A Bound for the Torsion in the K-Theory of Algebraic Integers

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A Kazuya Kato, qui
marche sous la lune.

Abstract. Let m be an integer bigger than one, A a ring of algebraic integers, F its fraction field, and $K_m(A)$ the m-th Quillen K-group of A. We give a (huge) explicit bound for the order of the torsion subgroup of $K_m(A)$ (up to small primes), in terms of m, the degree of F over $\mathbf{Q}$, and its absolute discriminant.

Let F be a number field, A its ring of integers and $K_m(A)$ the m-th Quillen K-group of A. It was shown by Quillen that $K_m(A)$ is finitely generated. In this paper we shall give a (huge) explicit bound for the order of the torsion subgroup of $K_m(A)$ (up to small primes), in terms of m, the degree of F over $\mathbf{Q}$, and its absolute discriminant.

Our method is similar to the one developed in [13] for $F = \mathbf{Q}$. Namely, we reduce the problem to a bound on the torsion in the homology of the general linear group $\mathrm{GL}_N(A)$. Thanks to a result of Gabber, such a bound can be obtained by estimating the number of cells of given dimension in any complex of free abelian groups computing the homology of $\mathrm{GL}_N(A)$. Such a complex is derived from a contractible CW-complex $\widetilde{W}$ on which $\mathrm{GL}_N(A)$ with compact quotient. We shall use the construction of $\widetilde{W}$ given by Ash in [1] . It consists of those hermitian metrics h on A^N which have minimum equal to one and are such that their set $M(h)$ of minimal vectors has rank equal to N in F^N. To count cells in $\widetilde{W}/\mathrm{GL}_N(A)$, one will exhibit an explicit compact subset of $A^N \otimes_{\mathbf{Z}} \mathbf{R}$ which, for every $h \in \widetilde{W}$, contains a translate of $M(h)$ by some matrix of $\mathrm{GL}_N(A)$ (Proposition 2). The proof of this result relies on several arguments from the geometry of numbers using, among other things, the number field analog of Hermite's constant [4].

The bound on the K-theory of A implies a similar upper bound for the étale cohomology of $\mathrm{Spec}\,(A[1/p])$ with coefficients in the positive Tate twists of $\mathbf{Z}_p$, for any (big enough) prime number p.
However, this bound is quite large since it is doubly exponential both in m and, in general, the discriminant of F. We expect the correct answer to be polynomial in the discriminant and exponential in m (see 5.1).
The paper is organized as follows. In Section 1 we prove a few facts on the geometry of numbers for A, including a result about the image of A^* by the regulator map (Lemma 3), which was shown to us by H. Lenstra. Using these, we study in Section 2 hermitian lattices over A, and we get a bound on $M(h)$ when h lies in $\widetilde{W}$. The cell structure of $\widetilde{W}$ is described in Section 3. The main Theorems are proved in Section 4. Finally, we discuss these results in Section 5, where we notice that, because of the Lichtenbaum conjectures, a lower bound for higher regulators of number fields would probably provide much better upper bounds for the étale cohomology of $\mathrm{Spec}\,(A[1/p])$. We conclude with the example of $K_8(\mathbf{Z})$ and its relation to the Vandiver conjecture.

1 Geometry of algebraic numbers

1.1

Let F be a number field, and A its ring of integers. We denote by $r = [F : \mathbf{Q}]$ the degree of F over $\mathbf{Q}$ and by $D = |\mathrm{disc}\,(K/\mathbf{Q})|$ the absolute value of the discriminant of F over $\mathbf{Q}$. Let r_1 (resp. r_2) be the number of real (resp. complex) places of F. We have $r = r_1 + 2\,r_2$. We let $\Sigma = \mathrm{Hom}\,(F, \mathbf{C})$ be the set of complex embeddings of F. These notations will be used throughout.
Given a finite set X we let $\#\,(X)$ denote its cardinal.

1.2

We first need a few facts from the geometry of numbers applied to A and A^*. The first one is the following classical result of Minkowski:

Lemma 1. *Let L be a rank one torsion-free A-module. There exists a non zero element $x \in L$ such that the submodule spanned by x in L has index*

$$\#\,(L/Ax) \leq C_1\,,$$

where

$$C_1 = \frac{r!}{r^r} \cdot 4^{r_2}\,\pi^{-r_2}\,\sqrt{D}$$

in general, and $C_1 = 1$ when A is principal.

Proof. The A-module L is isomorphic to an ideal I in A. According to [7], V §4, p. 119, Minkowski's first theorem implies that there exists $x \in I$ the norm of which satisfies

$$|N(x)| \leq C_1\,N(I)\,.$$

Here $|N(x)| = \#\,(A/Ax)$ and $N(I) = \#\,(A/I)$, therefore $\#\,(I/Ax) \leq C_1$. The case where A is principal is clear. q.e.d.

1.3

The family of complex embeddings $\sigma : F \to \mathbf{C}$, $\sigma \in \Sigma$, gives rise to a canonical isomorphism of real vector spaces of dimension r

$$F \otimes_{\mathbf{Q}} \mathbf{R} = (\mathbf{C}^{\Sigma})^{+}\,,$$

where $(\cdot)^{+}$ denotes the subspace invariant under complex conjugation. Given $\alpha \in F$ we shall write sometimes $|\alpha|_\sigma$ instead of $|\sigma(\alpha)|$.

Lemma 2. *Given any element $x = (x_\sigma) \in F \otimes_{\mathbf{Q}} \mathbf{R}$, there exists $a \in A$ such that*

$$\sum_{\sigma\in\Sigma} |x_\sigma - \sigma(a)| \leq C_2\,,$$

with

$$C_2 = \frac{4^{r_1}\,\pi^{r_2}}{r^{r-2}\,r!}\,\sqrt{D}$$

in general, and

$$C_2 = 1/2 \qquad \text{if} \qquad F = \mathbf{Q}\,.$$

Proof. Define a norm on $F \otimes_{\mathbf{Q}} \mathbf{R}$ by the formula

$$\|x\| = \sum_{\sigma\in\Sigma} |x_\sigma|\,.$$

The additive group A is a lattice in $F \otimes_{\mathbf{Q}} \mathbf{R}$, and we let $\mu_1, \ldots, \mu_r$ be its successive minima. In particular, there exist $a_1, \ldots, a_r \in A$ such that $\|a_i\| = \mu_i$, $1 \leq i \leq r$, and $\{a_1, \ldots, a_r\}$ are linearly independent over $\mathbf{Z}$. Any $x \in F \otimes_{\mathbf{Q}} \mathbf{R}$ can be written

$$x = \sum_{i=1}^{r} \lambda_i\, a_i\,, \quad \lambda_i \in \mathbf{R}\,.$$

Let $n_i \in \mathbf{Z}$ be such that $|n_i - \lambda_i| \leq 1/2$, for all $i = 1, \ldots, r$, and

$$a = \sum_{i=1}^{r} n_i\, a_i\,.$$

Clearly

$$\|x - a\| \leq \sum_{i=1}^{r} |\lambda_i - n_i|\,\|a_i\| \leq \frac{1}{2}\,(\mu_1 + \cdots + \mu_r) \leq \frac{r}{2}\,\mu_r\,. \tag{1}$$

On the other hand, we know from the product formula that, given any $a \in A - \{0\}$,

$$\prod_{\sigma \in \Sigma} |\sigma(a)| \geq 1 \,. \tag{2}$$

By the inequality between arithmetic and geometric means this implies

$$\|a\| = \sum_{\sigma \in \Sigma} |\sigma(a)| \geq r \,,$$

hence

$$\mu_i \geq r \qquad \text{for all} \quad i = 1, \ldots, r \,. \tag{3}$$

Minkowski's second theorem tells us that

$$\mu_1 \ldots \mu_r \leq 2^r \, W \, 2^{-r_2} \sqrt{D} \tag{4}$$

([7], Lemma 2, p. 115), where W is the euclidean volume of the unit ball for $\| \cdot \|$ in $F \otimes_{\mathbf{Q}} \mathbf{R}$. (Note that the covolume of A is $\sqrt{D}$.) The volume W is the euclidean volume of those elements $(x_i, z_j) \in \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ such that

$$\sum_{i=1}^{r_1} |x_i| + 2 \sum_{j=1}^{r_2} |z_j| \leq 1 \,.$$

One finds ([7], Lemma 3, p. 117)

$$W = 2^{r_1} \, 4^{-r_2} (2\pi)^{r_2} / r! \,. \tag{5}$$

From (3) and (4) we get

$$\mu_r \leq 2^r \, W \sqrt{D} \, 2^{-r_2} \, r^{-(r-1)} \,. \tag{6}$$

The lemma follows from (1), (5) and (6). q.e.d.

