Specialization of the p-adic Polylogarithm to p-th Power Roots of Unity

Dedicated to Professor Kazuya Kato
for his fiftieth birthday

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Abstract. The purpose of this paper is to calculate the restriction of the p-adic polylogarithm sheaf to p-th power torsion points.

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1 Introduction

Fix a rational prime p. The classical polylogarithm sheaf, constructed by Beilinson and Deligne, is a variation of mixed Hodge structures on the projective line minus three points. The p-adic polylogarithm sheaf is its p-adic analogue, and is expected to be the p-adic realization of the motivic polylogarithm sheaf. In our previous paper [Ban1], we explicitly calculated the p-adic polylogarithm sheaf on the projective line minus three points, and calculated its specializations to the d-th roots of unity for *d prime to p*. The purpose of this paper is to extend this calculation to the d-th roots of unity for *d divisible by p*. In particular, we prove that the specialization of the p-adic polylogarithm sheaf to d-th roots of unity is again related to special values of the p-adic polylogarithm function defined by Coleman [Col].

Let $K = \mathbb{Q}_p(\mu_d)$, with ring of integers $\mathcal{O}_K$. Let $\mathbb{G}_m = \operatorname{Spec} \mathcal{O}_K[t, t^{-1}]$ be the multiplicative group over $\mathcal{O}_K$. Denote by $S(\mathbb{G}_m)$ the category of *syntomic coefficients* on $\mathbb{G}_m$. This category is a rough p-adic analogue of the category of variation of mixed Hodge structures. Since p is in general ramified in K, we

will use the definition in [Ban2], which is a generalization of the definition in [Ban1] to the case when p is ramified in K.
In order to describe the polylogarithm sheaf, it is first necessary to introduce the logarithmic sheaf $\mathcal{L}og$, which is a pro-object in $S(\mathbb{G}_m)$. The first property we prove for this sheaf is that it satisfies the *splitting principle*, even at roots of unity whose order is divisible by p.

Proposition (= Proposition 5.1) *Let $z \neq 1$ be a d-th root of unity in K, and let $i_z : \operatorname{Spec} \mathcal{O}_K \hookrightarrow \mathbb{G}_m$ be the closed immersion defined by $t \mapsto z$. Then*

$$i_z^* \mathcal{L}og = \prod_{j \geq 0} K(j).$$

Let $\mathbb{U} = \mathbb{G}_m \setminus \{1\}$. In our previous paper, following the method of [HW1] Definition III 2.2, we constructed the polylogarithm extension

$$\mathrm{pol} \in \operatorname{Ext}^1_{S_{\mathrm{syn}}(\mathbb{U})}(K(0), \mathcal{L}og).$$

We first consider the case when z is a d-th root of unity, where d is an integer of the form $d = Np^r$ with $(N, p) = 1$ and $N > 1$. In this case, we have a natural map $i_z : \operatorname{Spec} \mathcal{O}_K \to \mathbb{U}$. Let $i_z^* \mathrm{pol}$ be the image of pol in

$$\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), i_z^* \mathcal{L}og) = \prod_{j \geq 0} \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$$

with respect to the pull-back map

$$\operatorname{Ext}^1_{S(\mathbb{U})}(K(0), \mathcal{L}og) \xrightarrow{i_z^*} \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), i_z^* \mathcal{L}og).$$

Our main result is concerned with the explicit shape of $i_z^* \mathrm{pol}$.
For integers $j \geq 1$, let $\mathrm{Li}_j(t)$ be the p-adic polylogarithm function defined by Coleman ([Col] VI, the function denoted $\ell_j(t)$). It is a locally analytic function defined on $\mathbb{P}^1(\mathbb{C}_p) \setminus \{1, \infty\}$ satisfying $\mathrm{Li}_j(0) = 0$. On the open unit disc $\{z \in \mathbb{C}_p \mid |z|_p < 1\}$, the function is given by the usual power series

$$\mathrm{Li}_j(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^j}.$$

To deal with the specialization at points in the open unit disc around one, we also consider the locally analytic function

$$\mathrm{Li}_{j,c}(t) = \mathrm{Li}_j(t) - c^{1-j} \, \mathrm{Li}_j(t^c),$$

where c is an integer > 1.
Our main theorem may be stated as follows:

Theorem 1 (= Theorem 7.3) *Let z be a d-th root of unity, where d is an integer of the form $d = Np^r$ with $(N, p) = 1$ and $N > 1$. Then we have*

$$i_z^* \operatorname{pol} = \left((-1)^j \operatorname{Li}_j(z)\right)_{j\geq 1} \in \prod_{j\geq 0} \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)),$$

where we view $(-1)^j \operatorname{Li}_j(z)$ as elements of $\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ through the isomorphism

$$\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)) \cong K. \tag{1}$$

Remark 1 *The above is compatible with the results of Somekawa [So] and also Besser-de Jeu [BdJ] on the calculation of the syntomic regulator.*

Remark 2 *In [Ban1], we proved that when d is prime to p,*

$$i_z^* \operatorname{pol} = \left((-1)^j \ell_j^{(p)}(z)\right)_{j\geq 1},$$

where $\ell_j^{(p)}(t)$ is a locally analytic function on $\mathbb{P}^1(\mathbb{C}_p) \setminus \{1, \infty\}$, whose expansion on the open unit disc around 0 is given by

$$\ell_j^{(p)}(t) = \sum_{n\geq 1, (n,p)=1} \frac{t^n}{n^j}.$$

The difference between this formula and the formula of the previous theorem comes from the choice of the isomorphism (1). (See Remark 7.2 for details.)

For the case when z is a p^r-th root of unity, let $c > 1$ be an integer and let $[c] : \mathbb{G}_m \to \mathbb{G}_m$ be the multiplication by c map induced from $t \mapsto t^c$. We denote by $[c]^*$ the pull back morphism of syntomic coefficients. We define the modified polylogarithm to be

$$\operatorname{pol}_c = \operatorname{pol} - [c]^* \operatorname{pol},$$

which we prove to be an element in $\operatorname{Ext}^1_{S_{\mathrm{syn}}(\mathbb{U}_c)}(K(0), \mathcal{L}og)$ for

$$\mathbb{U}_c = \operatorname{Spec} \mathcal{O}_K \left[t, \frac{t-1}{t^c - 1}\right].$$

We note that this modification, which removes the singularity around one, is standard in Iwasawa theory.
Our theorem in this case is:

Theorem 2 (= Theorem 8.3) *Let z be a p^r-th root of unity. Then we have*

$$i_z^* \operatorname{pol}_c = \left((-1)^j \operatorname{Li}_{j,c}(z)\right)_{j\geq 1} \in \prod_{j\geq 0} \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)),$$

where i_z^* is the pull back of syntomic coefficient by the natural inclusion i_z : $\operatorname{Spec} \mathcal{O}_K \to \mathbb{U}_c$. Again, we view $\operatorname{Li}_{j,c}(z)$ as an element of $\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ through the isomorphism (1).

Contents

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Notation Let p be a rational prime. In this paper, we let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k. We denote by π a generator of the maximal ideal of $\mathcal{O}_K$. We let K_0 the maximal unramified extension of $\mathbb{Q}_p$ in K, and W its ring of integers. We denote by σ the Frobenius morphism on K_0 and W.

2 Review of the p-adic polylogarithm function

In this section, we will review the theory of p-adic polylogarithm functions due to Coleman [Col]. Since we will mainly deal with the value of the p-adic polylogarithm function at units in $\mathcal{O}_{\mathbb{C}_p}$, we will not need the full theory of Coleman integration.

As in [Col], we call any locally analytic homomorphism $\log : \mathbb{C}_p^\times \to \mathbb{C}_p^+$, such that $\frac{d}{dz}\log(1) = 1$, a branch of the logarithm. Throughout this paper, we fix once and for all a branch of the logarithm. Since we will only deal with the values of p-adic analytic functions at points outside the open unit disc where the functions have logarithmic poles, the results of this paper is *independent* of the choice of the branch.

We define the p-adic polylogarithm function $\ell_j^{(p)}(t)$ for $|t| < 1$ by

$$\ell_j^{(p)}(t) = \sum_{(n,p)=1} \frac{t^n}{n^j} \qquad (j \geq 1).$$

By [Col] Proposition 6.2, this function extends to a rigid analytic function on $\mathbb{C}_p \setminus \{z\,;\, |z-1|_p < p^{(p-1)^{-1}}\}$.

Proposition 2.1 ([Col] Section VI) *The p-adic polylogarithm function $\mathrm{Li}_j(t)$ (Denoted $\ell_j(t)$ in [Col]) is a locally analytic function on $\mathbb{P}^1(\mathbb{C}_p) \setminus \{1, \infty\}$ satisfying*

(i) $\mathrm{Li}_0(t) = t/(1-t)$

(ii) $\frac{d}{dt}\mathrm{Li}_{j+1}(t) = \frac{1}{t}\mathrm{Li}_j(t) \qquad (j \geq 0).$

(iii) $\ell_j^{(p)}(t) = \mathrm{Li}_j(t) - p^{-j}\mathrm{Li}_j(t^p) \qquad (j \geq 1).$

Definition 2.2 *(i) For any integer j, we define the function $u_j(t)$ by*

$$u_j(t) = \begin{cases} \frac{1}{j!}\log^j(t) & (j \geq 0) \\ 0 & (j < 0). \end{cases}$$

Note that if z is a root of unity in $\mathbb{C}_p$, then $u_j(z) = 0$ $(j \neq 0)$.

