The Cold War and the Maximum Principle of Optimal Control

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Abstract. By the end of World War II, the next global confrontation emerged: the confrontation between the USA and the USSR and their allies, so between the West and the East with their antagonistic fundamental political values and their ideological contradiction. This development may be seen as a consequence of Marxism-Leninism and its claim for the world revolution or as a consequence of the political and economical structure of the USA with its permanent pursuit of new markets. All this had had also consequences for mathematicians, because the flow of information, though not completely cut, was not as easy as before. Looking positively at side effects, however, the isolated research may have not been burdened by traditional thinking and that may have been fruitful. Russian mathematicians around Pontryagin in the Steklov Institute got, with the maximum principle, new results beyond former frontiers while the Americans around Hestenes at the RAND corporation were captured by the tradition of the Chicago School of the Calculus of Variations. Nevertheless, both groups paved the way for a new field in mathematics called Optimal Control – and their protagonists fell out with each other inside their groups.

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With the advent of the Cold War mathematicians were immediately involved in the new global confrontation. A mathematical challenge of those times with

This article is an easy-to-read and considerably shortened version of the authors' paper entitled *The Maximum Principle of Optimal Control: A History of Ingenious Ideas and Missed Opportunities* [see Pesch and Plail (2009)], enriched by some anectodes. The conclusions therein and also here are extracted from the second author's monograph on the development of optimal control theory from its commencements until it became an independent discipline in mathematics; see Plail (1998).

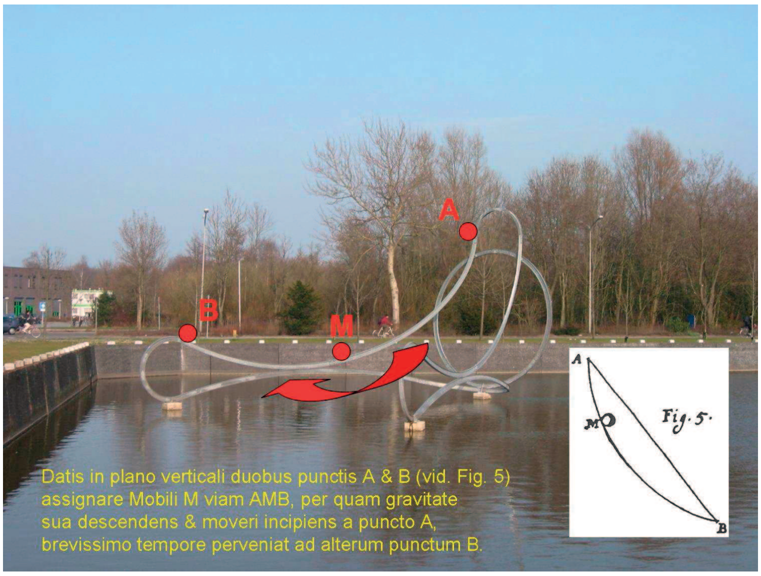

Figure 1: Johann Bernoulli's price question of 1696 and its solution which was realized in the Zernike Science Park of the University of Groningen. This monument was erected in 1996 to honor one of the most famous former members of its faculty, Johann Bernoulli, who had been a professor there from 1695 to 1705.

which they were confronted was: What is the optimal trajectory of an aircraft that is to be steered from a given cruise position into a favorable position against an attacking hostile aircraft? This problem became later known as the minimum-time-to-climb problem. It is the problem of determining the minimum-time aircraft trajectory between two fixed points in the range-altitude space.

On the first glance, the answer to this question seems be easy. Every mathematician would immediately recognize its similarity to the famous prize question of Johann Bernoulli from 1696: what is the curve of quickest descent between two given points in a vertical plane (Fig. 1).[1] This problem is considered to be the foundation stone of the Calculus of Variations to which so many famous mathematicians have contributed as the Bernoulli brothers Jacob and Johann, Euler, Lagrange, Legendre, Jacobi, Weierstrass, Hilbert, and Carathéodory to mention only a few. Hence the calculus of variations should help to find a solution. On the other hand there was something hidden in those problems which was new and could not be revealed by the calculus of variations.

[1]Bernoulli, Johann, Problema novum ad cujus solutionem Mathematici invitantur, *Acta Eruditorum*, pp. 269, 1696; see also *Johannis Bernoulli Basileensis Opera Omnia*, Bousquet, Lausanne and Geneva, Switzerland, Joh. Op. XXX (pars), t. I, p. 161, 1742.

