Absolute Derivations and Zeta Functions

Dedicated to Professor Kazuya Kato's 50th birthday

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Abstract. Just as the function ring case we expect the existence of the coefficient field for the integer ring. Using the notion of one element field in place of such a coefficient field, we calculate absolute derivations of arithmetic rings. Notable examples are the matrix rings over the integer ring, where we obtain some absolute rigidity. Knitting up prime numbers via absolute derivations we speculate the arithmetic landscape. Our result is only a trial to a proper foundation of arithmetic.

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Kronecker and many excellent arithmeticians attempted to study the arithmetic geometry by looking at the intimate analogy between $\mathbf{Z}$ and $\mathbf{F}_p[T]$. Although these two objects are similar in some respects, there exists a quite clear difference: the non-existence (or "invisibility") of the constant (coefficient) field of $\mathbf{Z}$. Zeta functions suggest to compare

$$\hat{\zeta}_{\mathbf{Z}}(s) = \frac{\det(R - (s - \frac{1}{2}))}{s(s-1)}$$

and

$$\hat{\zeta}_{\mathbf{F}_p[T]}(s) = \frac{1}{(1-p^{-s})(1-p^{-(s-1)})},$$

where $\hat{\zeta}$ denotes the "completed zeta function"; in the latter case we know good cohomologies with $\dim H^0(\mathbf{F}_p[T]) = 1$, $\dim H^1(\mathbf{F}_p[T]) = 0$, $\dim H^2(\mathbf{F}_p[T]) = 1$

and $H^i(\mathbf{F}_p[T]) = 0$ for $i > 2$. Up to now, we have not come across a cohomology theory such as $\dim H^0(\mathbf{Z}) = 1$, $\dim H^1(\mathbf{Z}) = \infty$ with a skew-hermitian operator $R : H^1(\mathbf{Z}) \to H^1(\mathbf{Z})$, $\dim H^2(\mathbf{Z}) = 1$ and $H^i(\mathbf{Z}) = 0$ for $i > 2$. Yet we can try to figure out the nature of the "constant field" $\mathbf{F}_1$ of $\mathbf{Z}$ (Manin [9], Deninger [1], Kurokawa [6]). As a first little step we calculate $\mathbf{F}_1$-derivations (in other words, "absolute derivations") of $\mathbf{Z}$ and allied objects here.

The authors thank Professor Kazuya Kato for his patient listening to our primitive tales in old days.
[One of our friends indicates the appearance of KAZUYA by looking at the leading alphabets of sentences in the introduction: the readers are welcome to find such an accidental coincidence.]

1 Absolute derivations

We define an absolute derivation of a ring R as a map $D : R \to R$ satisfying the condition (Leibniz rule)

$$D(ab) = D(a)b + aD(b) \quad \text{for all } a, b \in R.$$

If an absolute derivation D satisfies moreover the additivity property

$$D(a+b) = D(a) + D(b),$$

it is called a derivation of R. Here the word "absolute" indicates objects over "the one element field $\mathbf{F}_1$". Elements of the absolute mathematics are briefly described in § 2 below. We denote by $\mathrm{Der}_{\mathbf{F}_1}(R)$ the set of all absolute derivations of R, and by $\mathrm{Der}_{\mathbf{Z}}(R)$ the set of all derivations of R.

1.1

We first determine the absolute derivations of the most simple but the fundamental case $R = \mathbf{Z}$. For each prime p, define a map $\frac{\partial}{\partial p} : \mathbf{Z} \to \mathbf{Z}$ by

$$\frac{\partial}{\partial p}(x) = \frac{x}{p} \cdot \mathrm{ord}_p(x).$$

Here $\mathrm{ord}_p(x)$ denotes the p-order of $x \in \mathbf{Z}$, that is, the integer ℓ such that x is divisible by p^ℓ but is not divisible by $p^{\ell+1}$. Namely we have

$$\frac{\partial}{\partial p}(x) = \begin{cases} 0 & \text{if } p \nmid x \\ \ell p^{\ell-1} \cdot m & \text{if } x = p^\ell \cdot m \ (\ell \geq 1, p \nmid m) \end{cases}$$

and put $\frac{\partial}{\partial p}(0) = 0$. It is easy to see that $\frac{\partial}{\partial p}$ satisfies the Leibniz rule;

$$\frac{\partial}{\partial p}(xy) = \frac{\partial}{\partial p}(x)y + x\frac{\partial}{\partial p}(y),$$

whence $\frac{\partial}{\partial p} \in \mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z})$. In fact, since $\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$, we see that

$$\begin{aligned}\frac{\partial}{\partial p}(xy) &= \frac{xy}{p} \cdot \mathrm{ord}_p(xy) \\ &= \frac{xy}{p}\left(\mathrm{ord}_p(x) + \mathrm{ord}_p(y)\right) \\ &= \left(\frac{x}{p} \cdot \mathrm{ord}_p(x)\right) y + x\left(\frac{y}{p} \cdot \mathrm{ord}_p(y)\right) \\ &= \frac{\partial}{\partial p}(x)y + x\frac{\partial}{\partial p}(y).\end{aligned}$$

The following theorem shows these $\frac{\partial}{\partial p}$'s ($p$: prime numbers) span the set of the absolute derivations of $\mathbf{Z}$.

Theorem 1 *We have the following direct product decomposition:*

$$\mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z}) = \widehat{\bigoplus_{p:\mathrm{prime}}} \mathbf{Z}\frac{\partial}{\partial p} := \Big\{ \sum_p c_p \frac{\partial}{\partial p}\ ;\ c_p \in \mathbf{Z}\Big\} \subset \mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}),$$

where $\mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}) = \mathrm{Map}(\mathbf{Z}, \mathbf{Z})$.

Proof: Note first that the infinite sum $\sum_p c_p \frac{\partial}{\partial p}(x) \in \widehat{\bigoplus_p} \mathbf{Z}\frac{\partial}{\partial p}$ is well-defined since for each $x \in \mathbf{Z}$, $\frac{\partial}{\partial p}(x) = 0$ except the finite number of p. It is also easy to see that such an expression is unique. The fact that the sum of absolute derivations is also an absolute derivation, shows clearly that

$$\mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z}) \supset \widehat{\bigoplus_p} \mathbf{Z}\frac{\partial}{\partial p}.$$