(ii) For any integer $n \geq 1$, we define the function $D_n(t)$ by

$$D_n(t) = \sum_{j=0}^{n-1}(-1)^j \mathrm{Li}_{n-j}(t) u_j(t).$$

If z is a root of unity in $\mathbb{C}_p$, then $D_n(z) = \mathrm{Li}_n(z)$.

To deal with the torsion points of p-th power order, we need modified versions of the above functions.

Definition 2.3 *Let $c > 1$ be an integer prime to p. We let:*

(i) $\ell_{j,c}^{(p)}(z) = \ell_j^{(p)}(z) - c^{1-n}\ell_j^{(p)}(z^c) \quad (j \geq 1).$

(ii) $\mathrm{Li}_{j,c}(z) = \mathrm{Li}_{j,c}(z) - c^{1-n}\mathrm{Li}_{j,c}(z^c) \quad (j \geq 1).$

(iii)

$$D_{n,c}(z) = \sum_{j=0}^{n-1}(-1)^j \mathrm{Li}_{n-j,c}(t) u_j(t).$$

The above functions are locally analytic on the open unit disc around one.

3 The Category of Syntomic Coefficients

In this section, we will review the construction of the category of syntomic coefficients given in [Ban2] §4. Note that since we need to deal with the case when the prime p is ramified in K, the theory of [Ban1] is not sufficient.

Definition 3.1 *A syntomic datum* $\mathfrak{X} = (X, \overline{X}, j, \mathcal{P}_X, \phi_X, \iota)$ *consists of the following:*

(i) A proper smooth scheme $\overline{X}$*, separated an of finite type over* $\mathcal{O}_K$*, and an open immersion* $j : X \hookrightarrow \overline{X}$*, such that the complement* D *is a relative simple normal crossing divisor over* $\mathcal{O}_K$*.*

(ii) A formal scheme $\mathcal{P}_X$ *over* W*.*

(iii) For the formal completion $\overline{\mathcal{X}}$ *of* $\overline{X}$ *with respect to the special fiber, a closed immersion* $\iota : \overline{\mathcal{X}} \to \mathcal{P}_X \otimes_W \mathcal{O}_K$*, such that both* $\mathcal{P}_X$ *and the morphism* ι *are smooth in a neighborhood of* X_k*.*

(iv) A Frobenius map $\phi_X : \mathcal{P}_X \to \mathcal{P}_X$*, which fits into the diagram*

$$\begin{array}{ccccc} \overline{X}_k & \xrightarrow{\iota} & \mathcal{P}_X & \longrightarrow & \operatorname{Spf} W \\ {\scriptstyle F}\downarrow & & {\scriptstyle \phi_X}\downarrow & & {\scriptstyle \sigma^*}\downarrow \\ \overline{X}_k & \xrightarrow{\iota} & \mathcal{P}_X & \longrightarrow & \operatorname{Spf} W, \end{array} \tag{2}$$

where F *is the absolute Frobenius of* $\overline{X}_k$*.*

We will often omit j and ι from the notation and write

$$\mathfrak{X} = (X, \overline{X}, \mathcal{P}_X, \phi_X).$$

Example 3.2 *1. Let* $\mathbb{P}^1$ *be the projective line over* W *with coordinate* t*, and let* $\mathbb{P}^1_{\mathcal{O}_K} = \mathbb{P}^1 \otimes \mathcal{O}_K$*. We let* $\mathbb{G}_m$ *be the syntomic datum given by*

$$\mathbb{G}_m = \left(\mathbb{G}_{m\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \widehat{\mathbb{P}}^1, \phi\right),$$

where

(a) $\mathbb{G}_{m\mathcal{O}_K}$ *is the multiplicative group over* $\mathcal{O}_K$*, with natural inclusion* $j : \mathbb{G}_{m\mathcal{O}_K} \hookrightarrow \mathbb{P}^1_{\mathcal{O}_K}$*.*

(b) $\widehat{\mathbb{P}}^1$ *is the* p*-adic formal completion of* $\mathbb{P}^1$*.*

(c) $\iota : \widehat{\mathbb{P}}^1_{\mathcal{O}_K} \to \widehat{\mathbb{P}}^1 \otimes \mathcal{O}_K$ *is the identity.*

(d) ϕ *is the Frobenius given by* $\phi(t) = t^p$ *for the coordinate* t *on* $\widehat{\mathbb{P}}^1$*.*

2. *We let* $\mathbb{U}$ *be the syntomic datum given by*

$$\mathbb{U} = \left(\mathbb{U}_{\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \widehat{\mathbb{P}}^1, \phi\right),$$

where $\mathbb{U}_{\mathcal{O}_K} = \mathbb{P}^1_{\mathcal{O}_K} \backslash \{0, 1, \infty\}$, *with the natural inclusion* $j : \mathbb{U}_{\mathcal{O}_K} \hookrightarrow \mathbb{P}^1_{\mathcal{O}_K}$.

3. *We let* $\mathcal{O}_K$ *be the syntomic datum given by*

$$\mathcal{O}_K = (\operatorname{Spec} \mathcal{O}_K, \operatorname{Spec} \mathcal{O}_K, \operatorname{Spf} W, \sigma),$$

where j *and* ι *are the identity.*

Throughout this section, we fix a syntomic datum $\mathfrak{X}$. We will next review the definition of the category of *syntomic coefficients* $S(\mathfrak{X})$ on $\mathfrak{X}$. We will first define the categories $S_{\mathrm{dR}}(\mathfrak{X})$, $S_{\mathrm{rig}}(\mathfrak{X})$ and $S_{\mathrm{vec}}(\mathfrak{X})$. Let $X_K = X \otimes K$ and $\overline{X}_K = \overline{X} \otimes K$.

Definition 3.3 *We define the category* $S_{\mathrm{dR}}(\mathfrak{X})$ *to be the category consisting of objects the triple* $M_{\mathrm{dR}} := (M_{\mathrm{dR}}, \nabla_{\mathrm{dR}}, F^\bullet)$, *where:*

(i) M_{dR} *is a coherent* $\mathcal{O}_{\overline{X}_K}$ *module.*

(ii) $\nabla_{\mathrm{dR}} : M_{\mathrm{dR}} \to M_{\mathrm{dR}} \otimes \Omega^1(\log D_K)$ *is an integrable connection on* M_{dR} *with logarithmic poles along* $D_K = D \otimes K$.

(iii) $F^\bullet$ *is the Hodge filtration, which is a descending exhaustive separated filtration on* M_{dR} *by coherent sub-*$\mathcal{O}_{\overline{X}_K}$ *modules satisfying*

$$\nabla_{\mathrm{dR}}(F^m M_{\mathrm{dR}}) \subset F^{m-1} M_{\mathrm{dR}} \otimes \Omega^1_{\overline{X}_K}(\log D_K).$$

Let $X_k = X \otimes k$ be the special fiber of X and $\mathcal{X}$ the formal completion of X with respect to the special fiber. We denote by $\mathcal{X}_K$ the rigid analytic space over K associated to $\mathcal{X}$ ([Ber1] Proposition (0.2.3)) and by X_K^{an} the rigid analytic space over K associated to X_K (loc. cit. Proposition (0.3.3)). We will use the same notations for $\overline{X}$.

Definition 3.4 *We say that a set* $V \subset \overline{\mathcal{X}}_K$ *is a strict neighborhood of* $\mathcal{X}_K$ *in* X_K^{an}, *if* $V \cup (X_K^{\mathrm{an}} \setminus \mathcal{X}_K)$ *is a covering of* X_K^{an} *for the Grothendieck topology.*

For any abelian sheaf M on X_K^{an}, we let

$$j^\dagger M := \varinjlim_V \alpha_{V*} \alpha_V^* M,$$

where the limit is taken with respect to strict neighborhoods V of $\mathcal{X}_K$ in X_K^{an} with inclusion $\alpha_V : V \hookrightarrow \overline{\mathcal{X}}_K$. If M has a structure of a $\mathcal{O}_{X_K^{\mathrm{an}}}$-module, then $j^\dagger M$ has a structure of a $j^\dagger \mathcal{O}_{X_K^{\mathrm{an}}}$-module.

Definition 3.5 *We define the category $S_{\mathrm{vec}}(\mathfrak{X})$ to be the category consisting of objects the pair $M_{\mathrm{vec}} := (M_{\mathrm{vec}}, \nabla_{\mathrm{vec}})$, where:*

(i) M_{vec} *is a coherent* $j^\dagger \mathcal{O}_{X_K^{\mathrm{an}}}$ *module.*

(ii) $\nabla_{\mathrm{vec}} : M_{\mathrm{vec}} \to M_{\mathrm{vec}} \otimes \Omega^1_{X_K^{\mathrm{an}}}$ *is an integrable connection on* M_{vec}.

