On the Logarithmic Riemann-Hilbert Correspondence

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Abstract. We construct a classification of coherent sheaves with an integrable log connection, or, more precisely, sheaves with an integrable connection on a smooth log analytic space X over $\mathbf{C}$. We do this in three contexts: sheaves and connections which are equivariant with respect to a torus action, germs of holomorphic connections, and finally global log analytic spaces. In each case, we construct an equivalence between the relevant category and a suitable combinatorial or topological category. In the equivariant case, the objects of the target category are graded modules endowed with a group action. We then show that every germ of a holomorphic connection has a canonical equivariant model. Global connections are classified by locally constant sheaves of modules over a (varying) sheaf of graded rings on the topological space X_{log}. Each of these equivalences is compatible with tensor product and cohomology.

Keywords and Phrases: De Rham cohomology, Log scheme

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0 Introduction

Let $X/\mathbf{C}$ be a smooth proper scheme of finite type over the complex numbers and let X_{an} be its associated complex analytic space. The classical Riemann-Hilbert correspondence provides an equivalence between the category $L_{coh}(\mathbf{C}_X)$ of locally constant sheaves of finite dimensional $\mathbf{C}$-vector spaces V on X_{an} and the category $MIC_{coh}(X/\mathbf{C})$ of coherent sheaves (E, ∇) with integrable connection on $X/\mathbf{C}$. This correspondence is compatible with formation of tensor products and with cohomology: if an object V of $L_{coh}(\mathbf{C}_X)$ corresponds to an object (E, ∇) of $MIC_{coh}(X/\mathbf{C})$, there is a canonical isomorphism

$$H^i(X_{an}, V) \cong H^i(X, E \otimes \Omega^{\cdot}_{X/\mathbf{C}}), \tag{0.0.1}$$

where $E \otimes \Omega^{\cdot}_{X/\mathbf{C}}$ is the De Rham complex of (E, ∇).

When X is no longer assumed to be proper, such an equivalence and equation (0.0.1) still hold, provided one restricts to connections with regular singularities at infinity [3]. Among the many equivalent characterizations of this condition, perhaps the most precise is the existence of a smooth compactification $\overline{X}$ of X such that the complement $\overline{X} \setminus X$ is a divisor Y with simple normal crossings and such that (E, ∇) prolongs to a locally free sheaf $\overline{E}$ endowed with a connection with log poles $\nabla \colon \overline{E} \to \overline{E} \otimes \Omega^1_{\overline{X}/\mathbf{C}}(\log Y)$. In general there are many possible choices of $\overline{E}$, *some* of which have the property that the natural map

$$H^i(\overline{X}, \overline{E} \otimes \Omega^{\cdot}_{\overline{X}/\mathbf{C}}(\log Y)) \to H^i(X, E \otimes \Omega^{\cdot}_{X/\mathbf{C}}) \tag{0.0.2}$$

is an isomorphism.

In some situations, it is more natural to view the compactification data $(\overline{X}, \overline{E})$ as the fundamental object of study. To embody this point of view in the notation, let $(\underline{X}, Y)$ denote a pair consisting of a smooth scheme $\underline{X}$ over $\mathbf{C}$ together with a reduced divisor with strict normal crossings Y on $\underline{X}$, and let $X^* := \underline{X} \setminus Y$. Write $\mathcal{O}_X$ for $\mathcal{O}_{\underline{X}}$, and let M_X denote the sheaf of sections of $\mathcal{O}_X$ which become units on X^*. Then M_X is a (multiplicative) submonoid of $\mathcal{O}_X$ containing $\mathcal{O}_X^*$, and the natural map of sheaves of monoids $\alpha_X \colon M_X \to \mathcal{O}_X$ defines a "log structure"[6] on $\underline{X}$. The datum of $(\underline{X}, Y)$ is in fact equivalent to the datum of the "log scheme" $X := (\underline{X}, \alpha_X)$. The quotient monoid sheaf $\overline{M}_X := M_X/\mathcal{O}_X^*$ is exactly the sheaf of anti-effective divisors with support in Y. This sheaf is locally constant on a stratification of X and has finitely generated stalks, making it an essentially combinatorial object, which encodes in a convenient way much of the combinatorics of the geometry of $(\underline{X}, Y)$. For example, one can easily control the geometry of those closed subschemes of $\underline{X}$ which are defined by coherent sheaves of ideals K in the sheaf of monoids $\overline{M}_X$. Such a scheme $\underline{Z}$ inherits a log structure $\alpha_Z \colon M_Z \to \mathcal{O}_Z$ from that of X, and the sheaf of ideals K defines a sheaf of ideals K_Z in M_Z which is annihilated by α_Z. If one adds this extra datum to the package, one obtains an *idealized log scheme* (Z, α_Z, K_Z). Many of the techniques of logarithmic de Rham cohomology work as well for Z as they do for X, a phenomenon explained by the fact that (Z, α_Z, K_Z) is smooth over $\mathbf{C}$ in the category of idealized log schemes. Conversely, any fine saturated idealized log scheme X which is smooth over $\mathbf{C}$ (in the sense of Grothendieck's general notion of smoothness) is, locally in the étale topology, isomorphic to the idealized log scheme associated to the quotient monoid algebra $\mathbf{C}[P]$ by an ideal $K \subseteq P$, where P is a finitely generated, integral, and saturated monoid.

In [7], Kato and Nakayama construct, for any log scheme X of finite type over $\mathbf{C}$, a commutative diagram of ringed topological spaces

$$\begin{tikzcd} X_{an}^* \arrow[r, "j_{log}"] \arrow[dr, "j"'] & X_{log} \arrow[d, "\tau"] \\ & X_{an} \end{tikzcd}$$

The morphism τ is surjective and proper and can be regarded as a relative compactification of the open immersion j. We show in (3.1.2) that, if X is smooth, X_{an}^* and X_{log} have the same local homotopy type. Since τ is proper, it is much easier to work with than the open immersion j. The construction of X_{log} also works in the idealized case. Here X_{an}^* can be empty, hence useless, while its avatar X_{log} remains. These facts justify the use of the space X_{log} as a substitute for X_{an}^* as the habitat for log topology.

Let $X/\mathbf{C}$ be a smooth, fine, and saturated idealized log analytic space, let $\Omega^1_{X/\mathbf{C}}$ be the sheaf of log Kahler differentials, and let $MIC_{coh}(X/\mathbf{C})$ denote the category of coherent sheaves E on X equipped with an integrable (log) connection $\nabla \colon E \to E \otimes \Omega^1_{X/\mathbf{C}}$. One of the main results of [7] is a Riemann-Hilbert correspondence for a subcategory $MIC_{nilp}(X/\mathbf{C})$ of $MIC_{coh}(X/\mathbf{C})$. This consists of objects (E, ∇) which, locally on X, admit a filtration whose associated graded object "has no poles." (In the classical divisor with normal crossings case, such an object corresponds to the "canonical extension" of a connection with regular singular points and nilpotent residue map [3, II,5.2].) Kato and Nakayama establish a natural equivalence between $MIC_{nilp}(X/\mathbf{C})$

and a category $L_{unip}(X_{log})$ of locally constant sheaves of $\mathbf{C}$-modules on X_{log} with unipotent monodromy relative to τ. Note that if (E,∇) is an object of $MIC_{nilp}(X/\mathbf{C})$, then E is locally free, but that this is not true for a general (E,∇) in $MIC_{coh}(X/\mathbf{C})$.

Our goal in this paper is to classify the category $MIC_{coh}(X/\mathbf{C})$ of *all* coherent sheaves on X, with no restriction on E or its monodromy, in terms of suitable topological objects on X_{log}. These will be certain sheaves of $\mathbf{C}$-vector spaces plus some extra data to keep track of the choice of coherent extension. The extra data we need involve the exponents of the connection. These can be thought of in the following way. At a point x of X, one can associate to a module with connection (E,∇) its *residue* at x. This is a family of commuting endomorphisms of $E(x)$ parameterized by $T_{\overline{M},x} := \mathrm{Hom}(\overline{M}^{gp}_{X,x},\mathbf{C})$; it gives $E(x)$ the structure of a module over the symmetric algebra of $T_{\overline{M},x}$. The support of this module is then a finite subset of the maximal spectrum of $S^{\cdot}T_{\overline{M},x}$, which is just $\mathbf{C}\otimes\overline{M}^{gp}_{X,x}$. The *exponents* of the connection are the *negatives* of these eigenvalues; they are all zero for objects of $MIC_{nilp}(X/\mathbf{C})$. Let Λ denote the pullback of the sheaf $\mathbf{C}\otimes\overline{M}^{gp}_X$ to X_{log}, regarded as a sheaf of $\overline{M}_X$-sets induced from the *negative* of the usual inclusion $\overline{M}_X\to\mathbf{C}\otimes\overline{M}_X$. The map $\overline{M}_X\to\Lambda$ sending p to $-1\otimes p$ endows the pullback $\mathbf{C}^{log}_X$ of $\mathbf{C}[\overline{M}_X]/K_X$ to X_{log} with the structure of a Λ-graded algebra. It then makes sense to speak of sheaves of Λ-graded (or indexed) $\mathbf{C}^{log}_X$-modules. In §3, we describe a category $L_{coh}(\mathbf{C}^{log}_X)$ of "coherent" sheaves of Λ-graded $\mathbf{C}^{log}_X$-modules and prove the following theorem:

Theorem *Let $X/\mathbf{C}$ be a smooth fine, and saturated idealized log analytic space over the complex numbers. There is an equivalence of tensor categories:*

$$\mathcal{V}\colon MIC_{coh}(X/\mathbf{C})\longrightarrow L_{coh}(\mathbf{C}^{log}_X)$$

compatible with pullback via morphisms $X'\to X$.

As in [7], the equivalence can be expressed with the aid of a sheaf of rings $\tilde{\mathcal{O}}^{log}_X$ on X_{log} which simultaneously possesses the structure of a Λ-graded $\mathbf{C}^{log}_X$-module and an exterior differential:

$$d\colon \tilde{\mathcal{O}}^{log}_X\to\tilde{\Omega}^{1,log}_{X/\mathbf{C}} := \tilde{\mathcal{O}}^{log}_X\otimes_{\tau^{-1}\mathcal{O}_X}\tau^{-1}\Omega^1_{X/\mathbf{C}},$$

whose kernel is exactly $\mathbf{C}^{log}_X$. If (E,∇) is an object of $MIC_{coh}(X/\mathbf{C})$, $\tilde{E} := \tilde{\mathcal{O}}^{log}_X\otimes_{\tau^{-1}\mathcal{O}_X}\tau^{-1}E$ inherits a "connection"

$$\tilde{\nabla}\colon \tilde{E}\to\tilde{E}\otimes_{\tilde{\mathcal{O}}^{log}_X}\tilde{\Omega}^{1,log}_{X/\mathbf{C}},$$

and $\mathcal{V}(E,\nabla)$ is the Λ-graded $\mathbf{C}^{log}_X$-module $\tilde{E}^{\tilde{\nabla}}$. Conversely, if V is an object of

$L_{coh}(\mathbf{C}_X^{log})$, then $\tilde{V} := \tilde{\mathcal{O}}_X^{log} \otimes_{\mathbf{C}_X^{log}} V$ inherits a graded connection

$$\tilde{\nabla} := d \otimes \mathrm{id}\colon \tilde{V} \to \tilde{V} \otimes_{\tilde{\mathcal{O}}_X^{log}} \tilde{\Omega}^{1,log}_{X/\mathbf{C}}.$$

Pushing forward by τ and taking the degree zero parts, one obtains an $\mathcal{O}_X$-module which we denote by $\tau_*^\Lambda \tilde{V}$ and which inherits a (logarithmic) connection ∇; this gives a quasi-inverse to the functor $(E, \nabla) \mapsto \mathcal{V}(E, \nabla)$.

The equivalence provided by the theorem is also compatible with cohomology. A Poincaré lemma asserts that the map:

$$V \to E \otimes \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C}}$$

from V to the De Rham complex of $\tau^{-1}E \otimes \tilde{\mathcal{O}}_X^{log}$ is a quasi-isomorphism. An analogous topological calculation asserts that the map

$$E \otimes \Omega^{\cdot}_{X/\mathbf{C}} \to R\tau_*^\Lambda(\tau^{-1}E \otimes \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C}})$$

is a quasi-isomorphism, where $R\tau_*^\Lambda$ means the degree zero part of $R\tau_*$. One can conclude that the natural maps

$$H^i(X, E \otimes \Omega^{\cdot}_{X/\mathbf{C}}) \to H^i(X_{log}, E \otimes \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C},0}) \leftarrow\ H^i(X_{log}, V_0)$$

are isomorphisms. Note that in the middle and on the right, we take only the part of degree zero; this reflects the well-known fact that in general, logarithmic De Rham cohomology does not calculate the cohomology on the complement of the log divisor without further conditions on the exponents [3, II, 3.13]. The grading structure on the topological side obviates the unpleasant choice of a section of the map $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$ which is sometimes made in the classical theory [3, 5.4]; it has the advantage of making our correspondence compatible with tensor products.

The question of classifying coherent sheaves with integrable logarithmic connection is nontrivial even locally. A partial treatment in the case of a divisor with normal crossings is due to Deligne and briefly explained in an appendix to [4]. The discussion there is limited to the case of torsion-free sheaves and is expressed in terms of $\mathbf{Z}^r$-filtered local systems $(V, F.)$ of $\mathbf{C}$-vector spaces. In our coordinate-free formalism, $\overline{M}_X^{gp}$ replaces $\mathbf{Z}^r$, and the filtered local system $(V, F.)$ is replaced by its graded Rees-module $\oplus_m F_m V$.

Because some readers may be primarily concerned with the local problem, and/or may not appreciate logarithmic geometry, we discuss the local Riemann-Hilbert correspondence, in which the logarithmic techniques reduce to toroidal methods which may be more familiar, before the global one. We shall in fact describe this correspondence in two ways: one in terms of certain normalized representations of a "logarithmic fundamental group," and one in terms of equivariant nilpotent Higgs modules. Then the proof of the global theorem stated above amounts to formulating and verifying enough compatibilities so that one can reduce to the local case.

The paper has three sections, dealing with the Riemann-Hilbert correspondence in the equivariant, local, and global settings, respectively. The first section discusses homogeneous connections on affine toric varieties. Essentially, these are modules with integrable connection which are equivariant with respect to the torus action. These are easy to classify, for example in terms of equivariant Higgs modules. Once this is done, it is quite easy to describe an equivariant Riemann-Hilbert correspondence for such modules. It takes some more care to arrange the correspondence in a way that will be compatible with the global formulation we need later. The next section is devoted to the local Riemann-Hilbert correspondence. The main point is to show that the category of analytic germs of connections at the vertex of an affine toric variety is equivalent to the category of coherent equivariant connections (and hence also to the category of equivariant Higgs modules). There are two key ingredients: the first is the study of connections on modules of finite length (using Jordan normal form) and, by passing to the limit, of formal germs. Ahmed Abbes has pointed out the similarity between this construction and the technique of "decompletion" used by Fontaine in an analogous p-adic situation [5]. Our second key ingredient is a convergence theorem which shows that the formal completion functor is an equivalence on germs. Since our analytic spaces are only log smooth and our sheaves are not necessarily locally free, such a theorem is not standard. Instead of trying a dévissage technique to reduce to the classical case, we prove convergence from scratch, using direct estimates of the growth of terms of formal power series indexed by a monoid. In the last section, we globalize the Riemann-Hilbert correspondence by defining $\tilde{\mathcal{O}}_X^{log}$ and showing that it agrees, in a suitable sense, with the equivariant constructions in the first section. To illustrate the power of our somewhat elaborate main theorem, we show how it immediately implies a logarithmic version (3.4.9) of Deligne's comparison theorem [3, II, 3.13]. Our version says that the map (0.0.2) is an isomorphism provided that, at each $x \in X$, the intersection of the set of exponents of E (viewed as a subset of $\overline{\mathbf{C}} \otimes M_{X,x}$) with $\overline{M}^{gp}_{X,x}$ lies in $\overline{M}_{X,x}$. (In fact our result is slightly stronger, as well more general, than Deligne's original version.) We also explain how it immediately implies the existence of a logarithmic version of the Kashiwara-Malgrange V-filtration and of Deligne's meromorphic to analytic comparison theorem.

Since this paper seems long enough in its current state, we have not touched upon several obvious problems, which we expect present varying degrees of difficulty. These include a notion of regular singular points for modules with connection on a log scheme, and especially the functoriality of the Riemann-Hilbert correspondence with respect to direct images. We leave completely untouched moduli problems of log connections, referring to work by N. Nitsure in [9] and [10] on this subject.

The proofs given in the admirably short [7] use a dévissage argument, along with resolution of toric singularities, to reduce to the classical case of a divisor with normal crossings and a reference to [3]. Our point of view is that the monoidal models rendered natural by the log point of view are so convenient that it is natural and easy to give direct proofs, including proofs of the basic convergence results in the analytic setting. Thus our treatment is logically independent of [7] and even [3]. (Of course, these sources were fundamental inspirations.)

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1 An equivariant Riemann-Hilbert correspondence

1.1 Logarithmic and equivariant geometry

Smooth log schemes are locally modeled on affine monoid schemes, and the resulting toric geometry is a powerful tool in their analysis. We shall review the basic setup and techniques of affine monoid schemes (affine toric varieties) and then describe an equivariant Riemann-Hilbert correspondence for such schemes. This will be the main computational tool in our proof of the local and global correspondences in the next sections.

We start working over a commutative ring R, which later will become the field of complex numbers. All our monoids will be commutative unless otherwise stated. A monoid P is said to be *toric* if it is finitely generated, integral, and saturated and in addition P^{gp} is torsion free. If P is a monoid, we let $R[P]$ denote the monoid algebra of P over R, and write $e(p)$ or e_p for the element of $R[P]$ corresponding to an element p of P. If K is an ideal of P, we write $R[K]$ for the ideal of $R[P]$ generated by the elements of K and $R[P, K]$ for the quotient $R[P]/R[K]$. By an *idealized monoid* we mean a pair (P, K), where K is an ideal in a monoid P. Sometimes we simply write P for an idealized monoid (Q, K) and $R[P]$ for $R[Q, K]$.

We use the terminology of log geometry from, for example, [6]. Thus a *log scheme* is a scheme X, together with a sheaf of commutative monoids M_X on $X_{ét}$ and a morphism of sheaves of monoids α_X from M_X to the multiplicative monoid $\mathcal{O}_X$ which induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$. Then α induces an isomorphism from the sheaf of units M_X^* of M_X to $\mathcal{O}_X^*$; we denote by λ_X the inverse of this isomorphism and by $\overline{M}_X$ the quotient of M_X by $\mathcal{O}_X^*$. All our log schemes will be coherent, fine, and saturated; for the definitions and basic properties of these notions, we refer again to [6]. An *idealized log scheme* is a log scheme with a sheaf of ideals $K_X \subseteq M_X$ such that $\alpha_X(k) = 0$ for every local section k of K_X. A sheaf of ideals K_X of M_X is said to be *coherent* if it is locally generated by a finite number of sections, and we shall always assume this is the case. Morphisms of log schemes and idealized log schemes are defined in the obvious way. A morphism $f\colon X \to Y$ of fs idealized log schemes is *strict* if the induced map $f^{-1}\overline{M}_Y \to \overline{M}_X$ is an isomorphism, and it is *ideally strict* if the morphism $f^{-1}\overline{K}_Y \to \overline{K}_X$ is also an isomorphism.

If P is a monid, we let $\mathsf{A_P}$ denote the log scheme $\mathrm{Spec}(P \to R[P])$ and $\underline{\mathsf{A}}_\mathsf{P}$ its underlying scheme, *i.e.*, with trivial log structure. If A is an R-algebra, the set $\underline{\mathsf{A}}_\mathsf{P}(A)$ of A-valued points of $\underline{\mathsf{A}}_\mathsf{P}$ can be identified with the set of homomorphisms from the monoid P to the multiplicative monoid underlying A. This set has a natural monoid structure, and thus $\underline{\mathsf{A}}_\mathsf{P}$ can be viewed as a monoid object in the category of R-schemes. The canonical map $P \to P^{gp}$ induces a morphism $\mathsf{A}^*_P := \mathsf{A}_{\mathsf{P^{gp}}} = \underline{\mathsf{A}}_{\mathsf{P^{gp}}} \to \underline{\mathsf{A}}_\mathsf{P}$ which identifies A^*_P with the group scheme of units of $\underline{\mathsf{A}}_\mathsf{P}$. The natural morphism of log schemes $\mathsf{A_P} \to \underline{\mathsf{A}}_\mathsf{P}$ is injective on A-valued points, and its image coincides with the image of the map $\underline{\mathsf{A}}^*_\mathsf{P} \to \underline{\mathsf{A}}_\mathsf{P}$. If K is an ideal of P, the subscheme $\underline{\mathsf{A}}_{\mathsf{P,K}} := \mathrm{Spec}(R[P,K])$ it defines is invariant under the monoid action of $\underline{\mathsf{A}}_\mathsf{P}$ on itself, so that $\underline{\mathsf{A}}_{\mathsf{P,K}}$ defines an ideal of the monoid scheme $\underline{\mathsf{A}}_\mathsf{P}$. Also, K generates a (coherent) sheaf of ideals K_X in the sheaf of monoids M_X of $\mathsf{A_P}$, and the restrictions of M_X and K_X to $\underline{\mathsf{A}}_{\mathsf{P,K}}$ give it the structure of an idealized log scheme $\mathsf{A_{P,K}}$. It can be shown that, using Grothendieck's definition of smoothness via ideally strict infinitesimal thickenings as in [6], the ideally smooth log schemes over $\mathrm{Spec}\, R$ are exactly those that are, locally in the étale topology, isomorphic to $\mathsf{A_{P,K}}$ for some P and K. Note that these are the log schemes considered by Kato and Nakayama in [7].

Suppose from now on that P is toric. Then A^*_P is a torus with character group P^{gp}, and the evident map A^*_P to $\underline{\mathsf{A}}_\mathsf{P}$ is an open immersion. The complement F of a prime ideal $\mathfrak{p}$ of P is by definition a *face* of P. It is a submonoid of P, and there is a natural isomorphism of monoid algebras $R[F] \cong R[P]/\mathfrak{p}$, inducing an isomorphism $\underline{\mathsf{A}}_{\mathsf{P},\mathfrak{p}} \cong \underline{\mathsf{A}}_\mathsf{F}$. If k is an algebraically closed field, $\underline{\mathsf{A}}_{\mathsf{P},\mathfrak{p}}(k)$ is the closure of an orbit of the action of $\mathsf{A}^*_P(k)$ on $\underline{\mathsf{A}}_\mathsf{P}(k)$, and in this way the set of all faces of P parameterizes the set of orbits of $\underline{\mathsf{A}}_\mathsf{P}(k)$. In particular, the maximal ideal P^+ of P is the complement of the set of units P^* of P, and defines the minimal orbit of $\underline{\mathsf{A}}_\mathsf{P}$.

The map $P \to R$ sending every element of P^* to 1 and every element of P^+ to 0 is a homomorphism of monoids, and hence defines an R-valued point of $\underline{\mathsf{A}}_\mathsf{P}$, called the *vertex* of $\underline{\mathsf{A}}_\mathsf{P}$. The vertex belongs to $\underline{\mathsf{A}}_{\mathsf{P,K}}$ for every proper ideal K of P. By definition $\overline{P} := P/P^*$; and the surjection $P \to \overline{P}$ induces a strict closed immersion $\mathsf{A}_{\overline{\mathsf{P}}} \to \mathsf{A_P}$. The inclusion $P^* \to P$ defines a (log) smooth morphism $\mathsf{A_P} \to \mathsf{A_{P^*}}$; note that $\mathsf{A_{P^*}}$ is a torus and that $\mathsf{A}_{\overline{\mathsf{P}}}$ is the inverse image of its origin 1 under this map. Thus there is a Cartesian diagram:

$$\begin{tikzcd}
v \arrow[r] \arrow[d] & \underline{\mathsf{A}}_{\overline{\mathsf{P}}} \arrow[r] \arrow[d] & 1 \arrow[d] \\
\underline{\mathsf{A}}_{\mathsf{P},\mathsf{P}^+} \arrow[r] & \underline{\mathsf{A}}_{\mathsf{P}} \arrow[r] & \mathsf{A}_{\mathsf{P}^*}
\end{tikzcd}$$

The action of the torus A^*_P on $\underline{\mathsf{A}}_{\mathsf{P}}$ manifests itself algebraically in terms of a P^{gp}-grading on $R[P]$: $R[P]$ is a direct sum of R-modules $R[P] = \oplus\{A_p : p \in P^{gp}\}$, and the multiplication map sends $A_p \otimes A_q$ to A_{p+q}. Quasi-coherent sheaves on A_{P} which are equivariant with respect to the torus action correspond to P^{gp}-graded modules over $R[P]$.

More generally, if S is a P-set, there is a notion of an *S-graded $R[P]$-module.* This is an $R[P]$-module V together with a direct sum decomposition $V = \oplus\{V_s : s \in S\}$, such that for every $p \in P$, multiplication by $e_p \colon V \to V$ maps each V_s to V_{p+s}. For example, $R[S]$ is defined to be the free R-module generated by s in degree s, and if e_s is a basis in degree s and $p \in P$, $e_p e_s := e_{p+s}$. Morphisms of S-graded modules are required to preserve the grading. We denote by $Mod_R^S(P)$ the category of S-graded $R[P]$-modules and S-graded maps, and if K is an ideal of P, we denote by $Mod_R^S(P, K)$ the full subcategory consisting of those modules annihilated by K (*i.e.*, by the ideal of $R[P]$ generated by K). If the ring R is understood we may drop it from the notation.

Equivalently, one can work with *S-indexed R-modules.* Recall that the *transporter* of a P-set S is the category whose objects are the elements of S and whose morphisms from an object $s \in S$ to an object $s' \in S$ are the elements $p \in P$ such that $p+s = s'$, (with composition defined by the monoid law of P). Then an *S-indexed R-module* is by definition a functor F from the transporter of S to the category of R-modules. If F is an S-indexed R-module, then $\oplus\{F(s) : s \in S\}$ has a natural structure of an S-graded $R[P]$-module. This construction gives an isomorphism between the category of S-graded $R[P]$-modules and the category of S-indexed R-modules, and we shall not distinguish between these two notions in our notation. See also the discussion by Lorenzon [8].

If the action of P on S extends to a free action of P^{gp} on the localization of S by P we say that S is *potentially free.* If S is potentially free, then whenever s and s' are two elements of S and p is an element of P such that $s' = p + s$, then p is unique, and the transporter category of S becomes a pre-ordered set. In this case, an S-indexed module F for which all the transition maps are injective amounts to an S-filtered R-module, and the corresponding S-graded $R[P]$-module is torsion free.

In particular let $\phi \colon P \to Q$ be a morphism of monoids. Then Q inherits an action of P, and so it makes sense to speak of a *Q-graded $R[P]$-module.* The morphism ϕ also defines a morphism of monoid schemes $\underline{\mathsf{A}}_\phi \colon \underline{\mathsf{A}}_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{P}}$, and hence an action $\mu \colon \underline{\mathsf{A}}_{\mathsf{P}} \times \underline{\mathsf{A}}_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{P}}$ of $\underline{\mathsf{A}}_{\mathsf{Q}}$ on $\underline{\mathsf{A}}_{\mathsf{P}}$. A Q-grading on an

$R[P]$-module E then corresponds to an $\underline{\mathsf{A}}_{\mathsf{Q}}$-equivariant quasi-coherent sheaf $\tilde{E}$ on $\underline{\mathsf{A}}_{\mathsf{P}}$, *i.e.*, a quasi-coherent sheaf $\tilde{E}$ together with a linear map $\mu^*\tilde{E} \to pr_1^*\tilde{E}$ on $\underline{\mathsf{A}}_{\mathsf{P}} \times \underline{\mathsf{A}}_{\mathsf{Q}}$ satisfying a suitable cocycle condition. We shall be especially concerned with the case in which Q is a submonoid of $R \otimes P^{gp}$, or even all of $R \otimes P^{gp}$.

Remark 1.1.1 Let $\phi\colon P \to Q$ be a morphism of monoids, let S (resp. T) be a P-set (resp. a Q-set) and let $\psi\colon S \to T$ be a morphism of P-sets over ϕ. Then if E is an object of $Mod_R^S(P)$, the tensor product $R[Q] \otimes_{R[P]} E$ has a natural T-grading, uniquely determined by the fact that if $x \in E$ has degree s and $q \in Q$, then $e_q \otimes x$ has degree $q + \psi(s)$. This works because if $p \in P$, $(q + \phi(p)) + \psi(s) = q + \psi(p + s)$. We denote this T-graded $R[P]$-module by $\phi_\psi^*(E)$. If F is an object of $Mod_R^T(Q)$, then there is a natural map of $R[P]$-modules:

$$\phi_*^\psi(F) := \oplus_{s \in S} F_{\psi(s)} \to \phi_* F = \oplus_{t \in T} F_t.$$

Furthermore $\phi_*^\psi(F)$ is naturally S-graded, and the functor ϕ_*^ψ is right adjoint to the functor ϕ_ψ^*. For example, if P is the zero monoid and $T = Q^{gp}$, then the adjoint to the functor $Mod(R) \to Mod_R^{Q^{gp}}(Q)$ is the functor which takes a Q^{gp}-graded module to its component of degree zero. We denote this functor by π_*^Q.

Proposition 1.1.2 *Let P be an integral monoid, let S be a potentially free P-set, and view the orbit space S/P^* as a $\overline{P}$-set, so that the projection $\pi\colon S \to S/P^*$ is a morphism over the morphism $\pi\colon P \to \overline{P}$.*

1. *The base-change functor*

$$\pi_\pi^*\colon Mod_R^S(P) \to Mod_R^{S/P^*}(\overline{P})$$

is an equivalence of categories.

