# Siegel Varieties and $p$-Adic Siegel Modular Forms

*To John Coates for his sixtieth birthday*

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Abstract. In this paper, we present a conjecture concerning the classicality of a genus two overconvergent Siegel cusp eigenform whose associated Galois representation happens to be geometric, and more precisely, given by the Tate module of an abelian surface. This conjecture is inspired by the Fontaine-Mazur conjecture. It generalizes known results in the genus one case, due to Kisin, Buzzard-Taylor and Buzzard. The main difference in the genus two case is the complexity of the arithmetic geometry involved. This is why most of the paper consists in recalling (mostly with proofs) old and new results on the bad reduction of parahoric type Siegel varieties, with some consequences on their rigid geometry. Our conjecture would imply, in certain cases, a conjecture posed by H. Yoshida in 1980 on the modularity of abelian surfaces defined over the rationals.

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In a previous paper, we showed under certain assumptions (Theorem 4 of [26]) that a degree four symplectic Galois representation $\rho$ with singular Hodge-Tate weights which is congruent to a cohomological modular Galois representation (we say then that $\rho$ is residually cohomologically modular) is $p$-adically modular. The precise definitions of the expressions above can be found in [26] Sect.2 and 4. As a corollary, we obtain that certain abelian surfaces

$A_{/\mathbb{Q}}$ do correspond, if they are residually cohomologically modular, to overconvergent Siegel cusp forms of weight $(2,2)$ (see Theorem 8 of [26]), in the sense that their Galois representations coincide. This result fits a Generalized Shimura-Taniyama Conjecture due to H. Yoshida ([30], Section 8.2) according to which for any irreducible abelian surface $A$ defined over $\mathbb{Q}$, there should exist a genus two holomorphic Siegel cusp eigenform $g$ of weight $(2,2)$ such that $L(h^1(A),s) = L_{\mathrm{spin}}(g,s)$, where $L(h^1(A),s)$ is the Grothendieck $L$ function associated to the motive $h^1(A)$ and $L_{\mathrm{spin}}(g,s)$ is the degree four automorphic $L$ function associated to $g$ (with Euler factors defined via Hecke parameters rather than Langlands parameters, for rationality purposes). One should notice that this conjecture presents a new feature compared to the genus one analogue. Namely, contrary to the genus one case, the weight $(2,2)$ occuring here is not cohomological; in other words, the Hecke eigensystem of $g$ does not occur in the singular cohomology of the Siegel threefold (it occurs however in the coherent cohomology of this threefold). In particular, the only way to define the Galois representation $\rho_{g,p}$ associated to such a form $g$, either classical or overconvergent, is to use a $p$-adic limit process, instead of cutting a piece in the étale cohomology with coefficients of a Siegel threefold. This can be achieved in our case because $g$ fits into a two-variable Hida family of $p$-nearly ordinary cusp eigenforms. Note that, more generally, for a classical cusp eigenform $g$ of weight $(2,2)$ with (finite) positive slopes for its Hecke eigenvalues at $p$, one believes that two-variable Coleman families of cusp eigenforms passing through $g$ in weight $(2,2)$ could also be constructed, and this would allow a similar construction of $\rho_{g,p}$.

For our $p$-nearly ordinary overconvergent $g$, Theorem 8 of [26] states that the associated Galois representation $\rho_{g,p}$ does coincide with the $p$-adic realization of a motive $h^1(A)$. Therefore, $\rho_{g,p}$ is geometric; several results in the analogue situation for genus 1 (see [18], [6] and [7]) lead us to conjecture that this $g$ is actually classical.

The goal of the present paper is to generalize slightly and state precisely this conjecture (Sect.4.2). We also take this opportunity to gather geometric facts about Siegel threefolds with parahoric level $p$, which seem necessary for the study of the analytic continuation of such overconvergent cusp eigenforms to the whole (compactified) Siegel threefold; the rigid GAGA principle would then imply the classicity of such $g$. We are still far from fulfilling this program. However, we feel that the geometric tools presented here, although some of them can actually be found in the literature, may be useful for various arithmetic applications besides this one, for instance to establish the compatibility between global and local Langlands correspondence for cusp forms of parahoric level for $GSp(4,\mathbb{Q})$.

As a final remark, we should point out that there exist other Generalized Shimura-Taniyama Conjectures for submotives of rank 3 resp.4 of the motive $h^1(A)$ for certain abelian threefolds resp. fourfolds $A$ (see [3]). For those, Theorem 8 of [26] seems transposable; the question of classicity for the resulting overconvergent cusp eigenforms for unitary groups $U(2,1)$ resp. $U(2,2)$ could then be posed in a similar way. It would then require a similar study of the (rigid) geometry of Shimura varieties of parahoric type for the corresponding groups.

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Contents



# 1 Notations

Let

$$G = \mathrm{GSp}(4) = \{X \in \mathrm{GL}_4; {}^tXJX = \nu \cdot J\}$$

be the split reductive group scheme over $\mathbb{Z}$ of symplectic simitudes for the antisymmetric matrix $J$, given by its $2 \times 2$ block decomposition: $J = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}$ where $s$ is the $2 \times 2$ antidiagonal matrix whose non zero entries are 1. This group comes with a canonical character $\nu : X \mapsto \nu(X) \in \mathbb{G}_m$ called the similitude factor. The center of $G$ is denoted by $Z$, the standard (diagonal) maximal torus by $T$ and the standard (upper triangular) Borel by $B$; $U_B$ denotes its unipotent radical, so that $B = TU_B$. Let $\gamma_P = t_1/t_2$ resp. $\gamma_Q = \nu^{-1}t_2^2$ be the short, resp. the long simple root associated to the triple $(G, B, T)$. The standard maximal parabolic $P = MU$, associated to $\gamma_P$, is called the Klingen parabolic, while the standard maximal parabolic $Q = M'U'$, associated to $\gamma_Q$, is the Siegel parabolic. The Weyl group of $G$ is denoted $W_G$. It is generated by the two reflexions $s_P$ and $s_Q$ induced by conjugation on $T$ by $\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ resp. $\begin{pmatrix} 1 & & \\ & s & \\ & & 1 \end{pmatrix}$. Let us fix a pair of integers $(a, b) \in \mathbb{Z}^2$, $a \geq b \geq 0$; we identify it with a dominant weight for $(G, B, T)$, namely the character

$$T \ni t = diag(t_1, t_2, \nu^{-1}t_2, \nu^{-1}t_1) \mapsto t_1^a t_2^b$$

Let $V_{a,b}$ be a generically irreducible algebraic representation of $G$ associated to $(a,b)$ over $\mathbb{Z}$.
Let $\mathbb{A} = \mathbb{A}_f \times \mathbb{Q}_\infty$ be the ring of rational adeles. Fix a compact open subgroup $K$ of $G_f = G(\mathbb{A}_f)$; let $N \geq 1$ be an integer such that $K = K^N \times K_N$ with $K^N = G(\mathbb{Z}^N)$ maximal compact and $K_N = \prod_{\ell | N} K_\ell$ for local components $K_\ell$ to be specified later.
Let $\mathcal{H}^N$ be the unramified Hecke algebra outside $N$ (that is, the tensor product algebra of the unramified local Hecke algebras at all prime-to-$N$ rational primes); for each rational prime $\ell$ prime to $N$, one defines the abstract Hecke polynomial $P_\ell \in \mathcal{H}^N[X]$ as the monic degree four polynomial which is the minimal polynomial of the Hecke Frobenius at $\ell$ (see Remarks following 3.1.5 in [12]).
Let $C_\infty$ be the subgroup of $G_\infty = G(\mathbb{Q}_\infty)$ generated by the standard maximal compact connected subgroup $K_\infty$ and by the center $Z_\infty$.
For any neat compact open subgroup $L$ of $G(\mathbb{A}_f)$, the adelic Siegel variety of level $L$ is defined as: $S_L = G(\mathbb{Q})\backslash G(\mathbb{A})/LC_\infty$; it is a smooth quasi-projective complex 3-fold. If $L \subset L'$ are neat compact open subgroups of $G_f$, we have a finite etale transition morphism $\phi_{L,L'} : S_L \to S_{L'}$.