The following historical development will show that it is sometimes better to know too little than too much. Unbelievable? In mathematics?

1 The protagonists

Who were the mathematicians in this competition? Well, there were Magnus R. Hestenes (1906–1991), Rufus P. Isaacs (1914–1981), and Richard E. Bellman (1920–1984) in the "blue corner" (see Fig. 2) and Lev Semyonovich Pontryagin (1908–1988), Vladimir Grigorevich Boltyanskii (born 1925), and Revaz Valerianovich Gamkrelidze (born 1927) in the "red corner" (see Fig. 3).

All members of the blue corner later complained about their missed opportunities. In contrast, the names of all members of the red corner will for ever be connected with the maximum principle, since the proof of the maximum principle designated the birth of a new field in applied mathematics named optimal control, which has, and continues to have, a great impact on optimization theory and exciting applications in almost all fields of sciences.

2 How did it happen?

Initially, engineers attempted to tackle such minimum-time interception problems for fighter aircraft. Due to the increased speed of aircraft, nonlinear terms no longer could be neglected. However, linearisation was not the preferred method. The engineers confined themselves to simplified models and achieved improvements step by step. For example, Angelo Miele's (born 1922) solution for a simplified flight path optimization problem from the 1950s (with the flight path angle as control variable) exhibits an early example what later became known as bang – singular – bang switching structure (in terms of aerospace engineering: vertical zoom climb – a climb along a singular subarc – vertical dive). As early as 1946, Dmitry Yevgenyevich Okhotsimsky (1921–2005) solved the specific problem of a vertically ascending rocket to achieve a given final altitude with a minimum initial mass.[2] His solution consists of a motion with maximum admissible thrust, an ascent with an optimal relation between velocity and altitude, and finally a phase with thrust turned off.[3]

However, mathematicians like to have *general* solution methods, or at least solution methods for a large class of equivalent problems.

[2]This problem was firstly posed by Georg Karl Wilhelm Hamel (1877–1954) in 1927. Hamel's and Okhotsimsky's problem has to be distinguished from Robert Goddard's (1882–1945) earlier problem of 1919. In his problem the maximum altitude was sought which a rocket can reach with a given initial mass. The rocket pioneer Goddard is the eponym of the Goddard Space Flight Center in Greenbelt, Maryland.

[3]Okhotsimsky contributed to the planning of multiple space missions including launches to Moon, Mars and Venus – and the launch of the first Sputnik satellite in 1957.

Figure 2: The mathematicians at RAND: Magnus R. Hestenes, Rufus P. Isaacs, and Richard E. Bellman (Credits: Magnus R. Hestenes: Thanks to Dr. Ronald F. Boisvert, Mathematical and Computational Science Division of the Information Technology Laboratory at the National Institute of Standards and Technology in Gaithersburg, Maryland, who got this photo as part of a collection of photographs owned by John Todd (1911–2007), a professor of mathematics and a pioneer in the field of numerical analysis. John Todd worked for the British Admiralty during World War II. One of Todd's greatest achievements was the preservation of the Mathematical Research Institute of Oberwolfach in Germany at the end of the war. Rufus P. Isaacs: Painting by Esther Freeman. Thanks to Mrs. Rose Isaacs, Po-Lung Yu, and Michael Breitner; see P. L. Yu: An appreciation of professor Rufus Isaacs. *Journal of Optimization Theory and Applications* 27 (1), 1979, 1–6. Richard E. Bellman: `http://www.usc.edu/academe/faculty/research/ethical_conduct/index.html`.)

3 The traditionalists

After the end of World War II, the RAND coorporation (Research ANd Development) was set up by the United States Army Air Force at Santa Monica, California, as a nonprofit think tank focussing on global policy issues to offer research and analysis to the United States armed forces. Around the turn of the decade in 1950 and thereafter, RAND employed three great mathematicians, partly at the same time.

3.1 Magnus R. Hestenes

Around 1950, Hestenes wrote his two famous RAND research memoranda No. 100 and 102; see Hestenes (1949, 1950). In these reports, Hestenes developed a guideline for the numerical computation of minimum-time aircraft trajectories. In particular, Hestenes' memorandum RM-100 includes an early formulation of what later became known as the maximum principle: the optimal control vector (the angle of attack and the bank angle) has to be chosen in such a way that it maximizes the so-called Hamiltonian function along a minimizing trajectory.

In his report, we already find the clear formalism of optimal control problems with its separation into state and control variables. The state variables are determined by differential equations, here the equations of motion of an aircraft. The control variables represent the degrees of freedom which the pilot has in hand to steer the aircraft – and, if mathematicians are sitting behind him, to do this in an optimal way.