2. *If E is any object of $Mod_R^S(P)$, $\overline{E} := \pi_\pi^* E$, and $s \in S$ maps to $\overline{s} \in S/P^*$, then the natural map $E_s \to \overline{E}_{\overline{s}}$ is an isomorphism.*

Proof: Let I be kernel of the surjective map $R[P] \to R[\overline{P}]$. This is the ideal generated by the set of elements of the form $1 - e_u : u \in P^*$. If E is an object of $Mod_R^S(P)$, then $\overline{E} := \pi_\pi^* E \cong E/IE$. Since S is potentially free as a P-set, the action of the group P^* on S is free. Thus an element t of S/P^*, viewed as a subset of S, is a torsor under the action of P^*. Let $E_t := \oplus\{E_s : s \in t\}$. Then E_t has a natural action of $R[P^*]$ and $\overline{E}_t \cong E_t \otimes_{R[P^*]} R$, where $R[P^*] \to R$ is the map sending every element of P^* to 1_R. Let $t := \overline{s}$, and let J be the kernel of the augmentation map $R[P^*] \to R$. Since J and I have the same generators, $\overline{E}_t \cong E_t/JE_t$. For each $s' \in t$, there is a unique $u' \in P^*$ such that $s = u's'$, and multiplication by $e_{u'}$ defines an isomorphism $\iota_{s'}\colon E_{s'} \to E_s$. The sum of all these defines a morphism ι of R-modules $E_t \to E_s$. If $u \in P^*$ and $s' := us''$, then $u''u = u'$, and hence $\iota_{u''} \circ \cdot e_u = \iota_{u'}$. Thus ι factors through a morphism $\bar{\iota}$ of R-modules $E_t/JE_t \to E_s$. The inclusion $E_s \to E_t$ induces a section j: $\iota \circ j = \mathrm{id}$. Since the map $j\colon E_s \to E_t/JE_t$ is also evidently surjective, it is an isomorphism inverse to π_s. This proves (1.1.2.2), which implies that the functor π_π^* is fully faithful. One checks immediately that π_*^π is a quasi-inverse. □

With the notation of the proposition above, suppose that E is an S-graded $R[P]$-module. The map $\eta\colon R[P] \to R$ sending P to 1 can be thought of as a generic R-valued point of $\underline{\mathsf{A}}_{\mathsf{P}}$. Indeed, this map factors through $R[P^{gp}]$, and the above result shows that it induces an equivalence from the category of $S \otimes P^{gp}$-graded modules to the category of R-modules. Let E_η denote the R-module η^*E. For each $s \in S$, there is a map of R-modules

$$cosp_{s,\eta}\colon E_s \to E_\eta.$$

Corollary 1.1.3 *If E and $s \in S$ are as above, suppose that E is torsion free as an $R[P]$-module and also that it admits a set of homogeneous generators in degrees $t \le s$ (i.e., for each generating degree t, there exists $p \in P$ with $s = p + t$). Then the cospecialization map $cosp_{s,\eta}$ is an isomorphism.*

Proof: Let $E' := E \otimes R[P^{gp}]$. Since E is torsion free, the map from E to E' is injective. The proposition shows that for any $s' \in S \otimes P^{gp}$, the map $E'_{s'} \to E_\eta$ is bijective. So it suffices to see that the map $E_s \to E_\eta$ is surjective. Any $x' \in E'_{s'}$ is a sum of elements of the form $e_q x_q$, where $q \in P^{gp}$ and $x_q \in E$ is a homogeneous generator of some degree $t \le s$. Thus it suffices to show that if x' is equal to such an $e_q x_q$, then its image in E_η is in the image of E_s. Write $s = p + t$, with $p \in P$, so that $x' = e_q x_q = e_{q-p}(e_p x_q)$. Then $e_p x_q \in E_s$ has the same image in E_η as does x'. □

1.2 Equivariant differentials and connections

Let P be a toric monoid and let $X := \mathsf{A}_{\mathsf{P}}$; since X is affine, we may and shall identify quasi-coherent sheaves with $R[P]$-modules. We refer to [6] and [11] for the definitions and basic properties of the (log) differentials $\Omega^1_{X/R}$ and modules with connection on X/R. Recall in particular that $\Omega^1_{X/R}$ is the quasi-coherent sheaf on X corresponding to the $R[P]$-module

$$R[P] \otimes_{\mathbf{Z}} P^{gp} \cong R[P] \otimes_R \Omega_{P/R},$$

where $\Omega_{P/R} := R \otimes P^{gp}$. If $p \in P$, we sometimes denote by dp the class of $1 \otimes p^{gp}$ in $\Omega_{P/R}$ We write $\Omega^i_{P/R}$ for the ith exterior power of $\Omega_{P/R}$ and $T_{P/R}$ for its dual; we shall drop the subscripts if there seems to be no risk of confusion. An element p of P defines a global section $\beta(p)$ of M_X, and

$$dlog\,\beta(p) = dp = 1 \otimes p^{gp}$$

in $\Omega_{P/R} \subseteq \Omega^1_{X/R}$. Such an element p also defines a basis element e_p of $R[P]$, and $de_p = e_p dp \in \Omega^1_{X/R}$. The grading of $\Omega^1_{X/R}$ for which d is homogeneous of degree zero corresponds to the action of A^*_P on $\Omega^1_{X/R}$ induced by functoriality; under this action, $\Omega_{P/R} \subseteq \Omega^1_{X/R}$ is the set of invariant forms, *i.e.*, the component of degree zero. The dual $T_{P/R}$ of $\Omega_{P/R}$ can be thought of as the module of equivariant vector fields on $\mathsf{A_P}$. If E is an $R[P]$-module, a connection on the corresponding sheaf on X corresponds to a map

$$\nabla \colon E \to E \otimes_{R[P]} \Omega^1_{X/R} \cong E \otimes_R \Omega_{P/R},$$

and the Leibniz rule reduces to the requirement that

$$\nabla(e_p x) = e_p x \otimes dp + e_p \nabla(x).$$

for $p \in P$ and $x \in E$.

Remark 1.2.1 If K is an ideal of P, let $\mathsf{A_{P,K}}$ be the idealized log subscheme of $\mathsf{A_P}$ defined by K. Then the structure sheaf of $\mathsf{A_{P,K}}$ corresponds to $R[P,K]$ and $\Omega^1_{X/R}$ to $R[P,K] \otimes_R \Omega^1_{P/R}$. Thus the category of modules with integrable connection on $\mathsf{A_{P,K}}/R$ can be identified with the full subcategory of modules with integrable connection on $\mathsf{A_P}/R$ annihilated by K. This remark reduces the local study of connections on idealized log schemes to the case in which the ideal is empty.

Suppose now that S is a P-set and (E, ∇) is an S-graded $R[P]$-module with an integrable log connection. The S-grading on E induces an S-grading on $\Omega^1_{P/R} \otimes E$; we say that ∇ is *homogeneous* if it preserves the grading. Thus for each $s \in S$ and $p \in P$, there is a commutative diagram

$$\begin{tikzcd}
E_s \arrow[r, "\nabla + dp"] \arrow[d, "e_p"] & E_s \otimes_R \Omega_{P/R} \arrow[d, "e_p"'] \\
E_{p+s} \arrow[r, "\nabla"] & E_{p+s} \otimes_R \Omega_{P/R}
\end{tikzcd}$$

For example, the data of a homogeneous log connection on $R[S]$ amounts simply to a morphism of P-sets $d \colon S \to \Omega^1_{P/R}$. Note that such a morphism defines a pairing $\langle\ ,\ \rangle \colon T_{P/R} \times S \to R$.

Definition 1.2.2 *Let (P,K) be an idealized monoid and R a ring. Then a set of* exponential data *for (P,K) over R is an abelian group Λ together with homomorphisms $P \to \Lambda$ and $\Lambda \to \Omega_{P/R}$ whose composition is the map $p \mapsto dp$. The data are said to be* rigid *if for every nonzero $\lambda \in \Lambda$, there exists a $t \in T_{P/R}$ such that $\langle t, \lambda \rangle \in R^*$.*

Typical examples are $\Lambda = P^{gp}$, $\Lambda = R \otimes P^{gp}$, and $\Lambda = k \otimes P^{gp}$, where k is a field contained in R. Rigidity implies that $\Lambda \to \Omega$ is injective, and is equivalent to this if R is a field. Note that if R is flat over $\mathbf{Z}$, the map $P^{gp} \to \Omega$ is also injective.

We sometimes just write Λ for the entire set of exponential data. Given such data, P acts on Λ and it makes sense to speak of a Λ-graded $R[P]$-module with homogeneous connection. For example, $R[P]$ can be viewed as a Λ-graded $R[P]$-module, where e_p is given degree $\delta(p)$ as in (1.1.1), and the connection d is Λ-graded. Because the homomorphism $\Lambda \to \Omega_{P/R}$ is also a map of P-sets, $R[\Lambda]$ also has such a structure. Associated to the map $P \to \Lambda$ is a map from the torus A_Λ to A_P and a consequent action of A_Λ on A_P. Then a Λ-graded $R[P]$-module with a homegeneous connection corresponds to a quasi-coherent sheaf with a connection on A_P which are together equivariant with respect to this action.

Definition 1.2.3 *Let (P, K) be an idealized toric monoid and let*

$$P \xrightarrow{\delta} \Lambda \xrightarrow{\epsilon} \Omega_{P/R}$$

be a set of exponential data for P/R.

1. *$MIC^\Lambda(P, K/R)$ is the category of Λ-graded $R[P]$-modules with homogeneous connection and morphisms preserving the connections and gradings.*

2. *An object (E, ∇) of $MIC^\Lambda(P, K/R)$ is said to be* normalized *if for every $t \in T_{P/R}$ and every $\lambda \in \Lambda$ the endomorphism of E_λ induced by $\nabla_t - \langle t, \lambda\rangle$ is locally nilpotent. The full subcategory of $MIC^\Lambda(P, K/R)$ consisting of the normalized (resp. of the normalized and finitely generated) objects is denoted by $MIC^\Lambda_*(P, K/R)$ (resp. $MIC^\Lambda_{coh}(P, K/R)$).*

Remark 1.2.4 Let $MIC(P, K/R)$ be the category of $R[P, K]$-modules with integrable log connection but no grading. If the exponential data are rigid, the functor $MIC^\Lambda_*(P, K/R) \to MIC(P, K/R)$ is fully faithful. To see this, note first that, since the category $MIC^\Lambda(P, K/R)$ has internal Hom's, it suffices to check that if (E, ∇) is an object of $MIC^\Lambda_*(P, K/R)$ and $e \in E$ is horizontal, then $e \in E_0$. In other words, we have to show that ∇ is injective on E_λ if $\lambda \neq 0$. Since the data are rigid, if $\lambda \neq 0$ there exists a $t \in T$ such that $\langle t, \lambda\rangle$ is a unit, and the action of ∇_t on E_λ can be written as $\langle t, \lambda\rangle$ plus a locally nilpotent endomorphism. It follows that ∇_t is an isomorphism on E_λ.

When the choice of Λ is clear or fixed in advance, we shall permit ourselves to drop it from the notation. We also sometimes use the same letter to denote an element of P or Λ and its image in Λ or $\Omega_{P/R}$. This is safe to do if the maps $P \to \Lambda$ and $\Lambda \to \Omega_{P/R}$ are injective.

Example 1.2.5 The differential $d\colon R[P, K] \to R[P, K] \otimes_R \Omega_{P/R}$ defines an object of $MIC^\Lambda_{coh}(P/R)$, for any Λ. More generally, choose $\lambda \in \Lambda$, and let

L^λ denote the free Λ-graded $R[P,K]$-module generated by a single element x_λ in degree λ, with the connection d such that $d(e_p x_\lambda) = e_p x_\lambda \otimes (dp + \epsilon(\lambda))$. If $t \in T_{P/R}$, then $d_t(e_p x_\lambda) = \langle t, p+\lambda\rangle$. Since $e_p x_\lambda$ has degree $\delta(p)+\lambda$ and $d_t - \langle t, dp+\epsilon(\lambda)\rangle = 0$ in this degree, L^λ belongs to $MIC^\Lambda_{coh}(P,K/R)$. For λ and λ' in Λ there is a homogeneous and horizontal isomorphism $L^\lambda \otimes L^{\lambda'} \to L^{\lambda+\lambda'}$ sending $x_\lambda \otimes x_{\lambda'}$ to $x_{\lambda+\lambda'}$, and in this way one finds a ring structure on the direct sum $\oplus\{L^\lambda : \lambda \in \Lambda\}$, compatible with the connection. This direct sum is in some sense a universal diagonal object of $MIC^\Lambda_*(P,K/R)$. The ring $\oplus_\lambda L^\lambda$ can be identified with the tensor product of the monoid algebras $R[P,K]$ and $R[\Lambda]$, or with the quotient of the monoid algebra $R[P\oplus\Lambda]$ of $P\oplus\Lambda$ by the ideal generated by K. We shall also denote it by $R[P,K,\Lambda]$. Note the unusual grading: the degree of $e_p x_\lambda$ is $\delta(p)+\lambda$. The ring $R[P,K,\Lambda]$ admits another Λ grading, in which $e_p x_\lambda$ has degree λ. In fact it is naturally $\Lambda\oplus\Lambda$ graded. For convenience, shall set $\Lambda' := \Lambda$ and say that $e_p x_{\lambda'}$ has Λ-*degree* $\delta(p)+\lambda'$ and Λ'-*degree* λ'. When we need to save space, we shall let P stand for the pair (P,K) and just write $R[P,\Lambda]$ instead of $R[P,K,\Lambda]$.

Example 1.2.6 One can also construct a universal nilpotent object as follows. Let $\Omega := \Omega_{P/R}$, and for each $n \in \mathbf{N}$, let $\Omega \to \Gamma^n(\Omega)$ denote the universal polynomial law of degree n [2, Appendix A] over R. Thus, $\Gamma^n(\Omega)$ is the R-linear dual of the nth symmetric power of $T_{P/R}$, and $\Gamma^\cdot(\Omega) := \oplus_n \Gamma^n(\Omega)$ is the divided power polynomial algebra on Ω. It has an exterior derivative d which maps $\Gamma^n(\Omega)$ to $\Gamma^{n-1}(\Omega)\otimes\Omega$, defined by

$$d\omega_1^{[I_1]}\cdots\omega_n^{[I_n]} := \sum_i \omega_1^{[I_1]}\cdots\omega_i^{[I_i-1]}\cdots\omega_n^{[I_n]}\otimes\omega_i \tag{1.2.1}$$

Of course, if R is a $\mathbf{Q}$-algebra, $\Gamma^n(\Omega)$ can be identified with the nth symmetric power of Ω. Let $N(P,K) := R[P,K]\otimes_R \Gamma^\cdot(\Omega)$, graded so that $\Gamma^\cdot(\Omega)$ has degree zero, and let

$$\nabla\colon N(P,K) \to N(P,K)\otimes_R \Omega_{P/R} := d\otimes\mathrm{id} + \mathrm{id}\otimes d$$

be the extension of d satisfying the Leibniz rule with respect to $R[P]$. Then $N(P,K) \in MIC^\Lambda_*(P,K)$. Note that $N_*(P,K)$ has an exhaustive filtration $F.$, where $F_n := \sum_{i\le n} R[P,K]\otimes\Gamma^i(\Omega)$, and the associated graded connection is constant.

1.3 Equivariant Higgs fields

Let X be a smooth scheme over R, let $\Omega_{X/R}$ be its sheaf of Kahler differentials, and let $T_{X/R}$ be the dual of $\Omega_{X/R}$. Recall [13] that a *Higgs field* on a sheaf F of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-linear map $\theta\colon F \to F\otimes\Omega^1_{X/R}$ such that the composite $F \to F\otimes\Omega^1_{X/R} \to F\otimes\Omega^2_{X/R}$ vanishes. Such a θ is equivalent to an action of the symmetric algebra $S^\cdot T_{X/R}$ on F, and hence defines a sheaf of $\mathcal{O}_{\mathbf{T}^*_{X/R}}$-modules, where $\mathbf{T}^*_{X/R} := \mathbf{V}T_{X/R} := \operatorname{Spec}_X S^{\cdot}T_{X/R}$ is the cotangent bundle of X/R. One can prolong a Higgs field θ to a complex

$$K^{\cdot}(F,\theta) := F \to F \otimes \Omega^1_{X/R} \to F \otimes \Omega^2_{X/R} \to \cdots$$

with $\mathcal{O}_X$-linear boundary maps induced by θ, called the *Higgs complex* of (F,θ). All these constructions make sense with $T_{X/R}$ replaced by any locally free sheaf T of $\mathcal{O}_X$-modules, and we call (F,θ) an *$\mathcal{O}_X$-T-module* or *T-Higgs-module* in the general case.

One can define internal tensor products and duals in the category of T-Higgs modules in the same way one does for modules with connection. For example, if θ and θ' are T-Higgs fields on F and F' respectively, then $\theta \otimes \mathrm{id} + \mathrm{id} \otimes \theta'$ is the Higgs field on $F \otimes F'$ used to define the internal tensor product. If ω is a section of the dual Ω of T, the *ω-twist* of a T-Higgs field θ is the T-Higgs field $\theta + \mathrm{id} \otimes \omega$. An R-T module (F,θ) is said to be *nilpotent* if θ_t defines a locally nilpotent endomorphism of F for every $t \in T$. This means that the corresponding sheaf on $\mathbf{V}T$ is supported on the zero section.

A *Jordan decomposition* of a T-Higgs module (E,θ) is a direct sum decomposition $E \cong \oplus E_\omega : \omega \in \Omega$ such that each E_ω is invariant under θ and is the ω-twist of a nilpotent T-Higgs module. For example, if R is an algebraically closed field and E is finitely generated, then E can be viewed as a module of finite length over $S^{\cdot}T$ and its support is a finite subset of the maximal spectrum of $S^{\cdot}T$, which can be canonically identified with Ω. Thus E admits a canonical Jordan decomposition $E \cong \oplus E_\omega$.

The following simple and well-known vanishing lemma will play a central role.

Lemma 1.3.1 *Let (F,θ) be a T-Higgs module and suppose there exists a $t \in T$ such that θ_t is an automorphism of F. Then the Higgs complex $K^{\cdot}(F,\theta)$ is homotopic to zero, hence acyclic.*

Proof: Interior multiplication by t defines a sequence of maps

$$\rho^i \colon F \otimes \Omega^i \to F \otimes \Omega^{i-1}.$$

One verifies easily that $\kappa := d\rho + \rho d$ is $\theta_t \otimes \mathrm{id}$. Thus θ_t induces the zero map on cohomology, and since θ_t is an isomorphism, the cohomology vanishes. $\square$

We shall see that there is a simple relationship between equivariant Higgs fields and equivariant connections. In fact there are two constructions we shall use.

Definition 1.3.2 *Let P be an idealized toric monoid and $P \xrightarrow{d} \Lambda \xrightarrow{\epsilon} \Omega_{P/R}$ a set of exponential data for P.*

1. *$HIG^\Lambda(P/R)$ is the category of Λ-graded $R[P]$-$T_{P/R}$ modules. That is, the objects are pairs (E,θ), where E is a Λ-graded $R[P]$-module and*

$$\theta\colon E \to E \otimes_R \Omega_{P/R}$$

is a homogeneous map such that $\theta \wedge \theta = 0$, and the morphisms are the degree preserving maps compatible with θ.

2. *An object (E,θ) of $HIG^\Lambda(P/R)$ is* nilpotent *if for every $t \in T_{P/R}$, the endomorphism θ_t of E is locally nilpotent. The full subcategory of $HIG^\Lambda(P/R)$ consisting of nilpotent (resp., the nilpotent and finitely generated objects) is denoted by $HIG^\Lambda_*(P/R)$ (resp., $HIG^\Lambda_{coh}(P/R)$).*

EXAMPLE 1.3.3 If $\lambda \in \Lambda$, let L^λ be the free Λ-graded $R[P]$-module generated in degree λ by x_λ. Then there is a unique $T_{P/R}$-Higgs field θ on L^λ such that $\theta(e_p x_\lambda) = e_p x_\lambda \otimes \epsilon(\lambda)$ for each $p \in P$. The isomorphism $L^\lambda \otimes L^{\lambda'} \to L^{\lambda+\lambda'}$ sending $x_\lambda \otimes x_{\lambda'}$ to $x_{\lambda+\lambda'}$ is compatible with the induced Higgs fields, so we get a Higgs field θ on $R[P,\Lambda] = \oplus L^\lambda$, compatible with the ring structure. Similarly there is a unique Higgs field on $N(P) = R[P] \otimes \Gamma^{\cdot}(\Omega)$ such that

$$\omega_1^{[I_1]} \cdots \omega_n^{[I_n]} \mapsto \sum_i \omega_1^{[I_1]} \cdots \omega_i^{[I_i - 1]} \cdots \omega_n^{[I_n]} \otimes \omega_i$$

for all I.

Let (E,∇) be an object of $MIC^\Lambda(P/R)$. We can forget the $R[P]$-module structure of E and view it as an R-module. Since $T_{P/R}$ is a finitely generated free R-module, $\nabla\colon E \to E \otimes \Omega_{P/R}$ can be viewed as a $T_{P/R}$-Higgs field on E. If R is an algebraically closed field and E is finite dimensional over R, such fields are easy to analyze, using the Jordan decomposition. We can generalize this as follows.

LEMMA 1.3.4 *Let $P \to \Lambda \to \Omega_{P/R}$ be a rigid set of exponential data for an idealized monoid P.*

1. *Let (E,∇) be an object of $MIC(P,K/R)$. Suppose the corresponding $T_{P/R}$-Higgs module (E,∇) admits a Jordan decomposition $E = \oplus E_\lambda$. Then this direct sum decomposition gives E the structure of a Λ-graded $R[P,K]$-module, and with this structure, $(E,\nabla) \in MIC^\Lambda_*(P,K/R)$. Thus, $MIC^\Lambda_*(P,K/R)$ is equivalent to the full subcategory of $MIC^\Lambda(P,K/R)$ whose corresponding $T_{P/R}$-Higgs modules admit a Jordan decomposition.*

2. *If $(E,\nabla) \in MIC^\Lambda_*(P,K/R)$, then its de Rham complex is acyclic except in degree zero.*

Proof: Let $\theta_\lambda := \nabla - \mathrm{id} \otimes \lambda$. The Leibniz rule implies that for each $p \in P$ and $t \in T$, $\theta_{t,\lambda+p} \circ e_p = e_p \circ \theta_{t,\lambda}$. It follows that $\theta^n_{t,\lambda+p} \circ e_p = e_p \circ \theta^n_{t,\lambda}$, for every $n \geq 0$. If $x \in E_\lambda$, then x is killed by some power of $\theta_{t,\lambda}$, and hence $e_p x$ is killed by some power of $\theta_{t,\lambda+p}$. For any λ', $\theta_{t,\lambda'} = \theta_{t,\lambda+p} + \langle t, \lambda' - p - \lambda\rangle$. If $\lambda' \neq p+\lambda$, we can choose t so that $\langle t, \lambda' - p - \lambda\rangle$ is a unit, and hence $\theta_{t,\lambda+p}$ acts injectively on $E_{\lambda'}$. It follows that the degree λ' piece of $e_p x$ is zero. In other words, e_p maps E_λ to $E_{p+\lambda}$. This shows that $\oplus E_\lambda$ gives E the structure of a Λ-graded $R[P]$-module. Evidently each E_λ is invariant under ∇, and killed by K, and with this grading, $(E, \nabla) \in MIC^\Lambda_*(P, K/R)$. We have already remarked in (1.2.4) that $MIC^\Lambda_*(P, K/R)$ is a full subcategory of $MIC(P, K/R)$. The $T_{P/R}$-Higgs module associated to every object of $MIC^\Lambda_*(P, K/R)$ admits a Jordan decomposition, by definition, and the above argument show that the converse is also true. This proves (1).

Let (E, ∇) be an object of $MIC^\Lambda_*(P, K/R)$. Its de Rham complex is Λ-graded, and its component in degree λ can be viewed as the Higgs complex associated to the linear map $\nabla\colon E_\lambda \to E_\lambda \otimes \Omega_{P/R}$. If $\lambda \neq 0$, then there exists a $t \in T_P$ such that $\langle t, \lambda\rangle$ is not zero, hence a unit. Since E is normalized, $\nabla_t - \langle t, \lambda\rangle$ is nilpotent, and hence ∇_t an isomorphism, in degree λ. By (1.3.1), this implies that the complex $E \otimes \Omega^{\cdot}_{P/R}$ is acyclic in degree λ and proves (2). $\square$

In general, suppose that E is an object of $MIC^\Lambda(P/R)$. Then the degree λ component of ∇ is a Higgs field on E_λ. Then

$$\theta_\lambda := \nabla - \mathrm{id}_{E_\lambda} \otimes \lambda \quad : \; E_\lambda \to E_\lambda \otimes_R \Omega^1_{P/R}$$

is another Higgs field, and evidently (E, ∇) is normalized if and only if this field is nilpotent for every $\lambda \in \Lambda$. Moreover, $\theta := \oplus_\lambda \theta_\lambda$ is $R[P]$-linear, and endows E with the structure of an equivariant $R[P]$-$T_{P/R}$-module. This Higgs module structure can be viewed as the difference between the given connection ∇ and the "trivial" connection coming from the action of Λ. This simple construction evidently gives a complete description of the category of (normalized) equivariant connections in terms of the category of (nilpotent) equivariant Higgs modules, and it will play a crucial role in our proof of the equivariant Riemann-Hilbert correspondence.

We shall see that the above correspondence can be expressed in terms of a suitable "integral transform." As it turns out, this integral transform introduces a sign. To keep things straight, we introduce the following notation. Let

$$P \xrightarrow{\delta} \Lambda \xrightarrow{\epsilon} \Omega_{P/R}$$

be a set of exponential data for a toric monoid P. Let $P' := -P \subseteq P^{gp}$, let $\Lambda' := \Lambda$, let $\epsilon' := \epsilon$, and let $\delta'\colon P' \to \Lambda'$ be the composite of the inclusion

$-P \to P^{gp}$ with $\delta^{gp} \colon P^{gp} \to \Lambda$. Thus we have a commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "-\mathrm{id}"'] & \Lambda \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] & \Omega_{P/R} \arrow[d, "\mathrm{id}"] \\ P' \arrow[r, "-\delta'"] & \Lambda' \arrow[r, "\epsilon'"] & \Omega_{P'/R}. \end{tikzcd}$$

(Note that the vertical arrow on the right is the map induced by the identity map $P^{gp} \to P^{gp} = (-P)^{gp}$ and is the negative of the map induced by functoriality from the isomorphism $P' \to P$.)

In the context of the above set-up, there is a completely trivial equivalence between the categories $Mod_R^{\Lambda}(P, K)$ and $Mod_R^{\Lambda'}(P', K')$, where $K' := -K$. Namely, if $(E, \nabla) \in Mod_R^{\Lambda}(P, K)$, then for each $\lambda' \in \Lambda' = \Lambda$, let $E'_{\lambda'} := E_{-\lambda'}$. If $p' \in P'$, $-p' \in P$, and one can define

$$\cdot e_{p'} : E'_{\lambda'} \to E'_{\lambda'+p'}$$

to be multiplication by $e_{-p'}$. This gives $\oplus E'_{\lambda'}$ the structure of a Λ'-graded $R[P', K']$-module, and it is evident that the functor $E \mapsto E'$ is an equivalence. This is too trivial to require a proof, but since it will be very useful in our following constructions, it is worth stating for further reference.

Proposition 1.3.5 *Let (P, K) be an idealized toric monoid endowed with exponential data $P \xrightarrow{\delta} \Lambda \xrightarrow{\epsilon} \Omega$ and let $P' \xrightarrow{\delta'} \Lambda' \xrightarrow{\epsilon'} \Omega$ be the corresponding exponential data for (P', K').*

1. *The functor $Mod_R^{\Lambda}(P, K) \to Mod_R^{\Lambda'}(P', K')$ described above is an equivalence of categories, compatible with tensor products and internal Hom.*

2. *If $(E, \nabla) \in MIC^{\Lambda}(P, K/R)$, let E' be the object of $Mod_R^{\Lambda'}(P', K')$ corresponding to E, and define $\theta' \colon E' \to E' \otimes_R \Omega$ by the following diagram:*

$$\begin{tikzcd} E_\lambda \arrow[r, "="] \arrow[d, "\nabla - \mathrm{id} \otimes \epsilon(\lambda)"'] & E'_{-\lambda} \arrow[d, "\theta'"] \\ E_\lambda \otimes \Omega \arrow[r, "="] & E'_{-\lambda} \otimes \Omega \end{tikzcd}$$

Then θ' defines a Higgs field on E', and the corresponding functor $MIC^{\Lambda}(P, K/R) \to HIG^{\Lambda'}(P', K'/R)$ is an equivalence. Under this functor,an object (E, ∇) is normalized if and only if the corresponding (E', θ') is nilpotent.

$\square$

The value of the above proposition will be enhanced by the fact that its functors can be realized geometrically, using the ring $R[P,\Lambda]$ described in (1.3.3) and (1.2.5). (Here P stands for an idealized monoid (P,K).)

We have morphisms of monoids:

$$\begin{array}{rcl}
\phi\colon P \to P\oplus\Lambda & : & p \mapsto (p,0)\\
\eta\colon P\oplus\Lambda \to \Lambda & : & (p,\lambda) \mapsto \delta(p)+\lambda\\
\phi'\colon P' \to P\oplus\Lambda & : & p' \mapsto (-p',\delta'(p'))\\
\psi\colon P\oplus\Lambda \to \Lambda\oplus\Lambda & : & (p,\lambda) \mapsto (\delta(p)+\lambda,\lambda)\\
\pi\colon \Lambda \to \Lambda\oplus\Lambda & : & \lambda \mapsto (\lambda,0)\\
\pi'\colon \Lambda \to \Lambda\oplus\Lambda & : & \lambda \mapsto (0,\lambda)
\end{array}$$

These fit into commutative diagrams:

$$\begin{tikzcd}
P \arrow[r, "\phi"] \arrow[d, "\delta"'] & P\oplus\Lambda \arrow[d, "\psi"] \\
\Lambda \arrow[r, "\pi"] & \Lambda\oplus\Lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
P' \arrow[r, "\phi'"] \arrow[d, "\delta'"'] & P\oplus\Lambda \arrow[d, "\psi"] \\
\Lambda \arrow[r, "\pi'"] & \Lambda\oplus\Lambda
\end{tikzcd}$$

Note that $\eta\circ\phi' = 0$ and that $\underline{\mathsf{A}}_\phi$ corresponds to the projection $q\colon \underline{\mathsf{A}}_{\mathsf{P}}\times\underline{\mathsf{A}}_{\Lambda} \to \underline{\mathsf{A}}_{\mathsf{P}}$. Let $q' := \underline{\mathsf{A}}_{\phi'}$. Then there is a commutative diagram:

$$\begin{tikzcd}
\underline{\mathsf{A}}_{\Lambda} \arrow[r, "\underline{\mathsf{A}}_\eta"] \arrow[dr, "\underline{\mathsf{A}}_\delta"'] & \underline{\mathsf{A}}_{\mathsf{P}}\times\underline{\mathsf{A}}_{\Lambda} \arrow[r, "q'"] \arrow[d, "q=pr_1"] & \underline{\mathsf{A}}_{\mathsf{P}'} \\
 & \mathsf{A}_{\mathsf{P}} &
\end{tikzcd} \tag{1.3.1}$$

In this diagram, $\underline{\mathsf{A}}_\eta$ is a closed immersion, and identifies $\underline{\mathsf{A}}_{\Lambda}$ with ${q'}^{-1}(1_{\underline{\mathsf{A}}_{\mathsf{P}'}})$.

