Linear Programming Stories

The history of polyhedra, linear inequalities, and linear programming has many diverse origins. Polyhedra have been around since the beginning of mathematics in ancient times. It appears that Fourier was the first to consider linear inequalities seriously. This was in the first half of the 19^{th} century. He invented a method, today often called Fourier-Motzkin elimination, with which linear programs can be solved, although this notion did not exist in his time. If you want to know anything about the history of linear programming, I strongly recommend consulting Schrijver's book [5]. It covers all developments in deepest possible elaborateness.

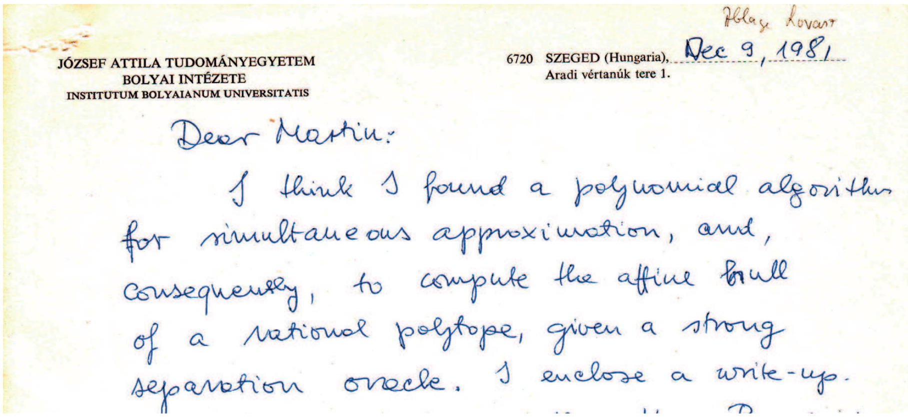
This section of the book contains some aspects that complement Schrijver's historical notes. The origins of the interior point method for linear programming are explored as well as column generation, a methodology that has proved of considerable practical importance in linear and integer programming. The solution of the Hirsch conjecture is outlined, and a survey of the development of computer codes for the solution of linear (and mixed-integer) programs is given. And there are two articles related to the ellipsoid method to which I would like to add a few further details.

According to the New York Times of November 7, 1979: "*A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics ...*". This obscure person was L. G. Khachiyan who ingeniously modified an algorithm, the ellipsoid method, developed for nonlinear programming by N. Z. Shor, D. B. Yudin, and A. S. Nemirovskii and proved in a very short paper [3] that this method solves linear programs in polynomial time. This was indeed a sensation. The ellipsoid method is a failure in practical computation but turned out to be a powerful tool to show the polynomial time solvability of many optimization problems, see [2].

One step in the ellipsoid method is the computation of a least volume ellipsoid containing a given convex body. The story of the persons behind the result that this ellipsoid, the Löwner-John ellipsoid, is uniquely determined and has very interesting properties, is told in this section. A second important ingredient of Khachiyan's modification is "clever rounding". A best possible approximation of a real number by a rational number with a bounded denominator can be achieved by computing a continued fraction. The history and some applications of this technique are covered also in a subsequent article.

When L. Lovász, A. Schrijver, and I were working on our book [2] we wanted to eliminate some "dirty tricks" that were needed to make the original version of the ellipsoid method work. The ellipsoid method produces successively shrinking ellipsoids containing the given polyhedron. It terminates due to a volume criterion, and thus it can only be applied to full-dimensional polyhedra. Since one usually does not know whether a given polyhedron is full-dimensional, one has to blow it up appropriately. How can one avoid this artificial blow up?

If a polyhedron is not full-dimensional (let us assume its dimension is one less than the space dimension), then it must lie in some hyperplane H. One observation is that, in such a case, the ellipsoid method produces shrinking ellipsoids that get very flat in the direction perpendicular to H. This means that, for these flat ellipsoids, the symmetry hyperplane belonging to the shortest axis must be very close to H. Is it possible to identify H by rounding the equation of this symmetry hyperplane? An immediate idea is to round each coefficient of the equation (using continued fractions), but this does not deliver what one wants. Simultaneous rounding, more precisely simultaneous Diophantine approximation, is needed. We searched all number theory books. There are important results of Dirichlet that could be applied, but no polynomial time algorithms. We were stuck. Then I obtained the following letter from Laci Lovász:


JÓZSEF ATTILA TUDOMÁNYEGYETEM
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6720 SZEGED (Hungaria), Dec 9, 1981
Aradi vértanúk tere 1.

Dear Martin:

I think I found a polynomial algorithm for simultaneous approximation, and, consequently, to compute the affine hull of a rational polytope, given a strong separation oracle. I enclose a write-up.

Laci's algorithm is based on an idea for finding short vectors in a lattice. At about the same time, several other mathematicians were addressing completely different problems that lead to the same type of questions Lovász answered. Among these were the brothers Arjen and Hendrik Lenstra with whom Laci teamed up and wrote the famous paper [4]. The algorithm described in [4] is now called LLL algorithm; it spurred enormous interest in many different areas of mathematics and computer science and found various extensions and improvements. The LLL algorithm virtually created a very lively subfield of mathematics, lattice basis reduction, and is already the subject of textbooks, see [1].

The brief story sketched here is nicely presented, including many other angles of this development and persons involved, in [6] and describes, in particular, the way the brothers Lenstra and some others recognized the importance of algorithmic basis reduction. From my personal point of view, this important development began with the successful attempt to handle an annoying detail of a linear programming algorithm.

Martin Grötschel

References

[1] M. R. Brenner, *Lattice Basis Reduction*, CRC Press, Boca Raton, 2012.

[2] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer, Berlin, 1988/1993.

[3] L. G. Khachiyan, *A polynomial algorithm in linear programming* (in Russian), Doklady Akademii Nauk SSSR 244 (1979), 1093–1096 (English translation: Soviet Mathematics Doklady 20 (1979), 191–194).

[4] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász. Factoring polynomials with rational coefficients, *Mathematische Annalen* 261 (1982), 515–534.

[5] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester, 1986/1998.

[6] I. Smeets et al., The History of the LLL-algorithm, in Phong Q. Nguyen (ed.) et al., *The LLL algorithm, Survey and applications*, Springer, 2010, pp. 1–17.