Hermitian Lattices and Bounds in K-Theory of Algebraic Integers

To Sasha Merkurjev, for his sixtieth birthday

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Abstract. Elaborating on a method of Soulé, and using better estimates for the geometry of hermitian lattices, we improve the upper bounds for the torsion part of the K-theory of the rings of integers of number fields.

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1 Introduction

Let F be a number field of degree d, with ring of integers $\mathcal{O}_F$ and discriminant D_F. We denote by $K_n(\mathcal{O}_F)$ the n-th K-theory group of $\mathcal{O}_F$, which was defined by Quillen and showed by him to be finitely generated. The rank of $K_n(\mathcal{O}_F)$ has been computed by Borel in [4]. In this article we consider the problem of finding an upper bound – in terms of n, d and D_F – for the order of the torsion part $K_n(\mathcal{O}_F)_{\mathrm{tors}}$. Such general bounds have been obtained by Soulé in [11]. Our Theorem 1.1 below sharpens Soulé's results.

As in Soulé's paper, our inequalities hold "up to small torsion". To state this precisely, for a finite abelian group A let us write $\mathrm{card}_\ell(A)$ for the order of A/B, where $B \subset A$ is the subgroup generated by elements of order $\leq \ell$.

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Theorem 1.1. *Let $F \neq \mathbb{Q}$ be a number field of degree d and with discriminant D_F. Then for any $n \geq 2$ we have*

$$\log \mathrm{card}_\ell K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq (2n+1)^{71n^4d^3} \cdot d^{293n^5d^5} \cdot |D_F|^{528n^5d^4},$$

where $\ell = \max(d+1, 2n+2)$.

To improve the readability, we have not tried to state here the best possible bounds that one could get with the method we use. We refer to the PhD thesis of the third author [7, Theorem 4.3] – in which the result was originally obtained – for slightly better estimates. However, it does not change the fact that the upper bounds are huge, and – although explicit – certainly unusable for practical computation. We shall insist here on the qualitative aspect of our result, which could be stated as follows.

Corollary 1.2. *There exist α and β, both polynomials in n and d, such that for any number field F of degree $d \geq 2$ we have*

$$\log \mathrm{card}_\ell K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq (nd)^\alpha |D_F|^\beta,$$

where $\ell = \max(d+1, 2n+2)$ and $n \geq 2$.

Compared to [11], our result improves the bound by an exponential factor: the previous bound for $\log \mathrm{card}_\ell K_n(\mathcal{O}_F)$ was at least $\exp(\alpha|D_F|^{1/2})$, for some polynomial $\alpha = \alpha(n,d)$ (see Proposition 4 in loc. cit. for the precise statement). The strategy is the following. The group $K_n(\mathcal{O}_F)$ can be related – via the Hurewicz map – to the integral homology $H_n(\mathrm{GL}(\mathcal{O}_F))$, and an upper bound (up to small torsion) for the order of $K_n(\mathcal{O}_F)$ can then be obtained through the study of the integral homology of $\mathrm{GL}_N(\mathcal{O}_F)$, with $N = 2n+1$ (cf. Section 3). The proof of Theorem 1.1 follows the method of Soulé, which uses Ash's well-rounded retract (cf. Section 2) to study these homology groups. This reduces the problem to finding good estimates concerning the geometry of hermitian lattices. Our approach to these estimates differs from that of Soulé (cf. Section 4), leading to the improved bounds in Theorem 1.1.
For $F = \mathbb{Q}$ our method does not bring any improvement, so that [11, Prop. 4 iv)] is still the best available general bound for $K_n(\mathbb{Z})$. We refer to [6, Theorem 1.3] for a different approach to the same problem for K_2, which gives better result than Corollary 1.2 in the case of totally imaginary fields. Note that all these results remain very far from the general bound conjectured by Soulé in [11, Sect. 5.1], which should take the following form for some constant $C(n,d)$:

$$\log \mathrm{card}\, K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq C(n,d) \log |D_F|. \tag{1.1}$$

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2 Hermitian metrics and the well-rounded retract

2.1 Notation

We keep the notation of the introduction. Let us denote by (r_1, r_2) the signature of the number field F. Let us write $F_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} F$. If Σ denotes the set of field embeddings $\sigma : F \to \mathbb{C}$, then $F_{\mathbb{R}}$ can be identified with the subspace $(\mathbb{C}^{\Sigma})^{+} \subset \mathbb{C}^{\Sigma}$ invariant under the involution $(x_\sigma) \mapsto (\overline{x_{\overline{\sigma}}})$, where $\overline{a}$ denotes the complex conjugation. This also provides an isomorphism $F_{\mathbb{R}} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