In the language of mathematics, Hestenes' problem reads as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\,\vec{v}) = \vec{T} + \vec{L} + \vec{D} + \vec{W}\,,$$
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \dot{W}(v, T, h)\,,$$

where the lift vector $\vec{L}$ and the drag vector $\vec{D}$ are known functions of the angle of attack α and the bank angle β; engineers have to give mathematicians this information. The weight vector $\vec{W}$ has the length w, m is the vehicle's mass assumed to be constant due to the short maneuver time. The thrust vector T is represented as a function of velocity $v = |\vec{v}|$ and altitude h. Then the trajectory is completely determined by the initial values of the position vector $\vec{r}$, the velocity vector $\vec{v}$ and the norm w of $\vec{W}$ as well as by the values of $\alpha(t)$ and $\beta(t)$ along the path.

The task now consists of determining the functions $\alpha(t)$ and $\beta(t)$, $t_1 \leq t \leq t_2$, in such a way that the flight time t_2 is minimized with respect to all paths which fulfill the differential equations and have prescribed initial and terminal conditions for $\vec{r}(t_1)$, $\vec{v}(t_1)$, $w(t_1)$, $\vec{r}(t_2)$, $\vec{v}(t_2)$, and $w(t_2)$.

3.2 Richard E. Bellman and Rufus P. Isaacs

Also in the early 1950s, Richard Bellman worked at RAND on multi-stage decision problems. Extending Bellman's principle of optimality,[4] it is possible to derive a form of a maximum principle. Bellman in his autobiography:

> *I should have seen the application of dynamic programming to control theory several years before. I should have, but I did not.*

One of Bellman's achievements is his criticism of the calculus of variations because of the impossibility of solving the resulting two-point boundary-value problems for nonlinear differential equations at that time.

Finally, Isaacs, the father of differential games, complained with respect to his "tenet of transition" from the early 1950s:

> *Once I felt that here was the heart of the subject ... Later I felt that it ... was a mere truism. Thus in (my book) "Differential Games" it is mentioned only by title. This I regret. I had no idea, that Pontryagin's principle and Bellman's maximal principle (a special case*

[4]based on Bellman's equation which can already be found in Carathéodory's earlier work of 1926. See Pesch (2012) and the references cited therein.

> *of the tenet, appearing little later in the RAND seminars) would enjoy such widespread citation.*

Indeed, Isaacs' tenet represents an even more general minimaximum principle. However, he had the greatness to understand:

> *The history of mathematics has often shown parallel evolution when the time was ripe.*

3.3 Priority quarrel in the blue corner

Concerning the matter of priority between Isaacs' tenet of transition and Bellman's principle of optimality, there was some level of contention between Isaacs and Bellman, as the following personal remembrance of Isaacs' colleague at RAND, Wendell H. Fleming, indicates:

> *One day in the early 1950s, Bellman was giving a seminar at RAND in which he solved some optimization problems by dynamic programming. At the end of Bellman's seminar lecture, Isaacs correctly stated that this problem could also be solved by his own methods. Bellman disagreed. After each of the two reiterated his own opinion a few times, Isaacs said: "If the Bellman says it three times, it must be true." This quote refers to a line from Lewis Carroll's nonsense tail in verse "The Hunting of the Snark". One of the main (and other absurd) characters in this tale is called the Bellman.*[5]

Last but not least, Hestenes also claimed in a letter to Saunders MacLane:

> *It turns out that I had formulated what is now known as the general optimal control problem. I wrote it up as a RAND report and it was widely circulated among engineers. I had intended to rewrite the results for publication elsewhere and did so about 15 years later.*

As a reason for the delay, he mentioned his workload as chairman at the University of Southern California and his duties at the Institute for Numerical Analysis.

3.4 Sometimes it may be better to know less

Hestenes was a student of Gilbert Ames Bliss (1876–1951) and an academic grandchild of Oskar Bolza (1857–1942)[6] from the famous Chicago School of

[5]The Hunting of the Snark (An Agony in 8 Fits) is usually thought of as a nonsense poem written by Lewis Carroll, the author of *Alice's Adventures in Wonderland*. This poem describes *with infinite humour the impossible voyage of an improbable crew to find an inconceivable creature*; cf. Martin Gardner: The Annotated Snark, Penguin Books, 1974.