## 2 Integral models and local models

Let $K$ be a compact open subgroup of $G(\widehat{\mathbb{Z}})$ such that $K(N) \subset K$. For any integer $M \geq 1$, we write $K_M$ resp. $K^M$ for the product of the local components of $K$ at places dividing $M$, resp. prime to $M$.
Let $p$ be a prime not dividing $N$ we denote by $I$, $\Pi_P$ resp. $\Pi_Q$ the Iwahori subgroup, Klingen parahoric, resp. Siegel parahoric subgroup of $G(\mathbb{Z}_p)$. We consider $K_B(p) = K \cap I \times K^p$, $K_P(p) = K \cap \Pi_P \times K^p$ and $K_Q(p) = K \cap \Pi_Q \times K^p$ and the corresponding Shimura varieties $S_B(p)$, $S_P(p)$ resp. $S_Q(p)$.
Let us consider the moduli problems

$$\mathcal{F}_\emptyset : \mathbb{Z}[\frac{1}{N}]-\mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{A, \lambda, \overline{\eta})_{/S}\}/\sim,$$

$$\mathcal{F}_B : \mathbb{Z}[\frac{1}{N}]-\mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{A, \lambda, \overline{\eta}, H_1 \subset H_2 \subset A[p])_{/S}\}/\sim,$$

$$\mathcal{F}_P : \mathbb{Z}[\frac{1}{N}]-\mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{A, \lambda, \overline{\eta}, H_1 \subset A[p])_{/S}\}/\sim$$

and

$$\mathcal{F}_Q : \mathbb{Z}[\frac{1}{N}]-\mathrm{Sch} \to Sets, \quad S \mapsto \{A, \lambda, \overline{\eta}, H_2 \subset A[p])_{/S}\}/\sim$$

where $A/S$ is an abelian scheme, $\lambda$ is a principal polarisation on $A$, $\overline{\eta}$ is a $K$-level structure (see end of Sect.6.1.1 of [12]), $H_i$ is a rank $p^i$ finite flat subgroup scheme of $A[p]$ with $H_2$ totally isotropic for the $\lambda$-Weil pairing.
As in Th.6.2.1 of [12] or [16] Prop.1.2, one shows

Theorem 1 *If $K$ is neat, the functors above are representable by quasiprojective $\mathbb{Z}[\frac{1}{N}]$-schemes $X_\emptyset$, $X_B(p)$, $X_P(p)$ and $X_Q(p)$. The first one is smooth over $\mathbb{Z}[\frac{1}{N}]$ while the others are smooth away from $p$; the functors of forgetfulness of the level $p$ structure provide proper morphisms $\pi_{B,\emptyset} : X_B(p) \to X_\emptyset$, $\pi_{P,\emptyset} : X_P(p) \to X_\emptyset$, and $\pi_{Q,\emptyset} : X_Q(p) \to X_\emptyset$ which are finite etale in generic fiber.*

We'll see that these morphisms are not necessarily finite hence not necessarily flat.

We'll also consider a moduli problem of level $\Gamma_1(p)$. Let $U_B$ be the unipotent radical of the Borel $B$ of $G$. Let $\mathcal{F}_{U_B}$ be the functor on $\mathbb{Q}-$Sch sending $S$ to $\{A, \lambda, \overline{\eta}, P_1, P_2)_{/S}\}/\sim$ where $P_1$ is a generator of a rank $p$ finite flat subgroup scheme $H_1$ of $A[p]$ while $P_2$ is a generator of the rank $p$ finite flat group scheme $H_2/H_1$ for $H_2$ a lagrangian of $A[p]$. Over $\mathbb{Q}$, it is not difficult to show that it is representable by a scheme $X_{U_B}(p)_\mathbb{Q}$.

Following [14] and [12] Sect.6.2.2, we define the $\mathbb{Z}[\frac{1}{N}]$-scheme $X_{U_B}(p)$ as the normalisation of $X_B(p)$ in $X_{U_B}(p)_\mathbb{Q}$; it comes therefore with a morphism $\pi_{U_B,B} : X_{U_B}(p) \to X_B(p)$ which is generically finite Galois of group $T(\mathbb{Z}/p\mathbb{Z})$.

Remark: All schemes above have geometrically connected generic fibers if and only if $\nu(K) = \widehat{\mathbb{Z}}^\times$. However, in general, the morphisms $\pi_{*,\emptyset}$ induce bijections between the sets of geometric connected components of $X_*(p)$ and $X_\emptyset$; therefore the descriptions of irreducible components of the special fiber at $p$ given below should be interpreted as relative to an arbitrary given connected component of the special fiber at $p$ of $X_\emptyset$.

We still denote by $X_*(p)$ the base change to $\mathbb{Z}_p$ of $X_*(p)_{/\mathbb{Z}[\frac{1}{N}]}$ $(* = \emptyset, B, P, Q)$.

The results that we will explain below are essentially due to de Jong [16], Genestier [11], Ngô-Genestier [22], Chai-Norman [9], C.-F. Yu [29]. As most of these authors, we make first use of the theory of local models [23], which allows to determine the local structure of $X_*(p)$; then, one globalizes using the surjectivity of the monodromy action due to [10]. This argument is sketched in [16] for $g = 2$ and developed for any genus and for any parahoric level structure in [29].

The determination of the local model and of its singularities has been done in case $* = B$ by de Jong [16], in case $* = P$ in [12] Sect.6.3 (inspired by [14]) and in case $* = Q$ in [12] Appendix. Let us recall the results.

## 2.1 The case $* = B$

We first recall the definition of the local model $M_B$ of $X_B(p)$ over $\mathbb{Z}_p$.

Let $St_0 = \mathbb{Z}_p^4$, with its canonical basis $(e_0, e_1, e_2, e_3)$, endowed with the standard unimodular symplectic form $\psi$: $\psi(x, y) = {}^t xJy$. We consider the standard diagram $St_2 \overset{\alpha_2}{\to} St_1 \overset{\alpha_1}{\to} St_0$ where $\alpha_{i+1}$ sends $e_i$ to $pe_i$ and $e_j$ to $e_j$ $(j \neq i)$. We endow $St_2$ resp. $St_0$ with the unimodular standard symplectic form $\psi$, which we prefer to denote $\psi_2$ resp. $\psi_0$. Let $\alpha^2 = \alpha_1 \circ \alpha_2$; then we have $\psi_0(\alpha^2(x), \alpha^2(y)) = p\psi_2(x, y)$.

Then, $M_B$ is the scheme representing the functor from $\mathbb{Z}_p$–Sch to Sets sending a scheme $S$ to the set of triples $(\omega_i)_{i=0,1,2}$, where $\omega_i$ is a direct factor of $St_i \otimes \mathcal{O}_S$, $\omega_0$ and $\omega_2$ are totally isotropic, and $\alpha_{i+1}(\omega_{i+1}) \subset \omega_i$ for $i = 0, 1$.
It is a closed subscheme of the flag variety over $\mathbb{Z}_p$ $G(St_2, 2) \times G(St_1, 2) \times G(St_0, 2)$. Let $\overline{\xi}_0 = (\overline{\omega}_2, \overline{\omega}_1, \overline{\omega}_0) \in M_B(\mathbb{F}_p)$ be the point given by $\overline{\omega}_2 = \langle e_0, e_1\rangle$, $\overline{\omega}_1 = \langle e_0, e_3\rangle$ and $\overline{\omega}_0 = \langle e_2, e_3\rangle$. Consider the affine neighborhood $U$ of $\overline{\xi}_0$ in $M_B$ given by $\omega_2 = \langle e_0 + c_{11}e_2 + c_{12}e_3, e_1 + c_{21}e_2 + c_{22}e_3\rangle$, $\omega_1 = \langle e_0 + b_{11}e_1 + b_{12}e_2, e_3 + b_{21}e_1 + b_{22}e_2\rangle$ and $\omega_0 = \langle e_2 + a_{11}e_0 + a_{12}e_1, e_3 + a_{21}e_0 + a_{22}e_1\rangle$.
We'll see below that it is enough to study the geometry of $U$ because this open set is "saturating" in $M_B$ (*i.e.* its saturation $G_BU$ for the action of the group $G_B$ of automorphisms of $M_B$ is $M_B$). Let us first study the geometry of $U$.
The equations of $U$ are $c_{11} = c_{22}$, $a_{11} = a_{22}$,
$pe_1 + c_{21}e_2 + c_{22}e_3 = c_{22}(e_3 + b_{21}e_1 + b_{22}e_2)$,
$e_0 + c_{11}e_2 + c_{12}e_3 = e_0 + b_{11}e_1 + b_{12}e_2 + c_{12}(e_3 + b_{21}e_1 + b_{22}e_2)$,
and similarly
$pe_0 + b_{11}e_1 + b_{12}e_2 = b_{12}(e_2 + a_{11}e_0 + a_{12}e_1)$,
$e_3 + b_{21}e_1 + b_{22}e_2 = e_3 + a_{21}e_0 + a_{22}e_1 + b_{22}(e_2 + a_{11}e_0 + a_{12}e_1)$.
Equating the coordinates of the two members, one gets the set of equations (2) of [16] Sect.5.
Putting $x = a_{11}$, $y = b_{12}$, $a = c_{12}$, $b = a_{12}$ and $c = b_{22}$, an easy calculation shows that $U = \operatorname{spec} \mathbb{Z}_p[x, y, a, b, c]/(xy - p, ax + by + abc)$. The special fiber $U_0 \subset M_B \otimes \mathbb{F}_p$ of $U$ is an affine threefold given by the equations $xy = 0$ and $ax + by + abc = 0$; it is the union of its four smooth irreducible components $Z_{00} = V(x, b)$, $Z_{01} = V(x, y + ac)$, $Z_{10} = V(y, a)$ and $Z_{11} = V(y, x + bc)$.
Let $R = \mathbb{Z}_p^{ur}[x, y, a, b, c]/(xy - p, ax + by + abc)$; then $\overline{\xi}_0$ has coordinates $(0, 0, 0, 0, 0)$ in $U_0(\overline{\mathbb{F}}_p)$. Let $\overline{\zeta}_0 = (\overline{x}_0, \overline{y}_0, \overline{a}_0, \overline{b}_0, \overline{c}_0)$ be an arbitrary point of $U_0(\overline{\mathbb{F}}_p)$. Note that $\overline{x}_0\overline{y}_0 = 0$ and $\overline{a}_0\overline{x}_0 + \overline{b}_0(\overline{y}_0 + \overline{a}_0\overline{c}_0) = \overline{b}_0\overline{y}_0 + \overline{a}_0(\overline{x}_0 + \overline{b}_0\overline{c}_0) = 0$.
Let $\mathfrak{m}_0$ be the maximal ideal of $R$ corresponding to $\overline{\zeta}_0$. The completion of $R$ at $\mathfrak{m}_0$ is given by the following easy lemma ([16] Section 5).

