Around Hilbert's 17th Problem

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The starting point of the history of Hilbert's 17th problem was the oral defense of the doctoral dissertation of Hermann Minkowski at the University of Königsberg in 1885. The 21 year old Minkowski expressed his opinion that there exist real polynomials which are nonnegative on the whole $\mathbb{R}^n$ and cannot be written as finite sums of squares of real polynomials. David Hilbert was an official opponent in this defense. In his "Gedächtnisrede" [6] in memorial of H. Minkowski he said later that Minkowski had convinced him about the truth of this statement. In 1888 Hilbert proved in a now famous paper [4] the existence of a real polynomial in two variables of degree six which is nonnegative on $\mathbb{R}^2$ but not a sum of squares of real polynomials. Hilbert's proof used some basic results from the theory of algebraic curves. Apart from this his construction is completely elementary. The first *explicit* example of this kind was given by T. Motzkin [10] only in 1967. It is the polynomial

$$M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2.$$

(Indeed, the arithmetic-geometric mean inequality implies that $M \geq 0$ on $\mathbb{R}^2$. Assume to the contrary that $M = \sum_j f_j^2$ is a sum of squares of real polynomials. Since $M(0,y) = M(x,0) = 1$, the polynomials $f_j(0,y)$ and $f_j(x,0)$ are constants. Hence each f_j is of the form $f_j = a_j + b_jxy + c_jx^2y + d_jxy^2$. Then the coefficient of x^2y^2 in the equality $M = \sum_j f_j^2$ is equal to $-3 = \sum_j b_j^2$. This is a contradiction.)

A nice exposition around Hilbert's construction and many examples can be found in [16]. Hilbert also showed in [4] that each nonnegative polynomial in two variables of degree four *is* a finite sum of squares of polynomials.

As usual we denote by $\mathbb{R}[x_1,\dots,x_n]$ and $\mathbb{R}(x_1,\dots,x_n)$ the ring of polynomials resp. the field of rational functions in $x_1,\dots,x_n$ with real coefficients.

The second pioneering paper [5] of Hilbert about this topic appeared in 1893. He proved by an ingenious and difficult reasoning that each nonnegative polynomial $p \in \mathbb{R}[x,y]$ on $\mathbb{R}^2$ is a finite sum of squares of rational (!) functions from $\mathbb{R}(x,y)$. Though not explicitly stated therein a closer look at Hilbert's proof shows even that p is a sum of *four* squares. For Motzkin's polynomial one has the identity

$$M(x,y) = \frac{x^2y^2(x^2+y^2+1)(x^2+y^2-2)^2+(x^2-y^2)^2}{(x^2+y^2)^2}$$

which gives a representation of M as a sum of four squares of rational functions.

Motivated by his previous work Hilbert posed his famous 17th problem at the International Congress of Mathematicians in Paris (1900):

Hilbert's 17th problem:
Suppose that $f \in \mathbb{R}(x_1, \dots, x_n)$ is nonnegative at all points of $\mathbb{R}^n$ where f is defined. Is f a finite sum of squares of rational functions?

A slight reformulation of this problem is the following: Is each polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ which is nonnegative on $\mathbb{R}^n$ a finite sum of squares of rational functions, or equivalently, is there an identity $q^2 f = \sum_j p_j^2$, where $q, p_1, \cdots, p_k \in \mathbb{R}[x_1, \dots, x_n]$ and $q \neq 0$. In the case $n = 1$ this is true, since the fundamental theorem of algebra implies that each nonnegative polynomial in one variable is a sum of two squares of real polynomials. As noted above, the case $n = 2$ was settled by Hilbert [5] itself. Hilbert's 17th problem was solved in the affirmative by Emil Artin [1] in 1927. Using the Artin-Schreier theory of ordered fields Artin proved