Recall from (1.2.5) that $R[P,\Lambda]$ is a $\Lambda\oplus\Lambda'$-graded ring, where $e_p x_{\lambda'}$ has degree $(\delta(p)+\lambda',\lambda')$. Thus, a Λ-Λ'-graded $R[P,\Lambda]$-module is an $R[P,\Lambda]$-module $\tilde{E}$ together with a direct sum decomposition into sub R-modules $\tilde{E}=\oplus\tilde{E}_{\lambda,\lambda'}$, such that multiplication by $e_p x_\mu$ maps $\tilde{E}_{\lambda,\lambda'}$ into $\tilde{E}_{\delta(p)+\lambda+\mu,\lambda'+\mu}$. The category of such objects (with bihomogeneous morphisms) will be denoted by $Mod_\Lambda^{\Lambda'}(P,\Lambda)$. The pair of morphisms (ϕ,π) induces a functor

$$q_\pi^*\colon Mod_R^\Lambda(P) \to Mod_\Lambda^{\Lambda'}(P,\Lambda) \quad : \quad E \mapsto E\otimes_{R[P]} R[P,\Lambda] \cong E\otimes_R R[\Lambda],$$

where $E\otimes_R R[\Lambda]$ is graded so that $e\otimes x_{\lambda'}$ has bidegree $(\lambda+\lambda',\lambda')$ if $e\in E$ has degree λ, as discussed in (1.1.1). Recall that its left adjoint, which we denote by q_*^π or $q_*^{\Lambda'}$, takes an object of $Mod_\Lambda^{\Lambda'}(P,\Lambda)$ to the Λ-graded $R[P]$-submodule consisting of the elements whose Λ'-degree is zero.

Recall that the connection ∇ on $R[P,\Lambda]$ sends $e_p x_\lambda$ to $e_p x_\lambda \otimes (dp + \epsilon(\lambda))$. In particular, $e_p x_{-p}$ is horizontal. This implies that, when $R[P,\Lambda]$ is regarded as an $R[P']$-module via q'^*, ∇ is $R[P']$-linear, and in fact defines an element of $HIG^{\Lambda'}(P'/R)$. More generally, if $(E,\nabla) \in MIC^\Lambda(P/R)$, the tensor product connection $\tilde\nabla$ on $q_\pi^*(E)$ is an equivariant Higgs field on the $R[P']$-module $q'_* q_\pi^*(E)$. On the other hand, if θ' is an equivariant Higgs field on a Λ'-graded $R[P']$-module E', the tensor product Higgs field $\tilde\theta := d \otimes \mathrm{id} + \mathrm{id} \otimes \theta'$ on $q'^*_{\pi'} E'$ is a connection over $R[P]$. Thus we have functors

$$
\begin{aligned}
q'^\Lambda_* q^* \colon MIC^\Lambda(P/R) &\to HIG^{\Lambda'}(P'/R) \\
q^{\Lambda'}_* q'^* \colon HIG^{\Lambda'}(P'/R) &\to MIC^\Lambda(P/R)
\end{aligned}
\tag{1.3.2}
$$

Remark 1.3.6 Let $R[P',\Lambda']$ be the ring constructed from $P' \to \Lambda'$ the same way $R[P,\Lambda]$ was constructed from $R[P,\Lambda]$. Then $R[P',\Lambda']$ is a Λ'-Λ-graded R-algebra, where $e_{p'} x_{\lambda'}$ has degree $(p'+\lambda', \lambda')$. The isomorphism of monoids $P \oplus \Lambda \to P' \oplus \Lambda$ sending (p,λ) to $(-p, p+\lambda)$ induces an isomorphism of $R[\Lambda]$-algebras

$$\iota\colon R[P,\Lambda] \to R[P',\Lambda'] \quad : \quad e_p x_\lambda \mapsto e_{-p} x_{p+\lambda}.$$

It takes elements of degree (λ,λ') to elements of degree (λ',λ). Its inverse ι' is constructed from the data $P' \to \Lambda'$ just as ι was constructed from $P \to \Lambda$, and the map $q'^*\colon R[P'] \to R[P,\Lambda]$ is just the inclusion $R[P'] \to R[P',\Lambda']$ followed by ι'. Since Λ' is a group, Proposition (1.1.2) implies that q_π^* is an equivalence: $Mod_R^\Lambda(P) \to Mod_\Lambda^{\Lambda'}(P,\Lambda/R)$, with quasi-inverse $q_*^{\Lambda'}$. Of course, ι^* is also an equivalence, and hence so are the functors in (1.3.2).

Proposition 1.3.7 *The equivalence in Proposition (1.3.5) is given by the functors (1.3.2).*

Proof: For any $\lambda \in \Lambda$, x_λ is a unit of $R[P,\Lambda]$ and $\iota(x_\lambda) = x_\lambda \in R[P',\Lambda']$. Then multiplication by $x_{-\lambda}$ induces an isomorphism $q_\pi^* E \to \iota^* q_\pi^* E$ which takes E_λ to $E'_{-\lambda}$; this is the isomorphism in (1.3.5.2). If $e \in E_\lambda$,

$$\tilde\nabla(x_{-\lambda} e) = x_{-\lambda}\nabla(e) + (\nabla x_{-\lambda})e = x_{-\lambda}(\nabla e - \epsilon(\lambda)e) = x_{-\lambda}\theta'(e).$$

This proves the commutativity of the diagram in (1.3.5.2). □

1.4 Equivariant Riemann-Hilbert

Now let $R = \mathbf{C}$, and let $P \to \Lambda \to \Omega$ be a rigid set of exponential data. The universal cover of the analytic torus $\underline{\mathsf{A}}_P^{*an}$ is the exponential map

$$\exp\colon \mathbf{V}\Omega^{an} \to \underline{\mathsf{A}}_P^{*an},$$

which we can describe as follows. Recall that $\mathbf{V}\Omega$ is the spectrum of the symmetric algebra $S^\cdot(\Omega)$, which is isomorphic to $\Gamma^\cdot(\Omega)$, since we are in characteristic zero. Thus the set of points of $\mathbf{V}\Omega^{an}$ is just $T := \mathrm{Hom}(P^{gp}, \mathbf{C})$, and an element ω of Ω defines a function on $\mathbf{V}\Omega^{an}$ whose value at $t \in T$ is $\langle t, \omega\rangle$. Then exp is the map taking an additive homomorphism $t\colon P^{gp} \to \mathbf{C}$ to the multiplicative homomorphism $\exp \circ t\colon P^{gp} \to \mathbf{C}^*$. The kernel of this map is the group $\mathrm{Hom}(P^{gp}, \mathbf{Z}(1))$, where $\mathbf{Z}(1)$ is the subgroup of $\mathbf{C}$ generated by $2\pi i$. Thus there is a canonical isomorphism:

$$\pi_1(P) := Hom(P^{gp}, \mathbf{Z}(1)) \cong \pi_1(\underline{\mathsf{A}}^{*an}_P) = Aut(\mathbf{V}\Omega^{an}/\underline{\mathsf{A}}^{*an}_P). \tag{1.4.1}$$

We shall now introduce an "equivariant Riemann-Hilbert transform" which classifies objects of $MIC^\Lambda_*(P)$ in terms of suitably normalized graded representations of the fundamental group $\pi_1(P)$.

Definition 1.4.1 *Let (P, K) be an idealized toric monoid with a rigid set of exponential data $P \to \Lambda \to \Omega_{P/\mathbf{C}}$. Then $L^\Lambda(P, K)$ is the category of pairs (V, ρ), where V is a Λ-graded $\mathbf{C}[P, K]$-module and ρ is a homogeneous action of $\pi_1(P)$ on V. An object (V, ρ) of $L^\Lambda(P, K)$ is said to be* normalized *if for every $\gamma \in \pi_1(P)$ and $\lambda \in \Lambda$, the action of $\rho_\gamma - \exp\langle\gamma, \lambda\rangle$ on V_λ is locally nilpotent. The full subcategory of $L^\Lambda(P, K)$ consisting of the normalized objects (resp. the normalized and finitely generated objects) is denoted by $L^\Lambda_*(P, K)$ (resp. $L^\Lambda_{coh}(P, K)$).*

Note that the normalization condition in the definition above is compatible with multiplication by elements of $\mathbf{C}[P]$. More precisely, if $\lambda \in \Lambda$, $p \in P$, and $\gamma \in \pi_1(P)$, then multiplication by e_p takes V_λ to $V_{p+\lambda}$, and $\rho_\gamma \circ (\cdot e_p) = (\cdot e_p) \circ \rho_\gamma$. Moreover, $\langle p, \gamma\rangle \in \mathbf{Z}(1)$, so $\exp\langle\gamma, p + \lambda\rangle = \exp\langle\gamma, p\rangle$.

Remark 1.4.2 If P is a finitely generated abelian free group and $\Lambda = \mathbf{C} \otimes P$, the category $L^\Lambda_{coh}(P)$ can be simplified: it is equivalent to the category of finite dimensional $\mathbf{C}$-vector spaces equipped with an action of $\pi_1(P)$. More generally, let P be any idealized toric monoid, let Λ be a subgroup of $\mathbf{C} \otimes P^{gp}$ containing P^{gp} and let $\overline{\Lambda}$ be the image of Λ in $\mathbf{C} \otimes \overline{P}^{gp}$. Note that $\pi_1(\overline{P}) \subseteq \pi_1(P)$. Let $\overline{L}^\Lambda_{coh}(P)$ denote the category of finitely-generated $\overline{\Lambda}$-graded-$\mathbf{C}[\overline{P}]$-modules W equipped with an action ρ of $\pi_1(P)$ such that for each $\gamma \in \pi_1(\overline{P})$ and each $\overline{\lambda} \in \overline{\Lambda}$, the action of $\rho_\gamma e^{-\langle\gamma, \overline{\lambda}\rangle}$ on $W_{\overline{\lambda}}$ is unipotent. Then the evident functor (tensoring with $\mathbf{C}[\overline{P}]$) is an equivalence of categories:

$$L^\Lambda_{coh}(P) \to \overline{L}^\Lambda_{coh}(P).$$

Here is a sketch of why this is so. To see that it is fully faithful, let V be an object of $L^\Lambda_{coh}(P)$ and let $\overline{V} := V \otimes_{\mathbf{C}[P]} \mathbf{C}[\overline{P}]$; it is enough to prove that the natural map $V_0^{\pi_1} \to \overline{V}_{\overline{0}}^{\pi_1}$ is an isomorphism. Let $\Lambda^* := \Lambda \cap (\mathbf{C} \otimes P^*)$ and let $V_{\Lambda^*} := \oplus V_\lambda : \lambda \in \Lambda^*$. Then V_{Λ^*} is a Λ^*-graded $\mathbf{C}[P^*]$-module, and $\overline{V}_{\overline{0}}$ is the quotient of V_{Λ^*} by IV_{Λ^*}, where I is the kernel of the map $\mathbf{C}[P^*] \to \mathbf{C}$ sending every element of P^* to 1. Note that I is the $\mathbf{C}$-submodule of $\mathbf{C}[P^*]$ generated by the set of all $e_u - e_v : u, v \in P^*$. We have an exact sequence:

$$0 \to IV_{\Lambda^*} \to V_{\Lambda^*} \to \overline{V}_{\overline{0}} \to 0,$$

which remains exact if we restrict to the subspace on which the action of $\pi_1(\overline{P})$ is unipotent. The coherence of V implies that the unipotent part of V_{Λ^*} is exactly V_{P^*}, and $IV_{\Lambda^*} \cap V_{P^*} = IV_{P^*}$. Thus there is an exact sequence;

$$0 \to IV_{P^*} \to V_{P^*} \to \overline{V}_{\overline{0}}^{un} \to 0.$$

That is, $\overline{V}_{\overline{0}}^{un} := V_{P^*}/IV_{P^*} \cong V_{P^*} \otimes_{\mathbf{C}[P^*]} \mathbf{C}$. Then by (1.1.2), the natural map $V_0 \to \overline{V}_{\overline{0}}^{un}$ is an isomorphism, and it follows that $V_0^{\pi_1} \to \overline{V}_{\overline{0}}^{\pi_1}$ is an isomorphism, as desired.

For the essential surjectivity, let W be an object of $\overline{L}_{coh}^{\Lambda}(P)$. For each $\overline{\lambda} \in \overline{\Lambda}$, $W_{\overline{\lambda}}$ is a finite dimensional $\mathbf{C}[\pi_1(P)]$-module, and hence can be written as a direct sum of submodules $W_{\overline{\lambda},\chi}$, where χ ranges over the set S of homomorphisms $\pi_1(P) \to \mathbf{C}^*$. If $\lambda \in \Lambda$, let $e^{\lambda} \colon \pi_1 \to \mathbf{C}^*$ be the homomorphism taking $\gamma \in \pi_1(P)$ to $e^{\langle \gamma, \lambda \rangle}$. By hypothesis, if $W_{\overline{\lambda},\chi} \neq 0$, the restriction of χ to $\pi_1(\overline{P})$ is $e^{\overline{\lambda}}$. This implies that there exists a $\lambda \in \Lambda$ which maps to $\overline{\lambda}$ and such that $e^{\lambda} = \chi$, and the set of such λ is a torsor under P^*. For each $\lambda \in \Lambda$, let $V_{\lambda} := W_{\overline{\lambda},e^{\lambda}}$, and for $p \in P$, let multiplication by $e_p \colon V_{\lambda} \to V_{p+\lambda}$ be multiplication by $e_{\overline{p}}$. Then $\oplus V_{\lambda}$ is the desired object of $L_{coh}^{\Lambda}(P)$. □

We can now define the equivariant Riemann-Hilbert correspondence:

$$\mathcal{V} \colon MIC_*^{\Lambda}(P, K) \to L_*^{\Lambda}(P', K').$$

Again we use the exponential data for P' deduced from the given exponential data for P. If (E, ∇) is an object of $MIC_*^{\Lambda}(P, K)$, let V be its corresponding $\mathbf{C}[P', K']$-module, as described in (1.3.5). View $-\nabla$ as defining a Higgs field on the underlying $\mathbf{C}$-module of V, and let ρ be the corresponding action of $\pi_1(P)$:

$$\rho_{\gamma} := \exp(-\nabla_{\gamma}) \text{ for } \gamma \in \pi_1(P) \subseteq T.$$

Note that ρ_{γ} preserves the Λ-grading. It also commutes with the action of $\mathbf{C}[P']$ on V. To see this, recall that if $p \in P$, $\nabla_{\gamma} \circ \cdot e_{-p} = \cdot e_{-p} \circ (\nabla_{\gamma} - \cdot\langle \gamma, p \rangle)$, by the Leibniz rule. Hence

$$\rho_{\gamma} \circ \cdot e_{-p} = \exp(-\nabla_{\gamma}) \circ \cdot e_{-p} = \cdot e_{-p} \circ \exp(\nabla_{-\gamma} + \cdot\langle \gamma, p \rangle) = e_{-p} \circ \rho_{\gamma} \circ \cdot \exp\langle \gamma, p \rangle,$$

and $\exp\langle \gamma, p \rangle = 1$. Note also that if $\gamma \in \pi_1(P)$, $\nabla_{\gamma} - \langle \gamma, \lambda \rangle$ is locally nilpotent on E_{λ}. Hence $\exp(\nabla_{\gamma})e^{-\langle \gamma, \lambda \rangle}$ is locally unipotent on E_{λ} and $\rho_{\gamma} e^{\langle \gamma, \lambda \rangle}$ is locally unipotent on $V_{-\lambda}$. Hence $\rho_{\gamma} - e^{-\langle \gamma, \lambda \rangle}$ is locally nilpotent on $V_{-\lambda}$, so $(V, \rho) \in L_*^{\Lambda}(P', K')$.

Proposition 1.4.3 *Let $P \to \Lambda \to \Omega_{P/\mathbf{C}}$ be a rigid set of exponential data for an idealized toric monoid (P,K), and let $P' \to \Lambda \to \Omega_{P/\mathbf{C}}$ be the corresponding exponential data for P'. The equivariant Riemann-Hilbert correspondence described above defines an equivalence of tensor categories*

$$\mathcal{V}\colon MIC_*^\Lambda(P,K) \to L_*^\Lambda(P',K').$$

If $(E,\nabla) \in MIC_*^\Lambda(P,K)$ and $(V,\rho) := \mathcal{V}(E,\nabla)$, then there is a canonical isomorphism

$$H^i_{DR}(E,\nabla) \cong H^i(\pi_1(P), V_0)$$

for all i. Moreover, if $\lambda \in \Lambda \setminus P^{gp}$, then $H^i(\pi_1(P), V_\lambda) = 0$ for all i.

Proof: It follows immediately from the construction that $\mathcal{V}$ is compatible with tensor product and duality, hence with internal Hom. To prove that it is fully faithful, it suffices to prove that if (E,∇) is an object of $MIC_*^\Lambda(P)$ and $V = \mathcal{V}(E,\nabla)$, the map $E_0^\nabla \to V_0^{\pi_1}$ is an isomorphism. For each $\gamma \in \pi_1$, ∇_γ defines a nilpotent endomorphism of E_0, and it will suffice to prove that if $e \in E_0$, $\nabla_\gamma(e) = 0$ if and only if $\rho_\gamma(e) = e$. This follows from the formulas:

$$\begin{aligned} \rho_\gamma &= \mathrm{id} - \nabla_\gamma + \frac{\nabla_\gamma^2}{2!} - \cdots \\ -\nabla_\gamma &= (\rho_\gamma - 1) - \frac{(\rho_\gamma - 1)^2}{2} + \cdots \end{aligned}$$

More generally, one has the following result, which implies the statement about cohomology.

Lemma 1.4.4 *Let (E,θ) be a nilpotent $T_{P/\mathbf{C}}$-Higgs module and let $V := E$ with the action of $\pi := \pi_1(P)$ defined by $\rho_\gamma := \exp(-\theta_\gamma)$ for $\gamma \in \pi_1(P)$. Then there are natural isomorphisms:*

$$H^i_{HIG}(E,\theta) \cong H^i(\pi, (V,\rho)).$$

for all i.

Proof: The category of representations of π is equivalent to the category of $\mathbf{Z}[\pi]$-modules, and if M is such a module, $H^i(\pi, M) \cong \mathrm{Ext}^i_{\mathbf{Z}[\pi]}(\mathbf{Z}, M)$, where $\mathbf{Z}$ is the trivial module. Let $P^\cdot$ be a finitely generated and projective resolution of $\mathbf{Z}$ over $\mathbf{Z}[\pi]$. As a sequence of $\mathbf{Z}$-modules, $P^\cdot$ is split, and hence it remains exact when tensored over $\mathbf{Z}$ with any ring R. It follow that, if V is an R-module, $\mathrm{Ext}^i_{R[\pi]}(R,V) \cong \mathrm{Ext}^i_{\mathbf{Z}[\pi]}(\mathbf{Z}, V)$ for every i. Applying this with $R = \mathbf{C}$, we see that $H^i(\pi, V) \cong \mathrm{Ext}^i_{\mathbf{C}[\pi]}(\mathbf{C}, V)$ for all i. If the action of π on V is unipotent, then V is in fact a module for the formal completion $\hat{\mathbf{C}}[\pi]$ of $\mathbf{C}[\pi]$ at the vertex. Since this completion is flat over $\mathbf{C}[\pi]$, it follows that the natural map

$$\mathrm{Ext}^i_{\mathbf{C}[\pi]}(\mathbf{C}, V) \to \mathrm{Ext}^i_{\hat{\mathbf{C}}[\pi]}(\mathbf{C}, V)$$

is an isomorphism.

Let $Y := \mathsf{A}_\pi = \operatorname{Spec} \mathbf{C}[\pi]$, let $T := \mathbf{C} \otimes \pi$, and suppose that E and V are as in the lemma. The exponential map induces an isomorphism of formal schemes $\hat{\mathbf{V}}T \to \hat{Y}$, where $\hat{\mathbf{V}}T$ is the formal completion of $\mathbf{V}T$ along the zero section and $\hat{Y}$ is the formal completion of Y at the vertex. Under this isomorphism, if $\gamma \in \pi$, $\exp^* \gamma = \mathrm{id} + \gamma + \gamma^2/2! + \cdots$. The Higgs module (E, θ) can be thought of as quasi-coherent sheaf on $\mathbf{V}T$. Since E is nilpotent, it is supported on the zero section, and, up to a sign, $V \cong \exp_* E$. By [1], the Higgs cohomology of E is $\operatorname{Ext}^i_{S^\cdot T}(\mathbf{C}, E)$, where $\mathbf{C}$ corresponds to the zero section of $\mathbf{V}T$. As before, this Ext remains the same when computed on the formal completion. Thus

$$H^i_{HIG}(E, \theta) \cong \operatorname{Ext}^i_{S^\cdot T}(\mathbf{C}, E) \cong \operatorname{Ext}^i_{\hat{S}^\cdot T}(\mathbf{C}, E) \cong \operatorname{Ext}^i_{\hat{\mathbf{C}}[\pi]}(\mathbf{C}, V) \cong H^i(\pi, V).$$

□

To prove that $\mathcal{V}$ is essentially surjective, let (V, ρ) be an object of $L_*(P', K')$, and for each $\lambda \in \Lambda$ let $E_\lambda := V_{-\lambda}$, so that $\oplus E_\lambda$ is a Λ-graded $\mathbf{C}[P]$-module. For $\gamma \in \pi_1$, $\rho_\gamma e^{\langle \gamma, \lambda \rangle}$ induces a unipotent automorphism u_γ of E_λ, and hence $\log u_\gamma := (u_\gamma - 1) - \frac{(u_\gamma - 1)^2}{2} + \cdots$ is well defined and nilpotent. Let $\nabla_\gamma := -\log u_\gamma + \langle \gamma, \lambda \rangle$. Then $\exp(-\nabla_\gamma) = \rho_\gamma$. Furthermore, $\nabla_{\gamma_1 + \gamma_2} = \nabla_{\gamma_1} + \nabla_{\gamma_2}$, and $\nabla_\gamma \circ e_p = e_p \nabla_\gamma + \langle \gamma, dp \rangle$. Thus $(E, \nabla) \in MIC^\Lambda_*(P, K)$ and $\mathcal{V}(E, \nabla) = (V, \rho)$, so that $\mathcal{V}$ is essentially surjective. □

Remark 1.4.5 If $(E, \nabla) \in MIC^\Lambda_*(P, K)$, then its cohomology vanishes except in degree zero. This is not true for objects of $L^\Lambda_*(P', K')$, and this is why we have to specify taking the degree zero part in the isomorphism on cohomology. On the other hand, if $\lambda \in \Lambda \setminus P^{gp}$, then the support of V_λ (regarded as a sheaf on A_π) does not meet the vertex, so its cohomology is zero.

There is an evident functor $L^\Lambda_{coh}(P) \to L^\Lambda_{coh}(P^{gp})$. Recall from (1.4.2) that in the latter category, the grading is superfluous, and that the functor can be viewed as the functor which takes V to $V \otimes_{\mathbf{C}[P]} \mathbf{C}$ via the map $\mathbf{C}[P] \to \mathbf{C}$ sending P to 1. This corresponds to evaluating a "generic point" and so we denote the corresponding module by V_η. There is a cospecialization map $V \to V_\eta$ and hence a map on cohomology.

Corollary 1.4.6 *Let V be a torsion free object of $L^\Lambda_{coh}(P)$ and let $D \subseteq \Lambda$ be the set of the degrees of a minimal set of homogeneous generators for V. Suppose that $D \cap P^{gp} \subseteq -P$. Then the natural map*

$$H^i(\pi_1(P), V_0) \to H^i(\pi_1(P), V_\eta)$$

is an isomorphism.

Proof: Let $V' := \sum \{V_\lambda : \lambda \in P^{gp}\}$. Remark (1.4.5) shows that the natural map $H^i(\pi_1(P), V') \to H^i(\pi_1(P), V)$ is an isomorphism, and the same is true for V_η. Thus we may as well assume that $V' = V$. But then Corollary (1.1.3) shows that the hypothesis on the degrees of the generators implies that the natural map $V_0 \to V_\eta$ is an isomorphism. □

As stated, Proposition (1.4.3) is too artificial to be of much value. We shall show that in fact it can be formulated in a more geometric manner which we can then use in our proof of the global Riemann-Hilbert correspondence.

Tensoring together the fundamental examples $\mathbf{C}[P,\Lambda]$ (1.2.5) and $N(P)$ (1.2.6), we obtain the $\mathbf{C}[P]$-algebra

$$J(P,\Lambda) := \mathbf{C}[P,\Lambda] \otimes_{\mathbf{C}} \Gamma^{\cdot}(\Omega) \cong \mathbf{C}[P,\Lambda] \otimes_{\mathbf{C}[P]} N(P).$$

It has a connection ∇ and a Higgs field θ as explained in Example (1.3.3).

The connection ∇ is in some sense the universal connection in Jordan normal form. Indeed, we shall see that $J(P,\Lambda)$ can be viewed as a ring of multivalued functions which is large enough to solve all the differential equations coming from objects of $MIC_*^{\Lambda}(P/R)$. This fact is the main computational tool underlying the equivariant Riemann-Hilbert correspondence. First let us attempt to explain its geometric meaning.

The map $\delta\colon P \to \Lambda$ induces a map $\underline{\mathsf{A}}_{\mathsf{\Lambda}} \to \underline{\mathsf{A}}_{\mathsf{P}}$. Let us write α for the canonical map from the analytic space $X^{an} \to X$.

The rings of functions $\mathbf{C}[P]$ and $\mathbf{C}[\Lambda]$ on $\underline{\mathsf{A}}_{\mathsf{P}}$ and $\underline{\mathsf{A}}_{\mathsf{\Lambda}}$ map to the ring of analytic functions on $\mathbf{V}\Omega^{an}$. For example, if $p \in P$ and $t \in T$,

$$\exp^*(e_p)(t) = \exp\langle t, dp\rangle.$$

Thus, the function associated to p is the logarithm of the function associated to e_p. Similarly, if $\lambda \in \Lambda$, then

$$\exp^*(x_\lambda)(t) := \exp\langle t, \delta\lambda\rangle,$$

so that we have maps exp from $\mathbf{V}\Omega^{an}$ to $\underline{\mathsf{A}}_P^{*an}$ and to $\underline{\mathsf{A}}_\Lambda^{*an}$. There is a commutative diagram (see (1.3.1))

$$\begin{tikzcd}
\mathbf{V}\Omega^{an} \arrow[r, "{(\exp,\exp)}"] \arrow[d, "\underline{\mathsf{A}}_\delta \circ \alpha \circ \exp"'] & \underline{\mathsf{A}}_P^{*an} \times \underline{\mathsf{A}}_\Lambda^{*an} \arrow[d, "\alpha \circ \mathrm{inc}"'] \arrow[dr] & \\
\underline{\mathsf{A}}_{\mathsf{\Lambda}} \arrow[r, "\underline{\mathsf{A}}_\eta"] \arrow[dr, "\underline{\mathsf{A}}_\delta"'] & \underline{\mathsf{A}}_{\mathsf{P}} \times \underline{\mathsf{A}}_{\mathsf{\Lambda}} \arrow[r, "q'"] \arrow[d, "q = pr_1"] & \underline{\mathsf{A}}_{\mathsf{P'}} \\
 & \mathsf{A}_{\mathsf{P}} &
\end{tikzcd}$$

Thus we obtain a map from $J(P,\Lambda)$ to the ring of analytic functions on $\mathbf{V}\Omega^{an}$. The group $\pi_1(P)$ acts on the ring of analytic functions on $\mathbf{V}\Omega^{an}$ by transport of structure, and preserves the subalgebra $\Gamma^{\cdot}(\Omega)$ of algebraic functions on $\mathbf{V}\Omega^{an}$ as well as the subring $\mathbf{C}[P,\Lambda]$. Let us make this explicit.

Lemma 1.4.7 *If $\gamma \in \pi_1(P)$, let ρ_γ act on $J(P,\Lambda)$ by*

$$\rho_\gamma := \exp(\theta_\gamma) = \exp \nabla_\gamma := e^{\nabla_\gamma} := \mathrm{id} + \frac{\nabla_\gamma}{1!} + \frac{\nabla_\gamma^2}{2!} + \cdots .$$

Then this action is compatible with the action on $\mathbf{V}\Omega$ via the exponential map and the diagram above.

Proof: The action of $\pi_1(P)$ on $\mathbf{V}\Omega^{an} = T$ is via translation: $\rho_\gamma(t) = t + \gamma$ if $\gamma \in \pi_1(P)$ and $t \in T := \mathrm{Hom}(P^{gp}, \mathbf{C})$. The induced action on the analytic functions on $\mathbf{V}\Omega^{an}$ is then by transport of structure, and in particular is by ring automorphisms. On the other hand, if $\gamma \in \pi_1(P)$ and $f_i \in J(P,\Lambda)$, then $\nabla_\gamma(f_1 + f_2) = \nabla_\gamma(f_1) + \nabla_\gamma(f_2)$, and $\nabla_\gamma(f_1 f_2) = \nabla_\gamma(f_1) f_2 + \nabla_\gamma(f_2) f_1$. It follows that $\exp(\nabla_\gamma)$ is also a ring automorphism of $J(P,\Lambda)$. Thus it suffices to check the compatibility of $\exp\nabla$ and ρ on a set of generators of the algebra $J(P,\Lambda)$. In particular, it suffices to check it for $\omega \in \Omega \subseteq \Gamma^{\cdot}(\Omega)$, $x_\lambda \in J(P,\Lambda)$, and $e_p \in \mathbf{C}[P]$. First of all, ∇_γ maps Ω to $\mathbf{C}$ and is zero on $\mathbf{C}$, and hence

$$e^{\nabla_\gamma}(\omega) = \omega + \nabla_\gamma(\omega) = \omega + \langle \gamma, \omega\rangle.$$

Thus

$$\begin{aligned} \langle t, e^{\nabla_\gamma}(\omega)\rangle &= \langle t,\omega\rangle + \langle\gamma,\omega\rangle \\ &= \langle t+\gamma,\omega\rangle \\ &= \langle \rho_\gamma(t),\omega\rangle \\ &= \langle t, \rho_\gamma(\omega)\rangle \end{aligned}$$

On the other hand, if $\lambda \in \Lambda$, $\nabla_\gamma(x_\lambda) = \langle\gamma,\lambda\rangle x_\lambda$, so $e^{\nabla_\gamma}(x_\lambda) = e^{\langle\gamma,\lambda\rangle} x_\lambda$. Pulling back to $\mathbf{V}\Omega^{an}$ and evaluating at t, we get

$$\begin{aligned} \langle t, e^{\nabla_\gamma}(x_\lambda)\rangle &= e^{\langle\gamma,\lambda\rangle}\langle t, x_\lambda\rangle \\ &= e^{\langle\gamma,\lambda\rangle} e^{\langle t,\lambda\rangle} \\ &= e^{\langle t+\gamma,\lambda\rangle} \\ &= \exp^*(x_\lambda)(t+\gamma) \\ &= \exp^*(x_\lambda)(\rho_\gamma t) \\ &= \exp^*(\rho_\gamma(x_\lambda))(t) \end{aligned}$$

Finally, if $p \in P$, $\rho_\gamma(e_p) = e_p$, and since $\nabla_\gamma e_p = \langle\gamma,p\rangle e_p$ and $\langle\gamma,p\rangle e_p \in \mathbf{Z}(1)$, e_p is also fixed by $\exp(\tilde\nabla_\gamma)$. This proves the compatibility of ρ with ∇. On the other hand, $\theta_\gamma(e_p x_\lambda \omega) = \nabla_\gamma(x_\lambda\omega) - \langle\gamma,p\rangle$, and $\langle\gamma,p\rangle \in \mathbf{Z}(1)$. Hence $\exp(\theta_\gamma) = \exp(\nabla_\gamma)$, and so ρ is also compatible with θ. □

Regarded as a $\mathbf{C}[P]$-module via the map q_*, $(J(P,\Lambda), d)$ is an object of $MIC_*^\Lambda(P)$. Regarded as a $\mathbf{C}[P']$-module via the map q'_*, $(J(P,\Lambda),\rho)$ is an object of $L^{\Lambda}(P')$, where $\rho := \exp(\nabla)$, since ∇ (hence ρ) is $\mathbf{C}[P']$-linear over q'_*. Let us check that it is normalized. Every element of degree λ' of $q'_*(J(P,\Lambda))$ can be written as a sum of elements of the form $e_p w x_{\lambda'}$ with $p \in P$, $w \in \Gamma^{\cdot}(\Omega_{P/\mathbf{C}})$, and $\lambda' \in \Lambda$, and

$$\rho_\gamma(e_p w x_{\lambda'}) = e^{\langle \gamma, p+\lambda' \rangle}(\exp \nabla_\gamma)(w) = e^{\langle \gamma, \lambda' \rangle}(\exp \nabla_\gamma) w.$$

Since $\exp \nabla_\gamma$ is locally unipotent on $\Gamma^{\cdot}(\Omega_{P/\mathbf{C}})$, $e^{\langle \gamma, \lambda' \rangle} \exp(\nabla_\gamma) - e^{\langle \gamma, \lambda' \rangle}$ is locally nilpotent. Note also that ρ_γ commutes with the connection ∇.