1.4

We also need a multiplicative analog of Lemma 2. Let $R(F)$ be the regulator of F, as defined in [7] V, § 1, p. 109. Let $s = r_1 + r_2 - 1$.

LEMMA 3. *Let (λ_σ), $\sigma \in \Sigma$ be a family of positive real numbers such that $\lambda_{\overline{\sigma}} = \lambda_\sigma$ when $\overline{\sigma}$ is the complex conjugate of σ. There exists a unit $u \in A^*$ such that*

$$\operatorname*{Sup}_{\sigma \in \Sigma} (\lambda_\sigma \, |u|_\sigma) \leq C_3 \left(\prod_{\sigma \in \Sigma} \lambda_\sigma \right)^{1/r} ,$$

with

$$C_3 = \exp\left(s(4\,r\,(\log 3\,r)^3)^{s-1}\,2^{r_2-1}\,R(F)\right).$$

Proof. We follow an argument of H. Lenstra. Let $H \subset \mathbf{R}^{r_1+r_2}$ be the s-dimensional hyperplane consisting of vectors $(x_1, \ldots, x_{r_1+r_2})$ such that $x_1+x_2+\cdots+x_{r_1+r_2}=0$. Choose a subset $\{\sigma_1, \ldots, \sigma_{r_1+r_2}\} \subset \Sigma$ such that $\sigma_1, \ldots, \sigma_{r_1}$ are the real embeddings of Σ and $\sigma_i \neq \overline{\sigma}_j$ if $i \neq j$. Given $\lambda = (\lambda_\sigma)_{\sigma\in\Sigma}$ as in the lemma, we let

$$\rho(\lambda) = (\log(\lambda_{\sigma_1}), \ldots, \log(\lambda_{\sigma_{r_1}}), 2\log(\lambda_{\sigma_{r_1+1}}), \ldots, 2\log(\lambda_{\sigma_{r_1+r_2}}))\,.$$

If $u \in A^*$ is a unit, and $\lambda = (|u|_\sigma)$, we have $\rho(\lambda) \in H$. We get this way a lattice

$$L = \{\rho(|u|_\sigma)\,,\ u \in A^*\}$$

in H.

Define a norm $\|\cdot\|$ on H by the formula

$$\|(x_i)\| = \operatorname{Sup}\,(\sup_{1\leq i\leq r_1} |x_i|, \sup_{r_1+1\leq i\leq r_1+r_2} |x_i|/2)\,.$$

It is enough to show that, for any vector $x \in H$ there exists $a \in L$ such that

$$\|x-a\| \leq \log(C_3)\,.$$

According to [14], Cor. 2, p. 84, we have (when $r \geq 2$)

$$\|a\| \geq \varepsilon$$

where

$$\varepsilon = r^{-1}(\log(3\,r))^{-3}\,,$$

for any $a \in L - \{0\}$. Therefore, using Minkowski's second theorem as in the proof of Lemma 2 we get that, for any $x \in H$ there exists $a \in L$ with

$$\|x-a\| \leq s\,2^{s-1}\,\varepsilon^{1-s}\,W\operatorname{vol}(H/L)\,,$$

where W is the euclidean volume of the unit ball for $\|\cdot\|$, where we identify H with $\mathbf{R}^s$ by projecting on the first s coordinates. Clearly $W \leq 2^{s+r_2-1}$ when, by definition (loc.cit.), $\operatorname{vol}(H/L)$ is equal to $R(F)$. The lemma follows.

1.5

We now give an upper bound for the constant C_3 of Lemma 3.

Lemma 4. *The following inequality holds*

$$R \leq 11\,r^2\,\sqrt{D}\log(D)^{r-1}\,.$$

Proof. Let κ be the residue at $s = 1$ of the zeta function of F. According to [11], Cor. 3, p. 333, we have

$$\kappa \leq 2^{r+1} D^a a^{1-r}$$

whenever $0 < a \leq 1$. Taking $a = \log(D)^{-1}$ we get

$$\kappa \leq e\, 2^{r+1} \log(D)^{r-1} . \tag{7}$$

On the other hand

$$\kappa = 2^{r_1} (2\pi)^{r_2} \frac{h(F)\, R(F)}{w(F)\sqrt{D}} , \tag{8}$$

where $h(F)$ is the class number of F and $w(F)$ the number of roots of unity in F ([7], Prop. 13, p. 300). Since $h(F) \geq 1$ we get

$$R(F) \leq w(F)\, 2^{-(r_1+r_2)} \pi^{-r_2} e\, 2^{r+1} \sqrt{D} \log(D)^{r-1} .$$

Since the degree over $\mathbf{Q}$ of $\mathbf{Q}(\sqrt[n]{1})$ is $\varphi(n)$, where φ is the Euler function, we must have

$$\varphi(w(F)) \leq r .$$

When $n = p^t$ is an odd prime power we have

$$\varphi(p^t) = (p-1)\, p^{t-1} \geq p^{t/2} .$$

Therefore

$$w(F) \leq 2\, r^2 .$$

Since

$$2^{-(r_1+r_2)} \pi^{-r_2} 2^r = \left(\frac{2}{\pi}\right)^{r_2} \leq 1$$

and $4\, e \leq 11$, the lemma follows.

2 Hermitian lattices

2.1

An *hermitian lattice* $\overline{M} = (M, h)$ is a torsion free A-module M of finite rank, equipped with an hermitian scalar product h on $M \otimes_{\mathbf{Z}} \mathbf{C}$ which is invariant under complex conjugation. In other words, if we let $M_\sigma = M \otimes_A \mathbf{C}$ be the complex vector space obtained from M by extension of scalars via $\sigma \in \Sigma$, h is given by a collection of hermitian scalar products h_σ on M_σ, $\sigma \in \Sigma$, such that $h_{\overline{\sigma}}(x, y) = h_\sigma(x, y)$ whenever x and y are in M.
We shall also write

$$h_\sigma(x) = h_\sigma(x, x)$$

and

$$\|x\|_\sigma = \sqrt{h_\sigma(x)}\,.$$

Lemma 5. *Let $\overline{M}$ be an hermitian lattice of rank N. Assume that M contains N vectors $e_1, \ldots, e_N$ which are F-linearly independent in $M \otimes_A F$ and such that*

$$\|e_i\| \leq 1$$

for all $i = 1, \ldots, N$. Then there exist a direct sum decomposition

$$M = L_1 \oplus \cdots \oplus L_N$$

where each L_i has rank one and contains a vector f_i such that

$$\#\,(L_i / A\, f_i) \leq C_1$$

and

$$\|f_i\| \leq (i-1)\, C_1\, C_2 + C_1^{1/r}\, C_3\,.$$

Here C_1, C_2, C_3 are the constants defined in Lemmas 1, 2, 3 respectively.