[6]Mathematicians like to track their academic relationships; cf. the Mathematics Genealogy Project: `http://genealogy.math.ndsu.nodak.edu/`.

the Calculus of Variations. Bolza in turn was a student of Felix Christian Klein (1849–1925) and Karl Theodor Wilhelm Weierstrass (1815–1897). He had attended Weierstrass' famous 1879 lecture course on the calculus of variations. This course might have had a lasting effect on the direction Bolza's mathematical interests have taken and that he has passed on to his descendants. In this tradition, Hestenes' derivation of his maximum principle fully relied on Weierstrass' necessary condition (and the Euler-Lagrange equation), in which the control functions are assumed to be continuous and to have values in an open control domain. These assumptions were natural for Hestenes' illustrative example of minimum time interception, but have obfuscated the potential of this principle.

It may be that Hestenes' deep knowledge of the calculus of variations, standing in the tradition of the Chicago School, was his drawback. This may have caused Hestenes not to find the hidden secrets behind those problems. Since certain optimal control problems such as Hestenes' interception problem can be classified as problems of the calculus of variations, this may have prevented him from separating his solution from that environment and generalizing his idea to problems with bounded controls. A major concerns namely was that, in aerospace engineering, the admissible controls cannot be assumed to lie always in open sets. The optimal controls may also run partly along the boundaries of those sets. This kind of problems were solved with short delay in the USSR.

Hence, it seems that sometimes it may be better to know less!

3.5 Merits

More important are Hestenes' merits. Hestenes indeed expressed Weierstrass' necessary condition as a maximum principle for the Hamiltonian. Herewith he had observed the importance of Weierstrass' condition for the theory of optimal control. Six years before the work at the Steklov Institute in Moscow began, the achievement of introducing a formulation that later became known as the general control problem was adduced by Hestenes in his Report RM-100. Nevertheless, this often has been credited to Pontryagin.[7]

Hestenes' report was considered to be hardly distributed outside RAND. However, there were many contacts between staff members of RAND engaged in optimal control and those "optimizers" outside RAND. Therefore, the content of RM-100 cannot be discounted as a flower that was hidden in the shade. The different circulation of Hestenes' RM-100 compared to Isaacs' RM-257, 1391, 1399, 1411, and 1486 may have been caused by the fact that Hestenes' memorandum contains instructions for engineers while Isaacs' memoranda were considered to be cryptic. To this Wendell H. Fleming meant:[8]

[7]First attemps to distinguish between state and control variables although not named this way can be found in Carathéodory's work; see Pesch (2012) and the references cited therein. For an extensive estimation of Hestenes' work considerating his surroundings and preconditions see Plail (1998).

[8]on the occasion of the bestowal of the Isaacs Award by the International Society of Dynamic Games in Sophia-Antipolis, France, in July 2006

Figure 3: The mathematicians at Steklov: Lev Semyonovich Pontryagin, Vladimir Grigor'evich Boltyanskii, and Revaz Valerianovich Gamkrelidze (Credits: Lev Semyonovich Pontryagin: `http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Pontryagin.html`. Vladimir Grigor'evich Boltyanskii: From Boltyanskii's former homepage at the Centro de Investigación en Matemáticas, Guanajuato, Mexico. Revaz Valerianovich Gamkrelidze: Photo taken by the first author at the Banach Center Conference on 50 Years of Optimal Control in Bedlewo, Poland, September, 2008.)

> *One criticism made of Isaacs' work was that it was not mathematically rigorous. He worked in the spirit of such great applied mathematicians as Laplace, producing dramatically new ideas which are fundamentally correct without rigorous mathematical proofs.*

4 The advant-gardists

4.1 Lev Semyonovich Pontryagin

Lev Semyonovich Pontryagin (1908–1988),[9] already a leading mathematician on the field of topology, decided to change his research interests radically towards applied mathematics around 1952. He was additionally encouraged by the fact that new serendipities in topology by the French mathematicians Leray, Serre and Cartan came to the fore. In addition, he also was pressured by M. V. Keldysh, director of the department of applied mathematics of the Steklov Institute, and by the organisation of the Communist Party at the institute to change his research direction. Maybe they wanted these mathematicians eventually to work for something more meaningful for the workers' and peasants' state than topology. Contact was then made with Colonel Dobrohotov, a professor at the military academy of aviation. In 1955, Pontryagin's group got together with members of the air force. As in the US, minimum time interception problems were discussed.

[9]Pontryagin lost his eyesight as the result of an explosion at the age of about 14. His mother wrote down his mathematical notes. Since she did not know the meaning or names of all these mathematical "hieroglyphs", they used a kind of a secret language to name them.