Now we can give description of the equivariant Riemann-Hilbert correspondence as an integral transform. If (E, ∇) is an object of $MIC^{\Lambda}(P/\mathbf{C})$, let J^*E be $\tilde{E} := E \otimes_{\mathbf{C}[P]} J(P,\Lambda)$ with the Λ-Λ'-grading and connection $\tilde{\nabla}$ as described in the discussion preceding (1.3.2), and with the action $\tilde{\rho}$ of $\pi_1(P)$ defined by $\mathrm{id}_E \otimes \rho$. If (V, ρ) is an object of $L_*^{\Lambda'}(P')$, let $J'^*(V)$ be $\tilde{V} := V \otimes_{R[P']} J(P,\Lambda)$, with the Λ-Λ'-grading as above, with $\tilde{\nabla} := \mathrm{id} \otimes d$, and with $\tilde{\rho}$ the tensor product action. In both cases, we end up with a $J(P,\Lambda)$-module endowed with a Λ-Λ'-grading, a connection, and an action of $\pi_1(P)$. Let q'^{∇}_* be the functor which takes such an object to its horizontal sections, regarded as a Λ'-graded $\mathbf{C}[P']$-module with an action of $\pi_1(P)$. Also, let q_*^{Λ', π_1} denote the part of Λ'-degree zero which is fixed by $\tilde{\rho}$, regarded a Λ-graded $\mathbf{C}[P]$-module with connection.

Theorem 1.4.8 *Let $P \to \Lambda \to \Omega_{P/\mathbf{C}}$ be a rigid set of exponential data for an idealized toric monoid and let $P' \to \Lambda \to \Omega_{P/\mathbf{C}}$ be the corresponding exponential data for P'.*

1. *The functors*
$$\mathcal{V} := q'^{\nabla}_* J^* \colon MIC_*^{\Lambda}(P) \to L_*^{\Lambda}(P')$$
and
$$\mathcal{E} := q_*^{\Lambda', \pi_1} J'^* \colon L_*^{\Lambda}(P') \to MIC_*^{\Lambda}(P)$$
are the functors in the equivariant Riemann-Hilbert correspondence (1.4.3).

2. *If $(E, \nabla) \in MIC_*^{\Lambda}(P)$, let $(\tilde{E}, \tilde{\nabla}, \tilde{\rho}) := J^*(E)$. Then in the category $L_*^{\Lambda}(P')$,*
$$H^i_{DR}(\tilde{E}, \tilde{\nabla}, \rho) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{V}(E, \nabla) & \text{if } i = 0. \end{cases}$$
Furthermore, the natural map $\mathcal{V}(E, \nabla) \otimes_{\mathbf{C}[P']} J(P,\Lambda) \to \tilde{E}$ is an isomorphism.

3. *If $(V, \rho) \in L_*^{\Lambda'}(P')$, let $(\tilde{V}, \tilde{\nabla}, \tilde{\rho}) := J^*(V, \rho)$. Then in the category $MIC_*^{\Lambda}(P)$,*
$$H^i(\pi_1(P), (\tilde{V}, \tilde{\nabla}, \rho))_{\Lambda'=0} = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{E}(V, \rho) & \text{if } i = 0. \end{cases}$$

Furthermore, the natural map $\mathcal{E}(V,\rho)\otimes_{\mathbf{C}[P]} J(P,\Lambda) \to \tilde{V}$ is an isomorphism.

We give the proof in the next section, where we deduce it from a more abstract construction which we call, for want of a better name, the "Jordan transform."

1.5 The Jordan transform

Most of the real work in this section makes sense over an arbitrary **Q**-algebra R, so we temporarily revert to this generality. To simplify the notation, we let P be an idealized toric monoid (previously denoted (P,K)), and we let $P \to \Lambda \to \Omega$ be a rigid set of exponential data. We have seen in (1.2.5) and (1.3.3) that $R[P,\Lambda]$ carries a connection ∇ and a Higgs field θ relative to $R[P]$. Note that this is *not* the Higgs field θ' constructed from ∇ as in (1.3.5). To emphasize the symmetric nature of the constructions, we now write ∇' for θ. Indeed, ∇ is a Higgs field relative to $R[P'] \subseteq R[P,\Lambda]$, and ∇' is a connection relative to $R[P']$. Note that ∇' and ∇ commute.

Let us summarize the structures $J(P,\Lambda) := R[P,\Lambda]\otimes_R \Gamma^{\cdot}(\Omega)$ carries.

1. It has a Λ-grading, where $e_p x_\lambda \omega^{[i]}$ has degree $p+\lambda$, and there is a Λ-graded homomorphism
$$q\colon R[P] \to J(P,\Lambda) \quad : \quad e_p \mapsto e_p x_0.$$
2. It has a second Λ-grading, (called the Λ'-grading) where $e_p x_\lambda \omega^{[i]}$ has Λ'-degree λ, and a Λ'-graded homomorphism
$$q' \quad : \quad R[P'] \to J(P,\Lambda) \quad : \quad e'_{p'} \mapsto e_{-p'} x_{p'}.$$
3. There is a map $\nabla\colon J(P,\Lambda) \to J(P,\Lambda)\otimes_R \Omega_{P/R}$ such that
$$\nabla\colon e_p x_\lambda \omega^{[i]} \mapsto e_p x_\lambda \omega^{[i]} \otimes (p+\lambda) + e_p x_\lambda \omega^{[i-1]}\otimes \omega.$$
Then $q_*(J(P,\Lambda),\nabla) \in MIC^{\Lambda}_*(P/R)$, and $q'_*(J(P,\Lambda),\nabla) \in HIG^{\Lambda'}(P'/R)$.
4. There is a map $\nabla'\colon J(P,\Lambda) \to J(P,\Lambda)\otimes_R \Omega_{P/R}$ such that
$$\nabla'\colon e_p x_\lambda \omega^{[i]} \mapsto e_p x_\lambda \omega^{[i]} \otimes \lambda + e_p x_\lambda \omega^{[i-1]}\otimes \omega.$$
Then $q_*(J(P,\Lambda),\nabla') \in HIG^{\Lambda}(P/R)$, and $q'_*(J(P,\Lambda),\nabla') \in MIC^{\Lambda'}_*(P'/R)$.

Note also that the set of elements of degree zero with respect to the Λ'-grading is just $R[P]\otimes \Gamma^{\cdot}(\Omega)$. Similarly, the set of elements of degree zero with respect to the Λ-grading is $R[P']\otimes \Gamma^{\cdot}(\Omega)$.

Let $MH^{\Lambda}_{\Lambda'}(P/R)$ denote the category of Λ-Λ'-graded $J(P,\Lambda)$-modules equipped with structures parallel to those of $J(P,\Lambda)$. In particular, an object $\tilde{E}$ of $MH^{\Lambda}_{\Lambda'}(P/R)$ is equipped with two commuting homogeneous maps:

$$\tilde{\nabla}, \tilde{\nabla}' \colon \tilde{E} \to \tilde{E} \otimes_R \Omega_{P/R}$$

where $\tilde{\nabla}$ is a homogeneous connection relative to $R[P]$ and a homogeneous Higgs structure relative to $R[P']$, and $\tilde{\nabla}'$ is a Higgs structure relative to $R[P]$ and a connection relative to $R[P']$.

Consider then the following functors:

1. If $(E,\nabla) \in MIC^{\Lambda}(P/R)$, let $J^*(E) := E \otimes_{R[P]} J(P,\Lambda)$, with the tensor product gradings, in which E is viewed as having Λ'-degree zero, and let $\tilde{\nabla} := \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla$ and $\tilde{\nabla}' := \mathrm{id}_E \otimes \nabla'$. Then $(J^*(E), \tilde{\nabla}, \tilde{\nabla}') \in MH^{\Lambda}_{\Lambda'}(P/R)$.
2. If $(E',\nabla') \in MIC^{\Lambda}(P'/R)$, let $J'^*(E') := E' \otimes_{R[P']} J(P,\Lambda)$ with the tensor product gradings, in which E' is viewed as having Λ'-degree zero, and let $\tilde{\nabla}' := \nabla' \otimes \mathrm{id} + \mathrm{id} \otimes \nabla'$, and $\tilde{\nabla} := \mathrm{id}_{E'} \otimes \nabla$. Then $(J'^*(E'), \tilde{\nabla}, \tilde{\nabla}') \in MH^{\Lambda}_{\Lambda'}(P/R)$.
3. If $(\tilde{E}, \tilde{\nabla}, \tilde{\nabla}') \in MH^{\Lambda}_{\Lambda'}(P/R)$, let $E := q^{\nabla'}_*(\tilde{E})$ (resp., $q^{\Lambda'}_*(\tilde{E})$) denote the elements which are killed by $\tilde{\nabla}'$ (resp., and of Λ'-degree zero.) Then E is a Λ-graded $R[P]$-module with a connection ∇ induced by $\tilde{\nabla}$, and $(E,\nabla) \in MIC^{\Lambda}(P/R)$.
4. If $(\tilde{E}, \tilde{\nabla}, \tilde{\nabla}') \in MH^{\Lambda}_{\Lambda'}(P/R)$, let $E' := q'^{\nabla}_*(\tilde{E})$ (resp., $E' := q'^{\Lambda}_*(\tilde{E})$) denote the elements which are killed by $\tilde{\nabla}$ (resp., and of Λ-degree zero.) Then E' is a Λ'-graded $R[P']$-module, with a connection ∇' induced by $\tilde{\nabla}'$, and $(E', \nabla') \in MIC^{\Lambda'}(P'/R)$.

Theorem 1.5.1 *Let $P \xrightarrow{\delta} \Lambda \xrightarrow{\epsilon} \Omega_{P/R}$ be a rigid set of exponential data for a toric idealized monoid. Then the functor $q'^{\nabla}_* J^*$ described above defines an equivalence of categories*

$$MIC^{\Lambda}_*(P/R) \to MIC^{\Lambda'}_*(P'/R).$$

This functor is compatible with tensor products and formation of cohomology, and has as quasi-inverse the functor $q^{\nabla'}_* J'^*$. Moreover:

1. *If $(E,\nabla) \in MIC^{\Lambda}_*(P/R)$ corresponds to $(E',\nabla') \in MIC^{\Lambda'}_*(P'/R)$, then for each λ there is a commutative diagram:*

$$\begin{tikzcd}
E_\lambda \arrow[r, "\cong"] \arrow[d, "-\nabla"'] & E'_{-\lambda} \arrow[d, "\nabla'"] \\
E_\lambda \otimes \Omega_{P/R} \arrow[r, "\cong"] & E'_{-\lambda} \otimes \Omega_{P/R}
\end{tikzcd}$$

2. *If* $(E,\nabla) \in MIC^{\Lambda}_*(P/R)$, *then*

$$H^i_{DR}(J^*(E),\tilde\nabla) = \begin{cases} E' := q'^{\nabla}_* J^*(E) & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

$$H^i_{HIG}(J^*(E),\tilde\nabla') = \begin{cases} E & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

Furthermore, the natural map $E' \otimes_{R[P']} J(P,\Lambda) \to J^*(E)$ *is an isomorphism.*

3. *If* $(E',\nabla') \in MIC^{\Lambda'}_*(P'/R)$, *then*

$$H^i_{HIG}(J'^*(E'),\tilde\nabla') = \begin{cases} E := q^{\nabla'}_* J'^*(E') & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

$$H^i_{DR}(J'^*(E'),\tilde\nabla) = \begin{cases} E' & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

Furthermore, the natural map $E \otimes_{R[P]} J(P,\Lambda) \to J'^*(E')$ *is an isomorphism.*

We begin with some preliminary lemmas.

Lemma 1.5.2 *Let* (E,∇) *be an object of* $MIC^{\Lambda}_*(P/R)$.

1. *Let* $K^\cdot$ *be the De Rham complex of* (E,∇) *and let* $K^\cdot_{\Lambda=0}$ *be its degree zero part (with respect to the* Λ*-grading). Then the map* $K^\cdot_{\Lambda=0} \to K^\cdot$ *is a quasi-isomorphism.*

2. *Let* $\tilde K^\cdot$ *be the Higgs complex of* $(J^*(E),\tilde\nabla')$, *and let* $\tilde K^\cdot_{\Lambda'=0}$ *be its degree zero part with respect to the* Λ'*-grading. Then the map* $\tilde K^\cdot_{\Lambda'=0} \to \tilde K^\cdot$ *is a quasi-isomorphism.*

Proof: The first statement is an immediate consequence of (1.3.4.2). Let

$$E'' := E \otimes_R \Gamma^\cdot(\Omega) \subseteq \tilde E := J^*(E,\nabla) \cong E \otimes_R \Gamma^\cdot(\Omega) \otimes R[\Lambda] \cong E'' \otimes_{R[P]} R[P,\Lambda].$$

Then $E'' = \tilde E_{\Lambda'=0}$, and the action of $\tilde\nabla'$ on E'' is nilpotent. For $\lambda' \in \Lambda$, the action of $\tilde\nabla'$ on the degree λ'-component of $\tilde E$ is $\nabla'_{E''} + \mathrm{id}\otimes\lambda'$. By (1.3.1), its Higgs complex is then acyclic if $\lambda' \neq 0$. This proves (2). $\square$

Lemma 1.5.3 *Let* T *be a free* R*-module with basis* $(t_1,\dots t_n)$, *and let* Ω *be the dual of* T, *with dual basis* $(\omega_1,\dots\omega_n)$. *If* (V,θ) *is a locally nilpotent* T*-Higgs module, let*

$$E'' := V \otimes_R \Gamma^\cdot(\Omega), \quad \text{and} \quad \nabla'' := \theta\otimes\mathrm{id} + \mathrm{id}\otimes d.$$

Let $\partial_i := \nabla''_{t_i}$ *and* $h\colon E'' \to E''$ *be* $\sum_I (-1)^{|I|}\omega^{[I]}\partial^I$, *where the sum is taken over all multi-indices* $I = (I_1,\dots I_n)$ *with* $I_i \in \mathbf{N}$.

1. *h is independent of the bases, and defines a projection operator with image $E''^{\nabla''}$.*

2. *$E''^{\nabla''}$ is invariant under* $\mathrm{id} \otimes d$*, and h induces an isomorphism $h' \colon V \to E''^{\nabla''}$ fitting into a commutative diagram:*

$$\begin{tikzcd}
V \arrow[r, "h'"] \arrow[d, "-\theta"'] & E''^{\nabla''} \arrow[r] \arrow[d] & V \otimes \Gamma^{\cdot}(\Omega) \arrow[d, "\mathrm{id} \otimes d"] \\
V \otimes \Omega \arrow[r, "h' \otimes \mathrm{id}"] & E''^{\nabla''} \otimes \Omega \arrow[r] & V \otimes \Gamma^{\cdot}(\Omega) \otimes \Omega
\end{tikzcd}$$

3. *The natural map $\Gamma^{\cdot}(\Omega) \otimes E''^{\nabla''} \to E''$ is an isomorphism, with inverse $\sum \omega^{[I]} \otimes h\partial^I$.*

4. *The De Rham cohomology $H^i_{DR}(E'')$ of E'' vanishes if $i > 0$.*

Proof: Most of this lemma is more or less standard, at least if one replaces the polynomial ring $\Gamma^{\cdot}(\Omega)$ by its formal completion at the origin. Notice first that for any $n > 0$, $\sum\{\omega^{[I]} \otimes t^I : |I| = n\}$ is the matrix for the canonical pairing between $\Gamma^n(\Omega)$ and $Sym^n(T)$. It follows that h (the Kasimir operator) is independent of the basis. The local nilpotence of the operators ∂_i implies that the operator h is well-defined, and the fact that it is a projection with image $E''^{\nabla''}$ is an immediate calculation. It is apparent from the definition that h' is injective. To see that it is surjective, write an arbitrary $e'' \in E''^{\nabla}$ as a sum $e'' = \sum \omega^{[I]} \otimes v_I$ with $v_I \in V$. Then e'' and $h'(v_0)$ are two elements of $E''^{\nabla''}$ which agree modulo the ideal $\Gamma^+(\Omega)$ of $\Gamma^{\cdot}(\Omega)$. It follows from the well-known complete version of this lemma that they agree in the formal completion at this ideal, and hence that they agree. This shows that h' is also surjective. Note that $\theta \circ h' = (h' \otimes \mathrm{id}) \circ \theta$. If $v \in V$, $\nabla'' h'(v) = 0$, and since $\nabla'' = \mathrm{id} \otimes d + \theta \otimes \mathrm{id}$,

$$\begin{aligned}
(\mathrm{id} \otimes d) \circ h'(v) &= -(\theta \otimes \mathrm{id}) \circ h'(v) \\
&= -(h' \otimes \mathrm{id}) \circ \theta(v)
\end{aligned}$$

This proves that the diagram in (2) commutes. Statement (3) is a straightforward calculation, and (4) then follows, since (3) reduces the computation of De Rham cohomology to the case of the trivial connection, which of course vanishes, by the Poincaré lemma in crystalline cohomology. □

Proof of Theorem (1.5.1) Let (E, ∇) be an object of $MIC^{\Lambda}_*(P/R)$ and let $(\tilde{E}, \tilde{\nabla}, \tilde{\nabla}')$ be $J^*(E, \nabla)$. Since (E, ∇) and $(J(P, \Lambda), \nabla)$ are normalized, so is $(\tilde{E}, \tilde{\nabla})$. We have

$$\tilde{E} := E \otimes_{R[P]} J(P, \Lambda) \cong E \otimes_R R[\Lambda] \otimes_R \Gamma^{\cdot}(\Omega).$$

Let $(V,\theta) := (E \otimes_R R[\Lambda], \tilde{\nabla})_{\Lambda=0}$ and let $E'' := \tilde{E}_\Lambda$ be the part of $\tilde{E}$ of Λ-degree zero. Thus

$$E'' := \tilde{E}_{\Lambda=0} \cong V \otimes_R \Gamma^{\cdot}(\Omega),$$

and (V,θ) is just the Higgs transform (1.3.7) of E. Since θ is nilpotent, (1.5.3) applies. Assembling the diagrams (1.3.5.2) and (1.5.3.2), we obtain a commutative diagram:

$$\begin{tikzcd}
E_\lambda \arrow[r, "\cong"] \arrow[d, "\nabla - \mathrm{id}\otimes\lambda"'] & V_{0,-\lambda} \arrow[r, "\cong"] \arrow[d, "\theta"'] & E''^{\nabla''}_{-\lambda} \arrow[d, "-\mathrm{id}\otimes\nabla_{\Gamma^{\cdot}(\Omega)}"] \\
E_\lambda \otimes \Omega \arrow[r] & V_{0,-\lambda}\otimes\Omega \arrow[r, "\cong"] & E''^{\nabla''}_{-\lambda} \otimes \Omega.
\end{tikzcd}$$

Now $E''_{-\lambda} \subseteq \tilde{E}_{0,-\lambda}$, and by definition

$$\tilde{\nabla}' := \mathrm{id}_E \otimes \nabla' = \mathrm{id}\otimes\nabla_{\Gamma^{\cdot}(\Omega)} + \mathrm{id}\otimes(-\lambda)$$

in these degrees. The diagram shows that the map $\nabla - \mathrm{id}\otimes\lambda\colon E_\lambda \to E_\lambda\otimes\Omega$ corresponds to the map $-\mathrm{id}\otimes\nabla'_{\Gamma^{\cdot}\Omega} = -\nabla' - \mathrm{id}\otimes\lambda$. Thus ∇ corresponds to $-\nabla'$, and we get the commutative diagram in (1). This diagram implies that $q'^{\nabla}_{*} J^*E$ belongs to $MIC^{\Lambda'}_{*}(P'/R)$.

It follows from (1.5.2) that the map from the de Rham complex $K''^{\cdot}$ of E'' to $\tilde{K}^{\cdot}$ is a quasi-isomorphism. Lemma (1.5.3) implies that $H^i_{DR}(E'') = 0$ if $i > 0$, and since $K''^{\cdot} \to \tilde{K}^{\cdot}$ is a quasi-isomorphism, the same is true of $H^i_{DR}(\tilde{K}^{\cdot})$. Lemma (1.5.3) also implies that the natural map $E''^{\nabla''} \otimes_R \Gamma^{\cdot}(\Omega) \to E''$ is an isomorphism. Now $E''^{\nabla''}$ is in fact an $R[P']$-module, and this isomorphism can be rewritten as an isomorphism

$$E''^{\nabla''} \otimes_{R[P']} \otimes R[P'] \otimes_R \Gamma^{\cdot}(\Omega) \to E''.$$

Tensoring with $R[\Lambda]$ and using the fact that the map $E''^{\nabla''} \to \tilde{E}^{\tilde{\nabla}}$ is an isomorphism, we see that the map

$$\tilde{E}^{\tilde{\nabla}} \otimes_{R[P']} \otimes R[P',\Lambda] \otimes_R \Gamma^{\cdot}(\Omega) \to E'' \otimes_R R[\Lambda]$$

is an isomorphism. But by Proposition (1.3.7), the natural map

$$E'' \otimes_R R[\Lambda] \to \tilde{E}$$

is an isomorphism. Hence the map

$$\tilde{E}^{\tilde{\nabla}} \otimes_{R[P']} \otimes J(P,\Lambda) \to \tilde{E}$$

is an isomorphism, proving the last statement of (2). The calculation of the Higgs cohomology of $(\tilde{E}, \tilde{\nabla}')$ is done in the same way as the de Rham cohomology. This completes the proof of (2), and (3) follows by symmetry.

Now suppose that $(E,\nabla) \in MIC^{\Lambda}_{*}(P/R)$ and let $(E',\nabla') := {q'}^{\nabla}_{*} J^*(E,\nabla)$. As we have seen, $(E',\nabla') \in MIC^{\Lambda'}_{*}(P'/R)$. By the last part of (2),

$$J(P,\Lambda) \otimes E' \cong \tilde{E},$$

and so $q_*{}^{\nabla'}(J(P,\Lambda) \otimes E') \cong q_*{}^{\nabla'}\tilde{E} \cong E$. This implies that the composite $MIC^{\Lambda}_{*}(P/R) \to MIC^{\Lambda'}_{*}(P'/R) \to MIC^{\Lambda}_{*}(P/R)$ is isomorphic to the identity. A similar argument works starting with $MIC^{\Lambda'}_{*}(P'/R)$. This completes the proof of the theorem. $\square$

Proof of (1.4.8) Let (E,∇) be an object of $MIC^{\Lambda}_{*}(P,\mathbf{C})$ and let $(V,\rho) := \mathcal{V}(E,\nabla)$. By construction, V is the $\mathbf{C}[P']$-module ${q'}^{\nabla}_{*} J^*(E,\nabla)$ of (1.5.1), and ρ is the map induced by $\tilde{\rho} := \mathrm{id}_E \otimes \rho_J$. Here ρ_J is the action of $\pi_1(P)$ on $J(P,\Lambda)$, which by (1.4.7) is $\mathrm{id}_E \otimes \exp\tilde{\nabla}' = \mathrm{id}_E \otimes \exp\theta$. The isomorphism $E \to V$ of (1) of (1.5.1) takes ∇' to $-\nabla$, and so the action ρ of (1.4.8) agrees with the action defined in (1.4.3). This proves (1) of (1.4.8), and (2) follows directly from (1.5.1.2). Conversely, let (V,ρ) be an object of $L^{\Lambda'}_{*}(P')$, and $\tilde{V} := V \otimes J(P,\Lambda)$. Then the action of π_1 on $\tilde{V}_{\Lambda'=0}$ is unipotent. Its logarithm is the nilpotent Higgs structure $\theta = -\nabla$, and so by (1.4.4), $q_*^{\Lambda',\pi_1}(\tilde{V}) = \tilde{V}^{\nabla} = E$. By (1.4.4), the Higgs cohomology of $\tilde{V}$ is the same as the group cohomology, and so (1.4.8.3) follows from (1.5.1.3). $\square$

Remark 1.5.4 A morphism of toric monoids $P \to Q$ induces a map $\Omega_{P/R} \to \Omega_{Q/R}$. A compatible morphism of exponential data is a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \Lambda_P \arrow[r] \arrow[d] & \Omega_{P/R} \arrow[d] \\ Q \arrow[r] & \Lambda_Q \arrow[r] & \Omega_{Q/R}. \end{tikzcd}$$

For example, if $\Lambda_P = P^{gp}$ or $k \otimes P^{gp}$ or $R \otimes P^{gp}$, there is an evident choice of $\Lambda_P \to \Lambda_Q$. Associated with such data are morphisms $R[P,\Lambda_P] \to R[Q,\Lambda_Q]$ and $J(P,\Lambda_P) \to J(Q,\Lambda_Q)$ and concomitant functors (with the subscripts on the Λ's omitted from the notation):

$$\begin{array}{rcl} MIC^{\Lambda}_{*}(P/R) & \to & MIC^{\Lambda}_{*}(Q/R) \\ HIG^{\Lambda}_{*}(P/R) & \to & HIG^{\Lambda}_{*}(Q/R) \\ MH^{\Lambda}_{\Lambda'}(P/R) & \to & MH^{\Lambda}_{\Lambda'}(Q/R) \end{array}$$

and, when $R = \mathbf{C}$,

$$L^{\Lambda}_{*}(P) \to L^{\Lambda}(Q).$$

It is easy to verify that the functors in (1.5.1) and (1.4.8) are compatible with these base change functors.

2 Formal and holomorphic germs

2.1 Exponents and the logarithmic inertia group

Let X be a smooth, fine, and saturated idealized log analytic space. If x is a point of X, let

$$\begin{aligned} I_x : &= \operatorname{Hom}(\overline{M}^{gp}_{X,x}, \mathbf{Z}(1)) \\ \Omega_{\overline{M}_x/\mathbf{C}} &:= \mathbf{C} \otimes \overline{M}^{gp}_{X,x} \\ T_{\overline{M}_x/\mathbf{C}} &:= \mathcal{H}om(\overline{M}^{gp}_{X,x}, \mathbf{C}) \cong \mathbf{C} \otimes I_x. \end{aligned}$$

The group I_x is called the *logarithmic inertia group at* x. It is the fundamental group of the torus $\mathsf{A}^*_{\overline{M}_{X,x}}$, and $T_{\overline{M}_x/\mathbf{C}}$ is the space of invariant vector fields on $\mathsf{A}^*_{\overline{M}_{X,x}}$.

It follows as in [12, 1.3.1] that there is a natural surjective map

$$\Omega^1_{X/\mathbf{C}}(x) \to \mathbf{C} \otimes \overline{M}^{gp}_{X,x}.$$

If (E, ∇) is a coherent sheaf with integrable connection on X, let $E(x) := E_x/m_x E_x$ be its fiber at x. Then there is a unique linear map ρ_x such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega^1_{X/\mathbf{C}} \\ \downarrow & & \downarrow \\ E(x) & \xrightarrow{\rho_x} & E(x) \otimes \Omega_{\overline{M}_x/\mathbf{C}} \end{array}$$

It follows from the integrability of ∇ that the endomorphisms of $E(x)$ defined by evaluating ρ_x at any two elements of $T_{\overline{M}_x/\mathbf{C}}$ commute. Thus ρ_x defines a $T_{\overline{M}_x/\mathbf{C}}$-Higgs field on $E(x)$, and $E(x)$ becomes a module over the symmetric algebra $S^{\cdot}T_{\overline{M}_x/\mathbf{C}}$. Since $E(x)$ is finite dimensional over $\mathbf{C}$, it is supported at a finite set of maximal ideals of this algebra, *i.e.*, at a finite set of elements of $\Omega_{\overline{M}_x}$.

Definition 2.1.1 *Let (E, ∇) be a coherent sheaf with integrable connection on X and let x be a point of X. Then the* residue *of (E, ∇) at x is the map ρ_x in the diagram above, and the* exponents *of (E, ∇) at x are the* negatives *of the elements in $\Omega_{\overline{M}_x/\mathbf{C}} = \mathbf{C} \otimes \overline{M}^{gp}_{X,x}$ lying in the support of the $\mathbf{C}$-$T_{\overline{M}_x}$ module defined by ρ_x.*

To understand the choice of the sign in the definition of exponents, consider the connection on the structure sheaf of the logarithmic affine line with $\nabla(1) :=$

$\lambda \otimes dt/t$, where $\lambda \in \mathbf{C}$. Then the corresponding $\mathbf{C}$-$T_{\overline{M}_x}$-module, has support at λ. On the other hand, the horizontal sections of the connection are the constant multiples of $t^{-\lambda}$, so it is $-\lambda$ which appears as an "exponent." Note that formation of the residue is compatible with tensor products. In particular, the set of exponents of the tensor product of two connections (E_1, ∇_1) and (E_2, ∇_2) is the set of sums $\lambda_1 + \lambda_2$, with λ_i an exponent of E_i.

Our main local theorem gives an equivalence between the category of analytic germs of log connections and the category of normalized homogeneous connections considered in §1. Fix a point x of X and let $\overline{M}_{X,x} \to \Lambda \to \Omega_{\overline{M}_x/\mathbf{C}}$ be a rigid set of exponential data for $\overline{M}_{X,x}$. Let $MIC^{\Lambda}_{coh}(X_x)$ denote the category of germs of coherent sheaves with integrable connection all of whose exponents lie in Λ. This category is closed under extensions, tensor products, and duals (because Λ is a group). If $P \to \Lambda \to \mathbf{C} \otimes P^{gp}$ is a rigid set of exponential data for a toric monoid P, then the image $\overline{\Lambda}$ of Λ in $\mathbf{C} \otimes \overline{P}^{gp}$ defines a set of exponential data for $\overline{P}$, and we sometimes write $MIC^{\Lambda}_{coh}(X_x)$ for $MIC^{\overline{\Lambda}}_{coh}(X_x)$.

