:

$$0 \to \widetilde{A}^{\mathrm{ord}}_r(k)_{\mathbb{T}} \xrightarrow{\iota_\infty} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \xrightarrow{\alpha} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \xrightarrow{\pi^*_\infty} X \to 0$$

for $X = \mathrm{Coker}(\alpha)$. We consider the short exact sequence:

$$0 \to \widetilde{A}^{\mathrm{ord}}_r(k)_{\mathbb{T}} \xrightarrow{\iota_\infty} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \to \mathrm{Coker}(\iota_\infty) \to 0$$

and another exact sequence:

$$0 \to \mathrm{Coker}(\iota_\infty) \xrightarrow{\alpha} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \xrightarrow{\pi^*_\infty} X \to 0.$$

Applying the dualizing functor: $M \mapsto M^* := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$, we get the following exact sequences:

$$0 \to \mathrm{Coker}(\iota_\infty)^* \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \xrightarrow{\iota^*_\infty} \widetilde{A}_r(k)^*_{\mathrm{ord},\mathbb{T}} \to \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{Coker}(\iota_\infty), \mathbb{Z}_p),$$

$$X^* \hookrightarrow \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \xrightarrow{\alpha^*} \mathrm{Coker}(\iota_\infty)^* \to \mathrm{Ext}^1_{\mathbb{Z}_p}(X,\mathbb{Z}_p) \to \mathrm{Ext}^1_{\mathbb{Z}_p}(\check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}},\mathbb{Z}_p).$$

Thus $\mathrm{Ext}^1_{\mathbb{Z}_p}(X,\mathbb{Z}_p)$ contains $\mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$. Computing $\mathrm{Ext}^1_{\mathbb{Z}_p}(M,\mathbb{Z}_p)$ by the injective resolution $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \xrightarrow{\pi} \mathbb{Q}_p/\mathbb{Z}_p \to 0$ (see [MFG, (4.10)]), we find

$$\mathrm{Ext}^1_{\mathbb{Z}_p}(M,\mathbb{Z}_p) = \mathrm{Coker}(\mathrm{Hom}_{\mathbb{Z}_p}(X,\mathbb{Q}_p) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathbb{Z}_p}(X,\mathbb{Q}_p/\mathbb{Z}_p)) = M[p^\infty]^\vee.$$

Since X is a $\mathbb{Z}_p$-module of finite type by Proposition 6.4, $\mathrm{Ext}^1_{\mathbb{Z}_p}(X,\mathbb{Z}_p) = X[p^\infty]^\vee$ is finite. Similarly $\mathrm{Ext}^1_{\mathbb{Z}_p}(\check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}},\mathbb{Z}_p) = \check{J}_\infty(k)^{\mathrm{ord},\mathbb{T}}[p^\infty]^\vee = \mathcal{G}_\mathbb{T}(k)^\vee_{tor}$ and hence if $\mathcal{G}_\mathbb{T}(k)^\vee_{tor} = 0$, $\mathrm{Ext}^1_{\mathbb{Z}_p}(X,\mathbb{Z}_p) = \mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$. Anyway, $\mathrm{Ker}(\iota^*_\infty)/\mathrm{Im}(\alpha^*)$ is finite.

We have $\mathrm{Coker}(\iota_\infty) \hookrightarrow \check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T}$. Again, we get, as Λ-modules,

$$\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{Coker}(\iota_\infty),\mathbb{Z}_p) \cong \mathrm{Coker}(\iota_\infty)[p^\infty]^\vee$$