Lemma 2.1

- *If $\overline{x}_0 + \overline{b}_0\overline{c}_0 \neq 0$, then if $\overline{y}_0 \neq 0$, $\widehat{R}_{\mathfrak{m}_0} \cong \mathbb{Z}_p^{ur}[[u, \beta, \gamma]]$,*
- *If $\overline{x}_0 + \overline{b}_0\overline{c}_0 \neq 0$ and $\overline{y}_0 = 0$, then $\widehat{R}_{\mathfrak{m}_0} \cong \mathbb{Z}_p^{ur}[[x, y, b, c]]/(xy - p)$,*
- *If $\overline{a}_0 \neq 0$, if $\overline{y}_0 = \overline{b}_0 = 0$ then $\widehat{R}_{\mathfrak{m}_0} \cong \mathbb{Z}_p^{ur}[[y, b, t, c]]/(ybt - p)$, and if $\overline{y}_0 \neq 0$ or $\overline{b}_0 \neq 0$, if $\overline{y}_0\overline{b}_0 = 0$ then $\widehat{R}_{\mathfrak{m}_0}$ is $\mathbb{Z}_p^{ur}[[y, b, t, c]]/(yt - p)$, or it is smooth if $\overline{y}_0\overline{b}_0 \neq 0$,*
- *If $\overline{c}_0 \neq 0$ and $\overline{x}_0 = \overline{b}_0 = \overline{a}_0 = \overline{y}_0 = 0$, if moreover $\overline{c}_0 \neq 0$, then $\widehat{R}_{\mathfrak{m}_0} \cong \mathbb{Z}_p^{ur}[[x, y, u, v, w]]/(xy - p, uv - p)$,*
- *If $\overline{x}_0 = \overline{b}_0 = \overline{a}_0 = \overline{y}_0 = \overline{c}_0 = 0$, that is, if $s_0 = x_0$ (defined above), then $\widehat{R}_{\mathfrak{m}_0} \cong \mathbb{Z}_p^{ur}[[x, y, a, b, c]]/(xy - p, ax + by + abc)$,*

*The other cases are brought back to those by permuting the variables $x$ and $y$ resp. $a$ and $b$.*

Proof: If $\overline{x}_0+\overline{b}_0\overline{c}_0\neq 0$, and $\overline{y}_0\neq 0$, we choose liftings $x_0,a_0,b_0,c_0\in\mathbb{Z}_p^{ur}$ and $y_0\in\mathbb{Z}_p^{ur\times}$ and introduce new variables $u,\alpha,\beta,\gamma$ by putting $y=y_0+u$ and $a=a_0+\alpha$, $b=b_0+\beta$, $c=c_0+\gamma$ (in case $\overline{b}_0=0$ for instance, we choose $b_0=0$ so that $\beta=b$, and similarly for $\gamma$). Then, the relation $ax+by+abc=0$ in $\widehat{R}_{\mathfrak{m}_0}$ reads $a(x+bc)+by=0$, so that the image of the variable $\alpha$ can be expressed as a series of the images of the variables $u,\beta,\gamma$; similarly, the relation $xy=p$ allows to express $x$ as a series of $u$; in conclusion, we have $\widehat{R}_{\mathfrak{m}_0}\cong\mathbb{Z}_p^{ur}[[u,\beta,\gamma]]$.
If $\overline{x}_0\neq 0=\overline{y}_0=0$, this reasoning shows that $\widehat{R}_{\mathfrak{m}_0}\cong\mathbb{Z}_p^{ur}[[x,y,\beta,\gamma]]/(xy-p)$.
If $\overline{a}_0\neq 0$, let us omit the centering at 0 of variables as above (needed for instance if $\overline{b}_0\neq 0$ or $\overline{y}_0\neq 0$). Let us write the relation $ax+by+abc=0$ as $x=-a^{-1}by-bc=b(-a^{-1}y-c)$. We introduce a new variable $t=-a^{-1}y-c$. Then we have $p=xy=bty$ so that $\widehat{R}_{\mathfrak{m}_0}\cong\mathbb{Z}_p^{ur}[[y,b,t,c]]/(ybt-p)$ unless, as mentioned, $\overline{b}_0\neq 0$ or $\overline{y}_0\neq 0$ where things become simpler.
If $\overline{x}_0=\overline{b}_0=\overline{a}_0=\overline{y}_0=0$ but $\overline{c}_0\neq 0$, then $(x+bc)(y+ac)=p+c(ax+by+abc)=p$; hence, putting $u=x+bc$ and $v=y+ac$, one defines a change of variables from the set of variables $(x,y,a,b,c)$ to $(x,y,u,v,c)$ (actually, as above, one should use $\gamma=c-c_0$ instead of $c$) and the conclusion follows.
The last case is clear.QED.

By the theory of local models, we have a diagram

$$\begin{array}{ccccc} & & \mathcal{W}_B & & \\ & \pi\swarrow & & \searrow f & \\ X_B(p) & & & & M_B \end{array}$$

where $\mathcal{W}_B$ classifies quintuples $(A,\lambda,H_1,H_2;\phi:St.\otimes\mathcal{O}_S\cong D(A.))$ over a scheme $S$ (see Sect.3 of [16], especially Prop.3.6, for the definition of $\phi$). One sees easily that it is representable by a $X_B(p)$-scheme $\pi:\mathcal{W}_I\to X_B(p)$. The morphism $f$ consists in transporting the Hodge filtration from the Dieudonné modules to $St.$ by $\phi$ and $\pi$ consists in forgetting $\phi$. Recall that those morphisms are smooth and surjective.
Given a point $z=(A_0\to A_1\to A_2,\lambda_0,\lambda_2;\phi)$ of $\mathcal{W}_B(\overline{\mathbb{F}}_p)$, the degree $p$ isogenies $A_0\to A_1\to A_2$ (defined by quotienting $A=A_0$ by $H_1$ and $H_2$) give rise to morphisms of filtered Dieudonné modules (writing $M_i$ for $D(A_i)_S$): $M_2\to M_1\to M_0$, sending $\omega_{i+1}$ into $\omega_i$. Let us consider the rank $p$ finite flat group schemes $G_0=H_1=\mathrm{Ker}\,(A_0\to A_1)$ and $G_1=H_2/H_1:\mathrm{Ker}\,(A_1\to A_2)$. Then, we have a canonical isomorphism
1) $\omega_i/\alpha(\omega_{i+1})\cong\omega_{G_i}$.
Recall that $\omega_{A_i^\vee}=\omega_i^\vee=M_i/\omega_i$, hence by Th.1, Sect.15 of [20]), if $G_i^\vee$ denotes the Cartier dual of $G_i$, we have
2) $\omega_{G_i^\vee}=M_i/(\omega_i+\alpha(M_{i+1}))$.
For $z\in\mathcal{W}_B(\overline{\mathbb{F}}_p)$ as above, let $x=\pi(z)=(A_0\to A_1\to A_2,\lambda_0,\lambda_2)$ and $s=f(z)=(\omega_2,\omega_1\omega_0)$.
We define $\sigma_i(s)=\dim\omega_i/\alpha(\omega_{i+1})$ and $\tau_i(s)=\dim M_i/(\omega_i+\alpha(M_{i+1}))$.
Then,

- if $G_i$ is $\mu_p$, $\sigma_i(s) = 1$ and $\tau_i(s) = 0$
- if $G_i$ is $\mathbb{Z}/p\mathbb{Z}$, $\sigma_i(s) = 0$ and $\tau_i(s) = 1$
- if $G_i$ is $\alpha_p$, $\sigma_i(s) = 1$ and $\tau_i(s) = 1$