Let $p_{\mathrm{dR}} : X_K^{\mathrm{an}} \to \overline{X}_K$ be the natural map.

Definition 3.6 *We define the functor*

$$\mathbf{F}_{\mathrm{dR}} : S_{\mathrm{dR}}(\mathfrak{X}) \to S_{\mathrm{vec}}(\mathfrak{X})$$

by associating to $M_{\mathrm{dR}} := (M_{\mathrm{dR}}, \nabla_{\mathrm{dR}}, F^\bullet)$ the module $j^\dagger(p_{\mathrm{dR}}^* M_{\mathrm{dR}})$ with the connection induced from ∇_{dR}. The functor $\mathbf{F}_{\mathrm{dR}}$ is exact, since it is a composition of exact functors ([Ber1] Proposition 2.1.3 (iii)).

Let $\mathcal{P}_{K_0}$ be the rigid analytic space over K_0 associated to $\mathcal{P}_X$ ([Ber1] (0.2.2)). As in loc. cit. Définitions (1.1.2)(i), we define the *tubular neighborhood* of $\overline{X}_k$ (resp. X_k) in $\mathcal{P}_{K_0}$ by

$$]\overline{X}_k[_{\mathcal{P}} := \mathrm{sp}^{-1}(\overline{X}_k) \qquad \big(\text{resp.} \quad]X_k[_{\mathcal{P}} := \mathrm{sp}^{-1}(X_k)\big),$$

where $\mathrm{sp} : \mathcal{P}_{K_0} \to \mathcal{P}_X$ is the *spécialization* [Ber1] (0.2.2.1). The tubular neighborhoods are rigid analytic spaces over K_0 with structures induced from that of $\mathcal{P}_{K_0}$.

Definition 3.7 *We say that a set $V \subset]\overline{X}_k[_{\mathcal{P}}$ is a strict neighborhood of $]X_k[_{\mathcal{P}}$ in $]\overline{X}_k[_{\mathcal{P}}$, if*

$$V \cup (]\overline{X}_k[_{\mathcal{P}} \backslash]X_k[_{\mathcal{P}})$$

is a covering of $]\overline{X}_k[_{\mathcal{P}}$ for the Grothendieck topology.

For any abelian sheaf M on $]\overline{X}_k[_{\mathcal{P}}$, we let

$$j^\dagger M := \varinjlim_{V} \alpha_{V*} \alpha_V^* M,$$

where the limit is taken with respect to strict neighborhoods V of $]X_k[_{\mathcal{P}}$ in $]\overline{X}_k[_{\mathcal{P}}$ with inclusion $\alpha_V : V \hookrightarrow]\overline{X}_k[_{\mathcal{P}}$. If M has a structure of a $\mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$-module, then $j^\dagger M$ has a structure of a $j^\dagger \mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$-module.
The Frobenius map $\phi_X : \mathcal{P}_X \to \mathcal{P}_X$ induces a natural morphism of rigid analytic spaces $\phi_X :]\overline{X}_k[_{\mathcal{P}} \to]\overline{X}_k[_{\mathcal{P}}$.

Definition 3.8 *We define the category $S_{\mathrm{rig}}(\mathfrak{X})$ to be the category consisting of objects the triple $M_{\mathrm{rig}} := (M_{\mathrm{rig}}, \nabla_{\mathrm{rig}}, \Phi_M)$, where:*

(i) M_{rig} *is a coherent* $j^\dagger \mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$*-module.*

(ii) $\nabla_{\mathrm{rig}} : M_{\mathrm{rig}} \to M_{\mathrm{rig}} \otimes \Omega^1_{]\overline{X}_k[_{\mathcal{P}}}$ *is an integrable connection on* M_{rig}.

(iii) Φ_M *is the Frobenius morphism, which is an isomorphism*

$$\Phi_M : \phi_X^* M_{\mathrm{rig}} \xrightarrow{\cong} M_{\mathrm{rig}}$$

of $j^\dagger \mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$*-modules compatible with the connection.*

The map $\iota : \overline{\mathcal{X}} \to \mathcal{P}_X \otimes_W \mathcal{O}_K$ induces a map of rigid analytic spaces

$$p_{\mathrm{rig}} : X_K^{\mathrm{an}} \to]\overline{X}_k[_{\mathcal{P}}. \tag{3}$$

Definition 3.9 *We define the functor*

$$\mathbf{F}_{\mathrm{rig}} : S_{\mathrm{rig}}(\mathfrak{X}) \to S_{\mathrm{vec}}(\mathfrak{X})$$

by associating to the object $M_{\mathrm{rig}} := (M_{\mathrm{rig}}, \nabla_{\mathrm{rig}}, \Phi_M)$ *the object*

$$\mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) := (p_{\mathrm{rig}}^* M_{\mathrm{rig}}, p_{\mathrm{rig}}^* \nabla_{\mathrm{rig}})$$

in $S_{\mathrm{vec}}(\mathfrak{X})$. *This functor is exact by definition.*

Definition 3.10 *We define the category of syntomic coefficients to be the category* $S(\mathfrak{X})$ *such that:*

(i) The objects of $S(\mathfrak{X})$ *consists of the triple* $\mathcal{M} := (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$, *where:*

  (a) M_{typ} *is an object in* $S_{\mathrm{typ}}(\mathfrak{X})$ *for* $\mathrm{typ} \in \{\mathrm{dR}, \mathrm{rig}\}$.

  (b) $\mathbf{p}$ *is an isomorphism*

$$\mathbf{p} : \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}) \xrightarrow{\cong} \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}})$$

  in $S_{\mathrm{vec}}(\mathfrak{X})$.

(ii) A morphism $f : \mathcal{M} \to \mathcal{N}$ *in* $S(\mathfrak{X})$ *is given by a pair* $(f_{\mathrm{dR}}, f_{\mathrm{rig}})$, *where* $f_{\mathrm{typ}} : M_{\mathrm{typ}} \to N_{\mathrm{typ}}$ *are morphisms in* $S_{\mathrm{typ}}(\mathfrak{X})$ *for* $\mathrm{typ} \in \{\mathrm{dR}, \mathrm{rig}\}$ *compatible with the comparison isomorphism* $\mathbf{p}$.

Example 3.11 *For each integer* $n \in \mathbb{Z}$, *we define the Tate object* $K(n)$ *in* $S(\mathfrak{X})$ *to be the set* $K(n) := (K(n)_{\mathrm{dR}}, K(n)_{\mathrm{rig}}, \mathbf{p})$, *where:*

(i) $K(n)_{\mathrm{dR}}$ *in* $S_{\mathrm{dR}}(\mathfrak{X})$ *is given by the rank one free* $\mathcal{O}_{\overline{X}_K}$*-module generated by* $e_{n,\mathrm{dR}}$, *with connection* $\nabla_{\mathrm{dR}}(e_{n,\mathrm{dR}}) = 0$ *and Hodge filtration*

$$\begin{cases} F^m K(n)_{\mathrm{dR}} = K(n)_{\mathrm{dR}} & m \le -n \\ F^m K(n)_{\mathrm{dR}} = 0 & m > -n. \end{cases}$$

(ii) $K(n)_{\mathrm{rig}}$ *in* $S_{\mathrm{rig}}(\mathfrak{X})$ *is given by the rank one free* $j^\dagger\mathcal{O}_{]\overline{X}_k[_{\mathcal{P}}}$*-module generated by* $e_{n,\mathrm{rig}}$*, with connection* $\nabla_{\mathrm{rig}}(e_{n,\mathrm{rig}}) = 0$ *and Frobenius*

$$\Phi(e_{n,\mathrm{rig}}) := p^{-n} e_{n,\mathrm{rig}}.$$

(iii) $\mathbf{p}$ *is the isomorphism given by* $\mathbf{p}(e_{n,\mathrm{dR}}) = e_{n,rig}$.