Theorem 2.1.2 *Let P be an idealized toric monoid with rigid exponential data Λ, let X be the log analytic space associated to A_{P}, and let $\hat{X}_v$ be the formal completion of X at its vertex v. Use $\overline{X}$ and similar notation for $\mathsf{A}_{\overline{\mathsf{P}}} \subseteq \mathsf{A}_{\mathsf{P}}$, where $\overline{P} := P/P^*$. Then the evident functors form a 2-commutative diagram:*

$$\begin{tikzcd}
 & MIC^{\Lambda}_{coh}(P/\mathbf{C}) \arrow[dl] \arrow[d] \arrow[dr] & \\
MIC^{\overline{\Lambda}}_{coh}(\overline{P}/\mathbf{C}) \arrow[r, "\overline{A}"] \arrow[dr, "C"] & MIC^{\overline{\Lambda}}_{coh}(\overline{X}_v) \arrow[d, "\overline{D}"] & MIC^{\Lambda}_{coh}(X_v/\mathbf{C}) \arrow[l, "B_{an}"'] \arrow[d, "D"] \\
 & MIC^{\overline{\Lambda}}_{coh}(\hat{\overline{X}}_v/\mathbf{C}) & MIC^{\Lambda}_{coh}(\hat{X}_v/\mathbf{C}), \arrow[l, "\hat{B}"']
\end{tikzcd}$$

in which all the labeled arrows are equivalences of tensor categories, compatible with De Rham cohomology.

The proof will occupy the rest of this section.

Remark 2.1.3 Let (E, ∇) be an object of $MIC^{\Lambda}_{coh}(X_v/\mathbf{C})$ and let (E', ∇) be the corresponding object of $MIC^{\overline{\Lambda}}_{coh}(\overline{P}/\mathbf{C})$. Then (E, ∇) and (E', ∇) have the same restriction to $\hat{\overline{X}}$, and in particular they have the same residue and exponents. That is, the residue ρ of E can be identified with the endomorphism of E'/P^+E' induced by ∇. Since ∇ is normalized, $\{\lambda : (E'/P^+E')_\lambda \neq 0\}$ is the same as the support of the $T_{\overline{P}/\mathbf{C}}$-Higgs module defined by ρ. Note that this set is just the set of degrees of any minimal set of generators for E'/P^+E'. Let (V, ρ) be the equivariant Riemann-Hilbert transform (1.4.3) of (E', ∇). Since the degrees of V are the negative of the degrees of E', it follows that the set of exponents of (E, ∇) is exactly the set of minimal degrees of V.

2.2 Formal germs

We begin with the functor C; without loss of generality we may and shall assume that $P = \overline{P}$. Then v corresponds to the maximal ideal of $\mathbf{C}[P]$ generated by P^+, and the completion of $\mathbf{C}[P]$ at this ideal can be identified with the formal power series ring $\mathbf{C}[[P]]$. This is the set of functions $a\colon P \to \mathbf{C}$, where for $a, b \in \mathbf{C}[[P]]$, $(a+b)_p := a_p + b_p$ and $(ab)_r := \sum\{a_p b_q : p + q = r\}$. To see that the sum is finite, choose a local homomorphism $\phi\colon P \to \mathbf{N}$, and observe that each $\{p \in P : \phi(p) \le n\}$ is finite. In fact, this set is the complement of an ideal K_n of P, and the set of such ideals $\{K_n : n \in \mathbf{N}\}$ is cofinal with the set of powers of P^+. If S is a free P set and V is a finitely generated S-graded $\mathbf{C}[P]$-module, the P^+-adic completion $\hat{V}$ of V can be identified with the product $\prod\{V_s : s \in S\}$. The action of P on S defines a partial ordering on S: $s \le t$ if there exists $p \in P$ with $p + s = t$; such a p is unique if it exists, and we write $t - s$ for this p. Then if $a \in \mathbf{C}[[P]]$ and $v \in \prod V_s$, $(av)_t := \sum a_{t-s} v_s$. The P-set $\Lambda \subseteq \mathbf{C} \otimes P^{gp}$ is only potentially free, but if V is a finitely generated Λ-graded $\mathbf{C}[P]$-module, there exists a finitely generated free P-subset S of Λ such that $V_\lambda = 0$ for $\lambda \notin S$, and we can identify $\hat{V}$ with $\prod\{V_s : s \in S\} \cong \prod\{V_\lambda : \lambda \in \Lambda\}$.

It is now easy to see that the functor C is compatible with cohomology, *i.e.*, that if (E, ∇) is an object of $MIC^{\Lambda}_{coh}(P/\mathbf{C})$, the natural map

$$(E \otimes \Omega^{\cdot}_{X/\mathbf{C}})_0 \to \hat{E} \otimes \Omega^{\cdot}_{X/\mathbf{C}}$$

from the degree zero part of its de Rham complex to its completion is a quasi-isomorphism. Indeed, $\Omega^1_{X/\mathbf{C},v} \cong \mathcal{O}_{X,v} \otimes_{\mathbf{C}} \Omega_{P/\mathbf{C}}$, and $\hat{E} \otimes \Omega^{\cdot}_{X/\mathbf{C}}$ can be identified with the product: $\prod_\lambda (E \otimes \Omega^{\cdot}_{P/\mathbf{C}})_\lambda$. For each λ, the degree λ part of the complex $E \otimes \Omega^{\cdot}_{P/\mathbf{C}}$ can be identified with the Higgs complex of the $T_{P/\mathbf{C}}$-Higgs module (E, ∇_λ). Since (E, ∇) is normalized (1.2.3), this complex is acyclic whenever $\lambda \neq 0$, by (1.5.2). Since infinite products in the category of vector spaces commute with cohomology, the cohomology of the product identifies with the cohomology of the degree zero part of $E \otimes \Omega^{\cdot}_{X/\mathbf{C}}$, as required. Since the functor C is compatible with the formation of internal Hom's, it follows that it is also fully faithful.

It remains to prove that C is essentially surjective. Let (E, ∇) be an object of $MIC^{\Lambda}_{coh}(\hat{X}_v/\mathbf{C})$. The connection

$$\nabla\colon E \to E \otimes_{\mathbf{C}[P]} \Omega^1_{X/\mathbf{C}} \cong E \otimes_{\mathbf{C}} \Omega_{P/\mathbf{C}}$$

can be regarded as a $\mathbf{C}$-$T_{P/\mathbf{C}}$-module structure on E, which is easy to analyze if E is finite dimensional over $\mathbf{C}$. Indeed, such an E admits a Jordan decomposition

$$(E, \nabla) \cong \oplus\{(E_\lambda, \nabla_\lambda) : \lambda \in \Omega\},$$

where each $(E_\lambda, \nabla_\lambda)$ has support in λ, and Lemma (1.3.4) applies. In fact, $E_\lambda = 0$ unless $\lambda \in \Lambda$ by (2.2.1) below. Thus $E \cong \oplus_\lambda (E_\lambda, \nabla_\lambda)$ is an object of $MIC^{\Lambda}_{coh}(P/\mathbf{C})$ and it is evident that its formal completion at v is (E, ∇). This shows that any such (E, ∇) is in the essential image of C.

For the general case, we use a limit argument and the following lemma.

Lemma 2.2.1 *Let (E, ∇) be an object of $MIC^{\Lambda}_{coh}(\hat{X}_v/\mathbf{C})$ such that E is finite dimensional over $\mathbf{C}$. Then the support of (E, ∇) as a $T_{P/\mathbf{C}}$-Higgs module is contained in the P-subset S of $\mathbf{C} \otimes P^{gp}$ generated by the support of $(E(v), \nabla)$, and in particular is contained in Λ. If K is an ideal of P, then the support of KE is contained in the K-translate of the support of E.*

Proof: If K is any ideal of P, then the ideal $\mathbf{C}[K]$ of $\mathbf{C}[P]$ it generates is invariant under ∇ and defines an element of $MIC^{\Lambda}_{coh}(P/\mathbf{C})$. Since $\nabla e_k = e_k \otimes dk$, the support of the corresponding Higgs module is the image of K in Λ. Since there is a surjective map $\mathbf{C}[K] \otimes E \to KE$, the support of KE is contained in the support of $\mathbf{C}[K] \otimes E$, which is the K-translate of the support of E. This proves the second statement. Since E has finite length, it is annihilated by P^{+n} for some $n \in \mathbf{Z}^+$, and we prove the first statement by induction on n. If $n = 1$, $E \cong E/P^+E = E(v)$ and the result is trivial. In the general case, note that P^+E is invariant under the connection and annihilated by $P^{+^{n-1}}$, so the induction hypothesis implies that the support of P^+E is contained in the P-subset of $\mathbf{C} \otimes P^{gp}$ generated by the support of $P^+E/P^{+2}E$. As we have just seen, this is contained in $P^+ + S \subseteq S$. Then the exact sequence $0 \to P^+E \to E \to E/P^+E \to 0$ shows that the support of E is contained in S as well. □

Now let (E, ∇) be any object of $MIC^{\Lambda}_{coh}(\hat{X}_v/\mathbf{C})$. Choose a local homomorphism $\phi \colon P \to \mathbf{N}$. Then ϕ extends uniquely to a $\mathbf{C}$-linear map $\mathbf{C} \otimes P^{gp} \to \mathbf{C}$ which we also denote by ϕ. Let $K^n := \{p \in P : \phi(p) \geq n\}$, and let $E_n := E/K^nE$. If $n' \geq n$ there is an exact sequence of modules with connection

$$0 \to K^nE/K^{n'}E \to E_{n'} \to E_n \to 0.$$

Each of these terms is finite dimensional over $\mathbf{C}$, and the $\mathbf{C}$-$T_{P/\mathbf{C}}$-module it defines has support in Λ. For every λ, the corresponding sequence:

$$0 \to (K^nE/K^{n'}E)_\lambda \to E_{n',\lambda} \to E_{n,\lambda} \to 0$$

is again exact. Let S be the support of E/P^+E and choose $m \in \mathbf{Z}$ so that $m < Re(\phi(s))$ for all $s \in S$. Suppose $(K^nE/K^{n'}E)_\lambda \neq 0$. Then by lemma (2.2.1), λ can be written as $p + s$ with $p \in K^n$ and $s \in S$, and

$$Re(\phi(\lambda)) = \phi(p) + Re(\phi(s)) > n + m.$$

Thus if $n \geq Re(\phi(\lambda)) - m$, $(K^nE/K^{n'}E)_\lambda$ vanishes and the map $E_{n',\lambda} \to E_{n,\lambda}$ is an isomorphism. Let E_λ be the inverse limit, *i.e.*, the stable value of $E_{n,\lambda}$ for

n large. Then ∇ maps E_λ to E_λ, and $\oplus(E_\lambda, \nabla_\lambda)$ is an object of $MIC_{coh}(P/\mathbf{C})$, whose completion at the vertex is (E, ∇).

This completes the proof that C is essentially surjective, and it follows from the diagram that the same is true of $\overline{D}$.

The fact the arrow $\hat{B}$ is an equivalence follows from the following slightly stronger result, which is a consequence of the fact there is no log structure in the transverse direction.

Lemma 2.2.2 *Let $X^\vee$ denote the formal completion of X along $\overline{X}$. Then the natural functor*

$$MIC^\Lambda_{coh}(X^\vee/\mathbf{C}) \to MIC^\Lambda_{coh}(\overline{X}/\mathbf{C})$$

is an equivalence, compatible with cohomology.

Proof: Since $\overline{X}/\mathbf{C}$ is smooth, the category $MIC^\Lambda_{coh}(\overline{X}/\mathbf{C})$ is equivalent to a full subcategory of the category of coherent crystals on $\overline{X}/\mathbf{C}$, and the same holds for $X/\mathbf{C}$ [6, 6.2]. Since $\overline{X} \to X$ is a strict closed immersion, the fact that the above functor is an equivalence follows formally from the properties of crystals: $X^\vee$ is a limit of strict infinitesimal thickenings of $\overline{X}$, and hence a crystal on $\overline{X}$ has a natural value on $X^\vee$, and in fact also on any strict infinitesimal thickening of $X^\vee$. To check the result on De Rham cohomology, one can work locally, using the fact that $X^\vee$ looks locally like $\overline{X} \times \operatorname{Spf} \mathbf{C}[[t_1, \ldots t_n]]$, and argue as in the classical case. $\square$

Since P is saturated, $\overline{P}^{gp}$ is a finitely generated free abelian group, and so the exact sequence $0 \to P^* \to P^{gp} \to \overline{P}^{gp} \to 0$ splits. Any splitting $\overline{P}^{gp} \to P^{gp}$ automatically maps $\overline{P}$ to P and induces a section of the map $\overline{X} \to X$. This implies that the functor $MIC^\Lambda_{coh}(P/\mathbf{C}) \to MIC^\Lambda_{coh}(\overline{P}/\mathbf{C})$ is essentially surjective. Since $\hat{B}$ is an equivalence, it follows from the diagram that D is also essentially surjective.

2.3 Convergent germs

Our first task is to establish a convenient description of the ring of germs of analytic functions at the vertex of $\underline{\mathsf{A}}_\mathsf{P}$ as a subring $\mathbf{C}\{P\}$ of $\mathbf{C}[[P]]$.

Proposition 2.3.1 *Let P be a fine sharp monoid, let v be the vertex of A^{*an}_P, and let T be the (necessarily finite) set of irreducible elements of P.*

1. *For $\delta \in \mathbf{R}^+$, let*

$$U_\delta := \{x \in \underline{\mathsf{A}}_\mathsf{P}(\mathbf{C}) : |x(t)| < \delta \text{ for all } t \in T\}.$$

Then $\{U_\delta : \delta \in \mathbf{R}^+\}$ forms a basis for the system of neighborhoods of v in $\mathsf{A}^{*an}_P(\mathbf{C})$ in the usual complex topology.

2. *If ϕ is a local homomorphism $P \to \mathbf{N}$ and $\alpha := \sum_p a_p e_p \in \mathbf{C}[[P]]$, then α converges in some neighborhood of v if and only if the set $\{\frac{\log|a_p|}{\phi(p)} : p \in P^+\}$ is bounded above.*

Proof: First suppose that $P = \mathbf{N}^n$. Then $X = \mathbf{C}^n$, v is the origin, U_δ is the polydisc about v of radius δ, and (1) is clear. If P is any fine sharp monoid, then T is finite and generates P as a monoid, and hence a bijection $\{1 \ldots n\} \to T$ induces a surjective homomorphism $\mathbf{N}^n \to P$ and a closed immersion $\underline{\mathsf{A}}_{\mathsf{P}} \to \mathbf{A}^n$. With respect to this closed immersion, U_δ is just the intersection of $\underline{\mathsf{A}}_{\mathsf{P}}(\mathbf{C})$ with the polydisc of radius δ about v. This proves (1) in general.

Suppose that $\alpha = \sum_p a_p e_p$, $c \in \mathbf{R}$, and $c \geq \phi(p)^{-1} \log|a_p|$ for every $p \in P^+$. Choose $\epsilon > 0$, let $\lambda_t := -(c+\epsilon)\phi(t)$ for each $t \in T$, and choose a positive number δ such that $\delta < e^{\lambda_t}$ for all t. Then U_δ is an open neighborhood of v in X, and if $x \in U_\delta$, $\log|x(t)| < \lambda_t$ for all t. Any $p \in P$ can be written $p = \sum n_t t$. It follows that for $x \in U_\delta$,

$$\begin{aligned}
\log|a_p x(p)| &= \log|a_p| + \log|x(p)| \\
&\leq c\phi(p) + \log|x(p)| \\
&\leq \sum_t n_t(c\phi(t) + \log|x(t)|) \\
&\leq \sum_i n_t(c\phi(t) + \lambda_t) \\
&\leq \sum_t n_t(-\epsilon\phi(t)) \\
&\leq -\epsilon\phi(p)
\end{aligned}$$

Thus $|a_p x(p)| \leq r^{\phi(p)}$, where $r := e^{-\epsilon} < 1$. As is well known, $\{p : \phi(p) = i\}$ has cardinality less than Ci^m for some C and m, so the set of partial sums of the series $\sum_p |a_p x(p)|$ is bounded by the set of partial sums of the series $\sum_i Ci^m r^i$. Since this latter series converges, so does the former.

Suppose on the other hand that $\{\phi(p)^{-1} \log|a_p| : p \in P^+\}$ is unbounded. For $c \in \mathbf{R}^+$, define $x_c \colon P \to \mathbf{C}$ by $x_c(p) := c^{-\phi(p)}$. Then $x_c \in \underline{\mathsf{A}}_{\mathsf{P}}(\mathbf{C})$, and if $\delta > 0$ and c is chosen large enough so that $\log c > (\phi(t))^{-1}(-\log\delta)$ for all $t \in T$, then $x_c \in U_\delta$. For every such c, there are infinitely many $p \in P^+$ such that $|a_p| > (c+1)^{\phi(p)}$. For any such p,

$$|a_p x_c(p)| \geq (1+c)^{\phi(p)} c^{-\phi(p)} = (1/c+1)^{\phi(p)} \geq 1,$$

so the series $\sum_p a_p x_c(p)$ cannot converge. □

Our next task is an existence and uniqueness result for formal and convergent solutions to certain differential equations. Recall that if $X = \mathsf{A}_{\mathsf{P}}$, a homomorphism $P \to \mathbf{N}$ defines an invariant vector field on X.

Proposition 2.3.2 *Let P be a sharp toric monoid, let $X := \mathsf{A_P}$, let v be its vertex, and let (E, ∇) be the germ of a coherent sheaf with integrable log connection on X^{an} at v. Suppose that $\phi\colon P \to \mathbf{N}$ is a local homomorphism such that $\phi(\lambda) < 0$ for every exponent λ of E at v.* [1] *Then ∇_ϕ acts bijectively on E and on $\hat{E}$.*

Let us first discuss the formal case. It suffices to prove that for each $n \in \mathbf{N}$, ∇_ϕ induces an automorphism of $E_n := E/K^nE$, where $K^n := \{p : \phi(p) \geq n\}$. Each E_n is finite dimensional over $\mathbf{C}$ and ∇ can be viewed as a $T_{P/\mathbf{C}}$-Higgs field on E_n. The support of E_n as a $T_{P/\mathbf{C}}$-Higgs module is a finite subset of $\Omega_{P/\mathbf{C}}$. By (2.2.1), its support is contained in the sub P-subset S of Λ generated by the support of the $T_{P/\mathbf{C}}$-Higgs module E/P^+E, *i.e.*, by the negative of the set of exponents. Thus $\phi(s) > 0$ for every $s \in S$, and hence ∇_ϕ is an automorphism of E_n.

To deal with convergence we must be more explicit. We have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla_\phi} & E \\ \downarrow & & \downarrow \\ \hat{E} & \xrightarrow{\hat{\nabla}_\phi} & \hat{E} \end{array}$$

It follows that $\nabla_\phi\colon E \to E$ is injective, and it remains to prove that it is surjective.

Let $(v_1, \dots v_n)$ be a subset of E whose reduction modulo P^+E forms a basis for E/P^+E, and let $V \subseteq E$ be its $\mathbf{C}$-linear span. Then V generates E as a module over the ring $\mathcal{O} := \mathcal{O}_{X^{an}_v}$. For each i, $\nabla_\phi(v_i) \in E$, and hence can be written (not necessarily uniquely) as a sum: $\sum a_{ij}v_j$, with $a_{ij} \in \mathcal{O}$. Let A denote the $n \times n$ matrix $\big(a_{ij}\big)$, and write A as a formal sum $\sum\{A_q e_q : q \in P\}$, where A_q is an $n \times n$ matrix in $\mathbf{C}$. For any $v \in V$,

$$\nabla_\phi(v) = \sum_q A_q(v)e_q$$

In particular, A_0 is the matrix of the endomorphism induced by ∇_ϕ on E/P^+E. The eigenvalues of this endomorphism are among those complex numbers of the form $\phi(s)$ for s in the support of $(E/P^+E, \nabla)$. By hypothesis, $\phi(p) + \phi(s) \neq 0$, for every $p \in P$ and s in this support. It follows that $A_0 + \phi(p)$ is invertible for every $p \in P$.

[1] One can show using a Baire category argument that a ϕ as in (2.3.2) exists if and only if the set of exponents does not meet P.

Any element v of $\hat{E}$ can be written as a formal sum $v = \sum v_q e_q$, with $v_q \in V$. Then:

$$\begin{aligned}
\nabla_\phi(v) &= \sum_q \Big(\nabla_\phi(v_q)e_q + v_q\langle\phi, de_q\rangle\Big) \\
&= \sum_q \sum_{q'} A_{q'}(v_q)e_{q'}e_q + \sum_q \phi(q)v_q e_q \\
&= \sum_p \Big(\sum_{q+q'=p} A_{q'}(v_q)\Big)e_p + \sum_p v_p\phi(p)e_p \\
&= \sum_p w_p e_p,
\end{aligned}$$

where

$$w_p := A_0(v_p) + \phi(p)v_p + \sum_{q<p} A_{p-q}(v_q).$$

Recall that $A_0 + \phi(p)$ is invertible; let B_p be its inverse. Then the above equation becomes:

$$B_p(w_p) = v_p + B_p \sum_{q<p} A_{p-q}(v_q).$$

In other words, if $w = \sum w_p e_p$, then the coefficients of $v = \hat{\nabla}_\phi^{-1}(w)$ are given recursively by the formula:

$$v_p = B_p(w_p) - B_p \sum_{q<p} A_{p-q}(v_q). \tag{2.3.1}$$

Note that the sum is finite since there are only finitely many q with $q < p$. We have to prove that if the series $\sum w_p e_p$ converges, so does the series $\sum v_p e_p$.

Since ϕ is local, there are only finitely many p with $\phi(p) \leq 2||A_0||$, and we can find a constant $M \geq 2$ such that $||B_p|| \leq M\phi(p)^{-1}$ for all these p. Let $\psi := \phi/M$. We claim that $||B_p|| \leq \psi(p)^{-1}$ for all $p \in P$. This is true by our choice of M if $\phi(p) \leq 2||A_0||$. If on the other hand $\phi(p) > 2||A_0||$, then

$$\begin{aligned}
||B_p|| &= ||\,(\phi(p) + A_0)^{-1}\,|| \\
&= \phi(p)^{-1}||1 - \phi(p)^{-1}A_0 + \phi(p)^{-2}A_0^2 - \cdots|| \\
&\leq \phi(p)^{-1}(1 + 1/2 + 1/4 + \cdots) \\
&\leq 2\phi(p)^{-1} \\
&\leq \psi(p)^{-1}.
\end{aligned}$$

Since A and w are convergent there exists a positive real number s such that $||A_p||$ and $||w_p||$ are less than $s^{\psi(p)}$ for all p. Moreover, since ∇_ϕ is $\mathbf{C}$-linear, we may without loss of generality assume that $||w_0|| \leq ||B_0||^{-1}$, so that $||v_0|| \leq 1$. Let $y_p := ||v_p s^{-\psi(p)}||$ for $p \in P$. It will suffice to show that there exists a t such that $y_p \leq t^{\psi(p)}$ for all p.

By the formula (2.3.1),

$$y_p \leq ||B_p||||w_p||s^{-\psi(p)} + ||B_p|| \sum_{q<p} ||A_{p-q}||s^{-\psi(p-q)}||v_q||s^{-\psi(q)}$$

Hence

$$y_p \leq \frac{1}{\psi(p)} + \frac{1}{\psi(p)} \sum_{q<p} y_q. \tag{2.3.2}$$

Let ϵ be the minimum of $\psi(P^+)$, and choose c so that $c\epsilon > 2$. Then let $a_0 := 1$ and for $p \in P^+$ define a_p inductively by setting

$$a_p := c \sum_{q<p} a_q (1 - \frac{\psi(q)}{\psi(p)}).$$

If $q < p$, $\psi(p) - \psi(q) \geq \epsilon$. Hence if p is any element of P^+,

$$\begin{aligned} a_p &= \sum_{q<p} ca_q (\frac{\psi(p) - \psi(q)}{\psi(p)}) \\ &\geq \sum_{q<p} \frac{c\epsilon a_q}{\psi(p)} \\ &\geq \sum_{q<p} \frac{2a_q}{\psi(p)} \geq \sum_{q<p} \frac{a_q}{\psi(p)} + \sum_{q<p} \frac{a_q}{\psi(p)} \\ &\geq \frac{1}{\psi(p)} + \frac{1}{\psi(p)} \sum_{q<p} a_q \end{aligned}$$

Note that $y_0 = ||v_0|| \leq 1 = a_0$. Then it follows by induction on p from the previous inequality and (2.3.2) that $y_p \leq a_p$ for all p. Thus it suffices to prove that there exists a t such that $a_p \leq t^{\phi(p)}$ for all ϕ, *i.e.*, that the series $\sum a_p e_p$ in fact lies in $\mathbf{R}\{P\}$. This will follow from the following lemma.

Lemma 2.3.3 *Let P be a fine sharp monoid, let $\phi\colon P \to (\mathbf{R}^{\geq}, +)$ be a local homomorphism, and let c be any positive real number. Define $a\colon P \to \mathbf{R}$ inductively setting $a(0) = 1$, and, if $p \in P^+$,*

$$a_p = c \sum_{q<p} a_q \Big(1 - \frac{\phi(q)}{\phi(p)}\Big)$$

Then $\sum a_p e_p$ belongs to the ring $\mathbf{R}\{P\}$ of germs of convergent elements of $\mathbf{R}[[P]]$, and is in fact independent of ϕ.

Proof: Let

$$f := \sum_{q \in P^+} e_q \in \mathbf{C}[[P]],$$

Evidently $f(x)$ converges for all x in $U_1 = \{x : q(x) < 1 \text{ for all } q \in P^+\}$, hence so does $g := \exp cf$. Write

$$g := \sum_{p \in P} b_p e_p.$$

Then

$$\begin{aligned}
dg &= cgdf \\
\sum_{p \in P^+} b_p e_p dp &= c\Big(\sum_{q \in P} b_q e_q\Big)\Big(\sum_{q' \in P^+} e_{q'} dq'\Big) \\
&= \sum_{q \in P, q' \in P^+} c b_q e_{q'+q} dq' \\
&= \sum_{p \in P^+} \Big(\sum_{q<p} c b_q d(p-q)\Big) e_p
\end{aligned}$$

Thus

$$\begin{aligned}
b_p dp &= \sum_{q<p} c b_q (dp - dq) \\
b_p \phi(p) &= \sum_{q<p} c b_q (\phi(p) - \phi(q))
\end{aligned}$$

Hence $a_p = b_p$, and therefore $\sum a_p e_p$ lies in $\mathbf{R}\{P\}$. ☐

We next show that the functors D and $\overline{D}$ are compatible with cohomology. Since this implies that they are fully faithful, this will complete the proof of the theorem. Since $\overline{D}$ is a special case of D, the following result suffices.

Proposition 2.3.4 *If (E, ∇) is an object of $MIC_{coh}(X_v/\mathbf{C})$, then the natural map*

$$E \otimes \Omega^{\cdot}_{X/\mathbf{C}} \to \hat{E} \otimes \Omega^{\cdot}_{X/\mathbf{C}}$$

is a quasi-isomorphism.

Proof: Since P is a toric monoid, its unit group is a finitely generated free group, say of rank r, and there is an isomorphism $P \cong \overline{P} \oplus \mathbf{Z}^r$. The vertex v of X is the point sending every element of P^* to 1 and every element of $\overline{P}^+$ to 0. Let $Q := \mathbf{N}^r \oplus \overline{P}$, let $X'' := \operatorname{Spec}(\overline{P} \to \mathbf{C}[Q])$, and let v'' be the point of X'' sending Q^+ to zero. Finally, let $X' := \mathsf{A}_{\mathbf{Q}}$, and let $f\colon X' \to X''$ be the map which is the identity on underlying analytic spaces and the inclusion on log structures. Thus

$$\begin{aligned}
X &\cong \operatorname{Spec}(\overline{P} \to \mathbf{C}[\overline{P}][t_1, t_1^{-1}, \ldots t_r, t_r^{-1}]) \cong \overline{X} \times \mathbf{G}_m^r \\
X' &\cong \operatorname{Spec}(\overline{P} \oplus \mathbf{N}^r \to \mathbf{C}[\overline{P}][x_1, \ldots x_r]) \cong \overline{X} \times \mathsf{A}_{\mathbf{N}^r} \\
X'' &\cong \operatorname{Spec}(\overline{P} \to \mathbf{C}[\overline{P}][x_1, \ldots x_r]) \cong \overline{X} \times \underline{\mathsf{A}}_{\mathbf{N}^r}
\end{aligned}$$

The homomorphism sending x_i to $t_i - 1$ and which is the identity on $\overline{P}$ defines a strict open immersion of log schemes $X \to X''$ sending v to v''. Replacing X' by a neighborhood of the vertex v' of X', we find a map $X' \to X$ which is an isomorphism on underlying analytic spaces and which sends v' to v. Thus we may and shall identify the stalk of E at v with the stalk of its pullback to X' at v'. (In other words, we have added some log structure to X to get X'.)

Lemma 2.3.5 *For each i, the stalk at v of natural map*

$$E \otimes \Omega^i_{X/\mathbf{C}} \to E \otimes \Omega^i_{X'/\mathbf{C}}$$

is injective. Furthermore, as submodules of $\hat{E} \otimes \Omega^i_{X'/\mathbf{C}}$,

$$E \otimes \Omega^i_{X/\mathbf{C}} = (\hat{E} \otimes \Omega^i_{X/\mathbf{C}}) \cap (E \otimes \Omega^i_{X'/\mathbf{C}})$$

at v.