It is therefore enough to prove that any $D \in \mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z})$ can be written as

$$D = \sum_p D(p)\frac{\partial}{\partial p}.$$

In order to see this we show that the absolute derivation D is completely determined by its values $D(p)$ on prime numbers $p = 2, 3, 5, \ldots$. By successive use of the Leibniz rule it is obvious to see that

$$D(p^\ell) = \ell p^{\ell-1} D(p).$$

Remark also that $D(0) = D(1) = D(-1) = 0$. Actually, the Leibniz rule shows

$$\begin{gathered} D(0) = D(0 \cdot 0) = D(0) \cdot 0 + 0 \cdot D(0) = 0, \\ D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1), \end{gathered}$$

whence it follows that $D(1) = 0$. Further,

$$0 = D(1) = D((-1) \cdot (-1)) = D(-1)(-1) + (-1)D(-1) = -2D(-1).$$

Since any non-zero $x \in \mathbf{Z}$ can be written as $x = \pm p_1^{i_1} p_2^{i_2} \cdots p_\ell^{i_\ell}$ by primes p_j, using the Leibniz rule again the assertion follows, that is, D is completely determined once the values $D(p)$ are given. To confirm the relation $D = \sum_p D(p)\frac{\partial}{\partial p}$ holds it suffices to show $D(q) = \left(\sum_p D(p)\frac{\partial}{\partial p}\right)(q)$ for each prime q. Since

$$\frac{\partial}{\partial p}(q) = \begin{cases} 1 & (p = q) \\ 0 & (p \neq q), \end{cases}$$

it is actually immediate to have

$$\Big(\sum_p D(p)\frac{\partial}{\partial p}\Big)(q) = \sum_p D(p)\frac{\partial}{\partial p}(q) = D(q).$$

This completes the proof. □

Remark 1 *It is easy to see that* $\mathrm{Der}_{\mathbf{Z}}(\mathbf{Z}) = 0$. *In fact, since* $D(0) = D(1) = D(-1) = 0$ *as above, the additivity asserts* $D(m) = 0$ *for* $m \in \mathbf{Z}$.

Remark 2 *We have for primes* p, q

$$\Big[\frac{\partial}{\partial p}, q\Big] = \delta_{pq},$$

where q *in the left-hand side is regarded as a multiplication operator, since*

$$\begin{aligned} \Big[\frac{\partial}{\partial p}, q\Big](x) &= \frac{\partial}{\partial p}(qx) - q\frac{\partial}{\partial p}(x) \\ &= \Big(\frac{\partial}{\partial p}(q)x + q\frac{\partial}{\partial p}(x)\Big) - q\frac{\partial}{\partial p}(x) = \frac{\partial}{\partial p}(q)x = \delta_{pq}\cdot x. \end{aligned}$$

1.2

We have the similar statement for $\mathbf{Z}[i]$ and $\mathbf{F}_p[T]$.
Let $\mathbf{Z}[i] = \{m + ni;\ \ m, n \in \mathbf{Z}\}$ $(i = \sqrt{-1})$ be the ring of Gaussian integers. Let $\{\pi\}$ be a complete set of representatives of the prime elements in $\mathbf{Z}[i]$. Define the map $\frac{\partial}{\partial \pi} \in \mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}[i])$ by

$$\frac{\partial}{\partial \pi}(x) = \frac{x}{\pi}\cdot \mathrm{ord}_\pi(x).$$

Then similar to the theorem above we obtain the following:

Theorem 2 *We have*

$$\mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z}[i]) = \widehat{\bigoplus_\pi} \mathbf{Z}[i]\frac{\partial}{\partial \pi} \subset \mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}[i]).$$

Moreover, for prime elements π, π' *we have*

$$\Big[\frac{\partial}{\partial \pi}, \pi'\Big] = \delta_{\pi\pi'}.$$

Now we consider the case $R = \mathbf{F}_p[T]$, the polynomial ring over the finite field $\mathbf{F}_p$. Any $f \in \mathbf{F}_p[T]$ can be factorized uniquely as

$$f = c \cdot h_1^{e_1} \cdot h_2^{e_2} \cdots h_\ell^{e_\ell},$$

where $c \in \mathbf{F}_p$, $e_i \in \mathbf{Z}_{\geq 0}$ and $h_1, h_2, \ldots, h_\ell$ are monic irreducible polynomials, that is, the prime elements in $\mathbf{F}_p[T]$. We define the order $\mathrm{ord}_h(f)$ in an obvious way; $\mathrm{ord}_{h_i}(f) = e_i$. Quite similarly as Theorem 1 we have the following theorem.

Theorem 3 *For a monic irreducible polynomial $h \in \mathbf{F}_p[T]$, define a map $\frac{\partial}{\partial h} : \mathbf{F}_p[T] \to \mathbf{F}_p[T]$ by*

$$\frac{\partial}{\partial h} f(T) = \frac{f(T)}{h(T)} \cdot \mathrm{ord}_h(f).$$

Then we have

$$\mathrm{Der}_{\mathbf{F}_1}(\mathbf{F}_p[T]) = \widehat{\bigoplus_{h:\text{monic irred.}}} \mathbf{F}_p[T] \frac{\partial}{\partial h}.$$

Remark 3 *We notice that $\frac{\partial}{\partial h}$ is different from the usual derivation. For example, $\frac{\partial}{\partial T}(T^2) = 2T$ and $\frac{\partial}{\partial T}(T^2+1) = 0$ here.*

1.3

For some general unique factorization domains, we have the following result:

Theorem 4 *Let R be a commutative unique factorization domain whose unit group $R^\times$ is a finitely generated abelian group. Fix a set of representative $P_0(R)$ of irreducible elements of $R \setminus (R^\times \cup \{0\})$ modulo $R^\times$, and a set of generators $P_1(R)$ of $R^\times$ modulo $R^\times_{\mathrm{tor}}$, where $R^\times_{\mathrm{tor}}$ is the subgroup of torsion elements. Put $P(R) = P_0(R) \cup P_1(R)$. Each element $a \in R \setminus \{0\}$ can be uniquely written as*

$$a = u \prod_{\pi \in P(R)} \pi^{m(\pi)},$$

where $m(\pi) \in \mathbf{Z}_{\geq 0}$ if $\pi \in P_0(R)$, $m(\pi) \in \mathbf{Z}$ if $\pi \in P_1(R)$, and $u \in R^\times_{\mathrm{tor}}$. We define

$$\frac{\partial}{\partial \pi}(a) = m(\pi) \frac{a}{\pi}$$

and

$$\frac{\partial}{\partial \pi}(0) = 0.$$

Then

$$\mathrm{Der}_{\mathbf{F}_1}(R) = \widehat{\bigoplus_{\pi \in P(R)}} R \frac{\partial}{\partial \pi}.$$