Example 3.12 (See [Ban1] Definition 5.1) *We define the logarithmic sheaf*

$$\mathcal{L}og^{(n)} := (L^{(n)}_{\mathrm{dR}}, L^{(n)}_{\mathrm{rig}}, \mathbf{p})$$

in $S(\mathbb{G}_m)$ *by:*

(i) $L^{(n)}_{\mathrm{dR}}$ *in* $S_{\mathrm{dR}}(\mathbb{G}_m)$ *is given by the rank* n *free* $\mathcal{O}_{\mathbb{P}^1_K}$*-module*

$$L^{(n)}_{\mathrm{dR}} = \prod_{j=0}^{n} \mathcal{O}_{\mathbb{P}^1_K} e_{j,\mathrm{dR}},$$

with connection $\nabla_{\mathrm{dR}}(e_{j,\mathrm{dR}}) = e_{j+1,\mathrm{dR}} \otimes d\log t$ *for* $0 \leq j \leq n-1$ *and* $\nabla(e_{n,\mathrm{dR}}) = 0$*, and Hodge filtration given by*

$$F^{-m} L^{(n)}_{\mathrm{dR}} = \prod_{j=0}^{m} \mathcal{O}_{\mathbb{P}^1_K} e_{j,\mathrm{dR}}.$$

(ii) $L^{(n)}_{\mathrm{rig}}$ *in* $S_{\mathrm{rig}}(\mathbb{G}_m)$ *is given by the rank* n *free* $j^\dagger\mathcal{O}_{]\mathbb{P}^1_k[_{\widehat{\mathbb{P}}^1}}$*-module*

$$L^{(n)}_{\mathrm{rig}} = \prod_{j=0}^{n} j^\dagger\mathcal{O}_{]\mathbb{P}^1_k[_{\widehat{\mathbb{P}}^1}} e_{j,\mathrm{rig}},$$

with connection $\nabla_{\mathrm{rig}}(e_{j,\mathrm{rig}}) = e_{j+1,\mathrm{rig}} \otimes d\log t$ *for* $0 \leq j \leq n-1$ *and* $\nabla(e_{n,\mathrm{rig}}) = 0$*, and Frobenius*

$$\Phi(e_{j,\mathrm{rig}}) := p^{-j} e_{j,\mathrm{rig}}.$$

(iii) $\mathbf{p}$ *is the isomorphism given by* $\mathbf{p}(e_{j,\mathrm{dR}}) = e_{j,\mathrm{rig}}$.

4 Morphisms of Syntomic Data

Definition 4.1 *Define a morphism between syntomic data* $u : \mathfrak{X} \to \mathfrak{Y}$ *to be a pair* $(u_{\mathrm{dR}}, u_{\mathrm{rig}})$ *such that:*

(i) $u_{\mathrm{dR}} : \overline{X} \to \overline{Y}$ *is a morphism of schemes over* $\mathcal{O}_K$.

(ii) $u_{\mathrm{rig}} : \mathcal{P}_X \to \mathcal{P}_Y$ *is a morphism of formal schemes over* W *compatible with the Frobenius, such that the diagram*

$$\begin{array}{ccc} \overline{X} \otimes k & \xrightarrow{\iota} & \mathcal{P}_X \otimes k \\ {\scriptstyle u_{\mathrm{dR}}}\downarrow & & {\scriptstyle u_{\mathrm{rig}}}\downarrow \\ \overline{Y} \otimes k & \xrightarrow{\iota} & \mathcal{P}_Y \otimes k \end{array} \tag{4}$$

is commutative.

REMARK 4.2 *Notice that in* (4)*, contrary to [Ban2] Definition 4.2 (iii), we do not impose the commutativity of the diagram*

$$\begin{array}{ccc} \overline{\mathcal{X}} & \xrightarrow{\iota} & \mathcal{P}_X \\ {\scriptstyle u_{\mathrm{dR}}}\downarrow & & {\scriptstyle u_{\mathrm{rig}}}\downarrow \\ \overline{\mathcal{Y}} & \xrightarrow{\iota} & \mathcal{P}_Y. \end{array} \tag{5}$$

EXAMPLE 4.3 *Let* z *be an element in* $\mathcal{O}_K^\times$*, and let* $\mathbb{G}_m$ *be the syntomic datum defined in Example 3.2.1. We denote by* z_0 *the Teichmüller representative of* z*. In other words,* z_0 *is a root of unity in* W *such that* $z \equiv z_0 \pmod{\pi}$*. Then*

$$i_z = (i_{\mathrm{dR}}, i_{\mathrm{rig}}) : \mathcal{O}_K \to \mathbb{G}_m$$

is a morphism of syntomic data, where $i_{\mathrm{dR}} : \operatorname{Spec} \mathcal{O}_K \to \mathbb{G}_{m\mathcal{O}_K}$ *and* $i_{\mathrm{rig}} : \operatorname{Spf} \mathcal{O}_K \to \widehat{\mathbb{P}}^1_W$ *are morphisms defined respectively by* $t \mapsto z$ *and* $t \mapsto z_0$*.*

Let $u = (u_{\mathrm{dR}}, u_{\mathrm{rig}}) : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of syntomic data. By [Ber1] (2.2.16), we have a functor $u^*_{\mathrm{rig}} : S_{\mathrm{rig}}(\mathfrak{Y}) \to S_{\mathrm{rig}}(\mathfrak{X})$.

LEMMA 4.4 *Let* $u : \mathfrak{X} \to \mathfrak{Y}$ *be a morphism of syntomic data, and let* $\mathcal{M} := (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ *be an object in* $S(\mathfrak{Y})$*. Then there exists a canonical and functorial isomorphism*

$$u^*(\mathbf{p}) : \mathbf{F}_{\mathrm{dR}}(u^*_{\mathrm{dR}} M_{\mathrm{dR}}) \to \mathbf{F}_{\mathrm{rig}}(u^*_{\mathrm{rig}} M_{\mathrm{rig}})$$

in $S_{\mathrm{vec}}(\mathfrak{X})$*.*

The above lemma is trivial if we assume the commutativity of (5).
Proof. Let $u_{\mathrm{vec}} : \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$ be the morphism of formal schemes induced from u_{dR}, and denote again by u_{vec} the map induced on the associated rigid analytic space. Then we have

$$\mathbf{F}_{\mathrm{dR}}(u^*_{\mathrm{dR}} M_{\mathrm{dR}}) = u^*_{\mathrm{vec}} \mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}).$$

Let $u_1 := \iota \circ u_{\mathrm{vec}}$ and $u_2 := (u_{\mathrm{rig}} \otimes 1) \circ \iota$ be maps of formal schemes

$$u_1, u_2 : \overline{\mathcal{X}} \to \mathcal{P}_Y \otimes \mathcal{O}_K.$$

Then $u_{\mathrm{vec}}^*\mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) = u_{1K}^*(M_{\mathrm{rig}} \otimes K)$ and $\mathbf{F}_{\mathrm{rig}}(u_{\mathrm{rig}}^* M_{\mathrm{rig}}) = u_{2K}^*(M_{\mathrm{rig}} \otimes K)$. Since (4) is commutative, u_1 and u_2 coincide on $\overline{X}_k$. Hence by [Ber1] Proposition (2.2.17), we have a canonical isomorphism

$$\epsilon_{1,2} : u_{1K}^*(M_{\mathrm{rig}} \otimes K) \xrightarrow{\cong} u_{2K}^*(M_{\mathrm{rig}} \otimes K). \tag{6}$$

The isomorphism of the lemma is the composition of the isomorphism

$$\mathbf{F}_{\mathrm{dR}}(u_{\mathrm{dR}}^* M_{\mathrm{dR}}) = u_{\mathrm{vec}}^*\mathbf{F}_{\mathrm{dR}}(M_{\mathrm{dR}}) \xrightarrow{\mathbf{p}\,\cong} u_{\mathrm{vec}}^*\mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}).$$

with $\epsilon_{1,2}$.

DEFINITION 4.5 *Let $u : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of syntomic data. Then*

$$u^* : S(\mathfrak{Y}) \to S(\mathfrak{X})$$

is the functor defined by associating to any object $\mathcal{M} := (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ the object

$$u^*\mathcal{M} = (u_{\mathrm{dR}}^* M_{\mathrm{dR}}, u_{\mathrm{rig}}^* M_{\mathrm{rig}}, u^*(\mathbf{p}))$$

in $S(\mathfrak{X})$.

5 THE SPLITTING PRINCIPLE

Let $\mathcal{L}og^{(n)}$ be the logarithmic sheaf defined in Example 3.12. In this section, we will extend the splitting principle of [Ban1] Proposition 5.2 to the points defined in Example 4.3.

PROPOSITION 5.1 (SPLITTING PRINCIPLE) *Let d be a positive integer, and let $z = \zeta_d$ be a primitive d-th root of unity in K. Let*

$$i_z = (i_{\mathrm{dR}}, i_{\mathrm{rig}}) : \mathcal{O}_K \to \mathbb{G}_m$$

be the morphism of syntomic data of Example 4.3 corresponding to z. Then we have an isomorphism

$$i_z^*\mathcal{L}og^{(n)} \cong \prod_{j=0}^{n} K(j)$$

in $S(\mathcal{O}_K)$.

The proof of the proposition will be given at the end of this section. In order to prove the proposition, it is necessary to explicitly calculate the map $i_z^*(\mathbf{p})$ of Lemma 4.4. For this purpose, we first review the Monsky-Washnitzer interpretation of overconvergent isocrystals and the explicit description of $\epsilon_{1,2}$ of (6) (See [Ber1] §2 and [T] §2 for details).