Proof. We proceed by induction on N. When $N = 1$, Lemma 1 tells us that $L_1 = M$ contains x_1 such that

$$\#\,(L_1 / A\, x_1) \leq C_1\,.$$

Let us write

$$x_1 = \alpha\, e_1$$

with $\alpha \in F^*$. Using Lemma 3, we can choose $u \in A^*$ such that

$$\operatorname*{Sup}_{\sigma \in \Sigma} |u\,\alpha|_\sigma \leq C_3 \left(\prod_\sigma |\alpha|_\sigma \right)^{1/r} = C_3\, N(\alpha)^{1/r} \leq C_3\, C_1^{1/r}\,.$$

The lemma follows with $f_1 = u\, x_1$.

Assume now that $N \geq 2$, and let $L_1 = M \cap F\, e_1$ in $M \otimes_A F$. As above, we choose $f_1 = a_{11}\, e_1$ in L_1 with $[L_1 : A\, f_1] \leq C_1$ and

$$\operatorname*{Sup}_{\sigma \in \Sigma} |a_{11}|_\sigma \leq C_3\, C_1^{1/r}\,.$$

The quotient $M' = L/L_1$ is torsion free of rank $N-1$. We equip M' with the quotient metric induced by h, we let $p : M \to M'$ be the projection, and $e_i' = p(e_i)$, $i = 2, \ldots, N$. Clearly

$$\|e_i'\| \leq 1$$

for all $i = 2, \ldots, N$.

We assume by induction that M' can be written

$$M' = L'_2 \oplus \cdots \oplus L'_N$$

and that L'_i contains a vector f'_i such that

$$n_i = \#\,(L'_i / A\, f'_i) \leq C_1$$

with

$$f'_i = \sum_{2 \leq j \leq i} a_{ij}\, e'_j\,, \tag{9}$$

$a_{ij} \in F$, and, for all $\sigma \in \Sigma$,

$$|a_{ij}|_\sigma \leq C_1\, C_2 \qquad \text{if } 2 \leq j < i \leq N\,,$$

and

$$|a_{ii}|_\sigma \leq C_1^{1/r}\, C_3\,,\ 2 \leq i \leq N\,.$$

Let $s : M' \to M$ be any section of the projection p. From (9) it follows that there exists $\mu_i \in F$ such that

$$s(f'_i) - \sum_{2 \leq j < i} a_{ij}\, e_j = \mu_i\, e_1\,.$$

Applying Lemma 2, we can choose $b_i \in A$ such that

$$\sum_{\sigma \in \Sigma} \left| \frac{\mu_i}{n_i} - b_i \right|_\sigma \leq C_2\,.$$

Define $t : M' \to M$ by the formulae

$$t(x) = s(x) - a(x)\, b_i\, e_1$$

whenever $x \in L'_i$, hence

$$n_i\, x = a(x)\, f'_i\,,$$

for some $a(x) \in A$, $2 \leq i \leq N$. If we take $f_i = t(f'_i)$, we get

$$f_i = s(f'_i) - n_i\, b_i\, e_1 = \sum_{2 \leq j < i} a_{ij}\, e_j + (\mu_i - n_i\, b_i)\, e_1$$

and, for all $\sigma \in \Sigma$,

$$|\mu_i - n_i\, b_i|_\sigma \leq n_i\, C_2 \leq C_1\, C_2\,.$$

We define $a_{i1} = \mu_i - n_i\, b_i$ and $L_i = t(L'_i)$. Then

$$M = L_1 \oplus \cdots \oplus L_N$$

satisfies our induction hypothesis:

$$\# (L_i / A\, f_i) \leq C_1$$

$$f_i = \sum_{1 \leq j \leq i} a_{ij}\, e_j \,,$$

$$|a_{ij}|_\sigma \leq C_1\, C_2 \qquad \text{when } j < i \,,$$

and

$$|a_{ii}|_\sigma \leq C_1^{1/r}\, C_3 \,, \qquad \text{for all } \sigma \in \Sigma \,,\ i = 1, \ldots, N \,.$$

Since

$$\|e_i\| \leq 1 \qquad \text{for all } i = 1, \ldots, N \,,$$

this implies

$$\|f_i\| \leq (i-1)\, C_1\, C_2 + C_1^{1/r}\, C_3 \,.$$

q.e.d.

2.2

Lemma 6. *Let $I \subset A$ be a nontrivial ideal. There exists a set of representatives $\mathcal{R} \subset A$ of A modulo I such that, for any x in $\mathcal{R}$,*

$$\sum_{\sigma \in \Sigma} |x|_\sigma \leq C_2 \left(\frac{r+3}{4} \right) N(I) \,,$$

where $N(I) = \# (A/I)$ and C_2 is the constant in Lemma 2.

Proof. According to the proof of Lemma 2, the $\mathbf{Z}$-module A contains a basis of r elements $e_1, \ldots, e_r$ such that

$$\sum_{\sigma} |e_i|_\sigma \leq \mu_r \leq \frac{2}{r}\, C_2 \,.$$

Therefore, by Lemma 5 applied to the field $\mathbf{Q}$, in which case $C_1 = C_3 = 1$ and $C_2 = 1/2$, there exists a basis (f_i) of A over $\mathbf{Z}$ such that,

$$\sum_{\sigma \in \Sigma} |f_i|_\sigma \leq \frac{2}{r}\, C_2 \left(\frac{i-1}{2} + 1 \right) \,.$$

Since the integer $n = N(I)$ belongs to I, the map $A/I \to A/n\,A$ is injective and we can choose $\mathcal{R}$ among those

$$x = \sum_{i=1}^{r} x_i\, f_i$$

such that $x_i \in \mathbf{Z}$ and $|x_i| \leq n/2$. In that case, if $x \in \mathcal{R}$, we have

$$\sum_{\sigma \in \Sigma} |x|_\sigma \leq \frac{n}{2} \sum_{\sigma, i} |f_i|_\sigma \leq n\, C_2\, \frac{r+3}{4} \,.$$

q.e.d.

2.3

LEMMA 7. *Let $\overline{M}$ be an hermitian lattice and assume that $M = L_1 \oplus L_2$ is the direct sum of two lattices of rank one. Let $f_i \in L_i$ be a non zero vector, and $n_i = \#(L_i/A\,f_i)$, $i = 1, 2$. Then there exists a vector $e_1 \in M$, and an isomorphism*

$$\psi : A\,e_1 \oplus L \to M$$

such that L contains a vector e_2 with

$$\#(L/A\,e_2) \leq n_1\,n_2\,,$$

$$\|e_1\| \leq n_2\,C_2\,\|f_1\| + \left(1 + C_2^2\,\frac{r+3}{4}\,n_1^r\right)\|f_2\|\,,$$

and

$$\|\psi(e_2)\| \leq n_2\,\|f_1\| + C_2\,\frac{r+3}{4}\,n_1^r\,\|f_2\|\,.$$

PROOF. The algebraic content of this lemma is [9], Lemma 1.7, p. 12. To control the norms in this proof we first define an isomorphism

$$u_i : L_i \to I_i$$

where I_i is an ideal of A. If $x \in L_i$, $u_i(x) \in A$ is the unique element such that

$$n_i\,x = u_i(x)\,f_i\,, \qquad i = 1, 2\,.$$

In particular $n_i = u_i(f_i)$.
Next, we choose an ideal J_1 in the class of I_1 which is prime to I_2. According to [9], proof of Lemma 1.8, we can choose

$$J_1 = \frac{x_0}{a_0}\,I_1\,,$$

where a_0 is any element of $I_1 - \{0\}$ and x_0 belongs to a set of representatives of A modulo $I_1\,J$, where $I_1\,J = a_0\,A$.
According to Lemma 6 we can assume that

$$\sum_{\sigma \in \Sigma} |x_0|_\sigma \leq C_2\left(\frac{r+3}{4}\right) N(I_1\,J) = C_2\left(\frac{r+3}{4}\right) N(a_0)\,.$$

The composite isomorphism

$$v_1 : L_1 \to J_1 \to I_1$$

maps f_1 to $n_1\,x_0/a_0$. We choose $a_0 = n_1$, hence $v_1(f_1) = x_0$ and

$$\sum_{\sigma \in \Sigma} |x_0|_\sigma \leq C_2\left(\frac{r+3}{4}\right) n_1^r\,.$$