We define $M_B(\overline{\mathbb{F}}_p)^{\mathrm{ord}}$ as the set of points $s$ such that $(\sigma_i(s), \tau_i(s)) \in \{(1,0),(0,1)\}$ for $i = 1, 2$.
One determines its four connected components and we check their Zariski closures are the irreducible components of $M_B(\overline{\mathbb{F}}_p)$ as follows. The calculations of the lemma above show that $M_B(\overline{\mathbb{F}}_p) \cap U$ is the union of the loci

- (1) $x = b = 0$,
- (2) $x = y + ac = 0$,
- (3) $y = a = 0$,
- (4) $y = x + bc = 0$,

Then, let us check that the component $x = b = 0$ is the Zariski closure of the locus $(m, m)$ where $H_1$ and $H_2/H_1$ are multiplicative. This component consists in triples $(\omega_2, \omega_1, \omega_0)$ such that the generators of $\omega_0$ satisfy $a_{11} = a_{12} = 0$, that is, by equations (1) of $U_0$ in Sect.6 of [16], such that $\omega_0 = \langle e_2, e_3\rangle$. Then one sees that $\alpha(\omega_1) = \langle b_{12}e_2, e_3 + b_{22}e_2\rangle$ has codimension 1 in $\omega_0$ if $b_{12} = 0$, and codimension 0 otherwise, while $\alpha(\omega_2) = \langle e_0 + c_{11}e_2 + c_{12}e_3, c_{21}e_2 + c_{22}e_3\rangle$ has codimension 1 if $c_{11} = 0$ and 0 otherwise.
On the other hand, $\alpha(M_1)$ is generated by $(e_1, e_2, e_3)$ so $M_0/\alpha(M_1)$ is generated by the image of $e_0$; since $\omega_0 = \langle e_2, e_3\rangle$, we see that $\tau_0(s) = 1$ for any $s \in Z_{00}$, while $\alpha(M_2)$ is generated by $(e_0, e_2, e_3)$ so that $M_1/\alpha(M_2)$ is generated by the image of $e_1$; since $\omega_1 = \langle e_0 + b_{12}e_2, e_3 + b_{22}e_2\rangle$, we see that $\tau_1(s) = 1$ also on $Z_{00}$. Hence the open dense locus defined by $b_{12} \neq 0$ and $c_{11} \neq 0$ is the ordinary locus of this component (that is, the set of points $s$ such that $(\sigma_i(s), \tau_i(s)) = (0, 1)$ $(i = 1, 2)$.
One can do similar calculations for the other components; to obtain the table at bottom of page 20 of [16] (note however that our labeling of the components is different).
This calculation proves the density of the ordinary locus in each irreducible component in $U_0$ and provides at the same time the irreducible components of the non-ordinary locus and of the supersingular locus. We find

Lemma 2.2 *The open subset $U_0$ of $M_B \otimes \mathbb{F}_p$ is an affine scheme with four irreducible components*

- *(1) $x = b = 0$, Zariski closure of the locus $(m, m)$ where $H_1$ and $H_2/H_1$ are multiplicative*
- *(2) $x = y + ac = 0$, Zariski closure of the locus $(m, e)$ where $H_1$ is multiplicative and $H_2/H_1$ is étale*

- *(3) $y = a = 0$, Zariski closure of the locus $(e, e)$ where $H_1$ and $H_2/H_1$ are étale*
- *(4) $y = x + bc = 0$, Zariski closure of the locus $(e, m)$ where $H_1$ is étale and $H_2/H_1$ is multiplicative.*

*The singular locus $U_0^{\rm sing}$ can be viewed as the union of two loci: "$H_1$ biconnected",whose equation is $x = y = 0$, and "$H_2/H_1$ biconnected", whose equation is $y + ac = x + bc = 0$. The intersection of those two is the supersingular locus $U_0^{\rm ssing}$.*
*The locus "$H_1$ biconnected" is the union of $U_0^{\rm ssing}$ and two 2-dimensional irreducible components*

- *(14) the locus $x = b = y = 0$,equation of the Zariski closure of the locus where $H_1$ is biconnected and $H_2/H_1$ is multiplicative,*
- *(23) the locus $y = x = a = 0$, equation of the Zariski closure of the locus where $H_1$ is biconnected and $H_2/H_1$ is étale,*

*where the label $(ij)$ denotes the irreducible 2-dimensional intersection of $(i)$ and $(j)$.*
*The supersingular locus $U_0^{\rm ssing}$ coincides with the intersection $(2) \cap (4)$ which is the union of one 2-dimensional component $x = y = c = 0$, which we denote by $(24)$ and one 1-dimensional component $a = b = x = y = 0$.*
*The locus "$H_2/H_1$ biconnected" is the union of $U_0^{\rm ssing}$ and of two irreducible components*

- *(12) $x = b = y + ac = 0$, equation of the Zariski closure of the locus where $H_1$ is multiplicative and $H_2/H_1$ is biconnected,*
- *(34) $y = a = x + bc = 0$, equation of the Zariski closure of the locus where $H_1$ is étale and $H_2/H_1$ is biconnected .*

*with the same convention $(ij) = (i) \cap (j)$ (here, those are irreducible 2-dimensional components);*
*Finally, the three irreducible components of the one-dimensional stratum associated to the four irreducible components of $U_0^{\rm sing}$ are*

- $x = y = a = b = 0$,
- $x = y = a = c = 0$,
- $x = y = b = c = 0$,

*They are all contained in $U_0^{\rm ssing}$. More precisely, the second and third are contained in $(24)$, and $U_0^{\rm ssing}$ is the union of the first and of $(24)$.*
*Thus, the supersingular locus of $M_B$ is not equidimensional, it is union of a two-dimensional irreducible component, namely the Zariski closure of the locus $(24)$, and a one-dimensional irreducible component, closure of $x = y = a = b = 0$.*

Let us consider the Iwahori group scheme $G_B$; it is a smooth group scheme over $\mathbb{Z}_p$ representing the functor $S \mapsto \mathrm{Aut}_S(St. \otimes \mathcal{O}_S)$. Its generic fiber is the symplectic group $G$ while its special fiber is extension of the upper triangular Borel $B$ by the opposite unipotent radical.
The complete list of the $G_B$-orbits in $M_B \otimes \mathbb{F}_p$ follows from the analysis above. There are thirteen such orbits. There are four 3-dimensional orbits (whose Zariski closures are the irreducible components), five 2-dimensional orbits, three 1-dimensional orbits, and one 0-dimensional orbit, intersection of all the closures of the other orbits. These orbits can be detected from the irreducible components as complement in an irreducible component of the union of the other components of smaller dimension. In [13] p.594, they are described in terms of thirteen alcoves in an apartment of the Bruhat-Tits building.
Let us explain now the property of saturation of $U$: $G_B \cdot U = M_B$. To prove this, we note that $U_0$ meets all the orbits of $G_B$ because it contains the smallest orbit, namely the point $\overline{\xi}_0$ defined above and that this point is in the closure of all the other orbits. (cf. the remark of [11] above Lemma 3.1.1). This observation, together with the previous lemma implies [16], [22]

PROPOSITION 2.3 *The scheme $M_B$ is flat, locally complete intersection over $\mathbb{Z}_p$. Its special fiber is the union of four smooth irreducible components. Its ordinary locus coincides with the regular locus and is dense; the singular locus has* 5 2*-dimensional irreducible components, all smooth, and two one-dimensional irreducible components, also smooth; the p-rank zero locus has* 3 *irreducible components, all smooth; one is* 2*-dimensional and two are* 1*-dimensional.*

The local and global geometry of $X_B(p)$ is mostly contained in the following:

THEOREM 2 *The scheme $X_B(p)$ is flat, locally complete intersection over $\mathbb{Z}_p$. The ordinary locus in the special fiber coincides with the regular locus; it is therefore dense in the special fiber $X_B(p) \otimes \mathbb{F}_p$; this scheme is the union of four smooth irreducible components $X^{mm}$, $X^{me}$, $X^{em}$, $X^{ee}$. They are the Zariski closures of their ordinary loci, which are given respectively by the following conditions on the filtration $0 \subset H_1 \subset H_2 \subset A[p]$: $H_2$ is multiplicative, $H_1$ is multiplicative and $H_2/H_1$ étale, $H_1$ is étale and $H_2/H_1$ is multiplicative, $H_2$ is étale. The singular locus of $X_B(p) \otimes \mathbb{F}_p$ is therefore the locus where either $H_1$ or $H_2/H_1$ is étale-locally isomorphic to $\alpha_p$.*
*There exists a semistable model $\widetilde{X}_B(p)$ of $X_B(p)$ over $\mathbb{Z}_p$ with a proper morphism $h : \widetilde{X}_B(p) \to X_B(p)$ whose generic fiber $h \otimes \mathbb{Q}_p$ is an isomorphism and whose special fiber $h \otimes \mathbb{F}_p$ is an isomorphism over the ordinary locus.*