For $x \in F_{\mathbb{R}}$, written as $x = (x_\sigma)_{\sigma \in \Sigma}$, we denote by $\overline{x} = (\overline{x_\sigma})$ the result of the complex conjugation applied component-wise. We denote by Tr the trace map from $F_{\mathbb{R}}$ to $\mathbb{R}$, defined by $\mathrm{Tr}(x) = \sum_{\sigma \in \Sigma} x_\sigma$. We will also use the absolute value of the norm map: $\mathcal{N}(x) = \prod_{\sigma \in \Sigma} |x_\sigma|$.

We fix a free $\mathcal{O}_F$-lattice L of finite rank $N \geq 1$. Let $V = F \otimes_{\mathcal{O}_F} L$ and $V_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} V$, so that $V_{\mathbb{R}}$ can be seen as a (left) $F_{\mathbb{R}}$-module. Let Γ be the group $\mathrm{GL}(L)$ of automorphism of L. By fixing a basis of L, we have the identification $\Gamma = \mathrm{GL}_N(\mathcal{O}_F)$. Then Γ is a discrete subgroup of the reductive Lie group $\mathrm{GL}(V_{\mathbb{R}}) = \mathrm{GL}_N(F_{\mathbb{R}})$. We shall denote the latter by G, and we will let it act on $V_{\mathbb{R}}$ on the left (and similarly for Γ on L).

2.2 Hermitian metrics

Let $h : V_{\mathbb{R}} \times V_{\mathbb{R}} \to F_{\mathbb{R}}$ be a hermitian form on $V_{\mathbb{R}}$, that is, $h = (h_\sigma)_{\sigma \in \Sigma}$ is $F_{\mathbb{R}}$-linear in the first variable and $h(y,x) = \overline{h(x,y)}$. The pair (L,h) is called a *hermitian lattice*. When $x = y$, we also write $h(x) = h(x,x)$. Note that $h(x)$ has only real components, and we say that h is positive definite if $h_\sigma(x,x) > 0$ for any nonzero $x \in V_{\mathbb{R}}$ and all $\sigma \in \Sigma$.

Let X be the (topological) space of positive definite hermitian forms on $V_{\mathbb{R}}$. The group $G = \mathrm{GL}(V_{\mathbb{R}})$ acts transitively on X in the following way: the element $\gamma \in G$ maps the form $h \in X$ to

$$(\gamma \cdot h)(x,y) = h(\gamma^{-1}x, \gamma^{-1}y). \tag{2.1}$$

The space X can be identified with the set of positive definite symmetric $N \times N$ matrices with coefficients in $F_{\mathbb{R}}$. Using this identification, it is not difficult to see that X is contractible.

To each $h \in X$ we associate the real quadratic form q_h on $V_{\mathbb{R}}$ (seen as a real vector space) defined for $x \in V_{\mathbb{R}}$ by:

$$q_h(x) = \mathrm{Tr}(h(x)). \tag{2.2}$$

Such a quadratic form $q_h : V_{\mathbb{R}} \to \mathbb{R}$ for $h \in X$ will be called a *hermitian metric*. Given h, we will denote by $\|\cdot\|_h$ the norm on $V_{\mathbb{R}}$ induced by q_h.

2.3 Ash's well-rounded retract

For $h \in X$ we set $m(L,h) \in \mathbb{R}_{>0}$ to be the minimum of $q_h(x)$ over the nonzero $x \in L \subset V$, and define

$$M(L,h) = \{x \in L \mid q_h(x) = m(L,h)\}. \tag{2.3}$$

Definition 2.1. We say that $h \in X$ (or (L,h)) is *well rounded* if $m(L,h) = 1$ and $M(L,h)$ generates V (as a vector space over F).

Let $\widetilde{W} \subset X$ be the subspace of well-rounded hermitian forms. Note that the action defined by (2.1) restricts to an action of $\Gamma = \mathrm{GL}_N(\mathcal{O}_F)$ on $\widetilde{W}$. In [2, p. 466–467], Ash defined a CW-complex structure on $\widetilde{W}$ that has the following properties: two points h and h' of $\widetilde{W}$ belong to the interior of the same cell $C(h) = C(h')$ if and only if $M(L,h) = M(L,h')$, and $C(h') \subset C(h)$ if and only if $M(L,h') \supset M(L,h)$. Moreover, we have that $M(L,\gamma \cdot h) = \gamma M(L,h)$. Then the action of Γ on $\widetilde{W}$ is compatible with the cell structure: an element $\gamma \in \Gamma$ maps the cell $C(h)$ to $C(\gamma \cdot h)$.