The direct sum of the inverses of v_1 and u_2 is an isomorphism

$$\varphi : J_1 \oplus I_2 \xrightarrow{\sim} L_1 \oplus L_2 = M\,.$$

Since J_1 and I_2 are prime to each other we have an exact sequence (as in [9] loc.cit.)

$$0 \longrightarrow J_1\,I_2 \longrightarrow J_1 \oplus I_2 \xrightarrow{p} A \longrightarrow 0$$

where p is the sum in A. Let

$$s : A \longrightarrow J_1 \oplus I_2$$

be any section of p and let $\alpha \in J_1$ be such that

$$s(1) = (\alpha, 1-\alpha)\,.$$

Let $\alpha = \lambda\, n_2\, x_0$ with $\lambda \in F$. Applying Lemma 2, we choose $a \in A$ such that

$$\sum_{\sigma\in\Sigma} |\lambda - a|_\sigma \leq C_2\,.$$

Since $n_2\, x_0$ lies in $J_1\, I_2$ the element

$$\beta = \alpha - a\, n_2\, x_0 = (\lambda - a)\, n_2\, x_0$$

lies in J_1, and $1-\beta$ lies in I_2. Since

$$\beta = v_1((\lambda - a)\, n_2\, f_1)$$

and

$$1 - \beta = u_2\left(\frac{1}{n_2} - (\lambda - a)\, x_0\, f_2\right)$$

we get

$$\begin{aligned} \|\varphi(\beta, 1-\beta)\| &\leq \|(\lambda - a)\, n_2\, f_1\| + \left\|\left(\frac{1}{n_2} - (\lambda - a)\, x_0\right) f_2\right\| \\ &\leq n_2\, C_2\, \|f_1\| + \left(1 + C_2^2 \left(\frac{r+3}{4}\right) n_1^r\right) \|f_2\|\,. \end{aligned}$$

We let $e_1 = \varphi(\beta, 1-\beta)$. On the other hand we define

$$L = J_1\, I_2 (\simeq L_1 \otimes L_2)$$

and map L to M by the composite map

$$L \xrightarrow{i} J_1 \oplus I_2 \xrightarrow{\varphi} M$$

where $i(x) = (x, -x)$. We choose

$$e_2 = n_2\, x_0 \in L$$

so that $\varphi \circ i\,(e_2) = (n_2\, v_1(f_1), x_0\, u_2(f_2))$ has norm

$$\|\varphi \circ i\,(e_2)\| \leq n_2\, \|f_1\| + C_2 \left(\frac{r+3}{4}\right) n_1^r\, \|f_2\| \, .$$

Furthermore we have isomorphisms

$$L \oplus A \overset{(i,s)}{\longrightarrow} J_1 \oplus I_2 \overset{\varphi}{\longrightarrow} M$$

and

$$\#\,(L/A\, e_2) = \#\,(J_1\, I_2/A\, e_2) \leq \#\,(J_1/A\, x_0) \times \#\,(I_2/A\, n_2) = n_1\, n_2 \, .$$

q.e.d.

2.4

PROPOSITION 1. *Let $\overline{M}$ be a rank N hermitian free A-module such that its unit ball contains a basis of $M \otimes_A F$. Then M has a basis $(e_1, \ldots, e_N)$ such that*

$$\|e_i\| \leq B_i$$

with $B_i = (i-1)\, C_2 + C_3$, $i = 1, \ldots, N$, when A is principal and

$$B_i = (1 + C_1\, C_2)(N\, C_2 + C_3) \left(1 + C_2\, \frac{r+3}{4}\right)^{\log_2(N)+2} C_1^{2(r+1)N/i}$$

in general. Here $\log_2(N)$ is the logarithm of N in base 2.

PROOF. When A is principal, $C_1 = 1$ and Proposition 1 follows from Lemma 5.
In general Lemma 5 tells us that

$$M = L_1 \oplus \cdots \oplus L_N$$

and L_i contains a vector f_i with $\#\,(L_i/A\, f_i) \leq C_1$ and

$$\|f_i\| \leq C_1((i-1)\, C_2 + C_3) \leq C_1\, (N\, C_2 + C_3) \, .$$

Let $k > 1$ be an integer and $\lambda > 0$ be a real number. We shall prove by induction N that, if M has a decomposition as above with

$$\#\,(L_i/A\, f_i) \leq k$$

and

$$\|f_i\| \leq k\, \lambda \, ,$$

then M has a basis $(e_1, \ldots, e_N)$ such that

$$\|e_i\| \leq \lambda \left(\frac{1}{k} + C_2\right) \left(1 + C_2\, \frac{r+3}{4}\right)^t k^{(r+1)(1+2+\cdots+2^t)} \, , \tag{10}$$

for all $i = 1, \dots, N$, where $t \geq 1$ is such that

$$\frac{N}{2^t} < i \leq \frac{N}{2^{t-1}} .$$

The case $N \leq 2$ follows from Lemma 7. If $N > 2$, let N' be the integral part of $N/2$. Applying Lemma 7 to every direct sum $L_i \oplus L_{N-i}$, $N/2 < i \leq N$, we get

$$M = M' \oplus \left(\bigoplus_{i=N'+1}^{N} A\, e_i \right)$$

with

$$\begin{aligned} \|e_i\| &\leq k\lambda \left(1 + C_2\, k + C_2^2\, \frac{r+3}{4}\, k^r\right) \\ &\leq \lambda \left(\frac{1}{k} + C_2\right) \left(1 + C_2\, \frac{r+3}{4}\right) k^{r+1} \end{aligned}$$

and M' is free, $M' = \overset{N'}{\underset{i=0}{\oplus}} L_i'$, and each L_i' contains a vector f_i' such that

$$[L_i' : A\, f_i'] \leq k^2$$

and

$$\|f_i'\| \leq \lambda \left(1 + C_2\, \frac{r+3}{4}\right) k^{r+1} .$$

By the induction hypothesis, M' has a basis (e_i), $1 \leq i \leq N'$, such that

$$\|e_i\| \leq \left(1 + C_2\, \frac{r+3}{4}\right) k^(r+1) \left(\frac{1}{k^2} + C_2\right) \left(1 + C_2\, \frac{r+3}{4}\right)^t (k^2)^{(r+1)(1+\cdots+2^t)}$$

whenever

$$\frac{N'}{2^t} < i \leq \frac{N'}{2^{t-1}} .$$

If

$$\frac{N}{2^{t+1}} < i \leq \frac{N}{2^t} ,$$

this inequality implies

$$\|e_i\| \leq \lambda \left(\frac{1}{k} + C_2\right) \left(1 + C_2\, \frac{r+3}{4}\right)^{t+1} k^{(r+1)(1+\cdots+2^{t+1})} .$$

Therefore M satisfies the induction hypothesis (10).
Since

$$1 + 2 + \cdots + 2^t = 2^{t+1} - 1 \leq \frac{2N}{i}$$

and $t \leq \log_2(N) + 1$, Proposition 1 follows by taking $k = C_1$ and

$$\lambda = C_1(N\, C_2 + C_3)$$

in (10). q.e.d.

2.5

Let $\overline{M}$ be a rank N hermitian free A-module. We let

$$m(h) = \mathrm{Inf}\,\{h(x)\,,\ x \in M - \{0\}\}$$

be the minimum value of h on $M - \{0\}$ and

$$M(h) = \{x \in M / h(x) = m(h)\}$$

be the (finite) set of minimal vectors of M. Let ω_N be the standard volume of the unit ball in $\mathbf{R}^N$.