Proof. Each $a \in R \setminus \{0\}$ is uniquely written as

$$a = u' \prod_{\pi \in P_0(R)} \pi^{m(\pi)}$$

with $u' \in R^\times$. Since $R^\times / R^\times_{\mathrm{tor}}$ is a free abelian group, we can write

$$u' = u \prod_{\pi \in P_1(R)} \pi^{m(\pi)}$$

with $u \in R^\times_{\mathrm{tor}}$ uniquely, where $P_1(R)$ is a finite set. It is clear that $\frac{\partial}{\partial \pi} \in \mathrm{Der}_{\mathbf{F}_1}(R)$ by definition.
Now, take any $X \in \mathrm{Der}_{\mathbf{F}_1}(R)$. We show

$$X = \sum_{\pi \in P(R)} X(\pi) \frac{\partial}{\partial \pi}.$$

Take an $a \in R \setminus \{0\}$. Express it as

$$a = u \prod_{\pi \in P(R)} \pi^{m(\pi)}$$

with $u \in R^\times_{\mathrm{tor}}$. Then, using $X(u) = 0$ we have

$$X(a) = \sum_{\pi \in P(R)} m(\pi) \frac{a}{\pi} X(\pi) = \sum_{\pi \in P(R)} X(\pi) \frac{\partial}{\partial \pi}(a).$$

Hence,

$$X = \sum_{\pi \in P(R)} X(\pi) \frac{\partial}{\partial \pi}. \quad \square$$

Example 1 (1) *If* $R = \mathbf{Z}[\sqrt{2}]$, *then* $P_1(R) = \{1 + \sqrt{2}\}$.

(2) *If* $R = \mathbf{Z}[T_1, \ldots, T_n]$, *then* $P_1(R) = \emptyset$.

1.4

We note on a special subset of $\mathrm{Der}_{\mathbf{F}_1}(R)$ for $R = \mathbf{Z}$ and $\mathbf{Z}[i]$.

Theorem 5 *Let* p *be a prime. Then* $\mathfrak{g}_p = \mathbf{Z}\frac{\partial}{\partial p}$ *is closed under the Lie bracket defined by the commutator* $[\cdot,\cdot]$ *of* $\mathrm{End}_{\mathbf{F}_1}(\mathbf{Z})$. *Similarly, for a prime element* π *in* $\mathbf{Z}[i]$, *the subset* $\mathfrak{g}_\pi = \mathbf{Z}[i]\frac{\partial}{\partial \pi}$ *of* $\mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z}[i])$ *is closed under the commutator of* $\mathrm{End}_{\mathbf{F}_1}(\mathbf{Z}[i])$.

Proof: The first assertion is easily confirmed by using the relations

$$[p^\ell \frac{\partial}{\partial p}, p^m \frac{\partial}{\partial p}] \quad = \quad (m-\ell)p^{\ell+m-1}\frac{\partial}{\partial p}.$$

The assertion for $\mathbf{Z}[i]$ can be proved similarly. $\square$

Remark 4 *Define $H = p\partial,\ E = -p^2\partial,\ F = \partial$ with $\partial = \frac{\partial}{\partial p}$. Then the formula above implies the following commutation relations*

$$[H, E] = E, \quad [H, F] = -F, \quad [E, F] = 2H.$$

These commutation relations coincide with those of standard generators $\{H,\ E,\ F\}$ of the simple Lie algebra $\mathfrak{sl}_2$. Hence, the operator $C := 2H^2 + EF + FE$ can be considered as an "absolute" Casimir operator (cf. [7]). This element C commutes with each H, E, F, and moreover it is not hard to see that C vanishes under the map $\mathfrak{g}_p \to \mathrm{End}_{\mathbf{F}_1}(\mathbf{Z})$. It would be interesting to study the "absolute Virasoro algebra" extending $\mathfrak{g}_p$ with its physical implications.

1.5

Let R be a (non-commutative) ring. For an element $a \in R$, we define an inner derivation D_a by $D_a(b) = ab - ba$. It is easy to see that $D_a \in \mathrm{Der}_{\mathbf{Z}}(R)$. We denote the set of all inner derivations by $\mathrm{InnDer}_{\mathbf{Z}}(R)$. We have

$$\mathrm{InnDer}_{\mathbf{Z}}(R) \subset \mathrm{Der}_{\mathbf{Z}}(R) \subset \mathrm{Der}_{\mathbf{F}_1}(R).$$

The main result of this subsection is the following:

Theorem 6 $\mathrm{Der}_{\mathbf{F}_1}(M_2(\mathbf{Z})) = \mathrm{InnDer}_{\mathbf{Z}}(M_2(\mathbf{Z}))$.

Let D be an absolute derivation of $M_2(\mathbf{Z})$.

Lemma 1 $D(0) = D(I_2) = D(-I_2) = 0$.

Proof: Actually, $D(0) = D(00) = 0$, $D(I_2) = D(I_2I_2) = 2D(I_2)$, $0 = D((-I_2)(-I_2)) = -2I_2D(-I_2)$. $\square$

Let $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $N' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. These elements satisfy the following relations $HN = N$, $NH = -N$, $H^2 = I_2$, $HN' = -N'$, $N'H = N'$, $NN' = E_{11}$, $N'N = E_{22}$.

Lemma 2 *(i) There exist $a, b \in \mathbf{Z}$ such that $D(N) = aH + bN$.*

(ii) There exist $a', b' \in \mathbf{Z}$ such that $D(N') = a'H + b'N'$.

Proof: $0 = D(0) = D(N^2) = D(N)N + ND(N)$. Then (i) follows easily. The argument for N' is the same. $\square$

Lemma 3 $D(H) = -2a'N - 2aN'$.

Proof: $D(N) = D(HN) = HD(N) + D(H)N$. Then $D(H)N = a(H - I_2)$. Also,

$$-D(N) = D(-N) = D(NH) = D(N)H + ND(H).$$

Then $ND(H) = -a(H + I_2)$. By these two conditions, the diagonal entries of $D(H)$ are both zero, and the $(2,1)$-entry of $D(H)$ is $-2a$.
By $D(HN') = D(-N')$, we have $D(H)N' = -a'(I+H)$. Then the $(1,2)$-entry of $D(H)$ is $-2a'$. $\square$

Lemma 4 *There exists $q \in \mathbf{Z}$ such that for all $c \in \mathbf{Z}$*

$$\begin{aligned} D(H + cN) &= D(H) + c(aH + qN), \\ D(I_2 + cN) &= c(aH + qN). \end{aligned}$$

Proof: Let $A_c := H + cN$ for $c \in \mathbf{Z}$. Since

$$D(N) = D(A_cN) = A_cD(N) + D(A_c)N$$

and (i) of Lemma 2, we have $D(A_c)N = acN + a(H - I_2)$. Also,

$$-D(N) = D(-N) = D(NA_c) = D(N)A_c + ND(A_c).$$

Then $ND(A_c) = -a(H+I_2) - acN$. By these two conditions, we have $D(A_c) = q_cN + acH - 2aN'$. By setting $c = 0$, we have $D(H) = q_0N - 2aN'$. Then $q_0 = -2a'$.
Notice

$$\begin{aligned} D(I_2 + (c' - c)N) &= D(A_cA_{c'}) \\ &= A_cD(A_{c'}) + D(A_c)A_{c'} \\ &= (c' - c)aH + (q_{c'} - q_c)N. \end{aligned}$$