Theorem 2.2 (Ash). *$\widetilde{W}$ is a deformation retract of X on which Γ acts with finite stabilizer Γ_σ for each cell σ of $\widetilde{W}$. The quotient $\Gamma\backslash\widetilde{W}$ is compact, of dimension* $\dim(X) - N$.

Proof. The proof of this statement follows from the argument of Ash given in the proof of the main theorem of [2], page 462. More precisely, the argument to prove that $\widetilde{W}$ is a deformation retract of X is the same as Ash uses for $W = \widetilde{W}/\Gamma$, in [2, §3 (i)]. The compactness of $\Gamma\backslash\widetilde{W}$ is proved in §3 (ii) of loc. cit., and the dimension is computed on page 466. □

Let $\mathcal{C}_\bullet$ be the complex of cellular chains on $\Gamma\backslash\widetilde{W}$. We can decompose it as $\mathcal{C}_\bullet = \mathcal{C}_\bullet^+ \cup \mathcal{C}_\bullet^-$, where Γ preserves (resp. does not preserve) the orientation of each $\sigma \in \mathcal{C}_\bullet^+$ (resp. $\sigma \in \mathcal{C}_\bullet^-$). It then follows from the spectral sequence described in [5, VII (7.10)] that up to prime divisors of the finite stabilizers Γ_σ, the homology of $\mathcal{C}_\bullet^+$ computes $H_\bullet(\Gamma)$. In particular, one has the following (cf. [11, Lemma 9]). See Section 1 for the definition of card_ℓ.

Corollary 2.3. *Let $\ell = 1 + \max(d,N)$. Then for any n we have:*

$$\mathrm{card}_\ell\, H_n(\Gamma)_{\mathrm{tors}} = \mathrm{card}_\ell\, H_n(\mathcal{C}_\bullet^+)_{\mathrm{tors}},$$

where $H_n(\cdot)_{\mathrm{tors}}$ denotes the torsion part of the integral homology.

3 Bounding torsion homology and K-theory

3.1 The Hurewicz map

For any $n > 1$ we consider the n-th Quillen K-group $K_n(\mathcal{O}_F) = \pi_n(B\,\mathrm{GL}(\mathcal{O}_F)^+)$ ("plus construction"). The Hurewicz map relating homotopy groups to homology provides a map $K_n(\mathcal{O}_F) \to H_n(\mathrm{GL}(\mathcal{O}_F)^+) =$

$H_n(\mathrm{GL}(\mathcal{O}_F))$. We know (see for instance [1, Theorem 1.5]) that its kernel does not contain elements of order p for $p > \frac{n+1}{2}$. Moreover, by a stability result of van der Kallen and Maazen (cf. [12, Theorem 4.11]) the homology of $\mathrm{GL}(\mathcal{O}_F) = \varinjlim \mathrm{GL}_N(\mathcal{O}_F)$ is equal to the homology of $\mathrm{GL}_N(\mathcal{O}_F)$ for any $N \geq 2n+1$. Let then $N = 2n+1$, and consider $\Gamma = \mathrm{GL}_N(\mathcal{O}_F)$. We deduce from Corollary 2.3 that for $\ell = \max(d+1, 2n+2)$ we have:

$$\mathrm{card}_\ell\, K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq \mathrm{card}_\ell\, H_n(\mathcal{C}^+_\bullet)_{\mathrm{tors}}. \tag{3.1}$$

3.2 Gabber's lemma

The abstract result that allows to obtain a bound for the right hand side of (3.1) is the following lemma. It was discovered by Gabber, and first appeared in Soulé [10, Lemma 1]. See Sauer [9, Lemma 3.2] for a more elementary proof.