which is a quotient of $\mathcal{G}_\mathbb{T}(k)^\vee_{tor}$ (see Lemmas 4.2, 4.3 and 6.5). Indeed, assuming finiteness of $\mathcal{G}_\mathbb{T}(k)$, the torsion part of $\check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T}$ is isomorphic to a submodule of $\mathcal{G}_\mathbb{T}(k)$ by Lemma 6.5; in particular, it has finite torsion (this proves (1a)). Without assuming finiteness of $\mathcal{G}_\mathbb{T}(k)$, the p-torsion part of $\mathrm{Coker}(\iota_\infty)$ is a Λ-submodule of a bounded p-torsion Λ–module $\mathcal{G}_\mathbb{T}(k)_{tor}$ by Lemma 6.5. Thus $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathrm{Coker}(\iota_\infty),\mathbb{Z}_p)$ is is a quotient of $\mathcal{G}_\mathbb{T}(k)^\vee_{tor}$ and killed by p^B for some $0 \geq B \in \mathbb{Z}$ (and this proves (1b)). In addition, $\mathrm{Coker}(i^*_\infty) = \mathrm{Coker}(\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \to \widetilde{A}_r(k)^*_{\mathrm{ord},\mathbb{T}})$ factors through the $\mathbb{Z}_p$-module $\widetilde{A}_r(k)^*_{\mathrm{ord},\mathbb{T}}$ of finite type, which lands in the bounded p-torsion module $\mathrm{Coker}(\iota_\infty)[p^\infty]^\vee$ (by Lemma 6.5); so, $\mathrm{Coker}(i^*_\infty)$ must have finite order (this shows (1c)). Therefore, the error term $\mathrm{Coker}(i^*_\infty)$ is a pseudo-null Λ-module, it is killed after localization at prime divisors of $\mathrm{Spec}(\Lambda)$. Thus we get all the assertions in (1).

The exact sequence in (1) tells us that $\check{J}_\infty(k)^*_{\mathrm{ord},\,\mathbb{T}}/\alpha(\check{J}_\infty(k)^*_{\mathrm{ord},\,\mathbb{T}})$ is isomorphic (up to finite modules) to the $\mathbb{Z}_p$-module $\widehat{A}_r(k)^*_{\mathrm{ord},\,\mathbb{T}}$ of finite type, which is a torsion Λ-module of finite type. Then by Nakayama's lemma, $\check{J}_\infty(k)^*_{\mathrm{ord},\,\mathbb{T}}$ is a Λ-module of finite type. This proves the assertion (3).

The extension modules appearing in the above proof of (1) is p-torsion Λ-module of finite type. Thus the sequence

$$0 \to X^* \to \check{J}_\infty(k)^*_{\mathrm{ord},\,\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\,\mathbb{T}} \to \widehat{A}_r(k)^*_{\mathrm{ord},\,\mathbb{T}} \to 0$$

is exact up to p-torsion error. By tensoring $\mathbb{Q}$ over $\mathbb{Z}$, we get the exact sequence (2):

$$X^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \hookrightarrow \check{J}_\infty(k)^*_{\mathrm{ord},\,\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \check{J}_\infty(k)^*_{\mathrm{ord},\,\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \widehat{A}_r(k)^*_{\mathrm{ord},\,\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The above sequence is the p-adic Banach dual sequence of the following exact sequence obtained from the sequence in (1) by tensoring $\mathbb{Q}$:

$$\widehat{A}_r(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \underset{\hookrightarrow}{\xrightarrow{\iota_\infty}} \check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\alpha} \check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \underset{\twoheadrightarrow}{\xrightarrow{\pi^*_\infty}} X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Indeed, equipping $\check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the Banach p-adic norm so that the closed unit ball is given by the image of $\check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T}$ in $\check{J}_\infty(k)^{\mathrm{ord}}_\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, the sequence is continuous (the first and the last term are finite dimensional $\mathbb{Q}_p$-vector spaces; so, there is a unique p-adic Banach space structure on them). The dual space of bounded functionals of each term is given by the $\mathbb{Q}_p$-dual of the corresponding space before tensoring $\mathbb{Q}$, which is given by $Y \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for $Y = \widehat{A}_r(k)^*_{\mathrm{ord},\ \mathbb{T}}$, $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}}$ and X^*, respectively. This proves (2). □

Corollary 6.7. *Let the notation be as in* (1) *of* Theorem 6.6. *Consider the set* $\Omega \subset \mathbb{T}$ *of prime factors (in* Λ*) of* $\gamma^{p^n} - 1$ *for* $n = 0, 1, 2, \ldots, \infty$. *Except for finitely many* $\alpha \in \Omega$, *we have* $\mathrm{Coker}(\alpha)^*_{\mathbb{T}} \otimes_\Lambda \Lambda_P = 0$ *for* $P = (\alpha) \in \mathrm{Spec}(\Lambda)$, *where* Λ_P *is the localization of* Λ *at* P.