Then $q_{c'+c} - q_c = q_{c'} - q_0$. This means that $\mathbf{Z} \ni c \mapsto q_c - q_0 \in \mathbf{Z}$ is an additive map. We set $q = q_1 - q_0$, then $q_c = cq + q_0$. This proves the lemma. $\square$

Lemma 5 *We have*

$$\begin{aligned} D(H + cN') &= D(H) + c(a'H - qN'), \\ D(I_2 + cN') &= c(a'H - qN'). \end{aligned}$$

Proof: By a similar argument, there exists $q' \in \mathbf{Z}$, independent of c, such that

$$\begin{aligned} D(H+cN') &= D(H)+c(a'H+q'N'), \\ D(I_2+cN') &= c(a'H+q'N') \end{aligned}$$

for all $c \in \mathbf{Z}$. In fact, let $A'_c := H + cN'$ for $c \in \mathbf{Z}$. Since

$$D(-N') = D(A'_c N') = A'_c D(N') + D(A'_c)N'$$

and (ii) of Lemma 2, we have $D(A'_c)N' = -a'cN' - a'(H+I_2)$. Also,

$$D(N') = D(N') = D(N'A'_c) = D(N')A'_c + N'D(A'_c).$$

Then $N'D(A'_c) = a'(H-I_2)+a'cN'$. By these two conditions, we have $D(A'_c) = q'_c N' + a'cH - 2a'N$. By setting $c = 0$, we have $D(H) = q'_0 N' - 2a'N$. Then $q'_0 = -2a$. The remaining is similar to the previous lemma.
Now we put $B = (I_2 + N)(I_2 - N')$. Then, $\det(B) = 1$ and $\mathrm{tr}(B) = 1$ show that $B^3 = -I_2$. Hence

$$0 = D(B^3) = D(B)B^2 + BD(B)B + B^2 D(B).$$

By multiplying $-B$ from the left, we have

$$BD(B)B^{-1} + B^{-1}D(B)B + D(B) = 0.$$

Taking the trace, $3\mathrm{tr}(D(B)) = 0$. On the other hand, calculate as

$$\begin{aligned} D(B) &= D(I_2+N)(I_2-N') + (I_2+N)D(I_2-N') \\ &= (aH+qN)(I-N') - (I+N)(a'H+q'N'), \end{aligned}$$

then $\mathrm{tr}(D(B)) = -q' - q$. Thus $q' = -q$. □

Proof of Theorem 6: We use the notation above, especially, $a, a', q \in \mathbf{Z}$. Let $Y = a'N - aN' + qE_{11} \in M_2(\mathbf{Z})$. We note that

$$\begin{aligned} &[Y,H] = -2a'N - 2aN', \\ &[Y,N] = aH + qN, \\ &[Y,N'] = a'H - qN'. \end{aligned}$$

We consider the corresponding inner derivation $D_Y \in \mathrm{InnDer}_{\mathbf{Z}}(M_2(\mathbf{Z}))$. Then $D(H) = D_Y(H)$, $D(I_2+cN) = D_Y(I_2+cN)$ and $D(I_2+cN') = D_Y(I_2+cN')$ for all $c \in \mathbf{Z}$. Recall the fact that the group $GL(2,\mathbf{Z})$ is generated by $\{H, I_2 + cN, I_2+cN' \mid c \in \mathbf{Z}\}$. Then we know that $D(A) = D_Y(A)$ for all $A \in GL(2,\mathbf{Z})$. We put $Z := D - D_Y \in \mathrm{Der}_{\mathbf{F}_1}(M_2(\mathbf{Z}))$. Then we have proved that $Z(A) = 0$ for all $A \in GL(2,\mathbf{Z})$, $Z(N) = bN$ for some $b \in \mathbf{Z}$. We set $K = N + N' \in GL(2,\mathbf{Z})$. Then $NKN = N$ implies that

$$Z(N)KN + NKZ(N) = Z(N).$$

This shows $bN = 0$ and so $b = 0$.
Now we take a non-zero integer $\lambda \in \mathbf{Z}$. Consider a matrix $A \in GL(2, \mathbf{Z})$ with $(2,1)$-entry of A is divisible by λ. We define $A' \in GL(2, \mathbf{Z})$ by

$$(E_{11} + \lambda E_{22})A = A'(E_{11} + \lambda E_{22}).$$

Then

$$\begin{aligned} Z(E_{11} + \lambda E_{22})A &= A'Z(E_{11} + \lambda E_{22}), \\ (E_{11} + \lambda E_{22})^{-1}Z(E_{11} + \lambda E_{22})A &= A(E_{11} + \lambda E_{22})^{-1}Z(E_{11} + \lambda E_{22}) \end{aligned}$$

for all A as above. Such an A is so many, this equality holds for all $A \in M_2(\mathbf{Z})$. This means that $(E_{11} + \lambda E_{22})^{-1}Z(E_{11} + \lambda E_{22})$ is a scalar matrix since it commutes with all the right multiplication. Therefore, there exists $\tau(\lambda) \in \mathbf{Z}$ such that

$$Z(E_{11} + \lambda E_{22}) = \tau(\lambda)(E_{11} + \lambda E_{22}).$$

On the other hand, we have $(E_{11} + \lambda E_{22})N = N$. Since $Z(N) = 0$, we have $Z(E_{11} + \lambda E_{22})N = 0$. This shows that $\tau(\lambda) = 0$. We have proved $Z(E_{11} + \lambda E_{22}) = 0$ for all non-zero $\lambda \in \mathbf{Z}$. By considering $K(E_{11} + \lambda E_{22})K$, we know that $Z(\lambda E_{11} + E_{22}) = 0$. This proves that $Z(A) = 0$ for all $A \in M_2(\mathbf{Z})$ with $\det(A) \neq 0$.
For any matrix C in $M_2(\mathbf{Z})$ of rank one, there exist $A, A' \in M_2(\mathbf{Z})$ with $\det A \neq 0$, $\det A' \neq 0$ such that $C = ANA'$. This shows that $Z(C) = 0$. Finally we have $Z(0) = 0$. □

This result would suggest a kind of rigidity or semi-simplicity of $M_2(\mathbf{Z})$ as an absolute algebra, but it is not quite sure, at this moment, that such a notion can be formulated rigorously. We also remark that some argument can be extended to other non-commutative algebras. For example, the following result is proved in [10]: Let $R \ni 1$ be a ring contained in the algebraic closure $\bar{\mathbf{Q}}$. Then for each $n \geq 2$,

$$\mathrm{Der}_{\mathbf{F}_1}(M_n(R)) = \mathrm{InnDer}_{\mathbf{Z}}(M_n(R)) = \{D_a \mid a \in M_n(R)\}.$$