Proposition 2. *Let $\overline{M} = (M, h)$ be as above. Assume that $m(h) = 1$ and that $M(h)$ spans the F-vector space $M \otimes_A F$. Then M has a basis $f_1, \ldots, f_N$ such that any $x \in M(h)$ is of the form*

$$x = \sum_{i=1}^{N} y_i\, f_i$$

with

$$\sum_{\sigma \in \Sigma} |y_i|_\sigma^2 \leq T_i\,,$$

$$T_i = r^{rN}\, C_3^{2rN+2}\, \gamma^N \prod_{j \neq i} B_j^2\,,$$

and

$$\gamma = 4^{r_1+r_2}\, \omega_N^{-2r_1/N}\, \omega_{2N}^{-2r_2/N}\, D\,.$$

Proof. From Proposition 1 we know that M has a basis $(e_1, \ldots, e_N)$ with $\|e_i\| \leq B_i$. Let $x \in M(h)$ be a minimal vector and (x_i) its coordinates in the basis (e_i).
Fix $i \in \{1, \ldots, N\}$ and $\sigma \in \Sigma$. Consider the square matrix

$$H_i = (h_\sigma(v_k, v_\ell))\,,$$

where $v_k = e_k$ if $k \neq i$ and $v_i = x$. Furthermore, let

$$H_\sigma = (h_\sigma(e_k, e_\ell))\,.$$

Since

$$|x_i|_\sigma^2 = \det(H_i)\,\det(H_\sigma)^{-1}$$

the Hadamard inequality implies

$$|x_i|_\sigma^2 \leq h_\sigma(x) \prod_{j \neq i} h_\sigma(e_j)\,\det(H_\sigma)^{-1}\,.$$

For any unit $u \in A^*$ we can replace e_i by $u^{-1} e_i$, and x_i by $y_i = u\, x_i$. We then have

$$\sum_{\sigma\in\Sigma} |y_i|_\sigma^2 \leq \sum_{\sigma\in\Sigma} h_\sigma(x) \prod_{j\neq i} h_\sigma(e_j)\, |u|_\sigma^2\, \det(H_\sigma)^{-1}\,. \tag{11}$$

Applying Lemma 3 to $\lambda_\sigma = \det(H_\sigma)^{-1/2}$ we find u such that, for all $\sigma \in \Sigma$,

$$|u|_\sigma^2 \det(H_\sigma)^{-1} \leq C_3^2 \prod_{\sigma\in\Sigma} \det(H_\sigma)^{-1}\,. \tag{12}$$

Since $\sum\limits_\sigma h_\sigma(x) = 1$ and $h_\sigma(e_j) \leq \|e_j\|^2 \leq B_j^2$, we deduce from (11) and (12) that

$$\sum_{\sigma\in\Sigma} |y_i|_\sigma^2 \leq C_3^2 \cdot \prod_{j\neq i} B_j^2 \cdot \prod_{\sigma\in\Sigma} \det(H_\sigma)^{-1}\,. \tag{13}$$

According to Icaza [4], Theorem 1, there exists $z \in L$ such that

$$\prod_{\sigma\in\Sigma} h_\sigma(z) \leq \gamma \prod_{\sigma\in\Sigma} \det(H_\sigma)^{1/N}$$

with

$$\gamma = 4^{r_1+r_2}\, \omega_N^{-2r_1/N}\, \omega_{2N}^{-2r_2/N}\, D\,.$$

Using Lemma 3 again and the fact that $m(h) = 1$, we find $v \in A^*$ such that

$$\begin{aligned} 1 \quad &\leq \quad h(vz) \leq r\, C_3^2 \prod_{\sigma\in\Sigma} h_\sigma(z)^{1/r} \\ &\leq \quad r\, C_3^2\, \gamma^{1/r} \prod_{\sigma\in\Sigma} \det(H_\sigma)^{1/rN}\,. \end{aligned}$$

From this it follows that

$$\prod_{\sigma\in\Sigma} \det(H_\sigma)^{-1} \leq (r\, C_3^2)^{rN}\, \gamma^N \tag{14}$$

and Proposition 2 follows from (13) and (14).

2.6

To count the number of vectors in $M(h)$ using Proposition 2 we shall apply the following lemma :

Lemma 8. *The number of elements a in A such that*

$$\sum_{\sigma\in\Sigma} |a|_\sigma^2 \leq T$$

is at most

$$B(T) = \mathrm{Sup}\,(T^{r/2}\,2^{r+3}, 1)\,.$$

PROOF. When $r_2 > 0$, this follows from [7], V § 1, Theorem 0, p. 102, by noticing that one can take $C_3 = 2^{r+3}$ in loc.cit. When $r_2 = 0$, the argument is similar.

3 REDUCTION THEORY

3.1

Fix an integer $N \geq 2$. Let

$$\Gamma = \mathrm{GL}_N(A)$$

and

$$G = \mathrm{GL}_N(F \otimes_{\mathbf{Q}} \mathbf{R})\,.$$

On the standard lattice $L_0 = A^N$ consider the hermitian metric h_0 defined by

$$h_0(x,y) = \sum_{\sigma\in\Sigma}\sum_{i=1}^{N} x_{i\sigma}\,\overline{y_{i\sigma}}$$

for all vectors $x = (x_{i\sigma})$ and $y = (y_{i\sigma})$ in $L_0 \otimes_{\mathbf{Z}} \mathbf{C} = (\mathbf{C}^N)^\Sigma$. Any $g \in G$ defines an hermitian metric $h = g(h_0)$ on L_0 by the formula

$$g(h_0)\,(x,y) = h_0(g(x), g(y))\,.$$

Let K be the stabilizer of h_0 and G and $X = K\backslash G$. We can view each $h \in X$ as a metric on L_0.
Following Ash [1], we say that a finite subset $M \subset L_0$ is *well-rounded* when it spans the F-vector space $L_0 \otimes_A F$. We let $\widetilde{W} \subset X$ be the space of metrics h such that $m(h) = 1$ and $M(h)$ is well-rounded. Given a well-rounded set $M \subset L_0$ we let $C(M) \subset \widetilde{W}$ be the set of metrics h such that

- $h(x) = 1$ for all $x \in M$
- $h(x) > 1$ for all $x \in L_0 - (M \cup \{0\})$.

As explained in [1], proof of (iv), pp. 466-467, $C(M)$ is either empty or topologically a cell, and the family of closed cells $\overline{C(M)}$ gives a Γ-invariant cellular decomposition of $\widetilde{W}$, such that

$$\overline{C(M)} = \coprod_{M' \supset M} C(M')\,.$$

Furthermore $\widetilde{W}/\Gamma$ is compact, of dimension $\dim(X) - N$.

3.2

Proposition 3. i) *For any integer $k \geq 0$, the number of cells of codimension k in $\widetilde{W}$ is at most*

$$c(k,N) = \binom{a(N)}{N+k}$$

where

$$a(N) = 2^{N(r+3)} \left(\prod_{i=1}^{N} T_i \right)^{r/2} ,$$

and T_i is as in Proposition 2.

ii) *Given a cell in $\widetilde{W}$, its number of codimension one faces is at most $a(N)^{N+1}$.*

Proof. Let Φ be the set of vectors $x = (x_i)$ in A^N such that, for all $i = 1, \ldots, N$,

$$\sum_{\sigma \in \Sigma} |x_i|_\sigma^2 \leq T_i \, .$$

Given $h \in \widetilde{W}$, Proposition 2 says that we can find a basis (f_i) of L_0 such that any x in $M(h)$ has its coordinates (x_i) bounded as above. If $\gamma \in \Gamma$ is the matrix mapping the standard basis of A^N to (f_i), this means that $M(\gamma(h)) = \gamma^{-1}(M(h))$ is contained in Φ.