REMARK:
The stratification of the special fiber of $M_B$ by the $G_B$-orbits (called the Kottwitz-Rapoport stratification) defines also a stratification of the special fiber of $X_B(p)$; the stratum $X_S$ associated to the (irreducible) stratum $S$ of $M_B$ is defined as $\pi(f^{-1}(S))$. The four orbits corresponding to the irreducible

components are connected because of the monodromy theorem of [10] (due to C.-F. Yu [29]). It has been pointed out to the author by A.Genestier that for the 2-dimensional orbits, no such connexity result is available yet by a $p$-adic monodromy argument. However, C.F. Yu explained to us how to prove that the $p$-rank one stratum does consist of four 2-dimensional irrreducible components as listed above for $M_B^{\mathrm{sing}}$. Indeed, for any $p$-rank one geometric closed point $x$ of $X_B(p) \otimes \mathbb{F}_p$, we have $A_x[p] = G_{1,1}[p] \times \mu_p \times \mathbb{Z}/p\mathbb{Z}$ where $G_{1,1}$ denotes the $p$-divisible group of a supersingular elliptic curve; hence the possibilities for the pairs $(H_1, H_2/H_1)$ are $(\alpha_p, \mu_p)$, $(\alpha_p, \mathbb{Z}/p\mathbb{Z})$, $(\mu_p, \alpha_p)$, $(\mathbb{Z}/p\mathbb{Z}, \alpha_p)$. This shows that the $p$-rank one stratum has exactly four connected components, so that the components of each type are irreducible.
For the supersingular locus $X_B(p)^{\mathrm{ss}}$, it is known by Li-Oort that the number of irreducible components is in general strictly greater than 3 (which is the number of irreducible components of $M_B^{\mathrm{ss}}$).
Proof: By [16] Sect.4, the morphisms $\pi : \mathcal{W}_B \to X_B(p)$ and $f : \mathcal{W}_I \to M_B$ are smooth and surjective and for any geometric point $x$ of $X_B(p)$, there exists a geometric point $s \in f(\pi^{-1}(\{x\})$ of $M_B$ and a local ring isomorphism

$$\widehat{\mathcal{O}}_{X_B(p),x} \cong \widehat{\mathcal{O}}_{M_B,s}$$

The description of the strictly henselian local rings $\widehat{\mathcal{O}}_{X_B(p),x}$ is therefore given by the list of Lemma 2.2. They are flat, complete intersection over $\mathbb{Z}_p^{ur}$.
The ordinary subcheme $X_B(p)^{\mathrm{ord}}$ of the special fiber $\mathbb{X}_B(p) \otimes \mathbb{F}_p$ is the locus where the connected component of $A[p]$ is of multiplicative type. By total isotropy of $H_2$ it follows easily that $X_B(p)^{\mathrm{ord}}(\overline{\mathbb{F}}_p) = \pi(f^{-1}(M_B^{\mathrm{ord}}))$. Therefore, $X_B(p)^{\mathrm{ord}}$ is the disjoint union of four open subsets $X^{mm,\mathrm{ord}}$, $X^{me,\mathrm{ord}}$, $X^{em,\mathrm{ord}}$, $X^{ee,\mathrm{ord}}$, defined by the conditions: "the type of the pair $(H_1, H_2/H_1)$ is $(m, m)$ resp. $(m, e)$, resp. $(e, m)$, resp. $(e, e)$, where $m$ means multiplicative and $e$ means étale". Let us denote by $X^{mm}$, $X^{me}$, $X^{em}$, $X^{mm}$ their Zariski closures in $X_B(p) \otimes \mathbb{F}_p$. By density of the ordinary locus, one has $X_B(p) \otimes \mathbb{F}_p = X^{mm} \cup X^{me} \cup X^{em} \cup X^{mm}$. Let us show that these four subschemes are smooth irreducible. For $i, j \in \{0, 1\}$, let $M_B^{\alpha\beta}$ ( $\alpha$ and $\beta$ in $\{m, e\}$) be the irreducible components of $M_B \otimes \mathbb{F}_p$ such that $M_B^{\alpha\beta} \cap U_0$ is the component $(\alpha, \beta)$ in Lemma 2.2; then we have $\pi(f^{-1}(M_B^{\alpha\beta})) = X^{\alpha\beta}$. Thus, the smoothness of the components $M_B^{\alpha\beta}$ of $M_B \otimes \mathbb{F}_p$ yields the smoothness of $X^{\alpha\beta} \cap \mathcal{U}_0$ for all $\alpha$ and $\beta$ in $\{m, e\}$. The connectedness of $X^{\alpha\beta}$ follows from a simple argument due to C.-F. Yu [29] which we repeat briefly, with a small correction (of the wrong statement (2.2) p.2595). let $A \to X_\emptyset$ be the universal abelian variety; let $X_\emptyset^o$ be the ordinary locus of $X_\emptyset \otimes \mathbb{F}_p$; then for any closed geometric point $\overline{x}$, by Sect.V.7 of [10] the monodromy representation $\pi_1(X_\emptyset^o, \overline{x}) \to GL_g(\mathbb{Z}_p)$ is surjective; this is equivalent to saying that the finite étale $X_\emptyset^o$-cover $Ig(p) = \mathrm{Isom}_{X_\emptyset^o}(\mu_p^2, A[p]^o)$ is connected. Consider the scheme $Ig_b(p) = \mathrm{Isom}_{X_\emptyset^o}((\mu_p^2 \times (\mathbb{Z}/p\mathbb{Z})^2, A[p])$ where the second member consists in symplectic isometries between the standard symplectic space (for the pairing given by the matrix $J$) and $A[p]$ endowed with the Weil pairing.

By extension of isomorphisms between lagrangians to symplectic isometries, we see that $Ig_b(p)$ is a purely inseparable torsor above $Ig(p)$ under the group scheme $\mu_p \otimes U(\mathbb{Z}/p\mathbb{Z})$ where $U$ denotes the unipotent radical of the Siegel parabolic. Hence $Ig_b(p)$ is connected. Now, for each connected component $X^{\alpha\beta,\mathrm{ord}}$ of $X_B(p)^{\mathrm{ord}}$, one can define a finite surjective morphism $Ig_b(p) \to X^{o,\alpha\beta}$. For instance for $X^{me,\mathrm{ord}}$, we define a filtration inside $\mu_p^2 \times (\mathbb{Z}/p\mathbb{Z})^2$ by $H_1^{me} = \mu_p \times 1 \times 0 \times 0 \subset H_2^{me} = \mu_p \times 1 \times \mathbb{Z}/p\mathbb{Z} \times 0$, and we define $f^{me}$ as sending $(A, \lambda, \xi) \in Ig(p)$ to $(A, \lambda, 0 \subset \xi(H_1^{me}) \subset \xi(H_2^{me}) \subset A[p]) \in X^{me,\mathrm{ord}}$. This shows the connectedness of $X^{me,\mathrm{ord}}$. A similar argument applies to the other components.

The construction of the $G_B$-equivariant semistable model $\widetilde{M}_B$ of $M_B$ has been done first by de Jong [16] by blowing-up $M_B$ along either of the irreducible components $(m,m)$ or $(e,e)$, while Genestier constructs a semistable scheme $\widetilde{\mathcal{L}}$ by three consecutive blowing-ups of the lagrangian grassmannian $\mathcal{L}$ in such a way that the resulting scheme has an action of $G_B$; then he shows that the isomorphism from the generic fiber of $\widetilde{\mathcal{L}}$ to that of $M_B$ extends to a proper morphism $\widetilde{\mathcal{L}} \to M_B$. He also shows [11] Construction 2.4.1 that the two constructions coincide: $\widetilde{M}_B = \widetilde{\mathcal{L}}$.

Then, both authors define $\widetilde{X}_B(p)$ as $(\mathcal{W}_B \times_{M_B} \widetilde{M}_B)/G_B$ (for its diagonal action). QED

Remark: The previous calculations show also that the proper morphism $\pi_{B,\emptyset}$ is not finite over the supersingular locus $C$ of $X_\emptyset$, for instance the inverse image $\pi_{B,\emptyset}^{-1}(CSS)$ of the (zero dimensional) superspecial locus $CSS \subset C$ coincides with the locus where the lagrangian $H_2$ coincides with the lagrangian $\alpha_p \times \alpha_p$ of $G_{1,1}[p] \times G_{1,1}[p]$, and $H_1 \subset H_2$; thus by [20] Sect.15, Th.2, the fiber of $\pi_{B,Q}$ at each superspecial point of $X_Q(p)$ is a projective line.

On the other hand, the morphism $\pi_{Q,\emptyset} : X_Q(p) \to X_\emptyset$ is finite.