Proof. Note that $\alpha \in \Omega$ (regarded as $\alpha \in \mathbb{T}$) satisfies the assumption (A) by Proposition 5.1 (P1) and that $\Lambda[\frac{1}{p}]$ is a principal ideal domain (as Λ is a unique factorization domain of dimension 2; see [CRT, Chapter 7]). Pick an isomorphism $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \Lambda[\frac{1}{p}]^R \oplus X^*_k$ of $\Lambda[\frac{1}{p}]$-modules with torsion $\Lambda[\frac{1}{p}]$-module X^*_k. Then for P outside the support of the $\Lambda[\frac{1}{p}]$-module X^*_k, by Theorem 6.6 (2),

$$K := \mathrm{Ker}(\alpha : \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \to \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}})$$

is killed by some p-power. Then by the assertion (1) of the above theorem, K is a $\mathbb{Z}_p$-module of finite type; hence K is finite. This shows the result. □

7. Closure of the global Λ-MW group in the local one

Let κ be a number field and $k = \kappa_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of κ for a prime $\mathfrak{p}|p$ of κ. Write W for the p-adic integer ring of k, and let Q be the quotient field of Λ. By [M55] or [T66], for an abelian variety $A_{/k}$ of dimension g, $\widehat{A}(k) = A(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ has torsion free part $\widehat{A}(k)_f$ isomorphic to the additive group W^g, and the torsion part $\widehat{A}(k)_{tor}$ is a finite group.
Write $F = \kappa$ or k. Recall the $\mathbb{T}$-component

$$\check{J}_\infty(F)^*_{\mathrm{ord},\mathbb{T}} := \check{J}_\infty(F)^*_{\mathrm{ord}} \otimes_{\mathbf{h}} \mathbb{T}$$

for a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$. By Theorem 6.6 (3), $\check{J}_\infty(F)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a $\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module of finite type. For simplicity, write $\mathbb{J}_{\mathbb{T}}(F) := \check{J}_\infty(F)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.
Let the notation be as in Corollary 6.7; in particular, Ω is the set of prime factors (in Λ) of $\gamma^{p^n} - 1$ for $n = 1, 2, \ldots, \infty$. Note that $\alpha \in \Omega \subset \mathbb{T}$ satisfies the condition (A) by Proposition 5.1 (P1). Then by Theorem 6.6 (2), this implies

$$\mathbb{J}_{\mathbb{T}}(F)/\alpha(\mathbb{J}_{\mathbb{T}}(F)) \cong \widehat{A}^{\mathrm{ord}}_r(F)^*_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Further localizing at each arithmetic point $P \in \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ with $P|(\gamma^{p^{r-1}} - 1)$, we get, for $\mathbb{J}_{\mathbb{T}_P}(F) = \mathbb{J}_{\mathbb{T}}(F) \otimes_{\mathbb{T}} \mathbb{T}_P$ for the localization $\mathbb{T}_P$ at P,

$$\mathbb{J}_{\mathbb{T}_P}(F)/\alpha(\mathbb{J}_{\mathbb{T}_P}(F)) \cong \widehat{A}^{\mathrm{ord}}_r(F)^*_{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Since $\Lambda[\frac{1}{p}]$ is a principal ideal domain, $\mathbb{J}_{\mathbb{T}}(F)$ is isomorphic to $\Lambda[\frac{1}{p}]^{m_F} \oplus X'_F$ for a torsion $\Lambda[\frac{1}{p}]$-module X'_F. Put $X_F := X'_F \oplus \mathcal{G}_?(k)^\vee$ and decompose