Let $\overline{C(M)}$ be a nonempty closed cell of codimension k in $\widetilde{W}$. For any $x \in L_0$, the equation $h(x) = 1$ defines a real affine hyperplane in the set of $N \times N$ hermitian matrices with coefficients in $(F \otimes_{\mathbf{Q}} \mathbf{C})^+$. The equations $h(x) = 1$, $x \in M$, may not be linearly independent, but, since $\overline{C(M)}$ has codimension k, M has at least $N+k$ elements. And since $M \subset M(h)$ for some $h \in \widetilde{W}$, there exists $\gamma \in \Gamma$ such that $\gamma^{-1}(M)$ is contained in Φ. Therefore, modulo the action of Γ, there are at most $\binom{\mathrm{card}(\Phi)}{N+k}$ cells $\overline{C(M)}$ of codimension k. From Lemma 7 we know that

$$\mathrm{card}(\Phi) \leq a(N) \, ,$$

therefore i) follows.

To prove ii), consider a cell $\overline{C(M)}$ and a codimension one face $\overline{C(M')}$ of $\overline{C(M)}$. We can write $M' = M \cup \{x\}$ for some vector x and there exists $\gamma \in \Gamma$ such that $\gamma(M') \subset \Phi$. Since M is well-rounded, the matrix γ is entirely determined by the set of vectors $\gamma(M)$, i.e. there are at most $\mathrm{card}(\Phi)^N$ matrices γ such that $\gamma(M) \subset \Phi$. Since $\gamma(x) \in \Phi$, there are at most $\mathrm{card}(\Phi)^{N+1}$ vectors x as above.

q.e.d.

3.3

Lemma 9. *Let $\gamma \in \Gamma - \{1\}$ and p be a prime number such that $\gamma^p = 1$. Then*

$$p \leq 1 + \mathrm{Sup}(r, N) \, .$$

Proof. Since γ is non trivial we have $P(\gamma) = 0$ where P is the cyclotomic polynomial

$$P(x) = X^{p-1} + X^{p-2} + \cdots + 1\,.$$

If F does not contain the p-th roots of one, P is irreducible, and therefore it divides the characteristic polynomial of the matrix γ over F, hence $p-1 \leq N$. Otherwise, F contains $Q(\mu_p)$, which is of degree $p-1$, therefore $p-1 \leq r$.

4 The main results

4.1

For any integer $n > 0$ and any finite abelian group A we let $\mathrm{card}_n(A)$ be the largest divisor of the integer $\#(A)$ such that no prime $p \leq n$ divides $\mathrm{card}_n(A)$. Let $N \geq 2$ be an integer. We keep the notation of § 3 and we let

$$\widetilde{w} = \dim(X) - N = r_1 \, \frac{N(N+1)}{2} + r_2 \, N^2 - N$$

be the dimension of $\widetilde{W}$. For any $k \leq \widetilde{w}$ we define

$$h(k,N) = a(N)^{(N+1)c(\widetilde{w}-k-1,N)}\,,$$

where $c(\cdot, N)$ and $a(N)$ are defined in Proposition 3.

Theorem 1. *The torsion subgroup of the homology of* $\mathrm{GL}_N(A)$ *is bounded as follows*

$$\mathrm{card}_{1+\sup(r,N)} H_k(\mathrm{GL}_N(A), \mathbf{Z})_{\mathrm{tors}} \leq h(k,N)\,.$$

Proof. We know from [1] that $\widetilde{W}$ is contractible and the stabilizer of any $h \in \widetilde{W}$ is finite. From Lemma 9 it follows that, modulo $\mathcal{S}_{1+\sup(r,N)}$, the homology of $\Gamma = \mathrm{GL}_N(A)$ is the homology of a complex $(C.,\partial)$, where C_k is the free abelian group generated by a set of Γ-representatives of those k-dimensional cells c in $\widetilde{W}$ such that the stabilizer of c does not change its orientation ([2], VII). According to Proposition 3, the rank of C_k is at most $c\,(\widetilde{w}-k, N)$ and any cell of $\widetilde{W}$ has at most $a(N)^{N+1}$ faces. Theorem 1 then follows from a general result of Gabber ([13], Proposition 3 and equation (18)).

4.2

For any integer $m \geq 1$ let

$$k(m) = h(m, 2\,m+1)\,.$$

Denote by $K_m(A)$ the m-th algebraic K-group of A.

Theorem 2. *The following inequality holds*

$$\mathrm{card}_{\sup(r+1,2m+2)} K_m(A)_{\mathrm{tors}} \leq k(m)\,.$$

Proof. As in [13], Theorem 2, we consider the Hurewicz map

$$H : K_m(A) \to H_m(\mathrm{GL}(A), \mathbf{Z})\,,$$

the kernel of which lies in $\mathcal{S}_n$, $n \leq (m+1)/2$. Since, according to Maazen and Van der Kallen,

$$H_m(\mathrm{GL}(A), \mathbf{Z}) = H_m(\mathrm{GL}_N(A), \mathbf{Z})$$

when $N \geq 2\,m + 1$, Theorem 2 is a consequence of Theorem 1.

4.3

Let p be an odd prime and $n \geq 2$ an integer. For any $\nu \geq 1$ denote by $\mathbf{Z}/p^\nu(n)$ the étale sheaf $\mu_{p^\nu}^{\otimes n}$ on $\mathrm{Spec}(A[1/p])$, and let

$$H^2(\mathrm{Spec}(A[1/p]), \mathbf{Z}_p(n)) = \varprojlim_{\nu} H^2(\mathrm{Spec}(A[1/p]), \mathbf{Z}_{p^\nu}(n))\,.$$

From [12], we know that this group is finite and zero for almost all p.

Theorem 3. *The following inequality holds*

$$\prod_{\substack{p \geq 4n-1 \\ p \geq r+2}} \mathrm{card}\, H^2(\mathrm{Spec}(A[1/p]), \mathbf{Z}_p(n)) \leq k(2n-2)\,.$$

Proof. According to [12], the cokernel of the Chern class

$$c_{n,2} : K_{2n-2}(A) \to H^2(\mathrm{Spec}(A[1/p]), \mathbf{Z}_p(n))$$

lies in $\mathcal{S}_{n+1}$ for all p. Furthermore, Borel proved that $K_{2m-2}(A)$ is finite. Therefore Theorem 3 follows from Theorem 2.

4.4

By Lemmas 1 to 7 and Propositions 1 to 3, the constant $k(m)$ is explicitly bounded in terms of m, r and D. We shall now simplify this upper bound.

Proposition 4. i) $\log\log k(m) \leq 220\, m^4 \log(m)\, r^{4r}\, \sqrt{D}\, \log(D)^{r-1}$

ii) *If F has class number one,*

$$\log\log k(m) \leq 210\, m^4 \log(m)\, r^{4r}\, \sqrt{D}\, \log(D)^{r-1}$$

iii) *If* $F = \mathbf{Q}(\sqrt{-D})$ *is imaginary quadratic*

$$\log\log k(m) \leq 1120\, m^4 \log(m) \log(D)\,;$$

if furthermore F *has class number one*

$$\log\log k(m) \leq 510\, m^4 \log(m) \log(D)\,.$$

iv) *When* $F = \mathbf{Q}$ *and* $m \geq 9$

$$\log\log k(m) \leq 8\, m^4 \log(m)\,;$$

furthermore

$$\log\log k(7) \leq 40\,545$$

and

$$\log\log k(8) \leq 70\,130\,.$$

PROOF. By definition

$$k(m) = h(m, 2\,m+1) = a(N)^{(N+1)c(\widetilde{w}-m-1,N)}$$

with $N = 2\,m+1$ and

$$c(\widetilde{w} - m - 1, N) = \binom{a(N)}{N + \widetilde{w} - m - 1}\,.$$

Since

$$\begin{aligned} N + \widetilde{w} - m - 1 &= r_1 \frac{N(N+1)}{2} + r_2\, N^2 - m - 1 \\ &\leq 2\,r\,m^2 + 3\,r\,m + r - 2\,m - 1 \end{aligned}$$

and since $a(2\,m+1)$ is very big, we get

$$\begin{aligned} \log\log k(m) &\leq (2\,r\,m^2 + 3\,r\,m + r - 2\,m - 1) \log a(2\,m+1) \\ &+ \log(2\,m+2) + \log\log a(2\,m+1) \\ &\leq r(2\,m^2 + 3\,m + 1) \log a(2\,m+1)\,. \end{aligned} \tag{15}$$