Theorem 1. *If $f \in \mathbb{R}[x_1, \cdots, x_n]$ is nonnegative on $\mathbb{R}^n$, then there are polynomials $q, p_1, \dots, p_k \in \mathbb{R}[x_1, \cdots, x_n]$, $q \neq 0$, such that*

$$f = \frac{p_1^2 + \cdots + p_k^2}{q^2}.$$

Artin's proof of this theorem is nonconstructive. For strictly positive polynomials f (that is, $f(x) > 0$ for all $x \in \mathbb{R}^n$) a constructive method was developed by Habicht [3]. It is based on Polya's theorem [13] which states that for each homogeneous polynomial p such that $p(x_1, \dots, x_n) > 0$ for all $x_1 \geq 0, \cdots, x_n \geq 0$ and $(x_1, \dots, x_n) \neq 0$, there exists a natural number N such that all coefficients of the polynomial $(x_1 + \cdots + x_n)^N p$ are positive. A quantitative version of Polya's theorem providing a lower estimate for the number N in terms of p was recently given by Powers and Reznick [14].

There is also a *quantitative* version of Hilbert's 17th problem which asks how many squares are needed. In mathematical terms it can be formulated in terms of the pythagoras number. For a ring K, the pythagoras number $p(K)$ is the smallest natural number m such that each finite sum of squares of elements of K is a sum of m squares. If there is no such number m we set $p(K) = \infty$. Clearly, $p(\mathbb{R}[x]) = p(\mathbb{R}(x)) = 2$. Recall that Hilbert [5] had shown that $p(\mathbb{R}(x,y)) \leq 4$. The landmark result on the quantitative version of Hilbert's 17th problem was published in 1967 by A. Pfister [11] who proved

Theorem 2. $p(\mathbb{R}(x_1, \cdots, x_n)) \leq 2^n$.

That is, by Theorems 1 and 2, each nonnegative polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is a sum of at most 2^n squares of rational functions. Pfister's proof was based on the theory of multiplicative forms (see, e.g., [12]), now also called Pfister forms.

The next natural question is: What is value of the number $p(\mathbb{R}(x_1, \dots, x_n))$? For $n \geq 3$ this is still unknown! It is not difficult to prove that the sum $1 + x_1^2 + \dots + x_n^2$ of $n+1$ squares is not a sum of m squares with $m < n+1$. Therefore

$$n + 1 \leq p(\mathbb{R}(x_1, \dots, x_n)) \leq 2^n.$$

Using the theory of elliptic curves over algebraic function fields it was shown in [2] that Motzkin's polynomial is not a sum of 3 squares. Hence $p(\mathbb{R}(x_1, x_2)) = 4$.

Artin's theorem triggered many further developments. The most important one in the context of optimization is to look for polynomials which are nonnegative on sets defined by polynomial inequalities rather than the whole $\mathbb{R}^n$. To formulate the corresponding result some preliminaries are needed. Let us write $\sum_n^2$ for the cone of finite sums of squares of polynomials from $\mathbb{R}[x_1, \dots, x_n]$.

In what follows we suppose that $F = \{f_1, \dots, f_k\}$ is a finite subset of $\mathbb{R}[x_1, \dots, x_n]$. In real algebraic geometry two fundamental objects are associated with F. These are the *basic closed semialgebraic set*

$$K_F = \{x \in \mathbb{R}^n : f_1(x) \geq 0, \cdots, f_k(x) \geq 0\}$$

and the *preorder*

$$T_F := \left\{ \sum_{\varepsilon_i \in \{0,1\}} f_1^{\varepsilon_1} \cdots f_k^{\varepsilon_k} \sigma_\varepsilon;\ \sigma_\varepsilon \in \sum\nolimits_n^2 \right\}.$$

Note that the preorder T_F depends on the set F of generators for the semialgebraic set K_F rather than the set K_F itself.

Obviously, all polynomials from T_F are nonnegative on the set K_F, but in general T_F does not exhaust the nonnegative polynomials on K_F. The Positivstellensatz of Krivine-Stengle describes all nonnegative resp. positive polynomials on the semialgebraic set K_F in terms of *quotients* of elements of the preorder T_F.