$X_F = \bigoplus_{\mathfrak{P}} \Lambda[\frac{1}{p}]/\mathfrak{P}^{e_F(\mathfrak{P})}$ for maximal ideals $\mathfrak{P}$ of $\Lambda[\frac{1}{p}]$. Put $\mathrm{Char}_{\Lambda[\frac{1}{p}]}(X_F) = \prod_{\mathfrak{P}} \mathfrak{P}^{e_F(\mathfrak{P})}$. If $P \in \mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}}_p)$ is prime to $\mathrm{Char}_{\Lambda[\frac{1}{p}]}(X_F)$,

$$\mathbb{J}_{\mathbb{T}}(F)/\alpha(\mathbb{J}_{\mathbb{T}}(F)) \cong (\Lambda/P)^{m_F} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In other words, the Λ_P/P-dimension of $\mathbb{J}_{\mathbb{T}}(F)/\alpha(\mathbb{J}_{\mathbb{T}}(F))$ is independent of P for most of P. We formulate this fact for $F = k$ as follows:

Theorem 7.1. *Let the notation be as above. Write W for the $\mathfrak{p}$-adic integer ring of k and Q for the quotient field of Λ. Then the Q-vector space $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_\Lambda Q$ has dimension equal to $g = \mathrm{rank}_{\mathbb{Z}_p} W \cdot \mathrm{rank}_\Lambda \mathbb{T}$.*

Proof. We use the notation introduced in Corollary 6.7. Pick $\alpha \in \Omega$, and let $A \subset J_r[\alpha]$ be the identity connected component. Define $J_r \twoheadrightarrow B$ to be the dual quotient of $A \hookrightarrow J_r$. By the control Theorem 6.6 (2), we have $\mathbb{J}_{\mathbb{T}}(k)/\alpha\mathbb{J}_{\mathbb{T}}(k) \cong \check{A}(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for all $\alpha \in \Omega$. Moreover, we have $\dim_{\kappa(P)} \mathbb{J}_{\mathbb{T}}(k)/\alpha\mathbb{J}_{\mathbb{T}}(k) = m_k$ outside a finite set $S \subset \Omega$. The set S is made of prime factors in Ω of $\mathrm{Char}_{\Lambda[\frac{1}{p}]}(X^*_k)$. Note that $m_k = \mathrm{rank}_{\Lambda[\frac{1}{p}]} \mathbb{J}_{\mathbb{T}}(k) = \dim_Q \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_\Lambda Q$; so, we compute $\mathrm{rank}_{\Lambda[\frac{1}{p}]} \mathbb{J}_{\mathbb{T}}(k)$.

By [M55] or [T66], we have $\widehat{A}(k) \cong W^{\dim A} \times \Delta$ for a finite p-abelian group Δ. Regarding $A(k)$ as a p-adic Lie group, we have a logarithm map $\log : A(k) \to Lie(A_{/k})$. For a ring R, write $h(R)$ (resp. $h_r(R)$) for the scalar extension to R of

$$\mathbb{Z}[T(n)|n = 1, 2, \dots] \subset \mathrm{End}(A_{/\mathbb{Q}}) \cong \mathrm{End}(B_{/\mathbb{Q}})$$
$$\text{(resp. } \mathbb{Z}[T(n)|n = 1, 2, \dots] \subset \mathrm{End}(J_{r/\mathbb{Q}})).$$