Lemma 3.1 (Gabber). *Let $A = \mathbb{Z}^a$ with the standard basis $(e_i)_{i=1,\ldots,a}$ and $B = \mathbb{Z}^b$, so that $B \otimes \mathbb{R}$ is equipped with the standard Euclidean norm $\|\cdot\|$. Let $\phi : A \to B$ be a $\mathbb{Z}$-linear map and let $\alpha \in \mathbb{R}$ such that $\|\phi(e_i)\| \leq \alpha$ for each $i = 1,\ldots,a$. If we denote by Q the cokernel of ϕ, then*

$$|Q_{\mathrm{tors}}| \leq \alpha^{\min(a,b)}\,.$$

Corollary 3.2. *Suppose that the cellular complex $\Gamma\backslash\widetilde{W}$ has at most α_k faces for any $k \geq 0$, and that any k-cell has at most β codimension 1 faces. Then*

$$H_k(\mathcal{C}^+_\bullet)_{\mathrm{tors}} \leq \beta^{\frac{1}{2}\min(\alpha_{k+1},\alpha_k)}.$$

Proof. For a cell $c \in \mathcal{C}^+_{k+1}$, its image by the boundary map ∂ is a sum of at most β k-cells, so that $\|\partial c\| \leq \sqrt{\beta}$. Thus, by Lemma 3.1 $\mathrm{coker}(\partial)_{\mathrm{tors}}$ is bounded by $\beta^{\frac{1}{2}\min(\alpha_{k+1},\alpha_k)}$ and a fortiori so is $H_k(\mathcal{C}^+_\bullet)_{\mathrm{tors}}$. □

3.3 Counting the cells

Suppose that the finite subset $\Phi \subset L$ has the following property:

> for any well-rounded pair (L,h), there exists $\gamma \in \Gamma = \mathrm{GL}_N(\mathcal{O}_F)$ such that $\gamma M(L,h) \subset \Phi$.

In other words, Φ contains a representative of every element of $\Gamma\backslash\widetilde{W}$. Since $C(h)$ has codimension j, where $N+j$ is the cardinality of $M(L,h)$, it follows immediately that the number of cells of codimension j in $\Gamma\backslash\widetilde{W}$ is bounded by the binomial coefficient $\binom{\mathrm{card}(\Phi)}{N+j}$. For large $\mathrm{card}(\Phi)$ we lose little by bounding this binomial coefficient by $\mathrm{card}(\Phi)^{N+j}$. Recall that $\Gamma\backslash\widetilde{W}$ has dimension

$\dim(X)-N$, so that for a k-cell of codimension j we have $N+j=\dim(X)-k$. For the dimension of X we have (where (r_1,r_2) is the signature of F):

$$\dim(X) = r_1\frac{N(N+1)}{2} + r_2 N^2 \tag{3.2}$$

$$\leq d\,\frac{N(N+1)}{2}. \tag{3.3}$$

Thus, for the number of k-cells in $\Gamma\backslash\widetilde{W}$ we can use the following upper bound:

$$\alpha_k = \mathrm{card}(\Phi)^{d\cdot\frac{N(N+1)}{2}-k} \tag{3.4}$$

By a similar counting argument, Soulé shows in [11, proof of Prop. 3] that there are at most $\beta = \mathrm{card}(\Phi)^{N+1}$ faces of codimension 1 in any given cell (not necessarily top dimensional) on $\Gamma\backslash\widetilde{W}$.

3.4 Bounds for K-theory in terms of Φ

Let $\ell = \max(d+1, 2n+2)$. By Corollary 3.2 and (3.1) we have that $\mathrm{card}_\ell\, K_n(\mathcal{O}_F)_{\mathrm{tors}}$ is bounded by $\beta^{\frac{1}{2}\alpha_{n+1}}$, where α_{n+1} and β can be chosen as in Section 3.3 (with $N=2n+1$). This gives (using now logarithmic notation):

$$\log \mathrm{card}_\ell\, K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq (n+1)\log(\mathrm{card}(\Phi))\,\mathrm{card}(\Phi)^{e(d,n)}, \tag{3.5}$$

where $\Phi\subset L$ has the property defined in Section 3.3, and

$$e(d,n) = d(2n^2+3n+1)-n-1. \tag{3.6}$$

This reduces the problem to finding such a set $\Phi\subset L$ of size as small as possible. In [11] Soulé constructed a suitable set Φ using the geometry of numbers. In what follows, we will exhibit a smaller Φ by using better estimates on hermitian lattices.

4 Hermitian lattices and bounded bases

The goal of this section is to construct in any well-rounded lattice (L,h) a basis whose vectors have bounded length, with respect to the norm induced by h. The method in an adaptation of the idea used by Soulé in [11] (see Section 4.3 below), in which we incorporate the results from [3], corresponding to the rank one case.