From Proposition 3 and Proposition 2 we get

$$a(N) = 2^{N(r+3)} \left(\prod_{i=1}^{N} T_i \right)^{r/2}, \tag{16}$$

and

$$\prod_{i=1}^{N} T_i = (r^{rN}\, \gamma^N\, C_3^{2rN+2})^N \prod_{i=1}^{N} B_i^{N-1}\,. \tag{17}$$

According to Proposition 1

$$\prod_{i=1}^{N} B_i = \left[(1+C_1\,C_2)(N\,C_2+C_3)\left(1+C_2\,\frac{r+3}{4}\right)^{\log_2(N)+2}\right]^N \cdot C_1^{2(r+1)NH_N}\,, \quad (18)$$

where

$$H_N = \sum_{i=1}^{N} \frac{1}{i} \leq 1+\log(N)\,.$$

Assume $s \neq 0$. Then the upper bound C_3^* we get from Lemmas 3 and 4 for C_3 is much bigger than C_2. Therefore

$$\log(N\,C_2+C_3) \leq \log(N) + \log(C_3^*)\,. \quad (19)$$

We deduce from (15), (16), (17), (18), (19) that

$$\log\log k(m) \leq X_1 + X_2$$

with

$$X_1 = r(2\,m^2+3\,m+1)\,\frac{r}{2}\,(N(2\,r\,N+2)+N(N-1))\log(C_3^*)$$

and

$$\begin{aligned} X_2 &= r(2\,m^2+3\,m+1)\Bigg(N(r+3)\log(2) \\ &+ \frac{r}{2}\,N\Bigg(N\log(\gamma)+(N-1)\Bigg[\log(1+C_1\,C_2)+\log(N) \\ &+ (\log_2(N)+2)\log\left(1+C_2\,\frac{r+3}{4}\right) \\ &+ 2(r+1)(1+\log(N))\log(C_1)\Bigg]\Bigg)\Bigg)\,. \end{aligned} \quad (20)$$

Since $s \leq r-1$, Lemma 3 and Lemma 4 imply

$$\log(C_3^*) \leq 11r^2(r-1)(4\,r(\log 3\,r)^3)^{r-2}\,2^{r-1}\,\sqrt{D}\,\log(D)^{r-1}\,,$$

from which it follows that

$$X_1 \quad \leq \quad 208\,\log(m)\,m^4\,r^{4r}\,\sqrt{D}\,\log(D)^{r-1}$$

when $m \geq 2$ and $r \geq 2$.
To evaluate X_2 first notice that

$$4\,\omega_N^{-2/N} \leq 1+N/4$$

by [10], II, (1.5), Remark, hence

$$\begin{aligned}\log(\gamma) &\leq r_1 \log\left(1+\frac{N}{4}\right) + 2\, r_2 \log\left(1+\frac{N}{2}\right) + \log(D) \\ &\leq r \log(N) + \log(D) \qquad (21)\end{aligned}$$

since $N \geq 5$.
By the Stirling formula and Lemma 1, if $r \geq 2$,

$$\begin{aligned}\log(C_1) &= \log(r!) - r\log(r) + r_2 \log\left(\frac{4}{\pi}\right) + \frac{1}{2}\log(D) \\ &\leq 1 + \frac{1}{2}\log(r) + \frac{1}{2}\log(D)\,, \qquad (22)\end{aligned}$$

$$\log\left(1 + C_2\,\frac{r+3}{4}\right) \leq \mathrm{Sup}\left(\log(C_2) + \log\left(\frac{r+3}{4}\right) + 1, \log(2)\right)\,,$$

where

$$\begin{aligned}\log(C_2) + \log(\frac{r+3}{4}) &\leq r\log(4) - (r-2)\log(r) - \log(r!) \\ &+ \log(\frac{r+3}{4}) + \frac{1}{2}\log(D) \\ &\leq 2.4 + \frac{1}{2}\log(D)\,,\end{aligned}$$

so that

$$\log\left(1 + C_2\,\frac{r+3}{4}\right) \leq 3.4 + \frac{1}{2}\log(D)\,. \qquad (23)$$

We also have

$$\log(1 + C_1\, C_2) \leq \mathrm{Sup}(1 + \log(C_1) + \log(C_2), \log(2))$$

and

$$\begin{aligned}\log(C_1) + \log(C_2) &\leq -r\log(r) + r - (r-2)\log(r) + r\log(4) + \log(D) \\ &\leq 3.4 + \log(D)\,,\end{aligned}$$

so that

$$\log(1 + C_1\, C_2) \leq 4.4 + \log(D)\,. \qquad (24)$$

From (20), (21), (22), (23), (24) we get

$$X_2 \leq a\log(D) + b$$

with

$$\begin{aligned} a &= r(2\,m^2+3\,m+1)(2\,m+1)\Bigg(\frac{r}{2}\,((2\,m+1)+2\,m+m\log_2(2\,m+1)+m \\ &+ 2\,m\,(r+1)(1+\log(2\,m+1)))\Bigg) \leq 75\,r^3\,m^4\log(m) \end{aligned}$$

if $r \geq 2$ and $m \geq 2$.
Finally

$$\begin{aligned} b &= r(2\,m^2+3\,m+1)(2\,m+1)\Bigg((r+3)\log(2)+\frac{r}{2}\,(2\,m+1)\,r(\log(r)+\log(2\,m+1)) \\ &+ \frac{r}{2}\,(2\,m)\Bigg(4.4+\log(2\,m+1)+3.4(\log_2(2\,m+1)+2) \\ &+ 2\,(r+1)\,(1+\log(2\,m+1))\left(1+\frac{1}{2}\log(r)\right)\Bigg)\Bigg) \leq 148\,r^4\,m^4\log(m) \end{aligned}$$

when $r \geq 2$ and $m \geq 2$.
Therefore

$$\begin{aligned} \log\log k(m) &\leq 208\log(m)\,m^4\,r^{4r}\sqrt{D}\log(D)^{r-1}+75\,r^3\,m^4\log(m)\log(D) \\ &+ 148\,r^4\,m^4\log(m) \leq 220\,m^4\log(m)\,r^{4r}\sqrt{D}\log(D)^{r-1} \end{aligned}$$

when m, r and D are at least 2. This proves i).
If we assume that A is principal, we can take $C_1 = 1$ in Lemma 1 and $B_i = (i-1)\,C_2 + C_3$ in Proposition 1. Since $C_2 < C_3$ we get

$$\log\left(\prod_{i=1}^{N} B_i\right) \leq \log(N!) + N\log(C_3)$$

and

$$\log\log k(m) \leq X_1 + X_3$$

where

$$\begin{aligned} X_3 &= r(2\,m^2+3\,m+1)\Bigg[(r+3)(2\,m+1)\log(2)+\frac{r^2}{2}(2\,m+1)^2\log(r) \\ &+ \frac{r}{2}\,(2\,m+1)^2\log(\gamma)+\frac{r}{2}\,(2\,m)\log((2\,m+1)!)\Bigg] \\ &\leq 6\,m^4\,r^2\log(D)+2\,r^{4r}\,m^4\log(m)\,. \end{aligned}$$

Therefore

$$X_1 + X_3 \leq 210\,m^4\log(m)\,r^{4r}\sqrt{D}\log(D)^{r-1}\,.$$

Assume now that $r_1 + r_2 = 1$. Then $C_3 = 1$ and the term X_1 disappears from the above computation. Assume first that $F = \mathbf{Q}(\sqrt{-D})$. Since $r_2 = 1$ and $r_1 = 0$ we get

$$\log\log k(m) \leq (4\,m^2+3\,m+1)\log a(2\,m+1)\,.$$

Furthermore (18) becomes

$$\prod_{i=1}^{N} B_i \leq \left[(1+C_1\,C_2)(1+N\,C_2)\left(1+\frac{5}{4}\,C_2\right)^{\log_2(N)+2}\right]^N \cdot C_1^{6N(1+\log(N))}\,.$$