Proof: We can check the injectivity statement after passing to formal completions. Recall from (2.2.2) that, since E is a crystal on X, there is a coherent sheaf $\overline{E}$ on $\overline{X}$ such that $\hat{E} \cong \pi^* \hat{\overline{E}}$, where $\pi \colon X \to \overline{X}$ is the map induced by our chosen splitting of $P \to \overline{P}$. Let $Y := \mathsf{A}_{\mathbf{N}^r}$, so that $X' \cong \overline{X} \times Y$ and $X \cong \overline{X} \times \underline{Y}$ near v. Then $\Omega^1_{X/\mathbf{C}} \cong \Omega^1_{\overline{X}/\mathbf{C}} \oplus \Omega^1_{\underline{Y}/\mathbf{C}}$ and $\Omega^1_{X'/\mathbf{C}} \cong \Omega^1_{\overline{X}/\mathbf{C}} \oplus \Omega^1_{Y/\mathbf{C}}$; furthermore all these sheaves are free at v. It follows that $\hat{\overline{E}}$ and the cokernel of the map $\Omega^i_{X/\mathbf{C}} \to \Omega^i_{X'/\mathbf{C}}$ are tor-independent. This proves the injectivity. Note that $\Omega^i_{X/\mathbf{C}}$ and $\Omega^i_{X'/\mathbf{C}}$ are free, and $x\Omega^i_{X'/\mathbf{C}} \subseteq \Omega^i_{X/\mathbf{C}}$, where $x := x_1 \cdots x_r$. If $e \in \hat{E}$ and $xe \in E$, then it is clear from (2.3.1) that $e \in E$. Since X' and X have the same underlying analytic structure, it follows that $(\hat{E} \otimes \Omega^i_{X/\mathbf{C}}) \cap (E \otimes \Omega^i_{X'/\mathbf{C}}) = E \otimes \Omega^i_{X/\mathbf{C}}$. □

Choose a local homomorphism $\phi \colon Q \to \mathbf{N}$ and for $n \in \mathbf{N}$ let $K_n := \{q \in Q : \phi(q) \geq n\}$. Let $E^i := E_v \otimes \Omega^i_{X/\mathbf{C}}$ and let $E^{i\prime}_v := E_v \otimes \Omega^i_{X'/\mathbf{C}}$. Then $E^{\cdot\prime}$ is a complex, containing subcomplexes $E^{\cdot}$ and $K_n E^{\cdot\prime}$ for each n. There is a commutative diagram of exact sequences of complexes:

$$\begin{tikzcd}
0 \arrow[r] & E^{\cdot} \cap K_n E^{\cdot\prime} \arrow[r] \arrow[d] & E^{\cdot} \arrow[r] \arrow[d] & E^{\cdot}_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{E}^{\cdot} \cap K_n \hat{E}^{\cdot\prime} \arrow[r] & \hat{E}^{\cdot} \arrow[r] & \hat{E}^{\cdot}_n \arrow[r] & 0
\end{tikzcd}$$

In this diagram, the quotient $E^{\cdot}_n$ is contained in $E^{\cdot\prime}_n := E^{\cdot\prime}/K_n E^{\cdot\prime}$ and annihilated by a power of the maximal ideal at v. Thus, the arrow on the right is an isomorphism of complexes. Our goal is to prove that the central arrow is a quasi-isomorphism, and so it will suffice to prove that the arrow on the left is a quasi-isomorphism. In particular, the following lemma suffices.

Lemma 2.3.6 *With the above notation, suppose that $n > \phi(\lambda)$ for every exponent λ of $E(v')$ on X'. Then $E^{\cdot} \cap K_n E^{\cdot\prime}$ and $\hat{E}^{\cdot} \cap K_n \hat{E}^{\cdot\prime}$ are acyclic.*

Proof: The homomorphism $\phi\colon Q \to \mathbf{N}$ induces a homomorphism $\Omega_{Q/\mathbf{C}} \to \mathbf{C}$, which can be regarded as an equivariant vector field on X'. It also induces for each i a homomorphism $\Omega^i_{Q/\mathbf{C}} \to \Omega^{i-1}_{Q/\mathbf{C}}$, by interior multiplication. These maps extend to $\mathbf{C}[Q]$-linear maps $E \otimes \Omega^i_{X'/\mathbf{C}} \to E \otimes \Omega^{i-1}_{X'/\mathbf{C}}$ sending $E \otimes \Omega^i_{X/\mathbf{C}}$ to $E \otimes \Omega^{i-1}_{X/\mathbf{C}}$ and $K_n E \otimes \Omega^i_{X'/\mathbf{C}}$ to $K_n E \otimes \Omega^{i-1}_{X'/\mathbf{C}}$. Let $\kappa := d\rho + \rho d$, *i.e.*, the Lie derivative with respect to ϕ. Then κ defines a morphism of complexes $E^{\cdot\prime} \to E^{\cdot\prime}$ which preserves the subcomplexes $E^{\cdot}$ and $K_n E^{\cdot\prime}$ and (hence) passes to the completions. By construction, κ is homotopic to zero. So to prove the complexes are acyclic, it suffices to prove that κ is an isomorphism on each of them.

Note that if $q \in Q$,

$$\kappa(e_q) = d\rho(e_q) + \rho d(e_q) = \rho(e_q dq) = \phi(q) e_q.$$

Furthermore, κ is a derivation, *i.e.*,

$$\kappa(\eta \wedge \omega) = \kappa(\eta) \wedge \omega + \eta \wedge \kappa(\omega)$$

if $\eta \in E \otimes \Omega^{\cdot}_{X/\mathbf{C}}$ and $\omega \in \Omega^i_{X/\mathbf{C}}$. If ω is equivariant, *i.e.*, if it lies in $\Omega^i_{Q/\mathbf{C}}$, $\kappa(\omega) := d\rho\omega + \rho d\omega = 0$. Thus if $e \in E$ and $\omega \in \Omega^i_{Q/\mathbf{C}}$, $\kappa(e \otimes \omega) = \nabla_\phi(e) \otimes \omega$. In other words, viewed as a map

$$\kappa\colon E \otimes_{\mathbf{C}} \Omega^i_{Q/\mathbf{C}} \to E \otimes_{\mathbf{C}} \Omega^i_{Q/\mathbf{C}},$$

$\kappa := \nabla_\phi \otimes \mathrm{id}$. Lemma (2.3.2) implies that ∇_ϕ acts bijectively on E and $\hat{E}$, and hence κ induces an automorphism of $E \otimes \Omega^{\cdot}_{X'/\mathbf{C}}$ and of $\hat{E} \otimes \Omega^{\cdot}_{X'/\mathbf{C}}$.

Since $K_n E$ and its quotients also satisfy the hypothesis of (2.3.2), κ also induces automorphisms of $K_n E^{\cdot\prime}$ and of $K_n E^{\cdot\prime}/K_{n'} E^{\cdot\prime}$ whenever $n' \geq n$. The image of $E^{\cdot} \cap K_n E^{\cdot\prime}$ in $K_n E^{\cdot\prime}/K_{n'} E^{\cdot\prime}$ is a finite dimensional subspace invariant under κ, and hence κ also acts as an automorphism of this subspace. Taking the limit over n', we see that κ induces an automorphism of $\hat{E}^{\cdot} \cap K_n \hat{E}^{\cdot\prime}$. It follows that κ is injective on $E^{\cdot} \cap K_n E^{\cdot\prime}$. If $e \in E^{\cdot} \cap K_n E^{\cdot\prime}$, there is a unique $\hat{e} \in \hat{E}^{\cdot} \cap K_n \hat{E}^{\cdot\prime}$ such that $\kappa(\hat{e}) = e$. But $e \in K_n E^{\cdot\prime}$ and so there is a unique $f \in K_n E^{\cdot\prime}$ such that $\kappa(f) = e$. Thus

$$\hat{e} = f \in \hat{E}^{\cdot} \cap E^{\cdot\prime} \cap K_n \hat{E}^{\cdot\prime} = E^{\cdot} \cap K_n E^{\cdot\prime}.$$

This proves that κ is an isomorphism of $E^{\cdot} \cap K_n E^{\cdot\prime}$ and completes the proof of the lemma. □

□

3 X_{log} and the global Riemann-Hilbert correspondence

3.1 X_{log} and its universal covering

If X is an idealized log scheme of finite type over $\mathbf{C}$, let X_{an} or X^{an} denote the corresponding log analytic space. Let us say that an idealized log analytic space X is *ideally log smooth* if it admits a covering by open subsets each of which is isomorphic to an open subset of $\underline{\mathsf{A}}^{an}_{P,K}$ for some fine idealized monoid (P, K). For such spaces, the sheaf of ideals K can be recovered from the log structure as the inverse image in M_X of 0. We let $\mathbf{S^1}$ be the unit circle, *i.e.*, $\{z \in \mathbf{C} : |z| = 1\}$ and $\mathbf{R}^{\geq}$ the multiplicative monoid of nonnegative real numbers. Thus the multiplication map $\mathbf{R}^{\geq} \times \mathbf{S^1} \to \mathbf{C}$ defines a log structure on $\mathbf{C}$, and we let ξ_{log} denote the corresponding log scheme $\mathrm{Spec}(\mathbf{R}^{\geq} \times \mathbf{S^1} \to \mathbf{C})$. Note that this log scheme is not integral or even coherent.

Kato and Nakayama have constructed in [7] a commutative diagram of ringed spaces:

$$\begin{tikzcd}
X^*_{an} \arrow[r, "j_{log}"] \arrow[dr, "j"'] & X_{log} \arrow[d, "\tau"] \\
 & X_{an}
\end{tikzcd}$$

We refer to [7] for the definition, but recall that the set underlying X_{log} is the set of $\mathbf{C}$-morphisms of log schemes $\xi_{log} \to X$ and that τ is the obvious map which forgets the log structure. This map is proper, and the fiber over a point x is a torsor under the group $\mathrm{Hom}(\overline{M}_{X,x}, \mathbf{S^1})$. Since X is saturated, this space is (noncanonically) isomorphic to $(\mathbf{S^1})^{r(x)}$, where $r(x)$ is the rank of $\overline{M}^{gp}_{X,x}$. The fundamental group I_x of the fiber $\tau^{-1}(x)$ (the *logarithmic inertia group at* x) can be canonically identified with $\mathrm{Hom}(\overline{M}^{gp}_{X,x}, \mathbf{Z}(1))$. Since this group is abelian and the fiber is connected, the choice of base point can be ignored.

When $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$, the space X_{log} has a convenient explicit description. If P is a monoid, let $C(P)$ denote the set of morphisms of monoids $\rho \colon P \to \mathbf{R}^{\geq}$, with the structure of topological monoid inherited from that of $\mathbf{R}^{\geq}$. If K is an ideal in P, let $C(P, K)$ be the set of those $\rho \in C(P)$ sending K to 0. This is a closed submonoid of $C(P)$, and in fact is an ideal in the monoid $C(P)$. Let $S(P)$ denote the set of morphisms of monoids $\sigma \colon P \to \mathbf{S^1}$, or, equivalently, $P^{gp} \to \mathbf{S^1}$, with its structure of topological group. If P is toric, P^{gp} is a finitely generated free abelian group, so $S(P)$ is a torus. Then if $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$, there is a canonical isomorphism $X_{log} \cong C(P, K) \times S(P)$. When K is a proper ideal, the map $c_0 \colon P \to \mathbf{R}^{\geq}$ sending P^* to 1 and P^+ to 0 is a point of $C(P, K)$, and the pair $(c_0, 1)$ is a point of X_{log} lying over the vertex of X, which we call the vertex of X_{log}.

It will be useful for us to work with an explicit universal cover of X_{log} when $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$. Let $\mathbf{R}(1) \subseteq \mathbf{C}$ denote the set of purely imaginary numbers, which forms a topological group under addition, and let $Y(P)$ denote the set of homomorphisms of abelian groups from P^{gp} to $\mathbf{R}(1)$. Finally, let $\tilde{\mathsf{A}}^{log}_{P,K} := C(P,K) \times Y(P)$, with its natural structure of a topological monoid. If K is a proper ideal call $\tilde{v} := (c_0, 0)$ the vertex of $\tilde{\mathsf{A}}^{log}_{P,K}$.

Proposition 3.1.1 *Let K be a proper ideal in a toric monoid P and let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$ and $\tilde{X}^{log} := \tilde{\mathsf{A}}^{log}_{P,K}$. Then the map*

$$\zeta\colon \tilde{X}^{log} = C(P,K) \times Y(P) \to C(P,K) \times S(P) = X_{log} : (\rho, y) \mapsto (\rho, \exp \circ y)$$

is a universal covering sending the vertex of $\tilde{X}^{log}$ to the vertex of X_{log}, with covering group canonically isomorphic to $\pi_1(P) := \mathrm{Hom}(P^{gp}, \mathbf{Z}(1))$. When P is a group, there is a natural isomorphism $\tilde{X}^{log} \cong \mathbf{V}\Omega^{an}_P$, under which the covering map ζ corresponds to the covering map exp *defined at the beginning of section (1.4).*

Proof: It is clear that id $\times$ exp is a covering map taking the vertex to the vertex. The exact sequence $0 \to \mathbf{Z}(1) \to \mathbf{R}(1) \to \mathbf{S^1} \to 0$ induces an exact sequence

$$0 \to \pi_1(P) \to Y(P) \to S(P) \to 0,$$

and so the covering group of ζ is canonically isomorphic to $\pi_1(P)$. To finish the proof, it will suffice to show that $\tilde{\mathsf{A}}^{log}_{P,K}$ is contractible. Choose a local homomorphism $\delta\colon P \to \mathbf{N}$. Then for any $t \in [0,1]$, t^δ defines a homomorphism $P \to \mathbf{R}^{\geq}$ and so is a point of $C(P)$. (Here we are using the convention that $0^0 = 1$.) Consider the continuous map

$$\tilde{\mathsf{A}}^{log}_{P,K} \times I \to \tilde{\mathsf{A}}^{log}_{P,K} : (x,t) \mapsto x_t \tag{3.1.1}$$

sending $(x,t) := ((\rho, y), t)$ to $x_t := (\rho^t t^\delta, ty)$. When $t = 1$, the map $x \mapsto x_t$ is the identity, and when $t = 0$, it is the constant map to the vertex, since $0^{\delta(p)}$ is 1 if $p \in P^*$ and 0 otherwise. If P is a group, then each element $c\colon P \to \mathbf{R}^{\geq}$ of $C(P)$ factors through $\mathbf{R}^+$, and so we can define $\tilde{c}\colon P \to \mathbf{R}$ to be $\log \circ c$. Then $C(P)$ can be identified with $\mathrm{Hom}(P, \mathbf{R})$ and $\tilde{X}^{log}$ with $\mathrm{Hom}(P, \mathbf{R}) \times \mathrm{Hom}(P, \mathbf{R}(1)) \cong \mathrm{Hom}(P, \mathbf{C}) = \mathbf{V}\Omega^{an}_P$. With this identification, ζ corresponds to exp. $\square$

The complement $\mathfrak{p}$ of each face F of P not meeting K is a prime ideal of P containing K and defines a closed log subscheme of $\mathsf{A}_{\mathsf{P},\mathsf{K}}$ whose underlying scheme is isomorphic to $\underline{\mathsf{A}}_{\mathsf{F}}$. Let $X_{\mathfrak{p}}$ or X_F denote this log scheme; in fact $X_F \cong \mathrm{Spec}(P \to \mathbf{C}[F])$, where $P \to \mathbf{C}[F]$ is the obvious one on F and kills $\mathfrak{p}$. If x is a point of the dense open subset $\underline{X}^*_F = \underline{\mathsf{A}}^*_{\mathsf{F}}$ of $\underline{X}_F$, the map $P \to \overline{M}_{X,x}$ induces an isomorphism $P/F \to \overline{M}_{X,x}$. Thus the family of faces F not meeting K defines a canonical stratification of $\mathsf{A}_{\mathsf{P},\mathsf{K}}$ on which the log structure is constant. We call this stratification, as well as the stratification it induces by pullback to X_{log} and its universal cover, the *canonical log stratification.* For each F, $\tau^{-1}(X_F^*)$ is the set of $(\rho,\sigma) \in C(P,K) \times S(P)$ such that $\rho^{-1}(\mathbf{R}^+) = F$. This space is homotopy equivalent to all of X_{log}, and the fiber over a point x is isomorphic to $S(P/F)$. In particular, $I_x \cong \mathrm{Hom}(P/F, \mathbf{Z}(1))$, and there is an exact sequence:

$$1 \to I_x \to \pi_1(X_F^{log}) \to \pi_1(X_{F,an}^*) \to 1.$$

Note that these strata are preserved by the contraction (3.1.1) above. That is, if $x \in \tilde{X}_F^{log*}$, then $x_t \in \tilde{X}_F^{log*}$ for all $t > 0$.

The following result may help explain the geometric significance of the construction of $\tau_X \colon X_{log} \to X_{an}$: it can be regarded as a compactification of the inclusion $X_{an}^* \to X_{an}$ which doesn't change its local homotopy type.

Theorem 3.1.2 *Suppose that $X/\mathbf{C}$ is a fine, smooth, and saturated log scheme (so $K_X = \emptyset$.) Then the map $j_{log} \colon X_{an}^* \to X_{log}$ is aspheric. That is, any point z of X_{log} has a basis of neighborhoods U such that $j_{log}^{-1}(U)$ is contractible. Consequently:*

1. *There are natural isomorphisms $\mathbf{Z} \cong Rj_{log*}\mathbf{Z}$, and $R\tau_*\mathbf{Z} \cong Rj_*\mathbf{Z}$.*
2. *If V is a locally constant abelian sheaf on X_{an}^*, then $j_{log*}V$ is locally constant on X_{log} and $R^i j_{log*}V = 0$ for $i > 0$.*

Proof: This question is local in a neighborhood of z in X_{log}, and hence also in a neighborhood of its image x in X. Since $X/\mathbf{C}$ is smooth, by [6, 3.5] there exists a toric monoid P and a strict étale map $f \colon X \to \mathsf{A}_\mathsf{P}$. Thus the theorem follows from the following lemma. □

Lemma 3.1.3 *Let K be a proper ideal in a toric monoid P, let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$ and let z be a point of X_{log} lying over a point x of X. Then z has a cofinal system of open neighborhoods U such that for each face F of P such that $x \in X_F$, the intersection of U with the stratum $\tau^{-1}(X_F^*)$ is contractible.*

Proof: If $z = (\rho, \sigma)$, then $G := \rho^{-1}(\mathbf{R}^+)$ is the face of P corresponding to the log stratum containing x. Then $x \in X_G^* \subseteq X_F$, $G \subseteq F$, and X_G^* and X_F^* are contained in X_{P_G}, where P_G is the localization of P by G. Thus without loss of generality we may replace P by P_G. Then x lies in the minimal orbit X_{P^*}. Since this orbit and its inverse image in X_{log} are homogeneous, we may as well assume that z is the vertex v of X_{log}.

Fix a splitting of $P \to \overline{P}$ and choose finite sets of generators S^+ for $\overline{P}$ and S^* for P^*. For each $\epsilon > 0$, let $C_\epsilon(P,K)$ be the set of $\rho \in C(P,K)$ such that $\rho(s) < \epsilon$ for $s \in S^+$ and $|\rho(s) - 1| < \epsilon$ for $s \in S^*$. Similarly, let $S_\epsilon(P)$ denote the set of $\sigma \in S(P)$ such that $|\sigma(s) - 1| < \epsilon$ for all $s \in S$, and let $U_\epsilon := C_\epsilon(P,K) \times S_\epsilon(P)$. Then the family of these U_ϵ for $\epsilon > 0$ is a basis for the set of neighborhoods of v. If F is a face of P not meeting K, the inverse image of X_F in X_{log} can be identified with $C(F) \times S(P)$. Since F is a face of P, $P^* = F^*$, the splitting $P \cong P^* \oplus \overline{P}$ induces a splitting $F \cong F^* \oplus \overline{F}$, and $S^+ \cap \overline{F}$ is a set of generators for $\overline{F}$. Then the intersection of $\tau^{-1}(X_F)$ with U_ϵ becomes

$C_\epsilon(\overline{F})\times C_\epsilon(F^*)\times S_\epsilon(P)$, where $C_\epsilon(\overline{F})$ is the set of $\rho\colon \overline{F} \to \mathbf{R}^{\geq}$ such that $\rho(s) < \epsilon$ for all $s \in \overline{F} \cap S^+$ and $C_\epsilon(F^*)$ is the set of homomorphisms $F^* \to \mathbf{R}^{\geq}$ such that $|\rho(s) - 1| < \epsilon$ for $s \in S^*$. Then $\tau^{-1}(X^*_F) \cap U_\epsilon$ is $C^*_\epsilon(\overline{F}) \times C_\epsilon(F^*) \times S_\epsilon(P)$, where $C^*_\epsilon(\overline{F})$ is the set of $\rho \in C_\epsilon(\overline{F})$ which factor through $\overline{F}^{gp}$. Thus $C^*_\epsilon(\overline{F})$ is contained in the set $C^*(\overline{F})$ of homomorphisms $\overline{F}^{gp} \to \mathbf{R}^+$. Choosing a basis $(f_1, \dots f_n)$ for the finitely generated free abelian group $\overline{F}^{gp}$ and using the topological isomorphism $\log\colon \mathbf{R}^+ \to \mathbf{R}$, we may identify $C^*(\overline{F})$ with the Euclidean space $E := \mathbf{R}^n$. Then each $s \in S^+ \cap \overline{F}$ can be written as a linear combination of the elements f_i and defines an element $\hat{s}$ in the dual of E. For each s, the set $E_s := \{e \in E : \hat{s}(e) < \log \epsilon\}$ is convex. Thus $C^*_\epsilon(\overline{F})$ becomes identified with the intersection of the convex subsets $E_s : s \in S^+ \cap \overline{F}$, which is therefore convex, hence contractible. Since $C_\epsilon(F^*)$ and $S_\epsilon(P)$ are evidently also contractible if $\epsilon < 1$, the same is true of $U_\epsilon \cap \tau^{-1}(X^*_F)$. $\square$

3.2 $\mathbf{C}_X^{log}$ and logarithmic local systems

We shall use the space X_{log} to globalize the local classification (2.1.2) of log connections, as explained in the introduction. Our first task is to give a more precise formulation of the global Riemann-Hilbert correspondence which takes into account the fact that the sheaf $\overline{M}_X$ is not constant. This will require the notion of cospecialization for certain constructible sheaves.

We begin by recalling the simple case of sheaves on intervals. Let $I = [0, 1]$ be the closed unit interval and let F be a sheaf on I. If F is constant, then for any connected open subset U of I, the restriction map $F(I) \to F(U)$ is an isomorphism. Hence for any $a, b \in I$, the maps $F(I) \to F_a$ and $F(I) \to F_b$ are isomorphisms, and so there is a canonical isomorphism $F_a \to F_b$. More generally, suppose only that the restriction of F to $(0, 1]$ is constant. Then if $a > 0$, the restriction mapping $F((0, 1]) \to F((0, a))$ is bijective. Since F is a sheaf, the sequence

$$F(I) \longrightarrow F((0,1]) \times F([0,a)) \rightrightarrows F((0,a))$$

is exact, and it follows that the map $F(I) \to F([0, a))$ is an isomorphism. Since this is true for all $a > 0$, the map $F(I) \to F_0$ is also an isomorphism. Hence there is a natural map

$$cosp_{0,b} : F_0 \xrightarrow{\rho_{I,0}^{-1}} F(I) \xrightarrow{\rho_{I,1}} F_b \tag{3.2.1}$$

for any $b \in [0, 1]$. Even more generally, suppose F is a sheaf on $[0, 1]$ and that for some $c \in (0, 1)$ the restrictions of F to $(0, c]$ and to $[c, 1]$ are locally constant. Then they are constant, and since $\{(0, c], [c, 1]\}$ is a locally finite closed cover of $(0, 1]$, it follows that the restriction of F to $(0, 1]$ is also constant. Hence for any $b \in [c,1]$ there is a commutative diagram

$$\begin{tikzcd} F_0 \arrow[r, "cosp_{0,c}"] \arrow[dr, "cosp_{0,b}"'] & F_c \arrow[d, "cosp_{c,b}"] \\ & F_b \end{tikzcd}$$

Now let x and y be points in a topological space X and let F be a sheaf on X. By an *F-path (resp. strict F-path) from x to y* we shall mean a continuous function $\gamma\colon I \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and such that the restriction of $\gamma^{-1}F$ to $(0,1]$ (resp. and such that $\gamma^{-1}(F)$) is locally constant. Then the above construction defines a canonical cospecialization map (resp. isomorphism)

$$\gamma^*_{x,y}\colon F_x \to F_y.$$

If γ is an F-path from x to y and γ' is a strict F-path from y to z, then the concatenation $\gamma\gamma'$ is an F-path from x to z, and $\gamma^*_{x,z} = {\gamma'}^*_{y,z} \circ \gamma^*_{x,y}$. If X is a log scheme and x and y are points of X or X_{log}, we shall simply say *log path* instead of $\overline{M}_X$-path.

We shall need a toric version of this cospecialization construction. Let K be a proper ideal in a toric monoid P, let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$, and let $\zeta\colon \tilde{X}^{log} \to X_{log}$ be the universal cover constructed in Proposition (3.1.1). For each face F of P not meeting K, let X^*_F denote the corresponding (locally closed) log stratum of X, and let $j_F\colon X^*_F \to X$ denote the inclusion. Thus j_F factors through the closure X_F of X^*_F in X. We denote by $\tilde{X}^*_F$ the inverse image of X^*_F in $\tilde{X}^{log}$, with similar notation for X^{log}_F and $\tilde{j}_F$. We say that a sheaf W on X_{log} or $\tilde{X}^{log}$ is *log constructible* if its restriction to each log stratum is locally constant.

Let W be a log constructible sheaf on $\tilde{X}^{log}$. Since the log strata are simply connected, the restriction of W to each log stratum $\tilde{X}^*_F$ is constant, and we let $W_F := W(\tilde{X}^*_F)$. Lemma (3.1.3) implies that each point of $\tilde{X}_F$ admits a neighborhood basis of open sets whose intersection with $\tilde{X}^*_F$ is connected (even contractible). It follows that $\tilde{j}_{F*}\tilde{j}^*_F W$ is canonically isomorphic to the constant sheaf W_F on $\tilde{X}^{log}_F$. If G is a face of F, the canonical map $\tilde{j}^*_G W \to \tilde{j}^*_G\tilde{j}_{F*}\tilde{j}^*_F W$ induces a map

$$cosp_{G,F}\colon W_G \to W_F.$$

Furthermore, there is a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \tilde{j}_{F*}\tilde{j}^*_F W \arrow[d] \\ \tilde{j}_{G*}\tilde{j}^*_G W \arrow[r] & \tilde{j}_{G*}\tilde{j}^*_G\tilde{j}_{F*}\tilde{j}^*_F W. \end{tikzcd}$$

Since $\tilde{j}_{F*}\tilde{j}_F^* W$ is the constant sheaf with value W_F, the same argument as before shows that the vertical arrow on the right is an isomorphism. If H is a face of G, we can pull back this diagram to $\tilde{X}_H^*$ and take global sections to obtain a commutative diagram

$$\begin{tikzcd} W_H \arrow[d, "cosp_{H,G}"'] \arrow[dr, "cosp_{H,F}"] & \\ W_G \arrow[r, "cosp_{G,F}"] & W_F. \end{tikzcd}$$

It is well-known and easy to check that W is determined completely by the family of sets W_F and cospecialization maps. Indeed, the functor which takes a log constructible sheaf on $\tilde{X}^{log}$ to the corresponding family of sets and maps is easily seen to be an equivalence. We should perhaps remark that it is not difficult to show that if $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$ and W is a log constructible sheaf on $\tilde{X}^{log}$, then the natural map from $W(\tilde{X}^{log})$ to the stalk of W at the vertex is an isomorphism. This fact can be used to give another interpretation of the cospecialization maps.

Let V be a log constructible sheaf on X_{log} and let $\tilde{V}$ be its pullback to $\tilde{X}^{log}$. Then each $\tilde{V}_F$ is equipped with a natural action of $\pi_1(P)$, and the cospecialization maps are compatible with this action. In this way one obtains an equivalence between the category of log constructible sheaves on X_{log} and the category of families of $\pi_1(P)$-sets and compatible cospecialization maps. Let x and y be points of X_{log} and choose points $\tilde{x}$ and $\tilde{y}$ of $\tilde{X}$ lying over them. Let $F(x)$ (resp. $F(y)$) be the face of P corresponding to the log stratum containing x (resp. y). Then if $F(x) \subseteq F(y)$,

$$cosp_{\tilde{x},\tilde{y}} \colon V_x \longrightarrow V_y$$

is by definition the map such that the diagram

$$\begin{tikzcd} \tilde{V}_{F(x)} \arrow[r, "cosp_{F(x),F(y)}"] \arrow[d] & \tilde{V}_{F(y)} \arrow[d] \\ V_x \arrow[r, "cosp_{\tilde{x},\tilde{y}}"] & V_y \end{tikzcd}$$

commutes. Here the left vertical arrow is the composite $\tilde{V}_{F(x)} \cong \tilde{V}(\tilde{X}_F^*) \cong \tilde{V}_{\tilde{x}} \cong V_x$, and the right one is defined similarly. Note that the map $cosp_{\tilde{x},\tilde{y}}$ depends on the choices of $\tilde{x}$ and $\tilde{y}$ lying over x and y, and that $cosp_{\tilde{x},\tilde{y}}$ is an isomorphism if x and y lie in the same log stratum. Note also that if $\tilde{z}$ is a point of $\tilde{X}$ such that $F(y) \subseteq F(z)$, then there is a commutative diagram

$$\begin{tikzcd} V_x \arrow[d, "cosp_{\tilde{x},\tilde{y}}"'] \arrow[dr, "cosp_{\tilde{x},\tilde{z}}"] & \\ V_y \arrow[r, "cosp_{\tilde{y},\tilde{z}}"] & V_z \end{tikzcd}$$

Remark 3.2.1 Let V be a log constructible sheaf on $X_{log} := \mathsf{A}^{log}_{P,K}$ and let γ be a continuous map from the unit interval I to X_{log} such that the image of $(0,1]$ is contained in a single log stratum. Choose a point $\tilde{x}$ of $\tilde{X}$ lying over $x := \gamma(0)$, let $\tilde{\gamma}\colon I \to \tilde{X}$ be the lift of γ such that $\tilde{\gamma}(0) = \tilde{x}$, and let $\tilde{y} := \tilde{\gamma}(1)$. Then the following diagram commutes:

$$\begin{tikzcd} \gamma^{-1}V_0 \arrow[r, "cosp_{0,1}"] \arrow[d] & \gamma^{-1}V_1 \arrow[d] \\ V_{\tilde{x}} \arrow[r, "cosp_{\tilde{x},\tilde{y}}"] & V_{\tilde{y}} \end{tikzcd}$$

The notions of log stratification and log constructibility make sense more generally, at least locally. Let X be an ideally smooth fs log scheme over $\mathbf{C}$, let x be a $\mathbf{C}$-valued point of X, and consider the map $\alpha_{X,x}\colon M_{X,\overline{x}} \to \mathcal{O}_{X,\overline{x}}$. (We are temporarily writing $\overline{x}$ to remind ourselves that we are taking the stalks in the étale topology.) A point of the spectrum $\mathcal{O}_{X,\overline{x}}$ is said to be a *log branch at* x if it is the inverse image of a prime ideal in the monoid $M_{X,\overline{x}}$. Since X is, locally in some étale neighborhood of x, isomorphic to $\mathsf{A}_{\mathsf{P},\mathsf{K}}$ for some fs monoid P and some ideal K in P, there is a bijection between the set of log branches at x and the set of prime ideals of $M_{X,\overline{x}}$ containing $K_{X,x}$. Each log branch at x defines an irreducible and unibranch closed subscheme Z in some étale neighborhood of x, and the restriction of $\overline{M}_X$ and $\overline{K}_X$ to a dense open subset Z^o of Z is constant. We shall call a maximal such Z^o a *log stratum at* x. We use the same terminology for the inverse images of these sets in X_{log}.