4.1 Geometry of ideal lattices

Let $I\subset F_{\mathbb{R}}$ be a nonzero $\mathcal{O}_F$-submodule of the form $I=x\mathfrak{a}$, where $x\in F_{\mathbb{R}}$ and $\mathfrak{a}$ is a fractional ideal of F. We define the *norm* of I by the rule $\mathcal{N}(I)=\mathcal{N}(x)\mathcal{N}(\mathfrak{a})$. Let q_0 be the standard (positive definite) hermitian metric on $F_{\mathbb{R}}$, i.e., for

$x \in F_{\mathbb{R}}$: $q(x) = \mathrm{Tr}(x\overline{x})$. The pair (I, q_0) is an *ideal lattice* (over F) in the sense of [3, Def. 2.2]. Its determinant is given by (see [3, Cor. 2.4]):

$$\det(I, q_0) = \mathcal{N}(I)^2\,|D_F|. \tag{4.1}$$

Let us denote by $\|\cdot\|$ the norm on $F_{\mathbb{R}}$ induced by the hermitian metric q_0. Estimates for the geometry of ideal lattices have been studied in [3]. For our particular case (I, q_0), Proposition 4.2 in loc. cit. takes the following form.

Proposition 4.1. *Let F of degree d, with discriminant D_F, and consider the ideal lattice (I, q_0). Then for any $x \in F_{\mathbb{R}}$ there exists $y \in I$ such that $\|x - y\| \leq R$, where*

$$R = \frac{\sqrt{d}}{2}|D_F|^{1/d}\mathcal{N}(I)^{1/d}.$$

4.2 Three consequences

From Proposition 4.1 we deduce the three following lemmas.

Lemma 4.2. *Given $x = (x_\sigma) \in F_{\mathbb{R}}$, there exists $a \in \mathcal{O}_F$ such that*

$$\sum_{\sigma\in\Sigma} |x_\sigma - \sigma(a)| \leq C_2,$$

where

$$C_2 = \frac{d}{2}\,|D_F|^{1/d}. \tag{4.2}$$

Proof. First note the general inequality $(\sum_{i=1}^d b_i)^2 \leq d\cdot\sum_{i=1}^d b_i^2$, which follows from applying the summation $\sum_{i,j}$ on both sides of

$$2b_ib_j \leq b_i^2 + b_j^2.$$

This implies that for $a = y \in I$ as in Proposition 4.1 with $I = \mathcal{O}_F$, we have

$$\begin{aligned}\sum_{\sigma\in\Sigma} |x_\sigma - \sigma(a)| &\leq \sqrt{d\cdot\sum_{\sigma\in\Sigma}|x_\sigma - \sigma(a)|^2}\\ &= \sqrt{d}\,\|x - a\|\\ &\leq \frac{d}{2}|D_F|^{1/d}.\end{aligned}$$

□

Lemma 4.3. *Given $x = (x_\sigma) \in F_{\mathbb{R}}$, there exists $a \in \mathcal{O}_F$ such that*

$$\sup_{\sigma\in\Sigma} |\sigma(a)\,x_\sigma| \leq C_3\,\mathcal{N}(x)^{1/d},$$

where

$$C_3 = \sqrt{d}\cdot|D_F|^{1/d}. \tag{4.3}$$

Proof. We can suppose that $x \neq 0$. We consider the ideal lattice (I, q_0) with $I = x\mathcal{O}_F$. For R as in Proposition 4.1, we have that $F_{\mathbb{R}} = I + B_R(0)$, where $B_R(0)$ is the closed ball of radius R with respect to $\|\cdot\|$. In particular, the smallest (nonzero) vector $xa \in I = x\mathcal{O}_F$ has length $\leq 2R$. That is, there exists $a \in \mathcal{O}_F$ such that

$$\begin{aligned} \sup_{\sigma\in\Sigma} |\sigma(a) x_\sigma| &\leq \|xa\| \\ &\leq 2R; \end{aligned}$$

and the result follows. □

Lemma 4.4. *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_F$. Then there exists a set $\mathcal{R} \subset \mathcal{O}_F$ of representatives of $\mathcal{O}_F/\mathfrak{a}$ such that for any $x \in \mathcal{R}$ we have*

$$\sum_{\sigma\in\Sigma} |\sigma(x)| \leq C_2 \mathcal{N}(\mathfrak{a})^{1/d}.$$

Proof. Let us consider the ideal lattice $(I, q) = (\mathfrak{a}, q_0)$, and let R be as in Proposition 4.1. Then for any $x \in \mathcal{O}_F \subset F_{\mathbb{R}}$, there exists $y \in I$ such that $\|x - y\| \leq R$. But $x - y \equiv x\,(I)$, so that the closed ball $B_R(0)$ contains a representative of each class of $\mathcal{O}_F/\mathfrak{a}$. The inequality is then obtained as in the proof of Lemma 4.2. □