1.6 Absolute Hochschild cohomology

Theorem 6 can be stated as

$$H^1_{\mathbf{F}_1}(M_2(\mathbf{Z}), M_2(\mathbf{Z})) = 0,$$

where the left-hand side indicates the absolute Hochschild cohomology in the following sense. Let R be a ring and M be an R-bimodule. Let $C^0_{\mathbf{F}_1} = M$, $C^1_{\mathbf{F}_1} = Map(R, M)$, $C^2_{\mathbf{F}_1} = Map(R \times R, M)$, $C^n_{\mathbf{F}_1} = Map(R^n, M)$. We define a derivation $\delta^n : C^n_{\mathbf{F}_1} \to C^{n+1}_{\mathbf{F}_1}$ by

$$\begin{aligned}(\delta^0 m)(a) &= am - ma,\\ (\delta^1 f)(a_1, a_2) &= a_1 f(a_2) - f(a_1 a_2) + f(a_1) a_2,\\ (\delta^n f)(a_1, \ldots, a_{n+1}) &= a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1})\\ &\qquad + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}.\end{aligned}$$

Then $(\delta^*, C^*_{\mathbf{F}_1})$ is a chain complex of abelian groups. We call the cohomology groups $H^n_{\mathbf{F}_1}(R, M) = \mathrm{Ker}\delta^n / \mathrm{Im}\delta^{n-1}$ of this chain complex by the absolute Hochschild cohomology (cf. [8]). In particular,

$$H^0_{\mathbf{F}_1}(R, R) = \mathrm{Ker}\delta^0 = \{b \in R \mid ba = ab \text{ for all } a \in R\},$$

the center of R, and

$$H^1_{\mathbf{F}_1}(R, R) = \mathrm{Der}_{\mathbf{F}_1}(R) / \mathrm{InnDer}_{\mathbf{Z}}(R).$$

For example, Theorem 1 says

$$H^1_{\mathbf{F}_1}(\mathbf{Z}, \mathbf{Z}) = \widehat{\bigoplus_{p:\text{prime}}} \mathbf{Z} \frac{\partial}{\partial p}.$$

2 Absolute mathematics

We explain the background material of absolute mathematics, i.e., the mathematics over $\mathbf{F}_1$. As noted in Manin [9] the first appearance of $\mathbf{F}_1$ seems to be in $GL_n(\mathbf{F}_1) = S_n$ where S_n is the symmetric group of order n. This might be a folklore, but a much precise reference was supplied by Soulé [11] citing a paper [12] by Tits. There it seems that Tits conjectured $G(\mathbf{F}_1) = W(G)$ for each algebraic group G, where $W(G)$ is the Weyl group; in the case $G = GL_n$ we get $GL_n(\mathbf{F}_1) = W(GL_n) = S_n$ again.

We consider that $GL_n(\mathbf{F}_1) = S_n$ suggests to identify the category $\mathbf{Mod}(\mathbf{F}_1)$ of $\mathbf{F}_1$-modules with the category $\mathbf{Set}$ of sets. Let denote the free $\mathbf{F}_1$-module over a set X by $\mathbf{F}_1^{(X)}$. Then the more precise expectation is as follows: for objects X and Y of $\mathbf{Set}$, the corresponding objects of $\mathbf{Mod}(\mathbf{F}_1)$ are $\mathbf{F}_1^{(X)}$ and $\mathbf{F}_1^{(Y)}$ respectively with the corresponding morphisms

$$\mathrm{Hom}_{\mathbf{Set}}(X, Y) \cong \mathrm{Hom}_{\mathbf{Mod}(\mathbf{F}_1)}(\mathbf{F}_1^{(X)}, \mathbf{F}_1^{(Y)}).$$

Hence, especially for $X = \{1, 2, \ldots, n\}$ it would hold that

$$M_n(\mathbf{F}_1) = \mathrm{End}_{\mathbf{Mod}(\mathbf{F}_1)}(\mathbf{F}_1^n) = \mathrm{End}_{\mathbf{Set}}(\{1, 2, \ldots, n\})$$

and

$$GL_n(\mathbf{F}_1) = \mathrm{Aut}_{\mathbf{Mod}(\mathbf{F}_1)}(\mathbf{F}_1^n) = \mathrm{Aut}_{\mathbf{Set}}(\{1, 2, \ldots, n\}) = S_n,$$

which give $\#M_n(\mathbf{F}_1) = n^n$ and $\#GL_n(\mathbf{F}_1) = n!$. For example

$$M_2(\mathbf{F}_1) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

and

$$GL_2(\mathbf{F}_1) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} = S_2.$$

(Here we may omit 0's respecting $\mathbf{F}_1 = \{1\}$.)
Furthermore we identify the category $\mathbf{Alg}(\mathbf{F}_1)$ of $\mathbf{F}_1$-algebras with the category **Monoid** of monoids according to the following picture:

$$\left(\begin{tikzcd} & \mathbf{Alg}(\mathbf{Z}) \arrow[dl] \arrow[dr] & \\ \mathbf{Mod}(\mathbf{Z}) \arrow[dr] & & \mathbf{Alg}(\mathbf{F}_1) \arrow[dl] \\ & \mathbf{Mod}(\mathbf{F}_1) & \end{tikzcd} \right) = \left(\begin{tikzcd} & \mathbf{Ring} \arrow[dl] \arrow[dr] & \\ \mathbf{Ab} \arrow[dr] & & \mathbf{Monoid} \arrow[dl] \\ & \mathbf{Set} & \end{tikzcd} \right)$$

under the forgetful functors

$$\begin{tikzcd} & (R, \times, +) \arrow[dl] \arrow[dr] & \\ (R, +) \arrow[dr] & & (R, \times) \arrow[dl] \\ & R. & \end{tikzcd}$$

Then, the absolute derivations $\mathrm{Der}_{\mathbf{F}_1}(R)$ of a ring R studied in §1 would be understood by looking at the multiplicative monoid structure of the ring R. (Actually we do not forget completely the additive structure.) Now we state a problem to which the absolute mathematics may be applicable. Let X be a (projective smooth) scheme of finite type over $\mathrm{Spec}(\mathbf{Z})$. The Hasse zeta function is defined as

$$\zeta_X(s) = \prod_{x \in X_0} (1 - N(x)^{-s})^{-1}$$

where x runs over the set X_0 of closed points (0-dimensional points) of X and $N(x)$ is the cardinality of the residue field at x. It is expected that there exists the so-called gamma factor $\Gamma_X(s)$ and that the completed zeta function $\hat{\zeta}_X(s) = \zeta_X(s)\Gamma_X(s)$ is meromorphic in $s \in \mathbf{C}$ with the functional equation $s \leftrightarrow \dim(X) - s$. For our purpose it is convenient to formulate the following conjecture (cf. Kurokawa [4], Deninger [2]): There would exist cohomologies $H^m(X)$ for $m = 0, 1, \ldots, 2\dim(X)$ with skew-hermitian operators $R^m : H^m(X) \to H^m(X)$ satisfying

$$\hat{\zeta}_X(s) = \prod_{m=0}^{2\dim(X)} \det\left(R^m - (s - \frac{m}{2})\right)^{(-1)^{m+1}}.$$