Corollary 3.2.2 *Let X be an ideally smooth fs log scheme and let x be a point of X_{log}. Then x has an étale neighborhood U such that for every point y of U, there exists a log path from x to y.*

Proof: Without loss of generality we may assume that $X = \mathsf{A}_{\mathsf{P},\mathsf{K}}$, where P is a fine monoid and K is an ideal of P, and that x is the vertex of X_{log}. We may work in $\tilde{X}_{log}$ instead of X_{log}. Then the result follows from (3.1.1) and the discussion which follows it. $\square$

Definition 3.2.3 *Let X be an idealized log scheme and let V be a sheaf on X_{log}. Then V is said to be* log constructible *if for each $x \in X_{log}$, V is locally constant on the log strata at x.*

Observe that this condition is local in the analytic topology of X. That is, if V is a sheaf on X_{log} and X_{an} admits a cover by open sets U such that the restriction of V to each $\tau^{-1}(U)$ is log constructible, then V is log constructible. The sheaves $\overline{M}_X$ and $\overline{K}_X$ are always log constructible. Let V be a log constructible sheaf on X_{log}, let x and y be points of X_{log}, and let γ be a log path from x to y. Then X admits an étale open cover $\{U_\lambda\}$ which admits charts as above, and the restriction of γ to each $\gamma^{-1}(U_\lambda) \cap (0,1]$ factors through a log stratum. It follows that the restriction of $\gamma^{-1}V$ to $(0,1]$ is locally constant, so γ defines a cospecialization map

$$\gamma^*_{x,y} \colon V_x \to V_y.$$

In particular, V is locally constant on the fiber $\tau^{-1}(x)$ of each point x of X, and hence if $z \in \tau^{-1}(x)$, V_z has a natural action ρ of the fundamental group I_x of $\tau^{-1}(x)$.

By a *sheaf of exponential data* for X we mean a log constructible sheaf of subgroups $\Lambda \subseteq \mathbf{C} \otimes \overline{M}^{gp}_X$ containing $\overline{M}^{gp}_X$. In practice, it will suffice to take $\Lambda := \mathbf{C} \otimes \overline{M}^{gp}_X$, but for some purposes it might be preferable to use $\mathbf{Q} \otimes \overline{M}^{gp}_X$ or $\overline{M}^{gp}_X$. We also write Λ for $\tau^{-1}\Lambda$ to simplify the notation. Let $\mathbf{C}^{log}_X$ denote the pullback to X_{log} of the quotient of the sheaf of monoid algebras $\mathbf{C}[-\overline{M}_X]$ by the ideal generated by $-K_X$. This sheaf is also log constructible. The inclusion $-\overline{M}_X \to \Lambda$ defines an action of $-\overline{M}_X$ on Λ, so that one has a notion of a sheaf of Λ-graded $\mathbf{C}^{log}_X$-modules.

Definition 3.2.4 *Let $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ denote the category of Λ-graded sheaves V of $\mathbf{C}^{log}_X$-modules on X_{log} satisfying the following conditions:*

1. *V is log constructible.*
2. *For each $z \in X_{log}$, the stalk V_z of V at z is finitely generated over $\mathbf{C}^{log}_{X,z}$.*
3. *If x and y are points of X_{log} and γ is any log path from x to y, then the cospecialization map*

$$\gamma^*_{x,y} \colon V_x \otimes_{\mathbf{C}^{log}_{X,x}} \mathbf{C}^{log}_{X,y} \to V_y$$

is an isomorphism.

4. *If $z \in X_{log}$, $\gamma \in I_{\tau(z)}$, and $\lambda \in \Lambda_z$, then $\exp\langle\gamma,\lambda\rangle$ is the only eigenvalue of the action of ρ_γ on $V_{\lambda,z}$, i.e., $\rho_\gamma - \exp\langle\gamma,\lambda\rangle \colon V_{z,\lambda} \to V_{z,\lambda}$ is nilpotent.*

We shall say that a sheaf of Λ-graded $\mathbf{C}^{log}_X$-modules is *coherent* if it satisfies the above conditions. These perhaps need some explanation. Let $x = \tau(z)$, and note that in (4) of the above definition, $\gamma \in I_x = \mathrm{Hom}(\overline{M}^{gp}_{X,x}, \mathbf{Z}(1))$ and $\lambda \in \mathbf{C} \otimes \overline{M}^{gp}_{X,x}$, so that $\langle \gamma, \lambda \rangle \in \mathbf{C}$ makes sense. Moreover, if $m \in -\overline{M}_{X,x}$, $\exp\langle \gamma, m \rangle = 1$, so (4) is compatible with multiplication by elements of $\mathbf{C}^{log}_X$. Note also that (4) implies that the action of I_x on $V_{z,\lambda}$ is unipotent if $\lambda \in \overline{M}^{gp}_{X,x}$, and that (2) implies that each graded piece $V_{z,\lambda}$ is a finite dimensional $\mathbf{C}$-vector space. Using the compatibility of cospecialization with concatenation of log paths, one can easily check that condition (3), like the others, is local on X_{an}. Thus the category $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ is of local nature on X_{an}. Suppose V is log constructible, that X admits a toric chart above, and that γ is a log path from x to y for which (3) holds. Then it follows from the toric interpretation of the cospecialization map that if γ' is a log path from x to any point y' in the log stratum of y, ${\gamma'}^*_{x,y'}$ is also an isomorphism. Furthermore if a morphism $V \to V'$ in $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ induces an isomorphism on the stalks at some point z of X_{log}, then one sees easily from (3) and Corollary (3.2.2) that it induces an isomorphism in some neighborhood of z. Thus objects in $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ can be thought of as analogs of locally constant sheaves—of course, when the log structure is trivial, they are indeed locally constant.

Let us describe the category $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ explicitly when $X = \mathsf{A_P}$ for a toric monoid P endowed with a rigid set of exponential data $\Lambda \subseteq \mathbf{C} \otimes P^{gp}$. For each face F of P, the image Λ_F of Λ in $(\mathbf{C} \otimes P^{gp})/(\mathbf{C} \otimes F^{gp})$ defines a set of exponential data for P/F, and an inclusion $F \subseteq G$ induces a cospecialization map $\Lambda_F \to \Lambda_G$. We thus obtain a sheaf of exponential data on $\mathsf{A_P}$, also denoted by Λ. Let M be a $\overline{\Lambda}$-graded $\mathbf{C}[\overline{P}]$-module endowed with an action of $\pi_1(P)$ which satisfies the coherence conditions of (1.4.1), *i.e.*, an object of $\overline{L}^{\Lambda}_{coh}(P)$ (see (1.4.2)). For each face F of P, let $M_F := \mathbf{C}[P/F] \otimes_{\mathbf{C}[\overline{P}]} M$, with its natural structure of a Λ_F-grading and action of $\pi_1(P)$, and recall from (1.4.2) that $M \mapsto M_F$ is an equivalence when $F = P^*$. Since this construction is compatible with further dividing by faces, the family $\{M_F : F\}$ defines an object of $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$. Conversely, if V is an object of $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$, the restriction of V to $X^{log}_{P^*}$ is locally constant, and the evident maps

$$\Gamma(\tilde{X}^{log}_{P^*}, \tilde{V}) \longrightarrow \tilde{V}_{\tilde{v}} \longleftarrow V_v$$

are isomorphisms, where $v \in X_{log}$ and $\tilde{v} \in \tilde{X}^{log}$ are the vertices. The automorphism group of the covering $\tilde{X}^{log}_{P^*} \to X^{log}_{P^*}$ is $\pi_1(P)$, and it acts naturally on $\Gamma(\tilde{X}^{log}_{P^*}, \tilde{V}) \cong V_v$. Thus $\Gamma(\tilde{X}^{log}_{P^*}, \tilde{V})$ is an object of $\overline{L}^{\Lambda}_{coh}(P)$. This establishes the following equivalence, and the compatibilities which go along with it should be clear.

Proposition 3.2.5 *Let $X := \mathsf{A_P}$, where P is a toric monoid with rigid exponential data $\Lambda \subseteq \mathbf{C} \otimes P^{gp}$.*

1. *The functor*

$$V \mapsto \Gamma(\tilde{X}^{log}_{P^*}, \tilde{V}) \cong \tilde{V}_{\tilde{v}}$$

is an equivalence from the category $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ to the category $\overline{L}^{\Lambda}_{coh}(P)$. A quasi-inverse is the functor taking an object M of $\overline{L}^{\Lambda}_{coh}(P)$ to the sheaf corresponding to the family $\{M_F := \mathbf{C}[P/F] \otimes M\}$ described above.

2. *If $(\phi, \psi)\colon (P, \Lambda_P) \to (Q, \Lambda_Q)$ is a homomorphism of toric monoids and exponential data and $Y := \mathsf{A}_{\mathsf{Q}}$, the diagram of functors*

$$\begin{array}{ccc} \overline{L}^{\Lambda}_{coh}(P) & \xrightarrow{\phi^*_\psi} & \overline{L}^{\Lambda}_{coh}(Q) \\ \downarrow & & \downarrow \\ L^{\Lambda}_{coh}(\mathbf{C}^{log}_X) & \xrightarrow{f^*_{log}} & L^{\Lambda}_{coh}(\mathbf{C}^{log}_Y) \end{array}$$

is 2-commutative.

3. *If V is an object of $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ and F is a face of P, then the cospecialization map $cosp_{P^*,F}\colon \tilde{V}_{P^*} \to \tilde{V}_F$ identifies $\tilde{V}_F$ with the tensor product $\tilde{V}_P \otimes_{\mathbf{C}[\overline{P}]} \mathbf{C}[P/F]$.*

□

3.3 The ring $\tilde{\mathcal{O}}^{log}_X$

To globalize the constructions of (1.4.8) and (2.1.2), we shall construct a sheaf of rings $\tilde{\mathcal{O}}^{log}_X$ on X_{log}, combining the constructions in [7] and [8]. Let us begin by reviewing the first of these.

If Y and X are topological spaces, let Y_X denote the sheaf which to every open set U of X assigns the set of continuous functions $U \to Y$. Recall from [7] that there is a commutative diagram with exact rows, in which the squares on the right are Cartesian:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \tau^{-1}(\mathcal{O}_X) & \xrightarrow{\exp} & \tau^{-1}\mathcal{O}^*_X & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \epsilon} & & \Big\downarrow{\scriptstyle \lambda} & & \\ 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathcal{L} & \xrightarrow{\pi} & \tau^{-1}M^{gp}_X & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \tilde{h}} & & \Big\downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathbf{R}(1)_{X_{log}} & \xrightarrow{\exp} & \mathbf{S}^1_{X_{log}} & \longrightarrow & 0 \end{array} \qquad (3.3.1)$$

To understand this diagram, recall that a point of X_{log} lying over a point x of X is a homomorphism of monoids $\sigma\colon M_{X,x} \to \mathbf{S}^1$ such that

$$\sigma(m)|\alpha_X(m)(x)| = \alpha_X(m)(x)$$

for every $m \in M_{X,x}$. If m is a local section of M_X, then the map sending such a σ to $\sigma(m_x)$ is a continuous function of σ, and so defines a section $h(m)$ of $\mathbf{S}^1_{X_{log}}$. This defines the homomorphism h in the diagram, and by definition, $\mathcal{L}$ is the fiber product of $\tau^{-1}M_X^{gp}$ and $\mathbf{R}(1)_{X_{log}}$ over $\mathbf{S}^1_{X_{log}}$. This defines the bottom two rows of the diagram, and the top row is just the pullback of the exponential exact sequence on X_{an}. The map $h \circ \lambda$ is arg, (*i.e.*, the map $u \mapsto |u|^{-1}u$). Let $Im\colon \tau^{-1}\mathcal{O}_X \to \mathbf{R}(1)_X$ be the map taking a function to its imaginary part. Then $\exp \circ Im = \arg \circ \exp$, and so there is a unique map ϵ with $\tilde{h}\epsilon = Im$ making the diagram commute. Since the big right rectangle is Cartesian, so is the upper square. Chasing the diagram shows that there is an exact sequence

$$0 \to \tau^{-1}\mathcal{O}_X \xrightarrow{\epsilon} \mathcal{L} \to \tau^{-1}\overline{M}_X^{gp} \longrightarrow 0. \tag{3.3.2}$$

By definition, $\mathcal{O}_X^{log}$ is the universal $\tau^{-1}\mathcal{O}_X$-algebra equipped with a map $\mathcal{L} \to \mathcal{O}_X^{log}$ such that the diagram

$$\begin{tikzcd} \tau^{-1}\mathcal{O}_X \arrow[r, "\epsilon"] \arrow[dr] & \mathcal{L} \arrow[d] \\ & \mathcal{O}_X^{log} \end{tikzcd}$$

commutes.

The ring $\mathcal{O}_X^{log}$ is adequate to deal with connections whose exponents vanish. In order to deal with the general case we adopt a construction of Lorenzon [8]. Recall that the exact sequence

$$0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$$

defines a family of $\mathcal{O}_X^*$-torsors, hence invertible sheaves, indexed by $\overline{M}_X^{gp}$. If a and b are local sections of $\overline{M}_X^{gp}$, there is a map of the corresponding invertible sheaves $L_a \otimes L_b \to L_{a+b}$, and one obtains using these maps an $\overline{M}_X^{gp}$-indexed or graded $\mathcal{O}_X$-algebra $\mathcal{A}_X^{gp} := \oplus L_a$. The ring $\mathcal{O}_X^{log} \otimes \mathcal{A}_X^{gp}$ is sufficient to classify objects of $MIC^{\Lambda}_{coh}(X/\mathbf{C})$ when $\Lambda = \overline{M}_X^{gp}$, and the corresponding local systems have unipotent logarithmic monodromy. For the general case, we need to enlarge $\mathcal{A}_X^{gp}$ even more.

Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{C} \otimes \tau^{-1}\mathcal{O}_X \arrow[r] \arrow[d, "\mu"'] & \mathbf{C} \otimes \mathcal{L} \arrow[r] \arrow[d, "\tilde{\mu}"] & \mathbf{C} \otimes \tau^{-1}\overline{M}_X^{gp} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathcal{O}_X^* \arrow[r] & \tilde{M}_X \arrow[r, "\tilde{\pi}"] & \mathbf{C} \otimes \tau^{-1}\overline{M}_X^{gp} \arrow[r] & 0. \end{tikzcd} \tag{3.3.3}$$

Here the top row is obtained by tensoring the sequence (3.3.2) with $\mathbf{C}$, and the bottom row is just the pushout by the map μ sending $a \otimes f$ to $\exp(af)$. Finally let

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow M_X^\Lambda \longrightarrow \Lambda \longrightarrow 0 \tag{3.3.4}$$

be the pullback of the bottom row of the above diagram by the map $\Lambda \subseteq \mathbf{C} \otimes \tau^{-1} M_X^{gp}$. If it seems to be unnecessary to specify the exponential data we write M_X^{log} instead of M_X^Λ. It follows from the exactness of the middle row of diagram (3.3.1) that there is an injection $\tau^{-1} M_X^{gp} \to M_X^\Lambda$ which agrees with $\tilde{\mu}$ when composed with π.

Let $\mathcal{A}_X^\Lambda$ (or $\mathcal{A}_X^{log}$) denote the Λ-graded $\mathcal{O}_X$-algebra corresponding to the exact sequence (3.3.4).

Proposition 3.3.1 *Let $d_\mathcal{L} \colon \mathcal{L} \to \tau^{-1}\Omega^1_{X/\mathbf{C}}$ denote the composition of the map $\pi \colon \mathcal{L} \to \tau^{-1} M_X^{gp}$ with $dlog \colon \tau^{-1} M_X^{gp} \to \tau^{-1}\Omega^1_{X/\mathbf{C}}$.*

1. *If f is a section of $\tau^{-1}(\mathcal{O}_X)$, then $d_\mathcal{L}\epsilon(f) = df$ in $\tau^{-1}(\Omega^1_{X/\mathbf{C}})$.*

2. *There is a unique homomorphism $\tilde{M}_X \xrightarrow{dlog} \tau^{-1}\Omega^1_{X/\mathbf{C}}$ such that*

$$dlog\tilde{\mu}(a \otimes \ell) = a d_\mathcal{L}(\ell)$$

for every section ℓ of $\mathcal{L}$ and every $a \in \mathbf{C}$.

3. *There is a unique additive and homogeneous homomorphism*

$$\nabla \colon \mathcal{A}_X^{log} \to \mathcal{A}_X^{log} \otimes \tau^{-1}\Omega^1_{X/\mathbf{C}},$$

satisfying the Leibniz rule with respect to $\tau^{-1}\mathcal{O}_X$ and such that if x_m is the section of $\mathcal{A}_X^{log}$ corresponding to a section m of M_X^{log},

$$\nabla x_m = x_m \otimes dlog(m).$$

4. *There is a natural map of Λ-graded rings $\iota \colon \mathbf{C}_X^{log} \to \mathcal{A}_X^{log}$, whose image is annihilated by d.*

Proof: By definition, $d_\mathcal{L}\epsilon(f) = dlog\pi\epsilon(f) = dlog\lambda exp(f) = df$, as asserted in (1). Let $\eta \colon \mathbf{C} \otimes \tau^{-1}\Omega^1_{X/\mathbf{C}} \to \tau^{-1}\Omega^1_{X/\mathbf{C}}$ be multiplication. If $a_i \in \mathbf{C}$ and $f_i \in \tau^{-1}(\mathcal{O}_X)$ for $i = 1 \ldots n$, it follows that

$$\eta \circ (\mathrm{id} \otimes d_\mathcal{L}) \circ (\mathrm{id} \otimes \epsilon)(\sum a_i \otimes f_i) = \sum a_i df_i = d \sum a_i f_i.$$

In particular, this is zero if $\sum a_i f_i$ is locally constant. The kernel of the map $\mu \colon \mathbf{C} \otimes \mathcal{O}_X \to \mathcal{O}_X^*$ is generated by the set of sums $\sum a_i \otimes f_i$ such that $\sum a_i f_i \in \mathbf{Z}(1)$, and in particular any such sum is killed by $\eta \circ (\mathrm{id} \otimes d_\mathcal{L}) \circ (\mathrm{id} \otimes \epsilon)$. Since $\tilde{M}_X$ is the quotient of $\mathbf{C} \otimes \mathcal{L}$ by the image of this kernel, there exists a unique $dlog$ as in (2).

To verify the existence of ∇, suppose that $\lambda \in \Lambda$, and let $\mathcal{A}^{log}_{X,\lambda}$ be the degree λ part of $\mathcal{A}^{log}_X$. This is an invertible $\tau^{-1}\mathcal{O}_X$-module, and if $m \in M^{log}_X$ maps to λ, there is a corresponding basis x_m of $\mathcal{A}^{log}_{X,\lambda}$. Then there is a unique

$$\nabla\colon \mathcal{A}^{log}_{X,\lambda} \to \mathcal{A}^{log}_{X,\lambda} \otimes \tau^{-1}\Omega^1_{X/\mathbf{C}}$$

satisfying the Leibniz rule such that $\nabla x_m = x_m \otimes dlog(m)$. We must verify that ∇ is independent of the choice of m. If m' is another section of M^{log}_X mapping to λ, then $m = um'$ for some $u \in \mathcal{O}^*_X$. Let ∇' be defined using m' in place of m. Then $x_m = ux_{m'}$ and $dlog(m) = dlogu + dlogm'$. Hence

$$\begin{aligned}
\nabla' x_m &= \nabla'(ux_{m'})\\
&= du \otimes x_{m'} + u\nabla' x_{m'}\\
&= u^{-1}du \otimes x_m + udlog(m') \otimes x_{m'}\\
&= dlogu \otimes x_m + dlog(m') \otimes x_m\\
&= dlogm \otimes x_m\\
&= \nabla(x_m),
\end{aligned}$$

as required.

Let us continue to write the monoid law of M_X multiplicatively and that of $\overline{M}_X$ additively. A section m of M^{-1}_X defines an element m^{-1} of M_X; let $\iota(m) := \alpha(m^{-1})x_m \in \mathcal{A}^{log}_X$. If $u \in \mathcal{O}^*_X$ and $m' = um$, then

$$\alpha({m'}^{-1})x_{m'} = u^{-1}\alpha(m^{-1})ux_m = \iota(m).$$

Thus $\iota(m)$ depends only on the image $\overline{m}$ of m in $\overline{M}^{-1}_X$, and we write $\iota(\overline{m})$ instead of $\iota(m)$. Then ι defines a homomorphism of graded rings $\mathbf{C}[-\overline{M}_X] \to \mathcal{A}^{log}_X$ sending $e_{\overline{m}}$ to $\iota(\overline{m})$. Since $\alpha(m^{-1})$ vanishes if and only if $m \in K_X$, ι factors through an injective homomorphism $\mathbf{C}^{log}_X := \mathbf{C}[-\overline{M}_X]/\mathbf{C}[-\overline{K}_X]$, which we also denote by ι. Furthermore,

$$\begin{aligned}
d\iota(m) &= \nabla(\alpha(m^{-1})x_m)\\
&= d\alpha(m^{-1})x_m + \alpha(m^{-1})\nabla x_m\\
&= \alpha(m^{-1})dlog(m^{-1})x_m + \alpha(m^{-1})dlog(m)x_m\\
&= -\alpha(m^{-1}dlog(m)x_m + \alpha(m^{-1})dlog(m)x_m\\
&= 0
\end{aligned}$$

□

Definition 3.3.2 *Let $X/\mathbf{C}$ be a fine saturated idealized log scheme with a sheaf of exponential data $\Lambda \subseteq \mathbf{C} \otimes \overline{M}^{gp}_X$. Then $\tilde{\mathcal{O}}^{log}_X$ is the Λ-graded $\tau^{-1}\mathcal{O}_X$-algebra $\mathcal{A}^{log}_X \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}^{log}_X$, and*

$$d\colon \tilde{\mathcal{O}}^{log}_X \to \tilde{\Omega}^{1,log}_{X/\mathbf{C}} := \tilde{\mathcal{O}}^{log}_X \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}\Omega^1_{X/\mathbf{C}}$$

is the map defined by the usual rule for the tensor product connection, using the connections defined above on $\mathcal{A}_X^{log}$ and $\mathcal{O}_X^{log}$.

Remark 3.3.3 Let $f\colon X \to Y$ be a morphism of fs idealized log schemes which is compatible, in the obvious sense, with sheaves of exponential data Λ_X and Λ_Y. Then there is a commutative diagram of ringed spaces

$$\begin{tikzcd} X_{log} \arrow[r, "f_{log}"] \arrow[d, "\tau"'] & Y_{log} \arrow[d, "\tau"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

which is Cartesian if f is strict [7]. It is straightforward to verify that f induces a map

$$\tau^{-1}\mathcal{O}_X \otimes_{(f\tau)^{-1}\mathcal{O}_Y} f_{log}^{-1}A \to A_X,$$

where A is $\mathcal{O}^{log}$, $\mathcal{A}^{log}$, or $\tilde{\mathcal{O}}^{log}$, compatible with the connections. If f is strict and the map $f^{-1}\Lambda_Y \to \Lambda_X$ is an isomorphism, then the above map is also an isomorphism.

We shall need an explicit description of the ring $\tilde{\mathcal{O}}_X^{log}$ when X is the log scheme associated to a toric monoid P. Let

$$\zeta\colon \tilde{X}^{log} := C(P) \times Y(P) \to X_{log}$$

be the universal covering constructed in (3.1.1). If $p \in P$ and $\tilde{x} = (\rho, y) \in \tilde{X}^{log}$, let $\hat{p}(\tilde{x}) := y(p) \in \mathbf{R}(1)$. Then $\hat{p}$ is a continuous function from $\tilde{X}^{log}$ to $\mathbf{R}(1)$, *i.e.*, a global section of $\mathbf{R}(1)_{\tilde{X}^{log}}$. The element p also defines a global section $\beta(p)$ of M_X, and in the diagram (3.3.1) pulled back to $\tilde{X}^{log}$, $h\beta(p) = \exp\hat{p}$. Thus $\tilde{\beta}(p) := (\hat{p}, \beta(p))$ is a global section of $\zeta^{-1}\mathcal{L}$, and $\tilde{\beta}$ is a map $P \to \zeta^{-1}\mathcal{L}$. We shall abuse notation and write $\mathcal{O}_{\tilde{X}}$ for the sheaf $\zeta^{-1}\tau^{-1}\mathcal{O}_{X_{an}}$.

Lemma 3.3.4 *Let $X := \mathsf{A_P}$ and let $\zeta\colon \tilde{X}^{log} \to X_{log}$ be the universal covering.*

1. *The map $\tilde{\beta}$ described above fits into a cocartesian diagram:*

$$\begin{tikzcd} \beta^{-1}(\mathcal{O}_X^*) \arrow[r, "\tilde{\lambda}"] \arrow[d] & \mathcal{O}_{\tilde{X}} \arrow[d, "\epsilon"] \\ P^{gp} \arrow[r, "\tilde{\beta}"] & \zeta^{-1}\mathcal{L}. \end{tikzcd}$$

2. *Let ρ be the action of $\pi_1(P) = Aut(\tilde{X}^{log}/X_{log})$ on $\mathcal{O}_{\tilde{X}}$ and $\zeta^{-1}\mathcal{L}$ by transport of structure. Then for each $p \in P^{gp}$, $\rho_\gamma(\tilde{\beta}(p)) = \tilde{\beta}(p) + \langle\gamma, p\rangle$. In particular, if z is a point in X_{log} then the action of $I_{\tau(x)}$ on $\mathcal{L}_z$ is given by*
$$\rho_\gamma(\ell) = \ell + \langle\gamma, \pi\ell\rangle$$
for any $\ell \in \mathcal{L}_z$ and $\gamma \in I_{\tau(x)}$.

3. *Let $\tilde{I}$ be the sheaf of ideals in the algebra $\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp})$ generated by all elements of the form $\tilde{\lambda}(p) \otimes 1 - 1 \otimes p$ for p a local section of $\beta^{-1}\mathcal{O}^*_X$. Then the map $\tilde{\beta}$ induces an isomorphism of $\mathcal{O}_{\tilde{X}}$ -algebras*
$$(\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}))/\tilde{I} \to \zeta^{-1}\mathcal{O}^{log}_X.$$

Proof: In the diagram, $\beta^{-1}(\mathcal{O}^*_X)$ means the subsheaf of the constant sheaf P consisting of those elements of P which become units in M_X, open set by open set. Let p be a section of this sheaf on some open set $U \subseteq X_{an}$ and let $m := \beta(p) \in M^*_X(U)$ and $u := \alpha_X(m) \in \mathcal{O}^*_X(U)$. Then $\log|u| \in \mathbf{R}_X(U)$, $\hat{p} \in \mathbf{R}(1)(\tau\zeta)^{-1}(U)$, and $\tilde{\lambda}(p) := (\log|u|, \hat{p})$ is a section of $\zeta^{-1}(\mathcal{O}_X)$ such that $\exp\tilde{\lambda}(p) = u$. Then the diagram in (1) commutes. The fact that it is cocartesian follows from the exact sequence (3.3.2).

Recall that the action of $\pi_1(P)$ on $\tilde{X}^{log}$ is the action induced by translation and its inclusion as a subgroup. Thus if f is a function on $\tilde{X}^{log}$, $\rho_\gamma(f(\tilde{x})) = f(\tilde{x}) + f(\gamma\tilde{x})$ for each $\tilde{x} \in \tilde{X}^{log}$. Hence $\gamma^*(\hat{p}) = \hat{p} + \langle\gamma, p\rangle$ and $\gamma^*(\tilde{\beta}(p)) = \tilde{\beta}(p) + \langle\gamma, p\rangle$, and if $q \in \beta^{-1}(\mathcal{O}^*_X)$, $\rho_\gamma(\tilde{\lambda}(q)) = \tilde{\lambda}(q) + \langle\gamma, q\rangle$. This proves the formula for the action of ρ on $\zeta^{-1}\mathcal{L}$. Note that if $\gamma \in I_x$, then $\gamma^*(\tilde{\beta}(p)) - \tilde{\beta}(p)$ depends only on the image $\overline{\beta}(p)$ of p in $\overline{M}_X$. Let $\ell := \tilde{\beta}(p)$, and note that $\overline{\beta}(p) = \pi(\ell)$. This proves the formula for the action of I_x on $\mathcal{L}_z$, since the map $P \to \overline{M}^{gp}_X \cong \mathcal{L}_z/\mathcal{O}_{X,x}$ is surjective.