Already prepared since 1952 by a seminar on oscillation theory and automatic control that was conducted by Pontryagin and M. A. Aizerman, a prominent specialist in automatic control, it was immediately clear that a time optimal control problem was at hand there. However, to strengthen the applications, engineers were also invited. In particular, A. A. Fel'dbaum and A. J. Lerner focussed the attention on the importance of optimal processes of linear systems for automatic control.

Pontryagin quickly noticed that Fel'dbaum's method had to be generalized in order to solve the problems posed by the military. The first important step towards a solution was done by Pontryagin "during three sleepless nights". A little later already the first results could be published by Pontryagin and his co-workers Boltyanskii and Gamkrelidze in 1956.

Their early form of the maximum principle (of 1956) presents itself in the following form: Given the equations of motion

$$\dot{x}^i = f^i(x^1, \ldots, x^n, u^1, \ldots, u^r) = f^i(x, u)$$

and two points ξ_0, ξ_1 in the phase space $x^1, \ldots, x^n$, an admissible control vector u is to be chosen[10] in such way that the phase point passes from the position ξ_0 to ξ_1 in minimum time.

In 1956, Pontryagin and his co-workers wrote:

> *Hence, we have obtained the special case of the following general principle, which we call maximum principle: the function*
>
> $$H(x, \psi, u) = \psi_\alpha \, f^\alpha(x, u)$$
>
> *shall have a maximum with respect to u for arbitrary, fixed x and ψ, if the vector u changes in the closed domain $\bar{\Omega}$. We denote the maximum by $M(x, \psi)$. If the $2n$-dimensional vector (x, ψ) is a solution of the Hamiltonian system*
>
> $$\begin{aligned} \dot{x}^i &= f^i(x, u) = \frac{\partial H}{\partial \psi_i}\,, \ i = 1, \ldots, n\,, \\ \dot{\psi}_i &= -\frac{\partial f^\alpha}{\partial x^i}\,\psi_\alpha = -\frac{\partial H}{\partial x^i}\,, \end{aligned}$$
>
> *and if the piecewise continuous vector $u(t)$ fulfills, at any time, the condition*
>
> $$H\big(x(t), \psi(t), u(t)\big) = M\big(x(t), \psi(t)\big) > 0\,,$$
>
> *then $u(t)$ is an optimal control and $x(t)$ is the associated, in the small, optimal trajectory of the equations of motion.*

[10]The letter u stands for the Russian word for control: upravlenie.

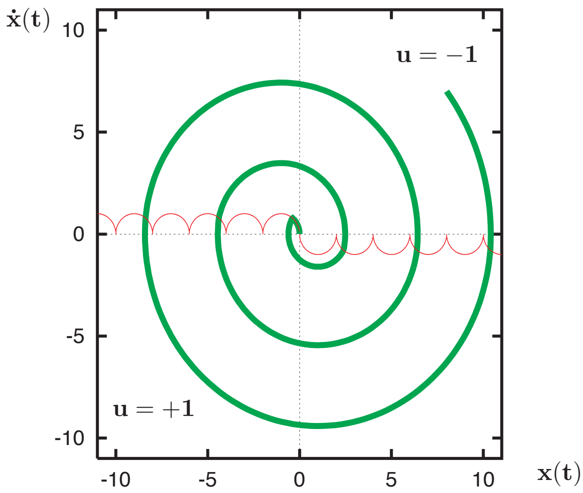

Figure 4: Phase diagram: optimal solution of the minimum-time harmonic oszillator problem: minimize the terminal time t_f subject to the differential equation $\ddot{x} + x = u$ with boundary conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, $x(t_f) = 0$, and $\dot{x}(t_f) = 0$, and control constraint $|u| \leq 1$. The problem allows a complete analytical solution and, moreover, a synthesis, i.e., for any given initial point $(x(0) = x_0\,, \dot{x}(0) = \dot{x}_0)$ in the phase plane, the origine $(x(t_f) = 0\,, \dot{x}(t_f) = 0)$ can be reached in minimum time t_f by a finite number of switches of the control u being of bang-bang type, i.e., it switches when ever the trajectories cross the garland-like switching curve. Thereby, the optimal control law satisfies a feedback law: the optimal value of u is -1 above and $+1$ below the switching curve while the phase trajectories piecewise consist of circles with shrinking radii.