4.3 Existence of bounded bases

Lemma 4.5 (Soulé). *Let $L = L_1 \oplus \cdots \oplus L_N$ be a decomposition of the hermitian lattice (L, h) into rank one lattices, and suppose that each L_i contains a vector f_i with $|L_i/\mathcal{O}_F f_i| \leq k$ and $\|f_i\|_h \leq k\lambda$, for some $k, \lambda > 1$. Then L has a basis $e_1, \ldots, e_N$ such that*

$$\|e_i\|_h \leq \lambda(1 + C_2)^{t+1} k^{(d+1)(4N-1)},$$

where $t = \lfloor \log_2(N) \rfloor + 1$.

Proof. The statement and its proof is essentially contained in the proof of [11, Prop. 1]. The main difference is that our constant C_2 is now smaller than C_2 in loc. cit. We can follow verbatim the same proof with the new C_2 except for the use of Lemma 6 (needed in Lemma 7) of loc. cit., which must be replaced by Lemma 4.4. Accordingly, the factor $(1 + C_2 \frac{r+3}{4})$ (where $r = d$) is replaced by $1 + C_2$. □

To obtain a bounded basis for (L, h) we need to find elements f_i that satisfy the condition of Lemma 4.5. This is done in the following proposition.

Proposition 4.6. *Let (L, h) be a free hermitian lattice over $\mathcal{O}_F$ of rank N, with $F \neq \mathbb{Q}$. We suppose that there exist $e_1, \ldots, e_N \in L$ that span $V = F \otimes_{\mathcal{O}_F} L$*

and such that $\|e_i\|_h \leq 1$ for $i = 1, \dots, N$. Then there exists a decomposition $L = L_1 \oplus \cdots \oplus L_N$ and elements $f_i \in L_i$ such that:

$$|L_i / f_i \mathcal{O}_F| \leq C_1 C_3^d \,;$$
$$\|f_i\|_h \leq i C_1 C_2 C_3^d,$$

where $C_1 = |D_F|^{1/2}$, and C_2 (resp. C_3) is defined in (4.2) (resp. (4.3)).

Proof. The proof proceeds by induction, and follows the line of argument of [11, proof of Lemma 5]. Let $N = 1$. By Lemma 1 in loc. cit., there exists $x \in L$ such that $|L/\mathcal{O}_F x| \leq C_1 = |D_F|^{1/2}$. Let us write $x = \alpha \cdot e_1$ for $\alpha \in F^\times$, where by assumption $\|e_1\|_h \leq 1$. By Lemma 4.3 applied to $\alpha \in F_\mathbb{R}$, there exists $a \in \mathcal{O}_F$ such $\sup_\sigma |\sigma(a\alpha)| \leq C_3 |N(\alpha)|^{1/d}$. In particular,

$$\begin{aligned} |N(a\alpha)| &\leq \left(\sup_{\sigma \in \Sigma} |\sigma(a\alpha)| \right)^d \\ &\leq C_3^d \, |N(\alpha)|, \end{aligned}$$

so that $|N(a)| \leq C_3^d$. We set $f_1 = a \cdot x$. Then

$$\begin{aligned} |L/\mathcal{O}_F f_1| &= |N(a)| \cdot |L/\mathcal{O}_F x| \\ &\leq C_3^d C_1. \end{aligned}$$

For the norm we have:

$$\begin{aligned} \|f_1\|_h^2 &= \mathrm{Tr}(h(f_1, f_1)) \\ &= \sum_{\sigma \in \Sigma} |\sigma(\alpha)|^2 \, h_\sigma(e_1, e_1) \\ &\leq \left(\sup_{\sigma \in \Sigma} |\sigma(\alpha)| \right)^2 \, \|e_1\|_h^2 \\ &\leq C_3^2 \cdot |N(\alpha)|^{2/d}. \end{aligned}$$