### 2.1.1 The case $* = U_B$

Recall that $U_B$ denotes the unipotent radical of $B$. The study of $X_{U_B}(p)$ can be deduced from that of $X_B(p)$ following the lines of [14] Sect.6, using Oort-Tate theory. More precisely, let $\mathcal{W}$ be the $G_B$-torsor considered above and $\mathcal{W}_U = f^{-1}(U)$ the inverse image of the affine open subset $U$ of $M_B$ (see beginning of 2.1). The locus where $H_1$ and $H_2/H_1$ are connected has equation $x = b = 0$. This locus can also be described by oort-Tate theory as follows. There exist two line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on $X_B(p)$ and two global sections $u_i \in H^0(X_B(p), \mathcal{L}_i^{\otimes(p-1)}$, $i = 1, 2$, together with scheme isomorphisms $H_1 \cong \mathrm{Spec}\,(\mathcal{O}_{X_B(p)}[T]/(T^p - u_1T))$, resp. $H_2/H_1 \cong \mathrm{Spec}\,(\mathcal{O}_{X_B(p)}[T]/(T^p - u_2T))$ such that the neutral sections correspond to $T = 0$; then the locus where $H_1$ and $H_2/H_1$ are connected is given by $u_1 = u_2 = 0$ in $X_B(p)$. Moreover, the (ramified) covering $X_{U_B}(p) \to X_B(p)$ is defined by $p-1$st roots $t_i$ of $u_i$. More precisely, when $\mathcal{L}$ is a line bundle on a scheme $X$ and $u$ is a global section of $\mathcal{L}$, one defines the scheme $X[u^{1/n}]$ as the closed subscheme of $\underline{\mathrm{Spec}}_X(\mathrm{Symm}^\bullet\mathcal{L})$ given by the (well-defined) equation $t^n = u$; it is finite flat over $X$.

Hereafter, we pull back the line bundles and sections $u_i$ to $\mathcal{W}_U$. The divisor $x=0$ has two irreducible components: $x=b=0$ and $x=y+ac=0$ along which $u_1$ has a simple zero. Moreover, $u_1/x$ is well defined and does not vanish on $\mathcal{W}_U$. Similarly, $u_2/(x+bc)$ is defined everywhere and does not vanish on $\mathcal{W}_U$. By extracting $p-1$st roots of these nowhere vanishing sections, one defines an etale covering $\mathcal{Z}\to\mathcal{W}_U$. Define $\mathcal{Z}_{U_B}=X_{U_B}\times_{X_B(p)}\mathcal{Z}$. On this scheme, the functions $x$ and $x+bc$ admit $p-1$st roots. Moreover, one has a diagram analogue to the local model theory:

$$X_{U_B}(p)\leftarrow\mathcal{Z}_{U_B}\to U'=U[f_1,f_2]/(f_1^{p-1}-x,f_2^{p-1}-(x+bc))$$

Lemma 2.4 *The two morphisms of the diagram above are smooth and surjective. The scheme $U'$ is a local model of $X_{U_B}(p)$.*

Proof: The morphism $\mathcal{Z}\to X_B(p)$ is smooth since it is the composition of an étale and a smooth morphism; the same holds therefore for its base change $\mathcal{Z}_{U_B}\to X_{U_B}(p)$. The smoothness of the other morphism is proved in a similar way, noticing that one also has $\mathcal{Z}_{U_B}=\mathcal{Z}\times_U U'$.
The surjectivity of $\mathcal{W}_U\to X_B(p)$ (hence of $\mathcal{Z}_{U_B}\to X_{U_B}(p)$) follows because $U$ is $G_B$-saturating. The surjectivity of $\mathcal{Z}_{U_B}\to U'$ comes from the surjectivity of $\mathcal{W}\to M_B$.

Corollary 2.5 *The singular locus of the reduced irreducible components of $X_{U_B}(p)$ is either empty or zero-dimensional.*

Let $T'$ be the diagonal torus of the derived group $G'$ of $G$.

Proposition 2.6 *The morphism $\pi_{U_B,B}:X_{U_B}(p)\to X_B(p)$ is finite flat, generically étale of Galois group $T'(\mathbb{Z}/p\mathbb{Z})$. The special fiber $X_{U_B}(p)\otimes\mathbb{F}_p$ of $X_{U_B}(p)$ has four irreducible components mapped by $\pi_{U_B,B}$ onto the respective irreducible components of $X_B(p)\otimes\mathbb{F}_p$; each irreducible component of $X_{U_B}(p)\otimes\mathbb{F}_p$ has prime to $p$ multiplicities and the singular locus of the underlying reduced subscheme of each component is at most zero dimensional.*

One can also describe a local model of the quasisemistable scheme $\widetilde{X}_{U_B}(p)=X_{U_B}(p)\times_{X_B(p)}\widetilde{X}_B(p)$. Namely, recall that the map $\widetilde{M}_B\to M_B$ restricted to the affine subscheme $U\subset M_B$ as before, is described (in de Jong's approach) as the blowing-up of $U$ along $x=b=0$. It is the union of two charts $V:(b,[x/b])$ and $V':(x,[b/x])$; the first is more interesting as it is $G_B$-saturating in the blowing-up. In $V$, one has $y=-([x/b]+c)$, hence after eliminating $y$, one finds a single equation for $V$ in the affine space of $a,b,c,[x/b]$, namely: $p=-ab[x/b]([x/b]+c)$. Therefore the inverse image $V_{U_B}$ of $V$ in $\widetilde{X}_{U_B}(p)$ has equations

$$p=-ab[x/b]([x/b]+c),\quad f_1^{p-1}=b.[x/b],\quad f_2^{p-1}=b\cdot([x/b]+c)$$

This scheme is not regular, but has toric, hence mild, singularities. The restriction of $\widetilde{\mathcal{Z}}_{U_B}$ above $V$ provides again a diagram

$$\widetilde{X}_{U_B}(p) \leftarrow \widetilde{\mathcal{Z}}_{U_B,V} \to V_{U_B}$$

with smooth and surjective arrows (for the left one, the surjectivity comes from the $G_B$-saturating character of $V$). Therefore, $V_{U_B}$ is a local model of $\widetilde{X}_{U_B}(p)$.

## 2.2 The case $* = P$

We follow the same method (see [12] Sect.6 for a slightly different proof). We keep the same notations (so $p$ is prime to the level $N$ of the neat group $K$). In order to study $X_P(p)$ over $\mathbb{Z}_p$, we consider the diagram of morphisms

$$\begin{array}{ccc} & \mathcal{W}_P & \\ \pi \swarrow & & \searrow f \\ X_P(p) & & M_P \end{array}$$

$\mathcal{W}_P$ is the $\mathbb{Z}_p$-scheme which classifies isomorphism classes of $(A, \lambda, \overline{\eta}, H_1, \phi)$ where $\phi : St. \otimes \mathcal{O}_S \to M.(A)$ is an isomorphism between two diagrams.
The first is $St. \otimes \mathcal{O}_S, \psi_0$ where $St_i = \mathbb{Z}_p^4$ $(i = 0, 1)$ and the diagram $St.$ consists in the inclusion $\alpha_1 : St_1 \to St_0$, $\alpha_1(e_0) = pe_0$ and $\alpha_1(e_i) = e_i$ $(i \neq 0)$, and as before, $\psi_0$ is the standard unimodular symplectic pairing on $St_0$ given by $J$.
The second is given by the inclusion of Dieudonné modules $D(A_1) \to D(A_0)$ associated to the $p$-isogeny $A_0 \to A_1$ where $A_0 = A$ and $A_1 = A/H_1$.
Let $G_P$ be the group scheme representing the functor $S \mapsto \mathrm{Aut}_S\,(St. \otimes \mathcal{O}_S)$; is is a smooth group scheme of dimension 11 over $\mathbb{Z}_p$ whose generic fiber is $G$ and the special fiber is an extension of the Klingen parahoric $P$ by the opposite unipotent radical. Then $\pi : cW_P \to X_P(p)$ is a $G_P$-torsor .
The local model $M_P$ is the projective $\mathbb{Z}_p$-scheme classifying isomorphism classes of pairs $(\omega_1, \omega_0)$ of rank 2 direct factors $\omega_i \subset St_i$ $(i = 0, 1)$ such that $\alpha_1(\omega_1) \subset \omega_0$ and $\omega_0$ is totally isotropic for $\psi_0$. The map $f$ send a point of $\mathcal{W}_P$ to the pair obtained by transporting the Hodge filtrations to $St. \otimes \mathcal{O}_S$ via the isomorphism $\phi$
We introduce again an open neighborhood $U$ of the point $\xi_0 = (\overline{\omega}_1, \overline{\omega}_0)$ in $M_P$ with $\overline{\omega}_1 = \langle e_0, e_3 \rangle$ and $\overline{\omega}_0 = \langle e_2, e_3 \rangle$. Its importance, as in the Iwahori case, stems from the fact that it is $G_P$-saturating $G_P U = M_P$ (same proof as above). It consists in the points $(\omega_1, \omega_0)$ where $\omega_1 = \langle e_0 + b_{11}e_1 + b_{12}e_2, e_3 + b_{21}e_1 + b_{22}e_2 \rangle$ and $\omega_0 = \langle e_2 + a_{11}e_0 + a_{12}e_1, e_3 + a_{21}e_0 + a_{22}e_1 \rangle$.
The condition $\alpha_1(\omega_1) \subset \omega_0$ yields the relations $p = b_{12}a_{11}$, $b_{11} = b_{12}a_{12}$, $0 = a_{21} + b_{22}a_{11}$ and $b_{21} = a_{22} + b_{22}a_{12}$. The isotropy relation yields $a_{11} = a_{22}$. By putting $x = a_{11}$, $y = b_{12}$, $z = a_{12}$, $t = b_{22}$, we find that $U = \mathrm{spec}\, R$ where $R = \mathbb{Z}_p[x, y, z, t]/(xy - p)$, so that for any maximal ideal $\mathfrak{m}_0$ corresponding to $(\overline{x}_0, \overline{y}_0, \overline{z}_0, \overline{t}_0)$ of $U(\overline{\mathbb{F}}_p)$, the completion $\widehat{R}_{\mathfrak{m}_0}$ is $\mathbb{Z}_p^{ur}[[x, y, z, t]]/(xy - p)$, if $\overline{x}_0\overline{y}_0 = 0$, and smooth otherwise. In any case, the local rings are $\mathbb{Z}_p$-regular.