We assume for now that z is an arbitrary element in $\mathcal{O}_K^\times$. We denote by z_0 the root of unity in W such that $z \equiv z_0 \pmod{\pi}$. Let $A = \Gamma(\mathbb{G}_{m\mathcal{O}_K}, \mathcal{O}_{\mathbb{G}_{m\mathcal{O}_K}}) = \mathcal{O}_K[t, t^{-1}]$. We fix a presentation

$$\mathcal{O}_K[x_1, \cdots, x_n]/I \cong A$$

over $\mathcal{O}_K$, which defines a closed immersion

$$\mathbb{G}_{m\mathcal{O}_K} \hookrightarrow \mathbb{A}^n_{\mathcal{O}_K}.$$

Then the intersections U_λ of $\mathbb{G}^{\mathrm{an}}_{mK}$ with the ball $B(0, \lambda^+) \subset \mathbb{A}^{n\,\mathrm{an}}_K$ for $\lambda \to 1^+$ form a system of strict neighborhoods (Definition 3.4) of $\widehat{\mathbb{G}}_{mK}$ in $\mathbb{G}^{\mathrm{an}}_{mK}$. For $\lambda > 1$, we let $A_\lambda = \Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$. Then $\lim_{\lambda \to 1^+} A_\lambda = A^\dagger \otimes K$, where $A^\dagger$ is the weak completion of A.
Let $M_{\mathrm{vec}} = (M_{\mathrm{vec}}, \nabla_{\mathrm{vec}})$ be an object in $S_{\mathrm{vec}}(\mathbb{G}_m)$. By [Ber1] Proposition 2.2.3, M_{vec} is of the form $j^\dagger(M_0, \nabla_0)$, where M_0 is a coherent module with integrable connection ∇_0 on a strict neighborhood U_λ. Let $M_\lambda = \Gamma(U_\lambda, M_0)$. Then for $\lambda' < \lambda$, the section $\Gamma(U_{\lambda'}, M_0)$ is given by $M_{\lambda'} = M_\lambda \otimes_{A_\lambda} A_{\lambda'}$, and

$$M := \Gamma(\mathbb{G}^{\mathrm{an}}_{mK}, M_{\mathrm{vec}}) = \varinjlim_{\lambda \to 1^+} M_\lambda. \tag{7}$$

M is a projective $A^\dagger \otimes K$-module with integrable connection $\nabla : M \to M \otimes \Omega^1_{A^\dagger \otimes K}$ induced from ∇_0.
Suppose the connection ∇_{vec} is *overconvergent*. By [Ber1] Proposition 2.2.13, for any $\eta < 1$, there exists $\lambda > 1$ such that

$$\left\| \frac{1}{i!} \nabla_\lambda(\partial_t^i)(m) \right\| \eta^i \to 0 \quad (i \to \infty) \tag{8}$$

for any $m \in M_\lambda$. Here, $\nabla_\lambda : M_\lambda \to M_\lambda \otimes \Omega^1_{A_\lambda/K}$ is the connection induced from ∇_0, ∂_t is the derivation by t, and $\| - \|$ is a Banach norm on M_λ.
Let $\mathcal{M} = (M_{\mathrm{dR}}, M_{\mathrm{rig}}, \mathbf{p})$ be an object in $S(\mathbb{G}_m)$. Then

$$M_{\mathrm{vec}} := \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) = (M_{\mathrm{rig}} \otimes_{K_0} K, \nabla_{\mathrm{rig}} \otimes_{K_0} K)$$

is an object in $S_{\mathrm{vec}}(\mathbb{G}_m)$. We have

$$i^*_{\mathrm{vec}} \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) = M \otimes_{i_{\mathrm{vec}}} K, \qquad \mathbf{F}_{\mathrm{rig}}(i^*_{\mathrm{rig}} M_{\mathrm{rig}}) = M \otimes_{i_{\mathrm{rig}}} K,$$

where M is as in (7), and $i_{\mathrm{vec}}, i_{\mathrm{rig}} : A^\dagger \otimes_{\mathcal{O}_K} K \to K$ are ring homomorphisms given respectively by $t \mapsto z$ and $t \mapsto z_0$. By [Ber1] 2.2.17 Remarque,

$$\epsilon_{1,2} : M \otimes_{i_{\mathrm{vec}}} K \xrightarrow{\cong} M \otimes_{i_{\mathrm{rig}}} K$$

of (6) is given explicitly by the Taylor series

$$\epsilon_{1,2}(m \otimes_{i_{\mathrm{vec}}} 1) = \sum_{i \geq 0} \frac{1}{i!} \nabla(\partial_t^i)(m) \otimes_{i_{\mathrm{rig}}} (z - z_0)^i. \tag{9}$$

The existence of the Frobenius Φ_M on $M_{\rm rig}$ insures that the connection $\nabla_{\rm rig}$ (hence $\nabla_{\rm vec}$) is overconvergent ([Ber1] Theorem 2.5.7). Since $|z - z_0| < 1$, the above series converges by (8).

Next, let

$$\mathcal{L}og^{(n)} := (L^{(n)}_{\rm dR}, L^{(n)}_{\rm rig}, \mathbf{p})$$

be the logarithmic sheaf of Example 3.12. As in (7), we $L = \Gamma(\mathbb{G}^{\rm an}_{mK}, L^{(n)}_{\rm vec})$ for $L^{(n)}_{\rm vec} = L^{(n)}_{\rm rig} \otimes_{K_0} K$. Then

$$L = \prod_{j=0}^{n} (A^\dagger \otimes K) e_j$$

for the basis $e_j = e_{j,{\rm rig}} \otimes 1$, and the connection is given by

$$\nabla(e_j) = e_{j+1} \otimes \frac{dt}{t} \quad (0 \leq j \leq n-1). \tag{10}$$

Let $u_j(t)$ be the function defined in Definition 2.2.

Proposition 5.2 *For integers $i, m \geq 0$, let $a^{(i)}_m$ be elements in $A^\dagger_K$ such that*

$$\nabla(\partial^i_t)(e_0) = \sum_{j=0}^{n} a^{(i)}_j e_j.$$

Then

$$\partial^i_t(u_m) = \sum_{j=0}^{n} a^{(n)}_j u_{m-j}.$$

In particular, we have

$$a^{(i)}_m(z_0) = \partial^i_t(u_m)(z_0). \tag{11}$$

Remark 5.3 *The definition of $a^{(i)}_j$ implies*

$$\nabla(\partial^i_t)(e_m) = \sum_{j=0}^{n-m} a^{(i)}_j e_{m+j}.$$

Proof. We will give the proof by induction on $i \geq 0$. Since $a^{(0)}_0 = 1$, the statement is true for $i = 0$. Suppose for an integer $i \geq 0$, we have

$$\partial^i_t(u_m) = \sum_{j=0}^{n} a^{(i)}_j u_{m-j}. \tag{12}$$

By comparing the definition of $a^{(i+1)}_j$ with the equality

$$\nabla(\partial^{i+1}_t)(e_0) = \nabla(\partial_t) \circ \nabla(\partial^i_t)(e_0) = \sum_{j=0}^{n} \left((\partial_t a^{(i)}_j) e_j + t^{-1} a^{(i)}_j e_{j+1} \right),$$

we obtain the equality

$$a_j^{(i+1)} = \partial_t a_j^{(i)} + t^{-1} a_{j-1}^{(i)}. \tag{13}$$

Similarly, from the hypothesis (12) and $\partial_t u_m = t^{-1} u_{m-1}$, we have

$$\partial_t^{i+1}(u_m) = \partial_t \circ \partial_t^i(u_m) = \sum_{j=0}^{n} \left((\partial_t a_j^{(i)}) u_{m-j} + t^{-1} a_j^{(i)} u_{m-j-1} \right).$$

This together with (13) gives the desired result. (11) follows from the fact that since z_0 is a root of unity, $u_m(z_0) = 0$ unless $m = 0$.

Corollary 5.4 *For any integers $i, m \geq 0$, we have*

$$\nabla(\partial_t^i)(e_m) \otimes_{i_{\mathrm{rig}}} 1 = \sum_{j=0}^{n-m} \left(e_{m+j} \otimes_{i_{\mathrm{rig}}} \partial_t^i(u_j)(z_0) \right).$$

Proof. The assertion follows immediately from Remark 5.3

Proposition 5.5 *We have*

$$\epsilon_{1,2}(e_m \otimes_{i_{\mathrm{vec}}} 1) = \sum_{j=0}^{n-m} \left(e_{m+j} \otimes_{i_{\mathrm{rig}}} u_j(z) \right)$$

for the map $\epsilon_{1,2} : L \otimes_{i_{\mathrm{vec}}} K \to L \otimes_{i_{\mathrm{rig}}} K$ of (9) associated to L.