Moreover, $|L/\mathcal{O}_F x| = |N(\alpha)| \cdot |L/\mathcal{O}_F e_1|$, so that $|N(\alpha)| \leq C_1$. This shows that $\|f_1\|_h \leq C_3 C_1^{1/d}$ and thus concludes the proof for $N = 1$.
The induction step is done exactly as in loc. cit., adapting the constants when necessary (C_1 to be replaced by $C_1 C_3^d$), to obtain the desired $f_i \in L_i$, i.e., with (using $F \neq \mathbb{Q}$ in the last inequality, so that $C_2 \geq 1$):

$$\begin{aligned} \|f_i\|_h &\leq (i-1) C_1 C_3^d C_2 + C_3 C_1^{1/d} \\ &\leq i C_1 C_2 C_3^d. \end{aligned}$$

□

We finally obtain the result about the existence of bounded bases. The assumption $N \geq 5$ is only here in order to simplify the statement.

Proposition 4.7. *Let (L,h) be a free hermitian lattice over $\mathcal{O}_F$ of rank $N \geq 5$, with $F \neq \mathbb{Q}$, and such that the subset $\{x \in L \mid \|x\|_h \leq 1\}$ spans $V = F \otimes_{\mathcal{O}_F} L$. Then there exists a basis $e_1, \dots, e_N$ of L such that $\|e_i\|_h \leq B$ for every $i = 1, \dots, N$, where*

$$B = \frac{4N^2}{2^N} d^{5Nd^2} |D_F|^{6N(d+1)}.$$

Proof. By Proposition 4.6 we can apply Lemma 4.5 with

$$k = C_1 C_3^d\,;$$
$$\lambda = N C_2.$$

This shows the existence of a basis $e_1, \dots, e_N$ with

$$\|e_i\|_h \leq N C_2 (1 + C_2)^{\lfloor \log_2(N) \rfloor + 2} (C_1 C_3^d)^{(d+1)(4N-1)}.$$

Since $C_2 \geq 1$, we have $(1 + C_2)^n \leq 2^n C_2^n$. Moreover, for $N \geq 5$ we have $\lfloor \log_2(N) \rfloor + 3 \leq N$. We deduce:

$$\begin{aligned} \|e_i\|_h &\leq 4N^2 C_2^N (C_1 C_3^d)^{(d+1)(4N-1)} \\ &= \alpha d^\beta |D_F|^\gamma, \end{aligned}$$

with (using $N \geq 5$ and $d \geq 2$):

$$\begin{aligned} \alpha &= 4\,\frac{N^2}{2^N}\,; \\ \beta &= N + \frac{d}{2}(d+1)(4N-1) \\ &\leq 5Nd^2\,; \\ \gamma &= \frac{N}{d} + \frac{3}{2}(d+1)(4N-1) \\ &\leq 6(d+1)N. \end{aligned}$$

This finishes the proof. □

5 Improved estimates for K-groups

5.1 A Bounded set Φ

The construction of a bounded set $\Phi \subset L$ will follow from this proposition.

Proposition 5.1. *Let (L,h) be a free well-rounded $\mathcal{O}_F$-lattice of rank $N \geq 5$, with $F \neq \mathbb{Q}$. Let $e_1, \dots, e_N$ and B be defined as in Proposition 4.7, and for $x \in M(L,h)$ write $x = \sum_i x_i e_i$, with $x_i \in \mathcal{O}_F$. Then for every $i = 1, \dots, N$ we have:*

$$\sum_{\sigma \in \Sigma} |\sigma(x_i)|^2 \leq T,$$

where

$$T = N^{Nd} d^{\frac{3}{2}Nd+1} B^{2(Nd-1)} |D_F|^{2N}.$$

Proof. Let $x \in M(L,h)$, i.e., $h(x) = 1$. For each $\sigma \in \Sigma$ let us consider the matrix $H_\sigma = (h_\sigma(e_i, e_j))$. Then the first argument in [11, proof of Prop. 2], based on the Hadamard inequality for positive definite matrix, shows that

$$|\sigma(x_i)|^2 \leq \det(H_\sigma)^{-1} h_\sigma(x) \prod_{j \neq i} h_\sigma(e_j).$$

Since $h_\sigma(e_j) \leq \|e_j\|_h^2 \leq B^2$, and similarly $h_\sigma(x) \leq 1$, we obtain:

$$\sum_{\sigma \in \Sigma} |\sigma(x_i)|^2 \leq B^{2(N-1)} \sum_{\sigma \in \Sigma} \det(H_\sigma)^{-1}. \tag{5.1}$$