Via transitive action of $G_P$ we conclude that $M_P$ is semistable, with special fiber a union of two smooth irreducible components $Z_0$ (locally: $x=0$) and $Z_1$ (locally: $y=0$).
In this situation, it is natural to consider only the maps

$$\sigma_0 : s \mapsto \dim \omega_0(s)/\alpha_1(\omega_1(s)) \quad \text{and}\, \tau_0 : s \mapsto \dim M_0/\omega_0(s) + \alpha_1(M_1)$$

as above; the regular locus $M_P^r$ of $M_P \otimes \mathbb{F}_p$ coincides with the locus where $(\sigma_0(s), \tau_0(s)) \in \{(0,1),(1,0)\}$.
As for $* = B$, we conclude that

Theorem 3 *The scheme $X_P(p)$ is flat, semistable over $\mathbb{Z}_p$. The ordinary locus in the special fiber is dense, strictly contained in the regular locus. The special fiber $X_B(p) \otimes \mathbb{F}_p$ is the union of two smooth irreducible components $X^m$ and $X^e$ where $X^m - X^e$ is the locus where $H_1$ is multiplicative, and $X^e - X^m$ is the locus where $H_1$ is étale. The singular locus of $X_P(p) \otimes \mathbb{F}_p$ is a smooth surface; it is the locus where $H_1$ is étale-locally isomorphic to $\alpha_p$.*

The proof of the density of the ordinary locus is as follows. The forgetful morphism $X_B(p) \to X_P(p)$ sends the ordinary locus of $X_B(p)$ onto the one of $X_P(p)$; hence the density of the first implies that of of the second. The singular locus is the intersection of the two components; it is the locus where $H_1$ is étale-locally isomorphic to $\alpha_p$.
Remark: We give an ad hoc proof of the density of the ordinary locus of $X_P(p) \otimes \mathbb{F}_p$ in[12] Prop.6.4.2.

### 2.3 The case $* = Q$

Again, the same method applies; however, in order to study $X_Q(p)$ over $\mathbb{Z}_p$ and find a semistable model $\widetilde{X}_Q(p) \to X_Q(p)$, we'll first perform calculations in the flavor of de Jong's method [16], as a motivation for Genestier's approach ([11] Sect.3.3.0 and 3.3.3 and [12] Appendix) which we will follow and further a little.
We consider the diagram of morphisms

$$\begin{tikzcd} & \mathcal{W}_Q \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ X_Q(p) & & M_Q \end{tikzcd}$$

where $\pi_Q : \mathcal{W}_Q \to X_Q(p)$ is the $X_Q(p)$-scheme classifying isomorphism classes of $(A, \lambda, \overline{\eta}, H_2, \phi)$ where $\phi : St. \otimes \mathcal{O}_S \to M.(A)$ is a symplectic isomorphism between two diagrams.
The first is $St. \otimes \mathcal{O}_S, \psi_0, \psi_2$ where $St_i = \mathbb{Z}_p^4$ $(i = 0, 2)$ and the diagram $St.$ consists in the inclusion $\alpha^2 : St_2 \to St_0$, $\alpha^2(e_i) = pe_i$ $(i = 0, 1)$ and $\alpha_1(e_i) = e_i$ $(i > 1)$, and as before, $\psi_0$ and $\psi_2$ both denote the standard unimodular symplectic pairing on $\mathbb{Z}_p^4$ given by $J$. Note that $\alpha^2$ is a symplectic similitude of similitude factor $p$: $\psi_2(\alpha^2(x), \alpha^2(y)) = p \cdot \psi_0(x, y)$.

Let $G_Q$ be the $\mathbb{Z}_p$-group scheme of automorphisms of $M_Q$. It acts on $\mathcal{W}_Q$ as well and $\pi_Q$ is a $G_Q$-torsor.
Let $\mathcal{L}$ be the grassmannian of lagrangian direct factors in $St_0$ over $\mathbb{Z}_p$. Following [11] and [12] Appendix, we shall construct a $G_Q$-equivariant birational proper morphism $\mathcal{L}^{(2)} \to \mathcal{L}$ over $\mathbb{Z}_p$, composition of two blowing-up morphisms along closed subschemes of the special fiber such that $\mathcal{L}^{(2)}$ is semistable and is endowed with a canonical $G_Q$-equivariant proper morphism $h : \mathcal{L}^{(2)} \to M_Q$ (an isomorphism in generic fiber). We shall call $h$ the Genestier morphism for $(GSp_4, Q)$. For the easiest case $(GSp_{2g}, P)$, see Prop.6.3.4. of [12].
As a motivation for the detailed construction below by two blowing-ups, we introduce the open subset $U$ of $M_Q$ consisting of pairs $(\omega_2, \omega_0) \in M_Q$ where $\omega_0$ is spanned by $e_3 + a_{21}e_0 + a_{11}e_1$ and $e_2 + a_{22}e_0 + a_{12}e_1$ (with $a_{12} = a_{21}$) and $\omega_2 = \langle e_1 + c_{21}e_2 + c_{11}e_3, \alpha e_0 + c_{22}e_2 + c_{12}e_3\rangle$ (with $c_{12} = c_{21}$), such that $\alpha^2(\omega_2) \subset \omega_0$; it is therefore isomorphic to the affine set of $\mathbf{A}^6_{\mathbb{Z}_p}$ consisting of pairs $(A, C)$ of $2 \times 2$ symmetric matrices such that $AC = p1_2$ by the map

$$(A, C) \mapsto \begin{pmatrix} s \\ sC \end{pmatrix}, \begin{pmatrix} sA \\ s \end{pmatrix}$$

Its special fiber has three irreducible components, given by $A = 0$, $B = 0$ and the Zariski closure of the locally closed set: $\operatorname{rk} A = \operatorname{rk} B = 1$. One then defines $\widetilde{U}$ in $\widetilde{M}_Q$ as the quotient by $\mathbb{G}_m$ of the affine open set of triples $(\lambda, A', \mu)$ such that $A' \neq 0$ is symmetric and $\lambda\mu\det A' = p$, the action of $\mathbb{G}_m$ being given by $t \cdot (\lambda, A', \mu) = ((t\lambda, t^{-1}A', t\mu)$. The map $(\lambda, A', \mu) \mapsto (A, C)$ given by $A = \lambda A'$, $C = \mu^t com(A')$ is the blowing-up of $U$ along the component $A = 0$.

Remark: One checks easily that $\widetilde{U}$ is also the blowing-up of $U$ along $C = 0$. Hence the projection is invariant under the symmetry $(A, C) \mapsto (C, A)$. This allows the definition of an involution $W$ on $\widetilde{U}$. This involution will extend to $\widetilde{M}_B$. See after Prop. below. Note however that the following construction is dyssymmetrical, and does not make explicit use of the open set $U$ defined above.

The first blowing-up $\mathcal{L}^{(1)}$ of the lagrangian grassmannian $\mathcal{L}$ over $\mathbb{Z}_p$ along the closure of $Q \cdot \overline{\omega}_{23}$ where $\overline{\omega}_{23}$ is the $\mathbb{F}_p$-lagrangian spanned by $e_2$ and $e_3$.
Note that by functoriality of the blowing-up, $\mathcal{L}^{(1)}$ is endowed with a natural action of $G_Q$ (which acts on $\mathcal{L}$ through the canonical morphism $G_Q \to G$ and leaves the center of blowing-up stable).
Namely, let us consider the affine open subset $\Omega_0$ of $\mathcal{L}$ consisting of the lagrangian planes $\omega_0 = \langle e_3 + a_{11}e_0 + a_{12}e_1, e_2 + a_{21}e_0 + a_{22}e_1\rangle$ (with $a_{12} = a_{21}$), the blowing-up $\mathcal{L}^{(1)}|\Omega_0$ is the closed $\mathbb{Z}_p$-subscheme of $\mathbb{A}^3 \times \mathbb{P}^3$ of points $(a_{11}, a_{12}, a_{22}; [A_{11}, A_{12}, A_{22}, S])$ such that

$$a_{11}A_{12} - a_{12}A_{11} = 0,\ a_{11}A_{22} - a_{22}A_{11} = 0, a_{12}A_{22} - a_{22}A_{12,} = 0$$

and

$$pA_{11} = a_{11}S,\ pA_{12} = a_{12}S,\ pA_{22} = a_{22}S.$$

The scheme $\mathcal{L}^{(1)}|\Omega_0$ can be described as the quotient by $\mathbb{G}_m$ of the locally closed $\mathbb{Z}_p$-subscheme $T_1$ of the affine space $\mathbb{A}^5$ defined in terms of the coordinates $(\lambda_0, P_0, A_{11}, A_{12}, A_{22})$ as the intersection of the closed subscheme $\lambda_0 P_0 = p$ with the complement of the closed subscheme $P_0 = A_{11} = A_{12} = A_{22} = 0$. The action of $\mathbb{G}_m$ is given by multiplication by $\lambda^{-1}$ on the first variable and by $\lambda$ on the rest.