The Lie algebra $Lie(A_{/\mathbb{Q}_p})$ is the dual of $\Omega_{B/\mathbb{Q}_p}$.
Note that $\Omega_{J_r/\mathbb{Q}} \cong \Omega_{X_r/\mathbb{Q}}$ (e.g., [GME, Theorem 4.1.7]). By q-expansion at the infinity cusp, we have an embedding $i : \Omega_{X_r/\mathbb{Q}} \hookrightarrow \mathbb{Q}[[q]]$ sending ω to $i(\omega)\frac{dq}{q}$. Writing $i(\omega) = \sum_{n=1}^\infty a(n,\omega)q^n$, we have $a(m, \omega|T(n)) = \sum_{0<d|(m,n),(d,Np)=1} d \cdot a(\frac{mn}{d^2}, \omega|\langle d\rangle)$ for the diamond operator $\langle d\rangle$ associated to $d \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$. From this, the pairing $\langle\cdot,\cdot\rangle : h_r(\mathbb{Q}) \times \Omega_{X_r/\mathbb{Q}} \to \mathbb{Q}$ given by $\langle H, \omega\rangle = a(1, \omega|H)$ is non-degenerate (see [GME, §3.2.6]). Thus we have

$$\Omega_{J_r/k} \cong \mathrm{Hom}_k(h_r(k), k) \quad \text{and} \quad \Omega_{A/k} \cong \mathrm{Hom}_k(h(k), k) \quad \text{as modules over } h_r(k),$$

since $h(k)$ is naturally a quotient of $h_r(k)$ and $B = J_r/\alpha(J_r)$ for $(\alpha) = \mathrm{Ker}(h_r(\mathbb{Z}_p) \twoheadrightarrow h(\mathbb{Z}_p))$ in $h_r(\mathbb{Z}_p)$. By the duality between $Lie(A_{/k})$ and $\Omega_{A/k}$, we have

$$Lie(A_{/k}) \cong h(k) \quad \text{as an } h(k)\text{-module.}$$

This leads to an isomorphism of h-modules:

$$\widehat{A}(k)^{\mathrm{ord}}_{\mathbb{T}} \otimes_{\mathbb{Z}_p} k \xrightarrow[\sim]{\log} Lie(A_{/k})_{\mathbb{T}} \cong (\mathbb{T}/(\alpha)\mathbb{T}) \otimes_{\mathbb{Z}_p} k$$

as $\mathbb{T}/(\alpha)\mathbb{T}$ is canonically isomorphic to a ring direct summand $h(\mathbb{Z}_p)^{\mathrm{ord}}_{\mathbb{T}}$ of $h(\mathbb{Z}_p)^{\mathrm{ord}}$ as $\mathbb{Z}_p$-algebras by the control theorem (cf. [GME, §3.2.6]). Thus

$$\mathrm{rank}_W \widehat{A}(k)^{\mathrm{ord}}_{\mathbb{T}} = [k : \mathbb{Q}_p]\, \mathrm{rank}_{\Lambda/(\alpha)} \mathbb{T}/(\alpha)\mathbb{T} = [k : \mathbb{Q}_p]\, \mathrm{rank}_\Lambda \mathbb{T}.$$

This proves the desired assertion, as $[k:\mathbb{Q}_p] = \mathrm{rank}_{\mathbb{Z}_p} W$. $\square$

We have a natural Λ-linear map

$$\check{J}_\infty(\kappa)^{\mathrm{ord}}_{\mathbb{T}} \xrightarrow{\iota} \check{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}} \text{ and } \check{J}_\infty(k)^{*}_{\mathrm{ord},\,\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\iota^*} \check{J}_\infty(\kappa)^{*}_{\mathrm{ord},\,\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

We would like to study their kernel and cokernel.

Take a reduced irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T})$. Let $\widetilde{\mathbb{I}}$ be the normalization of $\mathbb{I}$, and write $Q(\mathbb{I})$ for the quotient field of $\mathbb{I}$. Then $\mathbb{J}_F := \check{J}_\infty(F)^*_{\mathrm{ord},\widetilde{\mathbb{I}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module of finite type for $F = \kappa, k$. Note that $\widetilde{\mathbb{I}}[\frac{1}{p}]$ is a Dedekind domain. This we can decompose $\mathbb{J}_F = L_F \oplus X_F$ for a locally free $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module L_F of finite constant rank and a torsion module X_F isomorphic to $\bigoplus_{\mathfrak{P}} \widetilde{\mathbb{I}}[\frac{1}{p}]/\mathfrak{P}^{e_F(\mathfrak{P})}$ for finitely many maximal ideals $\mathfrak{P}$ of $\widetilde{\mathbb{I}}[\frac{1}{p}]$. We put $\mathrm{Char}_{\mathbb{I}}(X_F) = \prod_{\mathfrak{P}} \mathfrak{P}^{e_F(\mathfrak{P})}$.