For $\sum_\sigma \det(H_\sigma)^{-1}$ we can write, using the Hadamard inequality:

$$\begin{aligned}
\sum_{\sigma \in \Sigma} \det(H_\sigma)^{-1} &= \sum_{\sigma \in \Sigma} \left(\prod_{\sigma' \neq \sigma} \det(H_{\sigma'}) \prod_{\sigma' \in \Sigma} \det(H_{\sigma'})^{-1} \right) \\
&\leq \left(\sum_{\sigma \in \Sigma} \prod_{\sigma' \neq \sigma} \prod_{j=1}^{N} h_{\sigma'}(e_j) \right) \cdot \left(\prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1} \right) \\
&\leq d \cdot B^{2N(d-1)} \prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1}. \qquad (5.2)
\end{aligned}$$

According to Icaza [8, Theorem 1], there exists $z \in L$ such that

$$\prod_{\sigma \in \Sigma} \det(H_\sigma)^{-1} \leq \gamma^N \mathcal{N}(h(z))^{-N}, \tag{5.3}$$

where (cf. [11, Equ. (21)]):

$$\gamma \leq N^d |D_F|.$$

By applying Lemma 4.3 to $h(z) \in F_{\mathbb{R}}$, we find $a \in \mathcal{O}_F$ such that $h_\sigma(az) = \sigma(a) h_\sigma(z) \leq C_3 \mathcal{N}(h(z))^{1/d}$ for every $\sigma \in \Sigma$. Since (L,h) is well rounded, this implies:

$$d C_3 \mathcal{N}(h(z))^{1/d} \geq h(az) \geq 1, \tag{5.4}$$

so that $\mathcal{N}(h(z))^{-1} \leq d^d C_3^d = d^{\frac{3}{2}d} |D_F|$. Using this with (5.1), (5.2) and (5.3), this concludes the proof. $\square$

Corollary 5.2. *Let L be a free $\mathcal{O}_F$-lattice of rank $N \geq 5$, with $F \neq \mathbb{Q}$. Then there exists a subset $\Phi \subset L$ with the property given in Section 3.3 and such that*

$$\operatorname{card}(\Phi) \leq N^{3N^2 d^2} \cdot d^{5N^3 d^4} \cdot |D_F|^{9N^3 d^3}.$$

Proof. Let $f_1,\dots,f_N$ be any basis of L, and set, for T as in Proposition 5.1:

$$\Phi = \left\{ \sum_{i=1}^{N} x_i f_i \;\middle|\; x_i \in \mathcal{O}_F \text{ with } \sum_{\sigma\in\Sigma} |\sigma(x_i)|^2 \leq T \right\}.$$

According to [11, Lemma 8], the number of elements $x_i \in \mathcal{O}_F$ with $\sum_{\sigma\in\Sigma}|\sigma(x_i)|^2 \leq T$ is at most $T^{d/2}2^{d+3}$, so that $\operatorname{card}(\Phi)$ is bounded above by $T^{Nd/2}2^{N(d+3)}$. Expanding the constants T and B as in the statements of Propositions 5.1 and 4.7, we obtain the stated upper bound for $\operatorname{card}(\Phi)$.
Let h be a well-rounded hermitian metric on L. We can apply Proposition 5.1 to write every $x \in M(L,h)$ as $x = \sum x_i e_i$ for a bounded basis $e_1,\dots,e_N$ of L. The proposition implies that the transformation $\gamma \in \Gamma = \mathrm{GL}_N(\mathcal{O}_F)$ that sends the basis (e_i) to (f_i) is such that $\gamma \cdot x \in \Phi$. This means that Φ has the property defined in Section 3.3. ∎

5.2 Upper bounds for $K_n(\mathcal{O}_F)$

We finally come to the bounds for the K-groups of $\mathcal{O}_F$, as stated in Theorem 1.1. Let $\ell = \max(d+1, 2n+2)$. From Equation (3.5) we obtain

$$\log \operatorname{card}_\ell K_n(\mathcal{O}_F)_{\mathrm{tors}} \leq \operatorname{card}(\Phi)^{e(d,n)+n+1},$$

and note that for $n \geq 2$ we have $e(d,n)+n+1 \leq \frac{15}{4}n^2 d$. Theorem 1.1 now follows directly from Corollary 5.2, applied with $N = 2n+1 \leq \frac{5}{2}n$.

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