This condition was immediately verified to be successful by means of problems of the Bushaw-Fel'dbaum type, e.g., $\ddot{x} + x = u$. Such dynamical systems have to be steered from any point of the phase plane $\dot{x}$ vs. x to its origin in minimum time, where the set of admissible control values is bounded by $|u| \leq 1$. Just take x to be the distance between the aircraft and the missile, you immediately get an abstract planar aircombat problem. Its solution is described by Fig. 4.

4.2 Vladimir Grigor'evich Boltyanskii and Revaz Valerianovich Gamkrelidze

Their first theorem on the Maximum Principle was not correct in general cases. It is a necessary and sufficient condition only for linear problems (as proved by Gamkrelidze, 1957, 1958). Later in 1958 Boltyanskii showed that the maximum principle is only a necessary condition in the general case. He published the proof first separately, later on together with Pontryagin and Gamkrelidze in

1960. Boltyanskii's proof was very intricate and required substantial knowledge of different fields of mathematics. Indeed, Boltyanskii's proof greatly influenced the later development of the modern theory of extremal problems.[11]

The research efforts at the Steklov Institute led to a series of publications and culminated in their famous book of 1961 which became a standard work of optimal control theory until today. In 1962, Pontryagin, Boltyanskii, Gamkrelidze, and the fourth author of that book, Evgenii Frolovich Mishchenko (1922–2010), received the Lenin prize for their work.

Both Boltyanskii and Gamkrelidze concur in statements to the authors, that the somehow comparable conditions of the calculus of variations were not known during the development phase of the maximum principle, although Bliss' monograph of 1946 existed in a Russian translation from 1950.

Fortunately, the Pontryagin group did no know too much about the calculus of variations.

4.3 Priority quarrel in the red corner

Boltyanskii claimed the version of the maximum principle as a necessary condition to be his own contribution and described how Pontryagin hampered his publication. He said Pontryagin intended to publish the results under the name of four authors. After Boltyanskii refused to do so, he was allowed to publish his results in 1958 but said that he had to praise Pontryagin's contribution disproportionally and had to call the principle Pontryagin's maximum principle. According to Boltyanskii, Rozonoér, an engineer, was encouraged to publish a tripartite work on the maximum principle in *Avtomatika i Telemekhanika* in 1959, in order to disseminate the knowledge of the maximum principle in engineering circles and to contribute this way to the honour of Pontryagin as discoverer of the maximum principle.

This priority argument may be based on the fact that Pontryagin wanted to aim for a globally sufficient condition after Gamkrelidze's proof of a locally sufficient condition, and not to a necessary condition as it turned out to be after Boltyanskii's proof. Boltyanskii may have felt very uncomfortable to write in his monograph:

> *The maximum principle was articulated as hypothesis by Pontryagin. Herewith he gave the decisive impetus for the development of the theory of optimal processes. Therefore the theorem in question and the closely related statements are called Pontryagin's maximum principle in the entire world – and rightly so.*

Boltyanskii felt suppressed and cheated of the international recognition of his achievements. After the break-up of the USSR, Boltyanskii was able to extend his fight for the deserved recognition of his work.

[11] For precursors of Boltyankii's proof and their influences see Plail (1998).

Gamkrelidze held a different view:[12]

> *My life was a series of missed opportunities, but one opportunity, I have not missed, to have met Pontryagin.*

For traces of the Maximum Principle before the time covered here, see Plail (1998), Pesch and Plail (2009) as well as Pesch (2012) and the references cited therein.

4.4 Distinctions

Pontryagin received many honours for his work. He was elected a member of the Academy of Sciences in 1939, and became a full member in 1959. In 1941 he was one of the first recipients of the Stalin prize (later called the State Prize). He was honoured in 1970 by being elected Vice-President of the International Mathematical Union.

5 Résumé

Hestenes, Bellman, and Isaacs as well as Pontryagin and his co-workers Boltyanskii and Gamkrelidze have not exclusively contributed to the development of optimal control theory, but their works were milestones on the way to modern optimal control theory. Their works are examples for demanding mathematical achievements with a tremendous application potential, today no longer solely in the military sector or in aeronautics, but also for many industrial applications. Today the second step after the numerical simulation of complicated nonlinear processes often requires an optimization post-processing. Not seldom side conditions as differential equations and other constraints must be taken into account for real-life models. Optimal control definitely is the germ cell of all those new fields in continuous optimization that have recently developed such as optimal control with partial differential equations or shape, respectively topology optimization, which are continuously contributing to the accretive role of mathematics for the development of present and future key technologies.

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