Proof. Since $\log(z_0) = 0$, we have $\partial_t^i(u_j)(z_0) = 0$ for $i < j$. Substituting z to the Taylor expansion of $u_j(t)$ at $t = z_0$ gives the equality

$$u_j(z) = \sum_{i=j}^{\infty} \frac{1}{i!} \partial_t^i(u_j)(z_0)(z - z_0)^i.$$

The proposition now follows from the definition of $\epsilon_{1,2}$ (9) and Corollary 5.4.
Let us now return to the case when $z = \zeta_d$ is a primitive d-th root of unity.
Proof of Proposition 5.1. Since the connection is the only structure preventing $L_{\mathrm{dR}}^{(n)}$ and $L_{\mathrm{rig}}^{(n)}$ from splitting, we have

$$i_{\mathrm{dR}}^* L_{\mathrm{dR}}^{(n)} = \prod_{j=0}^{n} K e_{j,\mathrm{dR}} \qquad i_{\mathrm{rig}}^* L_{\mathrm{rig}}^{(n)} = \prod_{j=0}^{n} K_0 e_{j,\mathrm{rig}}.$$

It is sufficient to prove that the comparison isomorphism $i_z^*(\mathbf{p})$ respects the splitting. The isomorphism

$$\mathbf{p} : i_{\mathrm{dR}}^* L_{\mathrm{dR}}^{(n)} \to L \otimes_{i_{\mathrm{vec}}} K$$

is given by $e_{j,\mathrm{dR}} \mapsto e_{j,\mathrm{rig}}$. Since z is a torsion point, $u_j(z) = 0$ for $j \neq 0$. Hence by Proposition 5.5,

$$\epsilon_{1,2} : L \otimes_{i_{\mathrm{vec}}} K \to L \otimes_{i_{\mathrm{rig}}} K$$

maps $e_{j,\mathrm{rig}} \otimes_{i_{\mathrm{vec}}} 1$ to $e_{j,\mathrm{rig}} \otimes_{i_{\mathrm{rig}}} 1$. Hence $i_z^*(\mathbf{p}) = \epsilon_{1,2} \circ \mathbf{p}$ respects the splitting. We have

$$i_z^* \mathcal{L}og^{(n)} \cong \prod_{j=0}^{n} K(j)$$

in $S(\mathcal{O}_K)$ as desired.

REMARK 5.6 *The calculation of Proposition 5.5 shows that if z is an arbitrary element in $\mathcal{O}_K^\times$, then*

$$i_z^* \mathcal{L}og^{(n)} = (L_{z,\mathrm{dR}}^{(n)}, L_{z,\mathrm{rig}}^{(n)}, \mathbf{p}_z) \in S(\mathcal{O}_K),$$

where

$$L_{z,\mathrm{dR}}^{(n)} = \prod_{j=0}^{n} K e_{j,\mathrm{dR}}, \qquad L_{z,\mathrm{rig}}^{(n)} = \prod_{j=0}^{n} K_0 e_{j,\mathrm{rig}},$$

and

$$\mathbf{p}_z(e_{m,\mathrm{dR}}) = \sum_{j=0}^{n-m} e_{m+j,\mathrm{rig}} \otimes_{K_0} u_j(z).$$

6 THE SPECIALIZATION OF pol TO TORSION POINTS

In this section, we will first introduce the p-adic polylogarithmic extension pol calculated in [Ban1]. Then we will calculate its restriction to d-th roots of unity, where d is an integer of the form $d = Np^r$ with $(N, p) = 1$ and $N > 1$. The case $N = 1$ will be treated in Section 8.

Let $\mathbb{U}$ be the syntomic datum correspoinding to the projective line minus three points, as defined in Definition 3.2. The p-adic polylogarithm sheaf is an extension in $S(\mathbb{U})$ of the trivial object $K(0)$ by the logarithmic sheaf $\mathcal{L}og$ having a certain residue. In our previous paper, we determined the explicit shape of this sheaf.

THEOREM 6.1 ([BAN1] THEOREM 2) *The p-adic polylogarithmic extension $\mathrm{pol}^{(n)}$ is the extension*

$$0 \to \mathcal{L}og^{(n)} \to \mathrm{pol}^{(n)} \to K(0) \to 0$$

in $S(\mathbb{U})$, given explicitly by $\mathrm{pol}^{(n)} := (P_{\mathrm{dR}}^{(n)}, P_{\mathrm{rig}}^{(n)}, \mathbf{p})$, where:

(i) $P_{\mathrm{dR}}^{(n)}$ in $S_{\mathrm{dR}}(\mathbb{U})$ is given by

$$P_{\mathrm{dR}}^{(n)} = \mathcal{O}_{\mathbb{P}_K^1} e_{\mathrm{dR}} \bigoplus L_{\mathrm{dR}}^{(n)},$$

with connection $\nabla_{\mathrm{dR}}(e_{\mathrm{dR}}) = e_{1,\mathrm{dR}} \otimes d\log(t-1)$ and Hodge filtration given by the direct sum.

(ii) $P^{(n)}_{\mathrm{rig}}$ *in* $S_{\mathrm{rig}}(\mathbb{U})$ *is given by*

$$P^{(n)}_{\mathrm{rig}} = j^\dagger \mathcal{O}_{]\mathbb{U}_k[_{\widehat{\mathbb{P}}1}} e_{\mathrm{rig}} \bigoplus L^{(n)}_{\mathrm{rig}},$$

with connection $\nabla_{\mathrm{rig}}(e_{\mathrm{rig}}) = e_{1,\mathrm{rig}} \otimes d\log(t-1)$ *and Frobenius*

$$\Phi(e_{\mathrm{rig}}) := e_{\mathrm{rig}} + \sum_{j=1}^{n} (-1)^{j+1} \ell^{(p)}_j(t) e_{j,\mathrm{rig}}. \tag{14}$$

(iii) $\mathbf{p}$ *is the isomorphism given by* $\mathbf{p}(e_{\mathrm{dR}}) = e_{\mathrm{rig}} \otimes 1$.

REMARK 6.2 *In [Ban1] Theorem 2, the Frobenius is written as*

$$\Phi(e_{\mathrm{rig}}) := e_{\mathrm{rig}} + \sum_{j=1}^{n} (-1)^{j} \ell^{(p)}_j(t) e_{j,\mathrm{rig}}.$$

This is due to an error in the calculation of the proof. The correct Frobenius is the one given in (14).

Let z be a d-th root of unity, where d is an integer of the form $d = Np^r$ with $(N,p) = 1$ and $N > 1$, and let $z_0 \in W$ such that $z \equiv z_0 \pmod{\pi}$. The purpose of this section is to prove the following theorem.

THEOREM 6.3 *The specialization of the polylogarithm at z is explicitly given as follows:*

(i) $i_z^* P^{(n)}_{\mathrm{dR}} = K e_{\mathrm{dR}} \oplus \bigoplus_{j=0}^n K e_{j,\mathrm{dR}}$ *with the natural Hodge filtration.*

(ii) $i_z^* P^{(n)}_{\mathrm{rig}} = K_0 e_{\mathrm{rig}} \oplus \bigoplus_{j=0}^n K_0 e_{j,\mathrm{rig}}$ *with Frobenius*

$$\Phi(e_{\mathrm{rig}}) := e_{\mathrm{rig}} + \sum_{j=1}^{n} (-1)^{j+1} \ell^{(p)}_j(z_0) e_{j,\mathrm{rig}}.$$

(iii) $\mathbf{p}$ *is the isomorphism given by*

$$\mathbf{p}(e_{\mathrm{dR}}) = e_{\mathrm{rig}} \otimes 1 + \sum_{j=1}^{n} e_{j,\mathrm{rig}} \otimes (-1)^j (D_j(z) - D_j(z_0)),$$

where $D_j(t)$ is the function defined in Definition 2.2.

The proof of the theorem will be given at the end of this section. As in the case of $\mathcal{L}og$, we first consider the Monsky-Washnitzer interpretation of $\mathrm{pol}^{(n)}$. Let $B^\dagger_K = \Gamma(\mathbb{U}^{\mathrm{an}}_K, j^\dagger \mathcal{O}_{\mathbb{U}^{\mathrm{an}}_K})$,

$$P^{(n)}_{\mathrm{vec}} := \mathbf{F}_{\mathrm{rig}}(M_{\mathrm{rig}}) = (M_{\mathrm{rig}} \otimes_{K_0} K, \nabla_{\mathrm{rig}} \otimes_{K_0} K),$$

and $P^{(n)} = \Gamma(\mathbb{U}_K^{\mathrm{an}}, P_{\mathrm{vec}}^{(n)})$. Then we have

$$P^{(n)} = B_K^\dagger e \bigoplus \prod_{j=0}^n B_K^\dagger e_j$$

where $e = e_{\mathrm{rig}} \otimes 1$ and $e_j = e_{j,\mathrm{rig}} \otimes 1$, with connection $\nabla(e) = e \otimes d\log(1-t)$ and $\nabla(e_j) = e_{j+1} \otimes d\log t$.