The map $\tilde{\beta}$ followed by the inclusion is a homomorphism $P^{gp} \to \zeta^{-1}\mathcal{O}^{log}_X$, and by the universal property of the symmetric algebra, this map extends uniquely to a homomorphism of algebras $\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}) \to \zeta^{-1}\mathcal{O}^{log}_X$. For any local section q of $\beta^{-1}\mathcal{O}^*_X$, the commutativity of the square in (1) and the triangle (3.3) imply that $1 \otimes p$ and $\tilde{\lambda}(p) \otimes 1$ have the same image in $\zeta^{-1}\mathcal{O}^{log}_X$, so that this homomorphism annihilates $\tilde{I}$. On the other hand, the map
$$\mathcal{O}_{\tilde{X}} \oplus P^{gp} \to (\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}))/\tilde{I}$$
sending (f, p) to $f \otimes 1 + 1 \otimes p$ factors through $\zeta^{-1}\mathcal{L}$, since the square in (1) is cocartesian and since for any $q \in \beta^{-1}\mathcal{O}^*_X$ the elements $1 \otimes [q]$ and $\tilde{\lambda}(q) \otimes 1$ have the same image in $(\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}))/\tilde{I}$. By the universal property of $\mathcal{O}^{log}_X$, these maps extend uniquely to a map $\zeta^{-1}\mathcal{O}^{log}_X \to (\mathcal{O}_{\tilde{X}} \otimes S^{\cdot}(P^{gp}))/\tilde{I}$, which is the inverse to the map in (3). ☐

Proposition 3.3.5 *Let P be a toric monoid with exponential data $\Lambda \subseteq \mathbf{C} \otimes P^{gp}$ and a proper ideal $K \subseteq P$, and let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$.*

1. *Then there are natural maps, compatible with the connections and gradings, and actions of* $\pi_1(P)$*:*

$$\begin{array}{rcl}
\Gamma(\tilde{X}^{log}, \mathcal{O}_{\tilde{X}}) \oplus P^{gp} & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{L}) \\
\Gamma(\tilde{X}^{log}, \mathcal{O}^*_{\tilde{X}}) \oplus \Lambda & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\tilde{M}^{\Lambda}_X) \\
\Gamma(\tilde{X}^{log}, \mathcal{O}_{\tilde{X}}) \otimes \mathbf{Z}[\Lambda] & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{A}^{log}_X) \\
\Gamma(\tilde{X}^{log}, \mathcal{O}_{\tilde{X}}) \otimes \Gamma^{\cdot}[P^{gp}] & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{O}^{log}_X) \\
\Gamma(\tilde{X}^{log}, \mathcal{O}_{\tilde{X}}) \otimes_{\mathbf{C}[P]} J(P,\Lambda) & \to & \Gamma(\tilde{X}^{log}, \zeta^{-1}\tilde{\mathcal{O}}^{log}_X).
\end{array}$$

2. *Suppose that* P *is sharp and let* z *(resp.* v*) be the vertex of* X_{log} *(resp.* X*). Then these maps induce isomorphisms on stalks:*

$$\begin{array}{rcl}
\mathcal{O}_{X,v} \oplus P^{gp} & \to & \mathcal{L}_z \\
\mathcal{O}^*_{X,v} \oplus \Lambda & \to & \tilde{M}^{\Lambda}_{X,z} \\
\mathcal{O}_{X,v} \otimes \mathbf{Z}[\Lambda] & \to & \mathcal{A}^{log}_{X,z} \\
\mathcal{O}_{X,v} \otimes \Gamma^{\cdot}[P^{gp}] & \to & \mathcal{O}^{log}_{X,z} \\
\mathcal{O}_{X,v} \otimes_{\mathbf{C}[P]} J(P,\Lambda) & \to & \tilde{\mathcal{O}}^{log}_{X,z}
\end{array}$$

These isomorphisms are compatible with the connections, gradings, and actions of $I_v = \pi_1(P)$*.*

Proof: We have already constructed the first of the maps in statement (1), and the construction of the remaining maps is then straightforward. Let γ be an element of I_x, let p be an element of P^{gp} and let a be an element of $\mathbf{C}$. Then $\tilde{\mu}(a \otimes \tilde{\beta}(p))$ is a global section of $\zeta^{-1}\tilde{M}_X$, and

$$\begin{array}{rcl}
\rho_\gamma \tilde{\mu}(a \otimes \tilde{\beta}(p)) & = & \tilde{\mu}(a \otimes \rho_\gamma \tilde{\beta}(p)) \\
& = & \tilde{\mu}(a \otimes \tilde{\beta}(p) + a \otimes \langle \gamma, p \rangle) \\
& = & \tilde{\mu}(a \otimes \tilde{\beta}(p)) \mu(a \otimes \langle \gamma, p \rangle) \\
& = & \tilde{\mu}(a \otimes \tilde{\beta}(p)) \exp(a \langle \gamma, p \rangle) \\
& = & \tilde{\mu}(a \otimes \tilde{\beta}(p)) \exp\langle \gamma, a \otimes p \rangle
\end{array}$$

It follows that if λ is any element of Λ and $\tilde{m}$ is its image in $\Gamma(\tilde{X}^{log}, \zeta^{-1}\tilde{M}_X)$, then

$$\rho_\gamma(\tilde{m}) = \tilde{m} \exp\langle \gamma, \lambda \rangle$$

This shows that the second arrow is compatible with the actions of $\pi_1(P)$. Let $x_{\tilde{m}}$ be the basis element of $\mathcal{A}_{X,\lambda}$ corresponding to $\tilde{m}$ and let $u := \exp\langle \gamma, \lambda \rangle$. Then

$$\rho_\gamma(x_{\tilde{m}}) = x_{\rho_\gamma(\tilde{m})} = x_{u\tilde{m}} = u x_{\tilde{m}}.$$

This shows that the third arrow is compatible with the actions of $\pi_1(P)$. Furthermore, from the definition in (3.3.2), $\nabla x_{\tilde{m}} = x_{\tilde{m}} \otimes dlog\tilde{m}$, so it is also compatible with the connections. The sheaf $\mathcal{O}_X^{log}$ is generated over $\mathcal{O}_X$ by $\mathcal{L}$, on which we have already calculated the action of I_x, and it follows that its action on all of $\mathcal{O}_X^{log}$ is as described. The same argument works with the connections. The statement for $\tilde{\mathcal{O}}_X^{log}$ follows, as does part (2) of the proposition. □

3.4 Logarithmic Riemann-Hilbert

We can at last give the precise statement of the logarithmic Riemann-Hilbert correspondence.

Definition 3.4.1 *Let $X/\mathbf{C}$ be an fs smooth idealized log scheme and let $\Lambda \subseteq \mathbf{C} \otimes \overline{M}_X^{gp}$ be a set of exponential data for X. Let $MIC_{coh}^{\Lambda}(X_{an}/\mathbf{C})$ be the category of coherent sheaves of $\mathcal{O}_X$-modules on X_{an} equipped with an integrable logarithmic connection all of whose exponents lie in Λ.*

1. *If (E, ∇) is an object of $MIC_{coh}^{\Lambda}(X_{an})$, let*
$$\tilde{E} := \tilde{\mathcal{O}}_X^{log} \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}E,$$
with the induced connection $\tilde{\nabla}\colon \tilde{E} \to \tilde{E} \otimes \tilde{\Omega}^{1,log}_{X/\mathbf{C}}$, and let $\mathcal{V}(E,\nabla)$ be the sheaf of Λ-graded $\mathbf{C}_X^{log}$-modules $\tilde{E}^{\tilde{\nabla}}$.

2. *If V is an object of $L_{coh}^{\Lambda}(\mathbf{C}_X^{log})$, let $\tilde{V} := \tilde{\mathcal{O}}_X^{log} \otimes_{\mathbf{C}_X^{log}} V$, endowed with the connection $\tilde{\nabla} := d \otimes id$ and the tensor product Λ-grading, and let $(\mathcal{E}(V), \nabla) := \tau_*^{\Lambda}(\tilde{V}, \nabla)$, where the superscript Λ means the degree zero part with respect to the Λ-grading.*

Since the connection on $\tilde{E}$ is $\mathbf{C}_X^{log}$-linear and homogeneous, $\mathcal{V}(E,\nabla)$ is a sheaf of Λ-graded $\mathbf{C}_X^{log}$-modules. Thus the definition (1) above makes sense.

Theorem 3.4.2 *Let the notation be as in (3.4.1).*

1. *The functor $\mathcal{V}$ above is an equivalence of tensor categories*
$$MIC_{coh}^{\Lambda}(X_{an}) \to L_{coh}^{\Lambda}(\mathbf{C}_X^{log}),$$
with quasi-inverse $\mathcal{E}$.

2. *If $f\colon X \to Y$ is a morphism of smooth idealized fs log schemes and (E,∇) is an object of $MIC_{coh}^{\Lambda}(Y)$, then there is a natural isomorphism in $L_{coh}^{\Lambda}(\mathbf{C}_X^{log})$:*
$$f_{log}^*\mathcal{V}(E,\nabla) \cong \mathcal{V}(f^*E,\nabla).$$

3. *Let (E,∇) be an object of $MIC_{coh}^{\Lambda}(X_{an})$ and let $V := \mathcal{V}(E,\nabla)$.*

(a) The natural map

$$\tilde{\mathcal{O}}_X^{log} \otimes_{\mathbf{C}_X^{log}} V \to \tilde{E} := \tilde{\mathcal{O}}_X^{log} \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}E$$

is an isomorphism, compatible with the Λ-gradings and connections.

(b) The natural map

$$\mathcal{V}(E,\nabla) \to \tilde{E} \otimes_{\tilde{\mathcal{O}}_X^{log}} \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C}}$$

of complexes of abelian sheaves on X_{log} is a quasi-isomorphism.

(c) The natural map

$$E \otimes \Omega^{\cdot}_{X/\mathbf{C}} \to R\tau_*^{\Lambda}(\tilde{E} \otimes \tilde{\Omega}^{\cdot,log}_{X/\mathbf{C}})$$

is a quasi-isomorphism, where the superscript Λ means the degree zero part.

Proof: We will reduce the proof of the above global theorem to the local versions proved in the previous sections. Suppose that K is a proper ideal in a toric monoid P and let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$. Let E be an object of $MIC^{\Lambda}_{coh}(P, K/\mathbf{C})$, and let E_{an} be the corresponding object of $MIC(X_{an}/\mathbf{C})$. Then $\mathcal{V}(E_{an})$ is a sheaf of graded $\mathbf{C}_X^{log}$-modules on X_{log}. Its stalk at the vertex z is a Λ-graded $\mathbf{C}[-P]$-module. Since all the sheaves involved in the construction of $\mathcal{V}(E_{an})$ are locally constant on the fibers of τ, it also is locally constant on the fibers. On the other hand, the equivariant Riemann-Hilbert transform V of E is an object of $L^{\Lambda}(-P/\mathbf{C})$. Thus it is a Λ-graded $\mathbf{C}[-P]$-module, endowed with an action of $\pi_1(P)$. Recall that in (3.3.5) we constructed a map $J(P,\Lambda) \to \Gamma(\tilde{X}^{log}, \zeta^{-1}\tilde{\mathcal{O}}_X^{log})$. Tensoring with E, and observing that the resulting map is compatible with connections, we find a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{V}(E_{an})) \\ \downarrow & & \downarrow \\ E \otimes J(P,\Lambda) & \longrightarrow & \Gamma(\tilde{X}^{log}, \zeta^{-1}(E_{an} \otimes \tilde{\mathcal{O}}_X^{log})). \end{array}$$

Lemma 3.4.3 *Let $X := \mathsf{A}_{\mathsf{P},\mathsf{K}}$, let E be an object of $MIC^{\Lambda}_{coh}(P,K)$ and let $V \in L^{\Lambda}(-P,-K)$ be its equivariant Riemann-Hilbert transform (1.4.8). Let z be the vertex of X_{log}.*

1. *The map $V \to \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{V}(E_{an}))$ constructed above induces an isomorphism*

$$\overline{V} := V \otimes_{\mathbf{C}[-P]} \mathbf{C}[-\overline{P}] \cong \mathcal{V}(E_{an},\nabla)_z$$

in the category of $\overline{\Lambda}$-graded $\mathbf{C}[-\overline{P}]$-modules, compatible with the actions of $I_v \subseteq \pi_1(P)$.

2. *The natural map* $\mathcal{V}(E_{an},\nabla)_z \otimes_{\mathbf{C}_X^{log}} \tilde{\mathcal{O}}_{X,z}^{log} \to \tilde{E}_{an,z}$ *is an isomorphism.*

3. *The natural map* $\mathcal{V}(E_{an},\nabla)_z \to \tilde{E}_{an,z} \otimes \tilde{\Omega}^{\cdot}_{X/\mathbf{C}}$ *is a quasi-isomorphism.*

Proof: Let $\overline{X} := \mathsf{A}_{\overline{\mathsf{P}},\mathsf{K}}$ and let $(\overline{E},\nabla)$ be the image of (E,∇) in $MIC_{coh}^{\overline{\Lambda}}(\overline{P}/\mathbf{C})$. Because the functor B_{an} of (2.1.2) is fully faithful, the map $E_{an,v}^{\nabla} \to \overline{E}_{an,v}^{\nabla}$ is an isomorphism. Now $\tilde{E}_z := E \otimes \tilde{\mathcal{O}}_{X,z}^{log}$ is a direct limit of finitely generated modules with integrable connection. Applying the same remark to each of these and passing to the limit, we see that the map $\mathcal{V}(E_{an},\nabla)_z \to \mathcal{V}(\overline{E}_{an},\nabla)_z$ is also an isomorphism. This reduces the proof of the first and third statements to the case in which P is sharp. The second statement will also follow from the sharp case. Indeed, a section of $P \to \overline{P}$ induces a strict morphism $f\colon X \to \overline{X}$, so $f^*\tilde{\mathcal{O}}_{\overline{X}}^{log} \cong \tilde{\mathcal{O}}_X^{log}$. Thus for the remainder of the proof we may and shall assume that P is sharp.

Because the functor $\overline{A}$ of (2.1.2) is an equivalence, the map

$$E^{\nabla} \cong E_{an,v}^{\nabla} \tag{3.4.1}$$

is an isomorphism. Now $J(P,\Lambda)$ is a direct limit of objects J_a of $MIC_{coh}^{\Lambda}(P)$. For each a, $E \otimes J_a$ is an object of $MIC_{coh}^{\Lambda}(P)$, and so applying (3.4.1) to each of these produces an isomorphism

$$(E \otimes J_a)^{\nabla} \cong (E_{an,v} \otimes J_a)^{\nabla} \cong (E \otimes \mathcal{O}_{X_{an,v}} \otimes J_a)^{\nabla}.$$

Passing to the limit, we see by the last statement of (3.3.5) that the map

$$\big(E \otimes_{\mathbf{C}[P]} J(P,\Lambda)\big)^{\nabla} \to \big(E \otimes_{\mathbf{C}[P]} \mathcal{O}_{X_{an,v}} \otimes J(P,\Lambda)\big)^{\nabla} \to (E \otimes_{\mathbf{C}[P]} \tilde{\mathcal{O}}_{X^{log},v})^{\nabla}$$

is an isomorphism. By (1.4.8), the left hand side of this equation is the equivariant Riemann-Hilbert transform of E, which is in fact V, and the right side is by definition the stalk of $\mathcal{V}(E)$ at z. This proves (1). Recall from (1.4.8.2) that the natural map $V \otimes_{\mathbf{C}[-P]} J(P,\Lambda) \to E \otimes_{\mathbf{C}[P]} J(P,\Lambda)$ is an isomorphism. Statement (2) follows from this, after tensoring with $\mathcal{O}_{X_{an}}$. To prove (3), it will now suffice to show that $H^i(\tilde{E} \otimes \tilde{\Omega}^{\cdot}_X) = 0$ if $i > 0$. The same direct limit argument and Theorem (2.1.2) reduce this to the analogous computation in the category $MIC_{coh}^{\Lambda}(P/\mathbf{C})$, where it is a consequence of (1.4.8.2). ☐

We can now prove (2) of the theorem. Let (E,∇) be an object of $MIC_{coh}^{\Lambda}(Y_{an})$, and let $\tau_Y^*E := \tau^{-1}E \otimes \tilde{\mathcal{O}}_Y^{log}$, with a similar notation for X. As we have seen in (3.3.5), there is a natural map $f_{log}^{-1}\tilde{\mathcal{O}}_Y^{log} \to \tilde{\mathcal{O}}_X^{log}$, compatible with the exterior derivative and hence a natural and horizontal isomorphism

$$f_{log}^*\tau_Y^*E := f_{log}^{-1}\tau_Y^*E \otimes \tilde{\mathcal{O}}_X^{log} \cong \tau_X^* f^*E \tag{3.4.2}$$

Thus there is a natural map

$$\mathcal{V}(E) := (\tau_Y^*E)^{\nabla} \to f_{log*}(\tau_X^* f^*E)^{\nabla} = f_{log*}\mathcal{V}(f^*E).$$

By adjunction, we get a map

$$f_{log}^*\mathcal{V}(E) := f_{log}^{-1}\mathcal{V}(E)\otimes_{f^{-1}\mathbf{C}_Y^{log}}\mathbf{C}_X^{log}\to\mathcal{V}(f^*E),$$

which we are claiming is an isomorphism of $\mathbf{C}_X^{log}$-modules. It is clear from the local description (3.3.5) of $\tilde{\mathcal{O}}_X^{log}$ that it is faithfully flat over $\mathbf{C}_X^{log}$, so it suffices to prove that the map is an isomorphism after tensoring with $\tilde{\mathcal{O}}_X^{log}$. There is a commutative diagram

$$\begin{array}{ccc} f_{log}^*\mathcal{V}(E)\otimes_{\mathbf{C}_X^{log}}\tilde{\mathcal{O}}_X^{log} & \longrightarrow & \mathcal{V}(f^*E)\otimes_{\mathbf{C}_X^{log}}\tilde{\mathcal{O}}_X^{log} \\ \downarrow & & \downarrow \\ f_{log}^*\tau_Y^*E & \longrightarrow & \tau_X^*f^*E \end{array}$$

The lower horizontal map is the isomorphism (3.4.2) we started with, and we have already seen in (3.4.3) that the vertical arrows are isomorphisms. This implies that the arrow in (2) of the theorem is an isomorphism.

Lemma 3.4.4 *For any X as in the theorem and any $E\in MIC_{coh}^{\Lambda}(X_{an}/\mathbf{C})$, $\mathcal{V}(E)$ is log constructible (3.2.3).*

Proof: This can be verified locally in an analytic neighborhood of an arbitrary point x of X. Since $X/\mathbf{C}$ is fs and ideally log smooth, there exist a toric monoid P, an ideal K of P, and a strict étale map $X\to \mathsf{A}_{\mathsf{P},\mathsf{K}}$ sending x to the vertex. In the analytic topology, this map is locally an isomorphism, so we may and shall assume that $X=\mathsf{A}_{\mathsf{P},\mathsf{K}}$. By (2.1.2), there is a neighborhood of the vertex on which (E,∇) is isomorphic to the analytification of an object (M,∇) of $MIC_{coh}^{\Lambda}(P,K)$, so we may as well assume that (E,∇) *is* this analytification. We may also assume that $\Lambda=\mathbf{C}\otimes P^{gp}$. A splitting of $P\to\overline{P}$ induces a map $X\to\overline{X}$, and as we observed in (2.2.2), (E,∇) is isomorphic to the pullback of some $(\overline{E},\nabla)$ on $\overline{X}$, in some neighborhood U of v. By part (2) of the theorem, formation of $\mathcal{V}$ is compatible with pullback, and it follows that $\mathcal{V}(E,\nabla)$ is also pulled back from $\overline{X}$. Hence it is constant on $U\cap\mathsf{A}_{\mathsf{P}^*}$. But $\mathsf{A}_{\mathsf{P}^*}$ is the log stratum containing v. Since the same argument works in a neighborhood of every point, $\mathcal{V}(E,\nabla)$ is locally constant on the canonical stratification of X_{log}. □

Lemma 3.4.5 *The functor $\mathcal{V}$ of Theorem (3.4.2) maps $MIC_{coh}^{\Lambda}(X_{an})$ into $L_{coh}^{\Lambda}(\mathbf{C}_X^{log})$. In fact, suppose $X=\mathsf{A}_{\mathsf{P},\mathsf{K}}$, with P sharp, E is an object of $MIC^{\Lambda}(P,K)$ and V is its equivariant Riemann-Hilbert transform (1.4.8). Then the sheaf $\mathcal{V}(E_{an})$ is isomorphic to the object of $L^{\Lambda}(\mathbf{C}_X^{log})$ corresponding to V via the equivalence in (3.2.5).*

Proof: Let be E an object of $MIC_{coh}^{\Lambda}(P,K)$. We have seen in (3.4.4) that $\mathcal{V}(E_{an})$ is log constructible. To prove that it lies in $L(\mathbf{C}_X^{log})$ is a local question on X_{an}, so we may assume that $X = \mathsf{A}_{\mathsf{P,K}}$ and work in a neighborhood of the vertex. By (2.1.2), we may also assume that P is sharp. We claim that if $\tilde{x}$ is the vertex of $\tilde{X}^{log}$ and $\tilde{y}$ is any point of $\tilde{X}^{log}$, then

$$cosp^*_{\tilde{x},\tilde{y}} \colon \mathbf{C}^{log}_{X,y} \otimes \mathcal{V}(E_{an})_x \to \mathcal{V}(E_{an})_y$$

is an isomorphism. As we observed above, if this is true for $\tilde{y}$, it is also true for every other $\tilde{y}'$ in the same log stratum. Thus we may assume that $\tilde{y}$ is the vertex of $\tilde{Y}^{log}$, where $Y := \mathsf{A}_{\mathsf{P/F,K/F}}$ for some face F of P. The map $P \to P/F$ induces a map $i \colon Y \to X$; note that i does not map y to x. The equivariant Riemann-Hilbert transform W of $E \otimes \mathbf{C}[P/F]$ can be identified with $V \otimes \mathbf{C}[-P/F]$, and i^*E_{an} is the analytic sheaf with connection corresponding to $E \otimes \mathbf{C}[P/F]$. Thus it follows from (2) of theorem (3.4.2) and the functoriality of the constructions of (3.3.5) that there is a commutative diagram

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & \Gamma(\tilde{X}^{log}, \zeta^{-1}\mathcal{V}(E_{an})) \arrow[r] \arrow[d] & V_x \\
W \arrow[r] & \Gamma(\tilde{Y}^{log}, \zeta^{-1}i^*\mathcal{V}(E_{an})) \arrow[r] & W_y
\end{tikzcd}$$

Lemma (3.4.3) tells us that the composed horizontal maps are isomorphisms, and it follows from the definitions that the resulting map $V_x \to V_y$ is the cospecialization map $cosp^*_{\tilde{x},\tilde{y}}$. In particular, condition (3) of the definition (3.2.4) is satisfied. Since these maps are automatically compatible with the operations of $\pi_1(P)$, the lemma is proved. □

Lemma 3.4.6 *Let E be a coherent sheaf on X_{an} and let*

$$\tilde{E} := \tau^{-1}E \otimes_{\tau^{-1}\mathcal{O}_X} \tilde{\mathcal{O}}^{log}_X.$$

Then the natural map $E \to R\tau^{\Lambda}_*\tilde{E}$ is a quasi-isomorphism.

Proof: It suffices to prove that the stalk of this natural map at every point x of X is an isomorphism. Since τ is a proper morphism of paracompact Hausdorff spaces, the natural map

$$(R\tau^{\Lambda}_*\tilde{E})_x \to R\Gamma^{\Lambda}(\tau^{-1}(x), i^{-1}\tilde{E})$$

is an isomorphism, where $i \colon \tau^{-1}(x) \to X_{log}$ is the inclusion. Recall that the superscript Λ means taking the degree zero part in the grading, which commutes with cohomology. The degree zero part of $\tilde{E}$ is just $\tau^{-1}E \otimes \mathcal{O}^{log}_X$. Furthermore, the fiber $\tau^{-1}(x)$ is a torus and $\tilde{E}$ is locally constant on the fiber, so the sheaf cohomology is the same as group cohomology, computed with respect to the action of the fundamental group I_x on any stalk. Thus it suffices to show that $H^q(I_x, E_x \otimes \mathcal{O}^{log}_{X,z}) = 0$ if $q > 0$ and is E_x if $q = 0$. We may assume that

$X = \mathsf{A_P}$, with P a sharp toric monoid, so that $\mathcal{O}^{log}_{X,z} \cong \mathcal{O}_{X,z} \otimes_{\mathbf{C}} \Gamma^{\cdot}(\Omega)$. The action of I_x is unipotent, and its logarithm is a nilpotent T-Higgs field. Thus by (1.4.4) the group cohomology can be identified with the Higgs cohomology. But this Higgs complex is just E_x tensored with the Higgs complex of $\Gamma^{\cdot}(\Omega)$, which is a resolution of $\mathbf{C}$. $\square$

To prove that $\mathcal{V}$ is fully faithful, let E_i be objects of $MIC^{\Lambda}_{coh}(X_{an})$ and let $V_i := \mathcal{V}(E_i)$, for $i = 1, 2$. Since $\mathcal{V}$ is functorial, it induces a map of sheaves of $\mathbf{C}$-vector spaces $\mathcal{H}om(E_1, E_2) \to \tau_* \mathcal{H}om(V_1, V_2)$. It suffices to prove that this map of sheaves is an isomorphism, and to do this it suffices to check that its stalk at each point x is so. Then we may assume that $X = \mathsf{A_{P,K}}$, where P is a sharp toric monoid and that x is the vertex, and that each E_i comes from an object of $MIC^{\Lambda_x}_{coh}(P, K)$. Then the V_i can be identified with the corresponding equivariant Riemann-Hilbert transforms, and τ_* with the invariants under the log inertia group $\pi_1(P)$. Then the result follows from the full faithfulness of the equivariant Riemann-Hilbert transform.

To prove that $\mathcal{V}$ is essentially surjective, let V be an object of $L^{\Lambda}_{coh}(\mathbf{C}^{log}_X)$ and let x be a point of X. Then by (2.1.2) and (1.4.8), there exists an analytic neighborhood U of x and an object (E, ∇) of $MIC^{\Lambda}_{coh}(U)$ such that $\mathcal{V}(E, \nabla)_x \cong V_x$. Then in fact $\mathcal{V}(E, \nabla) \cong V$ on some possibly smaller neighborhood of x. We can glue these objects of MIC^{Λ}_{coh} using the gluing data coming from V and the full faithfulness of $\mathcal{V}$.

This completes the proof of (1) and (2) of the theorem. Parts (a) and (b) of (3) follow immediately from (3.4.3). For part (c), note that for each q, the natural map

$$E \otimes \Omega^q_{X/\mathbf{C}} \to R\tau^{\Lambda}_* \tilde{E} \otimes \tilde{\Omega}^{q,log}_{X/\mathbf{C}}$$

is a quasi-isomorphism, by (3.4.6), and hence the map in (c) is also a quasi-isomorphism. $\square$

Associated to the Λ-grading of the category $L^{\Lambda}_{coh}(X)$ is a Λ-filtration, where Λ is regarded as a sheaf of partially ordered sets, with the partial ordering induced by the action of $-\overline{M}_X$. This filtration carries over to the equivalent category $MIC^{\Lambda}_{coh}(X)$. Matthew Emerton has pointed out that this gives a log construction of the "Kashiwara-Malgrange V-filtration."

Corollary 3.4.7 *Any object (E, ∇) of $MIC^{\Lambda}_{coh}(X_{an})$ admits a unique and functorial decreasing filtration indexed by the sheaf of partially ordered set Λ, such that $\mathcal{V}(F^{\lambda}E) = \oplus_{\lambda' \geq \lambda} \mathcal{V}_{\lambda'}(E, \nabla)$. If (E', ∇) is a subobject of (E, ∇), then $E'_x \subseteq F^{\lambda}E_x$ if and only if all the exponents of E' at x are greater than or equal to λ in the partial ordering on Λ induced by the action of $-\overline{M}_{X,x}$.*

Remark 3.4.8 It is easy to see, for example from the compatibility of the local and global Riemann-Hilbert correspondence, that if (E, ∇) is an object of $MIC_{coh}(X_{an})$, then E is locally free (resp. torsion free, resp. reflexive) over $\mathcal{O}_X$ if and only if $\mathcal{V}(E, \nabla)$ is locally free (resp. ...) over $\mathbf{C}^{log}_X$.

As an illustration of the content of the main theorem (3.4.2), let us show how it easily implies a logarithmic version of Deligne's comparison theorem [3, II, 3.13]

Theorem 3.4.9 *Let $X/\mathbf{C}$ be an fs smooth log scheme (with no idealized structure) and let (E, ∇) be a torsion-free object of $MIC^{\Lambda}_{coh}(X_{an})$. At each point of x, let $S_x \subseteq \Lambda_x \subseteq \mathbf{C} \otimes \overline{M}^{gp}_{X,x}$ be the set of exponents of (E, ∇) at x. Suppose that for each such x, $S_x \cap \overline{M}^{gp}_{X,x} \subseteq \overline{M}_{X,x}$. Then the natural map*

$$H^*_{DR}(X_{an}, E) \to H^*_{DR}(X^*_{an}, E)$$

is an isomorphism.

Proof: Let $E^* := j^*E$, let $V := \mathcal{V}(E, \nabla)$, and let $V^* := \mathcal{V}(E^*, \nabla)$, which we can regard as a locally constant sheaf of $\mathbf{C}$-vector spaces on X^*. Theorem (3.4.2) provides a commutative diagram in the derived category:

$$\begin{tikzcd} E \otimes \Omega^{\cdot}_{X/\mathbf{C}} \arrow[r] \arrow[d] & Rj_*(E^* \otimes \Omega^{\cdot}_{X^*/\mathbf{C}}) \arrow[d] \\ R\tau^{\Lambda}_* V \arrow[r] & Rj_* V^*, \end{tikzcd}$$

in which the vertical arrows are quasi-isomorphisms. Thus it suffices to show that the bottom horizontal arrow is a quasi-isomorphism.

By (3.1.2), $V' := j_{log*}V^*$ is a local system of $\mathbf{C}$-vector spaces on X_{log}, and $j_{log*}V^* \cong Rj_{log*}V^*$. Thus it suffices to show that the natural map

$$R\tau^{\Lambda}_* V \to R\tau_* V'$$

is a quasi-isomorphism. This is a local question, and so we can restrict our attention to a neighborhood of a point x of X. If $\tilde{x} \in \tau^{-1}(x)$, then $V_{\tilde{x}}$ and $V'_{\tilde{x}}$ are equipped with actions of I_x, and we have to prove that the maps

$$H^i(I_x, V_{0,\tilde{x}}) \to H^i(I_x, V'_{\tilde{x}}) \tag{3.4.3}$$

are isomorphisms. Here V is a Λ_x-graded $\mathbf{C}[-\overline{M}_{X,x}]$-module, and the 0 means the degree zero part.

It follows from the coherence of V that $V'_{\tilde{x}}$ can be identified with the tensor product of $V_{\tilde{x}}$ over the map $\mathbf{C}[-\overline{M}_{X,x}] \to \mathbf{C}$ sending $\overline{M}_{X,x}$ to 1. It follows from (2.1.3) and the hypothesis on the exponents that the set of degrees of a set of generators for V intersected with $\overline{M}^{gp}_{X,x}$ is contained in $\overline{M}_{X,x}$. Hence Corollary (1.4.6) implies that (3.4.3) is an isomorphism.

□

Theorem 3.4.10 *Let $X/\mathbf{C}$ be a smooth fs log scheme (with no idealized structure) and let (E, ∇) be an object of $MIC^{\Lambda}_{coh}(X_{an}/\mathbf{C})$. Let $j\colon X^* \to X$ be the inclusion of the maximal open set where the log structure is trivial. Then the natural map*

$$j_* j_m^* E \otimes \Omega^{\cdot}_{X/\mathbf{C}} \to j_* j^* E \otimes \Omega^{\cdot}_{X/\mathbf{C}}$$

is a quasi-isomorphism, where $j_* j_m^*$ means the sheaf of sections of E with meromorphic poles along $X \setminus X^*$.

Proof: Fix a point x in X. It suffice to prove the theorem in a neighborhood of x. Thus we may assume that $X = \mathsf{A_P}$ for some toric monoid P. Let m be the sum of a minimal set of generators for P and let J be the ideal of P generated by m. The support of the corresponding closed subscheme of X is exactly the set where the log structure is nontrivial. The ideal I of $\mathcal{O}_X$ generated by $\beta(m)$ is an invertible sheaf of ideals, and its inverse defines an effective divisor D whose support is $X \setminus X^*$. Thus for any E, $j_* j_m^* E = \varinjlim E(nD)$. Since I comes from a sheaf of ideals in the monoid, it is stable under the connection d on $\mathcal{O}_X$. In particular, $\alpha(m)$ generates I and $d\alpha(m) = \alpha(m) dlog m \in I \otimes \Omega^1_{X/\mathbf{C}}$. By definition (2.1.1), $-\overline{m}_x$ is the unique exponent of this connection at x. Then the dual $\mathcal{O}_X(D)$ has a connection also, and its unique exponent is $\overline{m}_x$. If s is any element of $\overline{M}^{gp}_X$, $s + n\overline{m}_x \in \overline{M}_X$ for n sufficiently large. It follows that, locally on X, there exists an n such that $E(nD)$ satisfies the hypothesis of (3.4.9) for n sufficiently large. By the previous result, the map

$$E(nD) \otimes \Omega^{\cdot}_{X/\mathbf{C}} \to j_* j^* E \otimes \Omega^{\cdot}_{X/\mathbf{C}}$$

is a quasi-isomorphism for all n sufficiently large. Hence the same is true for the map from the direct limit. □

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