For an abelian variety A over κ, write $\overline{A(\kappa)} \subset \widehat{A}(k)$ for the p-adic closure of the image of $A(\kappa)$ in $\widehat{A}(k)$. Pick an arithmetic point $P \in \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ of weight 2. Suppose that the abelian variety A_P is realized in J_r and satisfies the condition (A). By Theorem 6.6 (2), the natural map

$$\mathbb{J}_F/P\mathbb{J}_F \to \widehat{A}_P(F)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism. Thus as long as $P \nmid \mathrm{Char}(X_k) \cdot \mathrm{Char}(X_\kappa)$, we have a surjective linear map

$$\widehat{A}_P(k)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \overline{A_P(\kappa)}^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \iota^*_P(\mathbb{J}_k/P\mathbb{J}_k) \tag{7.1}$$

$\mathbb{Q}_p$-dual to the inclusion

$$\overline{A_P(\kappa)}^{\mathrm{ord}}_{\widetilde{\mathbb{I}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subset \widehat{A}_P(k))^{\mathrm{ord}}_{\widetilde{\mathbb{I}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where $\iota^*_P = \iota^* \otimes \mathrm{id} : \mathbb{J}_k \otimes_{\mathbf{h}} \mathbf{h}/P = \mathbb{J}_k/P\mathbb{J}_k \to \widehat{A}_P(\kappa)_{\mathrm{ord},\,\widetilde{\mathbb{I}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ induced by ι^*. Put

$$r_k(F;\mathbb{I}) := \dim_{Q(\mathbb{I})} \mathbb{J}_F \otimes_{\widetilde{\mathbb{I}}[\frac{1}{p}]} Q(\mathbb{I}) = \mathrm{rank}_{\widetilde{\mathbb{I}}[\frac{1}{p}]} \mathbb{J}_F$$

for the quotient field $Q(\mathbb{I})$ of $\widetilde{\mathbb{I}}$.

We now assume

(a) Taking $r = r(P)$ and A_r to be A_P, the condition (A) holds for A_P for almost all arithmetic points $P \in \mathrm{Spec}(\mathbb{I})$ of weight 2.

By Proposition 5.1 (P2), the condition (A) holds for "all" arithmetic points $P \in \mathrm{Spec}(\mathbb{I})$ of weight 2 if $\mathbb{T} = \mathbb{I}$ and p is unramified in $\mathbb{T}/P$ for one arithmetic point $P \in \mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$. Indeed, as shown in [F02, Theorem 3.1], $\mathbb{T}$ is regular under this assumption (and the regularity guarantees the validity of (A) by Proposition 5.1 (P2)).

Pick a base arithmetic point $P_0 \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ of weight 2. The point P_0 gives rise to $f = f_{P_0} \in S_2(\Gamma_1(Np^{r+1}))$ with $B_0 = B_{P_0}$ and $A_0 = A_{P_0}$ satisfying $f|T(n) = P_0(T(n))f$ for all $n > 0$. By Theorem 6.6 (2), we have for $F = k, \kappa$,

(ct) $\mathbb{J}_F/P_0\mathbb{J}_F$ is isomorphic to the $\mathbb{Q}_p$-dual of $\widehat{A}_0(F)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Choosing P_0 outside $\mathrm{Char}_{\mathbb{I}}(X_\kappa)\cdot\mathrm{Char}_{\mathbb{I}}(X_k)$, we may assume the following condition for $F=k,\kappa$:

(dim) $\dim_{\mathbb{Q}_p(f)}\mathbb{J}_F/P_0\mathbb{J}_F = r_k(F;\mathbb{I})$.