We refer to Deninger [1], [2] for various investigations. For example, in the case $X = \mathrm{Spec}(\mathbf{Z})$, we expect

$$\hat{\zeta}(s) = \zeta(s)\pi^{-s/2}\Gamma(\frac{s}{2}) = \frac{\det(R^1 - (s - \frac{1}{2}))}{s(s-1)},$$

where we consider that $H^0(\mathrm{Spec}(\mathbf{Z}))$ and $H^2(\mathrm{Spec}(\mathbf{Z}))$ are one dimensional with the trivial R^0 and R^2. Of course $H^1(\mathrm{Spec}(\mathbf{Z}))$ should be infinite dimensional.

Problem Can such a cohomology $H^m(X)$ and an operator $R^m : H^m(X) \to H^m(X)$ be constructed via the absolute mathematics ? Is $H^m(X)$ interpreted as a cohomology of the associated absolute scheme $X^{\mathrm{abs}} \to \mathrm{Spec}(\mathbf{F}_1)$?

Some trials will be made in §3 and §4 below.

3 Zeta functions for absolute derivations

3.1

For a map $X : \mathbf{Z} \to \mathbf{C}$, we define the zeta function attached to X by the Dirichlet series:

$$\zeta(s, X) := \sum_{n=1}^{\infty} \frac{X(n)}{n^s}.$$

Lemma 6

$$\zeta(s, \frac{\partial}{\partial p}) = \frac{\zeta(s-1)}{p^s - p}.$$

Proof: We start with

$$\frac{\partial}{\partial p}(n) = \frac{n \cdot \mathrm{ord}_p(n)}{p}.$$

Then

$$\zeta(s, \frac{\partial}{\partial p}) = \frac{1}{p}\sum_{n=1}^{\infty} \mathrm{ord}_p(n) n^{-(s-1)}.$$

We express $n = p^k m$ with $(p, m) = 1$, and $k \geq 0$.

$$\begin{aligned}
\zeta(s, \frac{\partial}{\partial p}) &= \frac{1}{p} \sum_{m:(p,m)=1} m^{-(s-1)} \sum_{k=0}^{\infty} k p^{-k(s-1)} \\
&= \frac{1}{p} \times \zeta(s-1)(1 - p^{-(s-1)}) \times \frac{p^{-(s-1)}}{(1 - p^{-(s-1)})^2} \\
&= \zeta(s-1) \times \frac{1}{p^s - p}.
\end{aligned}$$

$\square$

Theorem 7 *For an $X \in \mathrm{Der}_{\mathbf{F}_1}(\mathbf{Z})$ of finite type, i.e., $X(p) = 0$ for all but finitely many primes p,*

$$\zeta(s, X) := \sum_{n=1}^{\infty} \frac{X(n)}{n^s} = \zeta(s-1) \sum_{p:\text{primes}} \frac{X(p)}{p^s - p}.$$

This can be extended to a meromorphic function on the whole complex plane. The special value at $s = 0$ is given by

$$\zeta(0, X) = \frac{1}{12} \sum_p \frac{X(p)}{p-1}.$$

Proof: It follows directly from Lemma 6 and $X = \sum_p X(p) \frac{\partial}{\partial p}$. $\square$

3.2 Quantum noncommutativity

We introduce the noncommutativity of primes as "$\zeta(0, [\frac{\partial}{\partial p}, \frac{\partial}{\partial q}])$".
First we give a rather explicit formula of the zeta of the commutator of absolute derivations.

Lemma 7

$$\zeta(s, [\frac{\partial}{\partial p}, \frac{\partial}{\partial q}]) = \frac{1}{pq} \zeta(s-1) \left((1 - q^{-(s-1)}) \sum_{k=1}^{\infty} p^k \frac{q^{-p^k(s-1)}}{(1 - q^{-p^k(s-1)})^2} \right.$$
$$\left. -(1 - p^{-(s-1)}) \sum_{k=1}^{\infty} q^k \frac{p^{-q^k(s-1)}}{(1 - p^{-q^k(s-1)})^2} \right).$$

Proof: We start with

$$\frac{\partial}{\partial p}(\frac{\partial}{\partial q}(n)) = \frac{n}{pq} \left(\mathrm{ord}_q(n)\mathrm{ord}_p(n) + \mathrm{ord}_q(n)\mathrm{ord}_p(\mathrm{ord}_q(n)) \right).$$

Then

$$[\frac{\partial}{\partial p}, \frac{\partial}{\partial q}](n) = \frac{n}{pq} \left(\mathrm{ord}_q(n)\mathrm{ord}_p(\mathrm{ord}_q(n)) - \mathrm{ord}_p(n)\mathrm{ord}_q(\mathrm{ord}_p(n)) \right).$$

We set $n = q^k m$ with $(q, m) = 1$, and $k \geq 0$. Then

$$\begin{aligned}
\zeta(s, [\frac{\partial}{\partial p}, \frac{\partial}{\partial q}]) &= \frac{1}{pq} \left(\sum_{n=1}^{\infty} \mathrm{ord}_q(n)\mathrm{ord}_p(\mathrm{ord}_q(n)) n^{-(s-1)} - \cdots \right) \\
&= \frac{1}{pq} \left(\sum_{m:(q,m)=1} m^{-(s-1)} \sum_{k=1}^{\infty} k \, \mathrm{ord}_p(k) q^{-k(s-1)} - \cdots \right) \\
&= \frac{1}{pq} \left((1 - q^{-(s-1)}) \zeta(s-1) \sum_{k=1}^{\infty} k \, \mathrm{ord}_p(k) q^{-k(s-1)} - \cdots \right)
\end{aligned}$$