PROPOSITION 6.4 *For integers $i, m > 0$, let $b_m^{(i)}$ be elements in $B_K^\dagger$ such that*

$$\nabla(\partial_t^i)(e) = \sum_{j=1}^n (-1)^j b_j^{(i)} e_j.$$

Then

$$\partial_t^i(D_m) = \sum_{j=1}^n (-1)^{m-j} b_j^{(i)} u_{m-j}.$$

In particular, we have

$$b_m^{(i)}(z_0) = \partial_t^i(D_m)(z_0). \tag{15}$$

Proof. The proof is again by induction on $i > 0$. We first consider the case when $i = 1$. In this case, $b_1^{(1)} = (1-t)^{-1}$. Since $\mathrm{Li}_{m-j}(t)$ and $u_j(t)$ satisfy the differential equations

$$\partial_t(\mathrm{Li}_j(t)) = \frac{1}{t}\,\mathrm{Li}_{j-1}(t) \quad (j \geq 1) \qquad \partial_t(u_j(t)) = \frac{u_{j-1}}{t} \quad (\forall j),$$

the definition of $D_m(t)$ (Definition 2.2) and the fact that $u_j(t) = 0$ for $j < 0$ implies that:

$$\begin{aligned}
\partial_t(D_m) &= \sum_{j=0}^{m-1} (-1)^j \partial_t(\mathrm{Li}_{m-j}(t) u_j(t)) \\
&= \sum_{j=0}^{m-1} \frac{(-1)^j}{t} \left(\mathrm{Li}_{m-j-1}(t) u_j(t) + \mathrm{Li}_{m-j}(t) u_{j-1}(t)\right) \\
&= \frac{(-1)^{m-1}}{t}\,\mathrm{Li}_0(t) u_{m-1}(t) = (-1)^{m-1} \frac{u_{m-1}(t)}{1-t} \\
&= (-1)^{m-1} b_1^{(1)}(t) u_{m-1}(t).
\end{aligned}$$

Hence the statement is true for $i = 1$. Suppose for an integer $i \geq 1$, we have

$$\partial_t^i(D_m) = \sum_{j=1}^n (-1)^{m-j} b_j^{(i)} u_{m-j}. \tag{16}$$

By comparing the definition of $b_j^{(i+1)}$ with the equality

$$\nabla(\partial_t^{i+1})(e_0) = \nabla(\partial_t) \circ \nabla(\partial_t^i)(e_0) = \sum_{j=1}^{n}(-1)^j \left((\partial_t b_j^{(i)})e_j + t^{-1}b_j^{(i)}e_{j+1}\right),$$

we obtain the equality

$$b_j^{(i+1)} = \partial_t b_j^{(i)} - t^{-1}b_{j-1}^{(i)} \qquad (i \geq 1, j > 1). \tag{17}$$

Similarly, from the hypothesis (16) and $\partial_t u_m = t^{-1}u_{m-1}$, we have

$$\begin{aligned}\partial_t^{i+1}(D_m) &= \partial_t \left(\sum_{j=1}^{i}(-1)^{m-j}b_j^{(i)}u_{m-j}\right)\\ &= \sum_{j=1}^{n}(-1)^{m-j}\left((\partial_t b_j^{(i)})u_{m-j} + t^{-1}b_j^{(i)}u_{m-j-1}\right).\end{aligned}$$

This together with (17) gives the desired result. (15) follows from the fact that since z_0 is a root of unity, $u_m(z_0) = 0$ unless $m = 0$.

PROPOSITION 6.5 *We have*

$$\epsilon_{1,2}(e \otimes_{i_{\mathrm{vec}}} 1) = e \otimes_{i_{\mathrm{rig}}} 1 + \sum_{j=1}^{n}\left(e_j \otimes_{i_{\mathrm{rig}}} (-1)^j(D_j(z) - D_j(z_0))\right)$$

for the map $\epsilon_{1,2} : P \otimes_{i_{\mathrm{vec}}} K \to P \otimes_{i_{\mathrm{rig}}} K$ *of* (9) *associated to* P.

Proof. Substituting z to the Taylor expansion of $D_j(t)$ at $t = z_0$ gives the equality

$$D_j(z) = \sum_{i=0}^{\infty}\frac{1}{i!}\partial_t^i(D_j)(z_0)(z - z_0)^i.$$

The proposition now follows from the definition of $\epsilon_{1,2}$ and Proposition 6.4.

7 THE MAIN RESULT (CASE $N > 1$)

The following lemma is well-known.

LEMMA 7.1 *There is a canonical isomorphism*

$$\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)) = K(j)_{\mathrm{dR}} \tag{18}$$

for $j > 0$.

Proof. Suppose $\widetilde{M} = (\widetilde{M}_{\mathrm{dR}}, \widetilde{M}_{\mathrm{rig}}, \widetilde{\mathbf{p}})$ is an extension of $K(0)$ by $K(j)$ in $S(\mathcal{O}_K)$. We have exact sequences

$$0 \to K(j)_{\mathrm{dR}} \to \widetilde{M}_{\mathrm{dR}} \to K(0)_{\mathrm{dR}} \to 0$$
$$0 \to K(j)_{\mathrm{rig}} \to \widetilde{M}_{\mathrm{rig}} \to K(0)_{\mathrm{rig}} \to 0.$$

Denote by $e_{j,\mathrm{dR}}$ and $e_{j,\mathrm{rig}}$ the basis of $K(j)_{\mathrm{dR}}$ and $K(j)_{\mathrm{rig}}$, and let $\widetilde{e}_{0,\mathrm{dR}}$ and $\widetilde{e}_{0,\mathrm{rig}}$ respectively be the liftings of $e_{0,\mathrm{dR}}$ and $e_{0,\mathrm{rig}}$ in $\widetilde{M}_{\mathrm{dR}}$ and $\widetilde{M}_{\mathrm{rig}}$. If we map $\widetilde{e}_{0,\mathrm{dR}}$ to $e_{0,\mathrm{dR}}$, then we have an isomorphism

$$\widetilde{M}_{\mathrm{dR}} \cong K(0)_{\mathrm{dR}} \bigoplus K(j)_{\mathrm{dR}}$$

in $S_{\mathrm{dR}}(\mathcal{O}_K)$. Next, since the quotient of M by $K(j)$ is isomorphic to $K(0)$, the Frobenius and $\widetilde{\mathbf{p}}$ is given by

$$\begin{aligned} \widetilde{\mathbf{p}}(\widetilde{e}_{0,\mathrm{dR}}) &= \widetilde{e}_{0,\mathrm{rig}} \otimes 1 + e_{j,\mathrm{rig}} \otimes a \\ \phi^*(\widetilde{e}_{0,\mathrm{rig}}) &= \widetilde{e}_{0,\mathrm{rig}} + c e_{j,\mathrm{rig}} \end{aligned}$$

for some $a \in K$ and $c \in K_0$. If we take $b \in K_0$ such that $(1 - \sigma/p^j)b = c$, then we have an isomorphism

$$\widetilde{M}_{\mathrm{rig}} \cong K(0)_{\mathrm{rig}} \bigoplus K(j)_{\mathrm{rig}}$$

in $S_{\mathrm{rig}}(\mathcal{O}_K)$ given by $\widetilde{e}_{0,\mathrm{rig}} \mapsto e_{0,\mathrm{rig}} - b e_{j,\mathrm{rig}}$. The above shows that we have an isomorphism

$$\widetilde{M} \cong \left(K(0)_{\mathrm{dR}} \bigoplus K(j)_{\mathrm{dR}},\, K(0)_{\mathrm{rig}} \bigoplus K(j)_{\mathrm{rig}},\, \mathbf{p} \right)$$

of extensions of $K(0)$ by $K(j)$ in $S(\mathcal{O}_K)$, where $\mathbf{p}$ is the isomorphism given by

$$\begin{aligned} \mathbf{p}(e_{0,\mathrm{dR}}) &= \widetilde{e}_{0,\mathrm{rig}} \otimes 1 + e_{j,rig} \otimes a \\ &= e_{0,\mathrm{rig}} \otimes 1 + e_{j,\mathrm{rig}} \otimes (a+b). \end{aligned}$$

The canonical map of the lemma is given by associating to $\widetilde{M}$ the element $(a+b)e_{j,\mathrm{dR}}$ in $K(j)_{\mathrm{dR}}$.
The inverse of this canonical map is constructed by associating to $w e_{j,\mathrm{dR}}$ in $K(j)_{\mathrm{dR}}$ the extension

$$\left(K(0)_{\mathrm{dR}} \bigoplus K(j)_{\mathrm{dR}},\, K(0)_{\mathrm{rig}} \bigoplus K(j)_{\mathrm{rig}},\, \mathbf{p} \right),$$

where

$$\mathbf{p}(e_{0,\mathrm{dR}}) = e_{0,\mathrm{rig}} \otimes 1 + e_{j,\mathrm{rig}} \otimes w.$$

This construction shows that the canonical map is in fact an isomorphism.

Remark 7.2 *Suppose $K = K_0$. Then by [Ban1] Theorem 1 and Example 2.8, we have an isomorphism*

$$\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)) \xrightarrow{\cong} H^1_{\mathrm{syn}}(\mathcal{O}_K, K(j)) = K(j)_{\mathrm{rig}}. \tag{19}$$

If M is an extension in $S(\mathcal{O}_K)$ corresponding to $ae_{j,\mathrm{dR}}$ in Lemma 7.1, then M maps by (19) *to $((1-p^{-j}\sigma)a)e_{j,\mathrm{rig}}$ in $K(j)_{\mathrm{rig}}$.*

The following theorem is Theorem 1 of the introduction.