Therefore

$$\begin{aligned}
\log\log k(m) &\leq (4\,m^2+3\,m+1)\Big[5\,N\log(2)+2N^2\log(2)\\
&+ N^2\log(\gamma)+N(N-1)\Big[\log(1+C_1\,C_2)\\
&+ \log(1+N\,C_2)+(\log_2(N)+2)\log\left(1+\frac{5}{4}\,C_2\right)\Big]\\
&+ 6\,N(1+\log N)\log(C_1)\Big]\,,
\end{aligned}$$

with $N=2\,m+1$. We have now

$$\gamma \leq \left(1+\frac{N}{2}\right)^2 D\,,$$

$$C_1=\frac{2}{\pi}\sqrt{D} \quad \text{and} \quad C_2=\frac{\pi}{2}\sqrt{D}\,.$$

This implies

$$\log\log k(m) \leq 597\,m^4\log(m)+256\,m^4\log(m)\log(D) \leq 1120\,m^4\log(m)\log(D)\,.$$

If $F=\mathbf{Q}(\sqrt{-D})$ is principal we can take $C_1=1$ and $B_i=(i-1)\,C_2+1$. We get

$$\log\log k(m) \leq 510\,m^4\log(m)\log(D)\,.$$

Finally, assume that $F=\mathbf{Q}$. Then

$$B_i=\frac{i+1}{2} \quad \text{since} \quad C_2=\frac{1}{2}\,, \quad \text{and} \quad \gamma\leq 1+\frac{N}{4}\,.$$

Therefore

$$\begin{aligned}
\log\log k(m) &\leq (2\,m^2+2\,m+1)\log a(2\,m+1)\\
&\leq (2\,m^2+2\,m+1)\left[4\,N\log(2)+\frac{N^2}{2}\log\left(1+\frac{N}{4}\right)\right.\\
&+ \left.\frac{N-1}{2}\log\left(\prod_{i=1}^{N}\frac{i+1}{2}\right)\right]\\
&\leq 8\,m^4\log(m)
\end{aligned}$$

if $m\geq 9$. We can also estimate $k(7)$ and $k(8)$ from this inequality above. This proves iv).

5 Discussion

5.1

The upper bound in Theorem 2 and Propostition 4 seems much too large. When $m = 0$, $\mathrm{card}\, K_0(A)_{\mathrm{tors}}$ is the class number $h(F)$, which is bounded as follows:

$$h(F) \leq \alpha \sqrt{D} \log(D)^{r-1} , \tag{25}$$

for some constant $\alpha(r)$ [11], Theorem 4.4, p. 153. Furthermore, when $F = \mathbf{Q}$, $m = 2n - 2$ and n is even, the Lichtenbaum conjecture predicts that $\mathrm{card}\, K_{2n-2}(\mathbf{Z})$ is the order of the numerator of B_n/n, where B_n is the n-th Bernoulli number. The upper bound

$$B_n \leq n^! \approx n^n$$

suggests, since the denominator of B_n/n is not very big, that $\mathrm{card}\, K_m(\mathbf{Z})_{\mathrm{tors}}$ should be exponential in m. We are thus led to the following:

Conjecture. *Fix $r \geq 1$. There exists positive constants α, β, γ such that, for any number field F of degree r on $\mathbf{Q}$,*

$$\mathrm{card}\, K_m(A)_{\mathrm{tors}} \leq \alpha \exp(\beta m^\gamma \log D) .$$

Furthermore, we expect that γ does not depend on r.

5.2

As suggested by A. Chambert-Loir, it is interesting to consider the analog in positive characteristic of the conjecture above. Let X be a smooth connected projective curve of genus g over the finite field with q elements, $\zeta_X(s)$ its zeta function and

$$P(t) = \prod_{i=1}^{2g} (1 - \alpha_i\, t) ,$$

where α_i are the roots of Frobenius acting on the fist ℓ-adic cohomology group of X. When $n > 1$, it is expected that the finite group $K_{2n-2}(X)$ has order the numerator of $\zeta_X(1-n)$, i.e. $P(q^{n-1})$. Since $|\alpha_i| = q^{1/2}$ for all $i = 1 \cdots 2g$, we get

$$P(q^{n-1}) \leq (1 + q^{n-1/2})^{2g} \leq q^{2ng} .$$

In the analogy between number fields and function fields, the genus g is known to be an analog of $\log(D)$. Therefore the bound above is indeed analogous to the conjecture in §5.1.

5.3

The upper bound for $k(m)$ in Proposition 4 i) is twice exponential in D. One exponential is due to our use of Lemma 3, where C_3 is exponential in D. Maybe this can be improved in general, and not only when $s = 0$.
The exponential in D occuring in Proposition 4 ii) might be due to our use of the geometry of numbers. Indeed, if one evaluates the class number $h(F)$ by applying naively Minkowski's theorem (Lemma 1), the bound one gets is exponential in D; see however [8], Theorem 6.5., for a better proof.

5.4

One method to prove (25) consists in combining the class number formula (see (7) and (8)) with a lower bound for the regulator $R(F)$. This suggests replacing the arguments of this paper by analytic number theory, to get good upper bounds for étale cohomology.
More precisely, let $n \geq 2$ be an integer, and let $\zeta_F(1-n)^*$ be the leading coefficient of the Taylor series of $\zeta_F(s)$ at $s = 1-n$. Lichtenbaum conjectured that

$$\zeta_F(1-n)^* = \pm\, 2^{r_1}\, R_{2n-1}(F)\, \frac{\prod\limits_p \operatorname{card} H^2(\operatorname{Spec}(A[1/p]), \mathbf{Z}_p(n))}{\prod\limits_p \operatorname{card} H^1(\operatorname{Spec}(A[1/p]), \mathbf{Z}_p(n))_{\mathrm{tors}}}\,, \tag{26}$$

where $R_{2n-1}(F)$ is the higher regulator for the group $K_{2n-1}(F)$. The equality (26) is known up a power of 2 when F is abelian over $\mathbf{Q}$ [5], [6], [3].
The order of the denominator on the right-hand side of (26) is easy to evaluate, as well as $\zeta_F(1-n)^*$ (since it is related by the functional equation to $\zeta_F(n)$).

Problem. *Can one find a lower bound for $R_{2n-1}(F)$?*

If such a problem could be solved, the equality (26) is likely to produce a much better upper bound for étale cohomology than Theorem 3. Zagier's conjecture suggests that this problem could be solved if one knew that the values of the n-logarithm on F are $\mathbf{Q}$-linearly independent.

5.5

To illustrate our discussion, let $F = \mathbf{Q}$ and $n = 5$. Then we have

$$H^2(\operatorname{Spec}(\mathbf{Z}[1/p]), \mathbf{Z}_p(5))/p = C^{(p-5)}\,,$$

where C is the class group of $\mathbf{Q}(\sqrt[p]{1})$ modulo p, and $C^{(i)}$ is the eigenspace of C of the i-th power of the Teichmüller character. Vandiver's conjecture predicts that $C^{(p-5)} = 0$ when p is odd. It is true when $p \leq 4.10^6$. Theorem 3 and Proposition 4 tell us that

$$\prod_p H^2(\mathrm{Spec}(\mathbf{Z}[1/p], \mathbf{Z}_p(5)) \leq k(8) \leq \exp\exp(70130).$$

If one could find either a better upper bound for the order of $K_8(\mathbf{Z})$ or a good lower bound for $R_9(\mathbf{Q})$, this would get us closer to the expected vanishing of $C^{(p-5)}$.

Notice that, using knowledge on $K_4(\mathbf{Z})$, Kurihara has proved that $C^{(p-3)} = 0$.

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