We set $k = p^l m'$ with $(p, m') = 1$ and $l \geq 0$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} k \operatorname{ord}_p(k) q^{-k(s-1)} &= \sum_{l=1}^{\infty} \sum_{m':(p,m')=1} p^l m' l q^{-p^l m'(s-1))} \\
&= \sum_{l=1}^{\infty} l p^l \left(\sum_{m'=1}^{\infty} m' q^{-p^l m'(s-1)} - \sum_{m'=1}^{\infty} (pm') q^{-p^{l+1} m'(s-1)} \right) \\
&= \sum_{l=1}^{\infty} l p^l \left(\frac{q^{-p^l(s-1)}}{(1-q^{-p^l(s-1)})^2} - p \frac{q^{-p^{l+1}(s-1)}}{(1-q^{-p^{l+1}(s-1)})^2} \right) \\
&= \sum_{l=1}^{\infty} l p^l \frac{q^{-p^l(s-1)}}{(1-q^{-p^l(s-1)})^2} - \sum_{l=0}^{\infty} l p^{l+1} \frac{q^{-p^{l+1}(s-1)}}{(1-q^{-p^{l+1}(s-1)})^2} \\
&= \sum_{l=1}^{\infty} p^l \frac{q^{-p^l(s-1)}}{(1-q^{-p^l(s-1)})^2}.
\end{aligned}$$

This proves the lemma. $\square$

Remark 5 *Notice a partial functional equation under $s \leftrightarrow 2-s$.*

Remark 6 *Let $\phi_p(z) = \sum_{n=1}^{\infty} \operatorname{ord}_p(n) z^n$. Then the Mellin transform of ϕ_p is $p\zeta(s+1, \frac{\partial}{\partial p})$, and the series in the lemma above is obtained as*

$$\begin{aligned}
\sum_{l=1}^{\infty} p^l \frac{q^{-p^l(s-1)}}{(1-q^{-p^l(s-1)})^2} &= \sum_{l=1}^{\infty} p^l \left. \frac{z^{p^l}}{(1-z^{p^l})^2} \right|_{z=q^{-(s-1)}} \\
&= z \frac{\partial}{\partial z} \sum_{l=1}^{\infty} \left. \frac{z^{p^l}}{1-z^{p^l}} \right|_{z=q^{-(s-1)}} \\
&= z \phi_p'(z) \big|_{z=q^{-(s-1)}}.
\end{aligned}$$

Now we give a rigorous definition of the quantum noncommutativity motivated by the lemma above.

Definition The quantum noncommutativity, abbreviated as QNC, of p and q is defined by

$$\mathrm{QNC}(p,q) = \frac{1}{12pq} \left((q-1) \sum_{k=1}^{\infty} p^k \frac{q^{-p^k}}{(1-q^{-p^k})^2} - (p-1) \sum_{k=1}^{\infty} q^k \frac{p^{-q^k}}{(1-p^{-q^k})^2} \right).$$

Numerically,

$$\begin{aligned}\mathrm{QNC}(2,3) &= 0.00220482..,\\ \mathrm{QNC}(2,5) &= 0.00172077..,\\ \mathrm{QNC}(2,7) &= 0.00124803..,\\ \mathrm{QNC}(3,5) &= 0.00031155...\end{aligned}$$

Note that $\mathrm{QNC}(p,p)=0$ and it seems that $\mathrm{QNC}(p,q)>0$ for $p<q$.
We remark that $\mathrm{QNC}(p,q)$ has a neat expression using the Jackson integral (q-integral). We recall the following standard notions in q-analysis [3]. For an appropriate function $f(x)$, we define the Jackson integral

$$\int_1^\infty f(x)d_qx := \sum_{k=1}^\infty (q^k-q^{k-1})f(q^k)$$

with a base q. For a real number x, we define the corresponding q-number

$$[x]_q := \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}.$$

Then the quantum noncommutativity is expressed as

$$\begin{aligned}\mathrm{QNC}(p,q) &= \frac{1}{12pq}\left((q-1)\sum_{k=1}^\infty p^k\frac{q^{-p^k}}{(1-q^{-p^k})^2}-(p-1)\sum_{k=1}^\infty q^k\frac{p^{-q^k}}{(1-p^{-q^k})^2}\right)\\ &= \frac{1}{12pq}\left(\frac{(q-1)}{(q^{1/2}-q^{-1/2})^2}\sum_{k=1}^\infty p^k\frac{1}{[p^k]_q^2}-\frac{(p-1)}{(p^{1/2}-p^{-1/2})^2}\sum_{k=1}^\infty q^k\frac{1}{[q^k]_p^2}\right)\\ &= \frac{1}{12(p-1)(q-1)}\left(\int_1^\infty\frac{d_px}{[x]_q^2}-\int_1^\infty\frac{d_qx}{[x]_p^2}\right).\end{aligned}$$

Question (A) Let $R=(r_{pq})_{p,q:\mathrm{primes}}$ with $r_{pq}=$"$\zeta(0,[\frac{\partial}{\partial p},\frac{\partial}{\partial q}])$"$=\mathrm{QNC}(p,q)$. Then does it hold that

$$\hat{\zeta}_{\mathbf{Z}}(s)=\frac{\det(1-R(s-\frac{1}{2}))}{s(s-1)}?$$

Here we may recall that

$$H^1_{\mathbf{F}_1}(\mathbf{Z},\mathbf{Z})_{\mathbf{C}}=\widehat{\oplus}_p\mathbf{C}\frac{\partial}{\partial p}.$$

(B) For a sheaf (automorphic or Galois) ρ of $\mathbf{Z}$, let $R_\rho=(r_{pq}(\rho))$ with

$$r_{pq}(\rho)=\left(\frac{\rho(p)+\rho(q)^*}{2}\right)r_{pq},$$

where $\rho(q)^*$ is the adjoint of $\rho(p)$. Then does it hold that

$$\hat{L}(s,\rho)=\frac{\det(1-R_\rho(s-\frac{1}{2}))}{s^{m(\rho)}(s-1)^{m(\rho)}}$$

where $m(\rho)$ is the multiplicity of the trivial representation in ρ ?

Remark 7 *Let P be a set of "generalized primes" with the zeta function*

$$\zeta_P(s) = \prod_p (1 - N(p)^{-s})^{-1},$$

whose conjectual functional equation is $s \mapsto d(P) - s$. Let $R : V \to V$ be a linear operator on a complex vector space V. Assume that there is a basis $\{v_p \mid p \in P\}$ of V indexed by P. Let $\{e_\mu \mid \mu \in \mathrm{Spect}(R)\}$ be an R-eigen basis of V with $Re_\mu = \mu e_\mu$. Thus

$$\oplus_\mu \mathbf{C} e_\mu = V = \oplus_p \mathbf{C} v_p.$$

Take a suitable test function W such that $W(R)$ has a trace. Then we have (under a suitable condition) a so-called "trace formula"

$$\sum_\mu W(\mu) = \sum_p M(p)$$

where $M(p) = M(p,p)$ is given by

$$W(R) v_p = \sum_q M(q,p) v_q.$$

When

$$W(\mu) = \log(\mu - (s - \frac{d(P)}{2}))$$

and

$$M(p) = \log((1 - N(p)^{-s})^{-1})$$

we would obtain the determinant expression

$$\begin{aligned} \zeta_P(s) &= \prod_p (1 - N(p)^{-s})^{-1} \\ &= \prod_\mu (\mu - (s - \frac{d(P)}{2})) \\ &= \mathrm{Det}(R - (s - \frac{d(P)}{2})). \end{aligned}$$

By this way we would have the analytic continuation of the zeta functions and the L-functions.