Theorem 7.3 *Let z be a torsion point of order $d = Np^r$, where $(N,p) = 1$ and $N > 1$. Then*

$$i_z^* \operatorname{pol}^{(n)} = ((-1)^j \operatorname{Li}_j(z) e_{j,\mathrm{dR}})_{j\geq 1}$$

in

$$\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), i_z^*\mathcal{L}og(1)) = \prod_{j=0}^{n} \mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)),$$

where we view $(-1)^j \operatorname{Li}_j(z)e_{j,\mathrm{dR}}$ as an element in $\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ through the isomorphism of lemma 7.1

Proof. By Theorem 6.3, the image of $i_z^* \operatorname{pol}^{(n)}$ in $\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ is the extension $\widetilde{M} = (M_{\mathrm{dR}}, \widetilde{M}_{\mathrm{rig}}, \widetilde{\mathbf{p}})$ given as follows: M_{dR} is the direct sum

$$M_{\mathrm{dR}} = K(0)_{\mathrm{dR}} \bigoplus K(j)_{\mathrm{dR}},$$

$\widetilde{M}_{\mathrm{rig}}$ is the extension of $K(0)_{\mathrm{rig}}$ by $K(j)_{\mathrm{rig}}$ with the Frobenius given by

$$\Phi(\widetilde{e}_{0,\mathrm{rig}}) = \widetilde{e}_{0,\mathrm{rig}} + (-1)^{j+1}\ell_j^{(p)}(z_0)e_{j,\mathrm{rig}}$$

for the lifting $\widetilde{e}_{0,\mathrm{rig}}$ of $e_{0,\mathrm{rig}}$ in $\widetilde{M}_{\mathrm{rig}}$, and $\widetilde{\mathbf{p}}$ is the isomorphism given by

$$\widetilde{\mathbf{p}}(e_{0,\mathrm{dR}}) = \widetilde{e}_{0,\mathrm{rig}} \otimes 1 + e_{j,\mathrm{rig}} \otimes (-1)^j(\operatorname{Li}_j(z) - \operatorname{Li}_j(z_0)).$$

This implies that, in the notation of Lemma 7.1, we have

$$\begin{aligned} a &= (-1)^j(\operatorname{Li}_j(z) - \operatorname{Li}_j(z_0)) \\ c &= (-1)^{j+1}\ell_j^{(p)}(z_0). \end{aligned}$$

Since z_0 is a root of unity prime to p, the Frobenius acts by $\sigma(z_0) = z_0^p$. Hence the Formula of Propisition 2.1 (iii) gives

$$\ell_j^{(p)}(z_0) = \left(1 - \frac{\sigma}{p^j}\right)\operatorname{Li}_j(z_0).$$

Again, in the notation of Lemma 7.1, we have

$$c = (-1)^{j+1}\operatorname{Li}_j(z_0).$$

Since $a + b = (-1)^j \operatorname{Li}_j(z)$, the construction of the canonical map shows that the image of $i_z^* \operatorname{pol}^{(n)}$ in $\mathrm{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ maps to $(-1)^j \operatorname{Li}_j(z)e_{j,\mathrm{dR}}$ in $K(j)_{\mathrm{dR}}$.

8 The main result (Case $N = 1$)

In this section, we will consider the specialization of the polylogarithm sheaf to p-th power roots of unity. As mentioned in the introduction, we will consider a slightly modified version of the polylogarithm. Let $c > 1$ be an integer prime to p, and let $\mathbb{U}^0_{c,\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[t, (1-t^c)^{-1}]$. We denote by $\mathbb{U}^0_c$ the syntomic data

$$\mathbb{U}^0_c = (\mathbb{U}^0_{c,\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \widehat{\mathbb{P}}^1, \phi).$$

The multiplication by $[c]$ map on $\mathbb{G}_{m\mathcal{O}_K}$ defines a morphism of syntomic datum

$$[c] : \mathbb{U}^0_c \to \mathbb{U}.$$

Definition 8.1 *We define the modified p-adic polylogarithmic* $\mathrm{pol}^{(n)}_c$ *by*

$$\mathrm{pol}^{(n)}_c = \mathrm{pol}^{(n)} - [c]^* \mathrm{pol}^{(n)} \in \operatorname{Ext}^1_{S(\mathbb{U}^0_c)}(K(0), \mathcal{L}og^{(n)}).$$

The explicit shape of $\mathrm{pol}^{(n)}$ given in Theorem 6.1 and the definition of the pull-back $[c]^*$ gives the following proposition. Let

$$\theta_c(t) = \frac{1-t^c}{1-t}.$$

Proposition 8.2 *The modified p-adic polylogarithmic* $\mathrm{pol}^{(n)}_c$ *is the extension in $S(\mathbb{U}^0_c)$, given explicitly by* $\mathrm{pol}^{(n)}_c := (P^{(n)}_{\mathrm{dR}}, P^{(n)}_{\mathrm{rig}}, \mathbf{p})$, *where:*

(i) $P^{(n)}_{\mathrm{dR}}$ *in* $S_{\mathrm{dR}}(\mathbb{U}^0_c)$ *is given by*

$$P^{(n)}_{\mathrm{dR}} = \mathcal{O}_{\mathbb{P}^1_K} e_{\mathrm{dR}} \bigoplus L^{(n)}_{\mathrm{dR}},$$

with connection $\nabla_{c,\mathrm{dR}}(e_{\mathrm{dR}}) = e_{1,\mathrm{dR}} \otimes d \log \theta_c(t)$ *and Hodge filtration given by the direct sum.*

(ii) $P^{(n)}_{\mathrm{rig}}$ *in* $S_{\mathrm{rig}}(\mathbb{U}^0_c)$ *is given by*

$$P^{(n)}_{\mathrm{rig}} = j^\dagger \mathcal{O}_{]\mathbb{U}^0_{c,k}[_{\widehat{\mathbb{P}}^1}} e_{\mathrm{rig}} \bigoplus L^{(n)}_{\mathrm{rig}},$$

with connection $\nabla_{c,\mathrm{rig}}(e_{\mathrm{rig}}) = e_{1,\mathrm{rig}} \otimes d \log \theta_c(t)$ *and Frobenius*

$$\Phi(e_{\mathrm{rig}}) := e_{\mathrm{rig}} + \sum_{j=1}^{n} (-1)^{j+1} \ell^{(p)}_{j,c}(t) e_{j,\mathrm{rig}},$$

(iii) $\mathbf{p}$ *is the isomorphism given by* $\mathbf{p}(e_{\mathrm{dR}}) = e_{\mathrm{rig}} \otimes 1$.

Let $\mathbb{U}_{c,\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[t, \theta_c(t)^{-1}]$, and denote by $\mathbb{U}_c$ the syntomic data

$$\mathbb{U}_c = (\mathbb{U}_{c,\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \widehat{\mathbb{P}}^1, \phi).$$

The explicit shape of $\mathrm{pol}_c^{(n)}$ given in the previous proposition shows that $\mathrm{pol}_c^{(n)}$ is in fact an object in $S(\mathbb{U}_c)$. In particular, we can specialize $\mathrm{pol}_c^{(n)}$ at points on the open unit disc around one.
Similar caluclations as that of Theorem 6.3 with $\ell_j^{(p)}$, $D_j^{(p)}$ and D_j replaced by $\ell_{j,c}^{(p)}$, $D_{j,c}^{(p)}$ and $D_{j,c}$ gives the following theorem, which is Theorem 2 of the introduction.

Theorem 8.3 *Let z be a p^r-th root of unity, and let $z_0 = 1$. Then the specialization of the modified polylogarithm at z is explicitly given as follows:*

(i) $i_z^* P_{\mathrm{dR}}^{(n)} = K e_{\mathrm{dR}} \oplus \bigoplus_{j=0}^n K e_{j,\mathrm{dR}}$ *with the natural Hodge filtration.*

(ii) $i_z^* P_{\mathrm{rig}}^{(n)} = K e_{\mathrm{rig}} \oplus \bigoplus_{j=0}^n K e_{j,\mathrm{rig}}$ *with Frobenius*

$$\Phi(e_{\mathrm{rig}}) := e_{\mathrm{rig}} + \sum_{j=1}^n (-1)^{j+1} \ell_{j,c}^{(p)}(z_0) e_{j,\mathrm{rig}}.$$

(iii) $\mathbf{p}_c$ *is the isomorphism given by*

$$\mathbf{p}_c(e_{\mathrm{dR}}) = e_{\mathrm{rig}} \otimes 1 + \sum_{j=1}^n e_{j,\mathrm{rig}} \otimes (-1)^j (D_{j,c}(z) - D_{j,c}(z_0)).$$

As a corollary, we obtain the following result.

Corollary 8.4 *Let z be a torsion point of order p^r. Then*

$$i_z^* \,\mathrm{pol}_c^{(n)} = ((-1)^j \operatorname{Li}_j(z) e_{j,\mathrm{dR}})_{j \geq 1}$$

in

$$\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), i_z^* \mathcal{L}og(1)) = \prod_{j=0}^n \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j)),$$

where we view $(-1)^j \operatorname{Li}_{j.c}(z) e_{j,\mathrm{dR}}$ as an element in $\operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), K(j))$ through the isomorphism of lemma 7.1

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