Here $\mathbb{Q}_p(f)$ is the quotient field of $\mathbb{I}/P_0\mathbb{I}$ and is generated by $P_0(T(n))$ for all n over $\mathbb{Q}_p$.

Since $A_0(\kappa)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a $\mathbb{Q}(f)$ vector space, if $A_0(\kappa)\otimes\mathbb{Q}\neq 0$, we have $\dim_{\mathbb{Q}(f)} A_0(\kappa)\otimes\mathbb{Q}>0$, which implies that $\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\neq 0$. Suppose

$$k=\mathbb{Q}_p \text{ and } (\widehat{A}_0(\kappa)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\neq 0.$$

Then $(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}$ is a finite dimensional vector subspace over $\mathbb{Q}_p$ of $\widehat{A}_0(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ stable under $T(n)$ for all n. Let us identify $P_0(T(n))\in\overline{\mathbb{Q}}_p$ with a system of eigenvalues of $T(n)$ occurring on $\widehat{A}_0(\kappa)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Then $(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}$ and $\widehat{A}_0(k)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ are $\mathbb{Q}_p(f)$-vector spaces. Thus we conclude

$$0<\dim_{\mathbb{Q}_p(f)}(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\le\dim_{\mathbb{Q}_p(f)}(\widehat{A}_0(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}=1,$$

which implies

$$0<\dim_{\mathbb{Q}_p(f)}(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}=\dim_{\mathbb{Q}_p(f)}\widehat{A}_0(k)^{\mathrm{ord}}_{\widetilde{\mathbb{I}}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p=1.$$

By (7.1), we get

$$\begin{aligned}&\dim_{\mathbb{Q}_p}\iota^*_{P_0}(\mathbb{J}_\kappa/P_0\mathbb{J}_\kappa)\\ &\quad=\dim_{\mathbb{Q}_p}(\overline{A_0(\kappa)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)=\dim_{\mathbb{Q}_p}(\widehat{A}_0(\mathbb{Q}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)=\dim_{\mathbb{Q}_p}\mathbb{Q}_p(f).\end{aligned}$$

In other words, by Theorem 6.6 (2), the kernel of the map ι^*: $K:=\mathrm{Ker}(\iota^*:\mathbb{J}_k\to\mathbb{J}_\kappa)$ for $k=\mathbb{Q}_p$ is a torsion $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module. Now we move weight 2 arithmetic points $P\in\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)\subset\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$. Then $K_P=K/PK$ covers surjectively

$$\mathrm{Ker}(\iota^*_P:\mathbb{J}_k/P\mathbb{J}_k\to\mathbb{J}_\kappa/P\mathbb{J}_\kappa).$$

By $\widetilde{\mathbb{I}}[\frac{1}{p}]$-torsion property of K, $K/PK=0$ for almost all points in $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, and we get

Corollary 7.2. *Let the notation and the assumption be as above. Suppose the condition* (a), (dim), $k=\kappa_{\mathfrak{p}}=\mathbb{Q}_p$ *and*

$$\dim_{\mathbb{Q}(f)}A_0(\kappa)>0.$$

Then except for finitely many arithmetic points of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ *weight* 2, *we have* $\dim_{\mathbb{Q}(f_P)}A_P(\kappa)>0$ *and*

$$\dim_{\mathbb{Q}_p(f_P)}(\overline{A_P(\kappa)}^{\mathrm{ord}}\otimes_{\mathbb{I}/P}\mathbb{Q}_p(f_P))=\dim_{\mathbb{Q}_p}\mathbb{Q}_p(f_P).$$

For general abelian variety $A_{/\mathbb{Q}}$, an estimate of $\dim_{\mathbb{Q}_p}\overline{A(\mathbb{Q})}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ relative to $\dim_{\mathbb{Q}}A(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Q}$ and a conjecture is given in [W11]. Here we studied the dimension over a family and showed its co-ordinary (or ordinary) part stays maximal for most of members of the family if one is maximal.

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