4 Towards absolute schemes

We try to set up the first step to $\mathbf{F}_1$-schemes. Recall that the usual scheme is coming from the affine scheme $\mathrm{Spec}_{\mathbf{Z}}(A) = \mathrm{Spec}(A)$ for a (commutative) ring A, where $\mathrm{Spec}_{\mathbf{Z}}(A)$ is the set of the prime ideals of A with the Zariski (Stone-Jacobson-Gelfand) topology. Since we consider an $\mathbf{F}_1$-algebra as a monoid, we

define Spec(M) for a monoid M. A typical example is the case $M = (A, \times)$ for a ring A.
Generalizing a bit, let X be an algebraic system having an associative multiplication with the identity element 1. We assume that X has the zero element 0 (satisfying $0 \cdot x = x \cdot 0 = 0$ for all $x \in X$). An equivalence relation α on X is called a congruence if α is compatible with the algebraic operations on X. For example, if $x \equiv x'$ and $y \equiv y' \mod \alpha$, then $xy \equiv x'y' \mod \alpha$. In other words, such an α is associated to the residue (quotient) algebraic system X/α. We denote by Cong(X) the set of all congruences on X.

Example 2 *A congruence on a (not necessary commutative) ring corresponds to a two-sided ideal.*

Example 3 *A congruence on a group corresponds to a normal subgroup.*

We say that a congruence α on X is a prime congruence if X/α is integral in the sense that X/α has no (non-zero) zero divisors: if $x \not\equiv 0$, $y \not\equiv 0 \mod \alpha$, then $xy \not\equiv 0 \mod \alpha$. We denote by Spec($X$) the set of the prime congruences on X with the following topology: the closed subsets are

$$V(\beta) = \{\alpha \leq \beta \mid \alpha \text{ is a prime congruence}\}$$

for $\beta \in \mathrm{Cong}(X)$. Here $\alpha \leq \beta$ means that $x \equiv 0 \mod \beta$ implies $x \equiv 0 \mod \alpha$. As in the usual case, it is checked that such $V(\beta)$'s satisfy the needed conditions for closed sets by

$$V(\mathrm{id}) = X, \quad V(\mathrm{triv}) = \emptyset,$$

$$\begin{aligned} &V(\beta_1) \cup \cdots \cup V(\beta_n) = V(\beta_1 \wedge \cdots \wedge \beta_n), \\ &\bigcap_{\lambda \in \Lambda} V(\beta_\lambda) = V(\sum_\lambda \beta_\lambda), \end{aligned}$$

where $X/\mathrm{triv} = \{1\}$, $X/\mathrm{id} = X$, and $\sum_\lambda \beta_\lambda$ denotes the congruence generated by β_λ's.
For a ring A, we define

$$(\mathrm{Spec}_{\mathbf{Z}}(A))^{\mathrm{abs}} = \mathrm{Spec}_{\mathbf{F}_1}(A) = \mathrm{Spec}((A, \times)).$$

[Notice that Spec($(A, \times)$) is <u>not</u> the set of usual "ideals" of $(A, \times)$.] Then we have the natural map

$$\mathrm{Spec}_{\mathbf{Z}}(A) \longrightarrow \mathrm{Spec}_{\mathbf{F}_1}(A)$$

since each congruence on the ring A induces a congruence on the multiplicative monoid $(A, \times)$. The "local-global" map

$$A \longrightarrow \prod_{\alpha \in \mathrm{Spec}_{\mathbf{F}_1}(A)} (A/\alpha)$$

refines the usual local global map

$$A \longrightarrow \prod_{\gamma \in \mathrm{Spec}_{\mathbf{Z}}(A)} (A/\gamma).$$

For example, $\mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z})$ and $\displaystyle\prod_{\alpha \in \mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z})} (\mathbf{Z}/\alpha)$ are both big and unusual.

Remark 8 *The absolute fundamental group $\pi_1(\mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z}))$ would be interesting from the view point of the Langlands conjecture since for each unramified automorphic representation π of $GL_n(\mathbf{A})$ ($\mathbf{A}$ being the adele ring of $\mathbf{Q}$) we may have an n-dimensional representation (local system)*

$$\rho : \pi_1(\mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z})) \to GL_n(\mathbf{C})$$

satisfying

$$L(s, \pi) = L(s, \rho).$$

Now we notice on the $\mathbf{F}_1$-tensor product. We define the $\mathbf{F}_1$-tensor product of rings A and B as

$$A \otimes_{\mathbf{F}_1} B = (A, \times) * (B, \times)$$

the free product of monoids under the identification

$$1_A = 1_B = 1 \text{ and } 0_A = 0_B = 0.$$

Theorem 8

$$\mathrm{Spec}_{\mathbf{F}_1}(A \otimes_{\mathbf{F}_1} B) = \mathrm{Spec}_{\mathbf{F}_1}(A) \times \mathrm{Spec}_{\mathbf{F}_1}(B).$$

Proof: We show

$$\mathrm{Spec}(M * N) \cong \mathrm{Spec}(M) \times \mathrm{Spec}(N)$$

by sending $\alpha * \beta$ to (α, β), where M and N are multiplicative monoids having 1 and 0. Here $\alpha * \beta$ is the natural congruence on $M * N$ coming from α and β. Since it is easy to see that $\alpha * \beta$ is a prime congruence on $M * N$ if α and β are prime congruences on M and N respectively by

$$(M * N)/(\alpha * \beta) \cong (M/\alpha) * (N/\beta),$$

it is sufficient to show that the map is surjective. Let γ be a prime congruence on $M * N$. Then we obtain a congruence α on M and a congruence β on N by restricting γ to M and N respectively, and it holds that $\gamma = \alpha * \beta$. Since

$$(M * N)/\gamma \cong (M/\alpha) * (N/\beta),$$

we know that α (resp. β) is a prime congruence on M (resp. N). $\square$

Thus we have

$$\mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z} \otimes_{\mathbf{F}_1} \mathbf{Z}) = \mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z}) \times \mathrm{Spec}_{\mathbf{F}_1}(\mathbf{Z})$$

where $\mathbf{Z} \otimes_{\mathbf{F}_1} \mathbf{Z} = \mathbf{Z} * \mathbf{Z}$, as expected in [4]–[7] and [9].

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