Theorem 3. *Let $f \in \mathbb{R}[x_1, \dots, x_n]$.*

(i) *$f(x) \geq 0$ for all $x \in K_F$ if and only if there exist $p, q \in T_F$ and $m \in \mathbb{N}$ such that $pf = f^{2m} + q$.*

(ii) *$f(x) > 0$ for all $x \in K_F$ if and only if there are $p, q \in T_F$ such that $pf = 1 + q$.*

This theorem was proved by G. Stengle [19], but essential ideas were already contained in J.-L. Krivine's paper [8]. In both assertions (i) and (ii) the 'if' parts are almost trivial. Theorem 3 is a central result of modern real algebraic geometry. Proofs based on the Tarski-Seidenberg transfer principle can be found in [15] and [9].

Let us set $f_1 = 1$ and $k = 1$ in Theorem 3(i). Then $K_F = \mathbb{R}^n$ and $T_F = \sum_n^2$. Hence in this special case Theorem 3(i) gives Artin's Theorem 1. The Krivine–Stengle Theorem 3(i) expresses the nonnegative polynomial f on K_F as a quotient of the two polynomials $f^{2m} + q$ and p from the preorder T_F. Simple examples show that the denominator polynomial p cannot be avoided in general. For instance, if $f_1 = 1$, $k = 1$, the Motzkin polynomial M is nonnegative on $K_F = \mathbb{R}^n$, but it is not in the preorder $T_F = \sum_n^2$. Replacing M by the polynomial $\tilde{M}(x,y) := x^4y^2 + x^2y^4 + 1 - x^2y^2$ we even get a strictly positive polynomial of this kind. (One has $\tilde{M}(x,y) \geq \frac{26}{27}$ for all $(x,y) \in \mathbb{R}^2$.) Letting $f_1 = (1-x^2)^3$, $k = n = 1$, the semialgebraic set K_F is the interval $[-1,1]$ and the polynomial $f = 1-x^2$ is obviously nonnegative on K_F. Looking at the orders of zeros of f at ± 1 one concludes easily that f is not in T_F. In view of these examples it seems to be surprising that strictly positive polynomials on a compact basic closed semialgebraic set always belong to the preorder. This result is the Archimedean Positivstellensatz which was proved by the author [17] in 1991.

Theorem 4. *Suppose that $f \in \mathbb{R}[x_1, \ldots, x_n]$. If the set K_F is compact and $f(x) > 0$ for all $x \in K_F$, then $f \in T_F$.*

The original proof given in [17] (see also [18], pp. 344–345) was based on the solution of the moment problem for compact semialgebraic sets. The first algebraic proof of Theorem 4 was found by T. Wörmann [20], see, e.g., [15] or [9].

By definition the preorder T_F is the sum of sets $f_1^{\varepsilon_1} \cdots f_k^{\varepsilon_k} \sum_n^2$. It is natural to ask how many terms of this kind are really needed. This question is answered by a result of T. Jacobi and A. Prestel in 2001. Let $g_1, \ldots, g_{l_k}$ denote the first $l_k := 2^{k-1} + 1$ polynomials of the following row of mixed products with no repeated factors of the generators $f_1, \ldots f_k$:

$$1,\ f_1, \ldots, f_k,\ f_1f_2, f_1f_3, \ldots, f_{k-1}f_k,\ f_1f_2f_3, \ldots, f_{k-2}f_{k-1}f_k,\ f_1f_2\cdots f_k.$$

Let S_F be the sum of sets $g_j \sum_n^2$, where $j = 1, \ldots, l_k$. Then Jacobi and Prestel [7] proved the following

Theorem 5. *If K_F is compact and $f \in \mathbb{R}[x_1, \ldots, x_n]$ satisfies $f(x) > 0$ for all $x \in K_F$, then $f \in S_F$.*

We briefly discuss this result. If $k = 3$, then $l_k = 5$ and $S_F = \sum_n^2 + f_1 \sum_n^2 + f_2 \sum_n^2 + f_3 \sum_n^2 + f_1f_2 \sum_n^2$, that is, the sets $g \sum_n^2$ for $g = f_1f_3, f_2f_3, f_1f_2f_3$ do not enter in the definition of S_F. If $k = 4$, then no products of three or four generators occur in the definition of S_F. Thus, if $k \geq 3$, Theorem 5 is an essential strengthening of Theorem 4.

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