Indeed, the quotient map $T_1 \to \mathcal{L}^{(1)}|\Omega_0$ is

$$(\lambda_0, P_0, A_{11}, A_{12}, A_{22}) \mapsto (a_{11}, a_{12}, a_{22}; [A_{11}, A_{12}, A_{22}, S])$$

where $a_{11} = \lambda_0 A_{11}$, $a_{12} = \lambda_0 A_{12}$, $a_{22} = \lambda_0 A_{22}$, $S = P_0$.

To take care of equation (1), following [11] Theorem, one forms the blow-up $\mathcal{L}^{(2)}$ of $\mathcal{L}^{(1)}$ along the strict transform $Z_{02}^{c,(1)}$ of the Zariski closure $Z_{02}^c$ of $Z_{02} = Q \cdot \overline{\omega}_{02}$ where $\overline{\omega}_{02}$ is the lagrangian spanned by $e_0$ and $e_2$.

The equations of $\mathcal{L}^{(2)}|\Omega_0$ can be determined as follows. First, one notes that $Z_{02}^{c,(1)}|\Omega_0$ is given as a $\mathbb{Z}_p$-subscheme of $\mathcal{L}^{(1)}|\Omega_0$ by the equations $A_{11}A_{22} - A_{12}^2 = P_0 = 0$. Its inverse image in $T_1$ is given by the same equations (this time, viewed in an affine space). Let $\delta = A_{11}A_{22} - A_{12}^2$.

Then, the blowing-up $T^{(2)}$ of $T_1$ along this inverse image is the subscheme of $T_1 \times \mathbb{P}^1$ with coordinates $(\lambda_0, P_0, A_{11}, A_{12}, A_{22}, [P_1, \delta_1])$ given by the equation $\delta P_1 = \delta_1 P_0$ (with $(P_1, \delta_1) \neq (0, 0)$).

Introducing $\lambda_1$ such that $P_0 = \lambda_1 P_1$, and $\delta = \lambda_1 \delta_1$, one can rewrite $T^{(2)}$ as the quotient by $\mathbb{G}_m$ of the affine locally closed subscheme $T_2$ of $\mathbb{A}^7$ with affine coordinates $(\lambda_0, \lambda_1, P_1, A_{11}, A_{12}, A_{22}, \delta_1)$ and equations $\lambda_0\lambda_1 P_1 = p$ and $\lambda_1\delta_1 = A_{11}A_{22} - A_{12}^2$ in the open subset of $\mathbb{A}^7$ intersection of the locus $(\lambda_1 P_1, A_{11}, A_{12}, A_{22}) \neq (0, 0, 0, 0)$ with $(\delta_1, P_1) \neq (0, 0)$; the action of $\mu \in \mathbb{G}_m$ being the trivial one on $\lambda_0$ and $A_{ij}$, the multiplication by $\mu^{-1}$ on $\lambda_1$ and the multiplication by $\mu$ on $P_1$ and $\delta_1$.

The quotient map is

$$(\lambda_0, \lambda_1, P_1, A_{11}, A_{12}, A_{22}, \delta_1) \mapsto (\lambda_0, P_0, A_{11}, A_{12}, A_{22}, [P_1, \delta_1])$$

with $P_0 = \lambda_1 P_1$.

We can thus write $\mathcal{L}^{(2)}|\Omega_0$ as a quotient $T_2/\mathbb{G}_m^2$, for the action of $(\lambda, \mu) \in \mathbb{G}_m^2$ on $(\lambda_0, \lambda_1, P_1, A_{11}, A_{12}, A_{22}, \delta_1) \in T_2$ by multiplication by $\lambda^{-1}$ on $\lambda_0$, $\mu^{-1}$ on $\lambda_1$, by $\lambda\mu$ on $P_1$, by $\lambda$ on $A_{ij}$ and $\lambda^2\mu$ on $\delta_1$.

The $\mathbb{Z}_p$-scheme $T_2$ is clearly semistable. It implies by Lemme 3.2.1 of [11] that $\mathcal{L}^{(2)}|\Omega_0$ is also semistable. Since $G_Q \cdot \mathcal{L}^{(2)}|\Omega_0 = \mathcal{L}^{(2)}$, the same holds for $\mathcal{L}^{(2)}$.

Let us consider the forgetful morphism $\pi_0 : M_Q \to \mathcal{L}$, $(\omega_2, \omega_0) \mapsto \omega_0$; the open subset $U'' = \pi^{-1}(\Omega_0) \subset M_Q$. This open set is not affine, it is dyssymmetrical, it contains the affine open set $U$ defined above.

We can now define the Genestier morphism $h$ on $\mathcal{L}^{(2)}|\Omega_0$. It is given by the $\mathbb{G}_m^2$-invariant map

$$T_2 \to U'', \quad (\lambda_0, \lambda_1, P_1, A_{11}, A_{12}, A_{22}, \delta_1) \mapsto (\omega_2, \omega_0)$$

where $\omega_0$ is given by $a_{ij} = \lambda_0 A_{ij}$ and $\omega_2$ is given in terms of its Plücker coordinates on the basis $(e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3)$: $[\delta_1, P_1 A_{11}, -P_1 A_{12}, P_1 A_{12}, P_1 A_{22}, \lambda_1 P_1^2]$. This point of $\mathbb{P}^5$ is well defined because if $\delta_1 = 0$, we have $P_1 \neq 0$ and if $\lambda_1 = 0$, one of the $A_{ij} \neq 0$. It is invariant by the action of $\mathbb{G}_m^2$ hence factors through $\mathcal{L}^{(2)}|\Omega_0$. Moreover it corresponds to an isotropic plane because the third and fourth coordinates are opposite. By [11] Sect.3 before Lemme3.1.1, the saturation of $\Omega_0$ under $G_Q$ is $\mathcal{L}$, hence by $G_Q$ equivariance, it is defined everywhere on $\mathcal{L}^{(2)}$. One sees easily the surjectivity of $h$ restricted to $\mathcal{L}^{(2)}|\Omega_0$ onto $U''$ (which consists of points in $\mathbb{P}^5$ $[u_0, u_1, u_2, -u_2, u_3, u_4]$ such that $u_0 u_4 = u_1 u_3 - u_2^2$), hence by $G_Q$-equivariance, to the whole of $M_Q$.

Definition 2.7 *We put $\widetilde{M}_Q = \mathcal{L}^{(2)}$, it is a semistable $\mathbb{Z}_p$-scheme; its special fiber has three smooth irreducible components. We define $\widetilde{X}_Q(p) = (\mathcal{W}_Q \times \widetilde{M}_Q)/G_Q$; it is a semistable model of $X_Q(p)$ over $\mathbb{Z}_p$ with smooth irreducible components; their number is at least three. It comes with a proper birational morphism $h_X : \widetilde{X}_Q(p) \to X_Q(p)$ which we call the Genestier morphism which is an isomorphism on the generic fiber.*

What precedes is a developed version of [12] Appendix, which may be useful to non expert algebraic geometers. We give now some new information on $h$ and $h_X$.
For any geometric point $s = (\omega_2, \omega_0)$ of the special fiber of $M_Q$, let $k = k(s)$ be the residue field; we define

$$\sigma(s) = \dim \omega_0/\alpha^2(\omega_2), \quad \tau(s) = \dim M_0/(\alpha^2(M_2) + \omega_0)$$

Let $x = (A, \lambda, H_2)$ be a geometric point of $X_Q(p)$ corresponding to $s$. Note that $\sigma(s)$ is the $p$-rank of the connected component $H_2^0$ of the group scheme $H_2$, while $\tau(s)$ is the $p$-rank of the connected component of the Cartier dual $H_2^t$ of $H_2$. It can be identified by the Weil pairing to $A[p]/H_2$. From this it is easy to verify that the condition

$$(\mathrm{Ord}) \qquad (\sigma(s), \tau(s)) \in \{(0,2), (2,0), (1,1)\}$$

is equivalent to the ordinarity of the point $x$. Let $M_Q^{\mathrm{ord}}$ be the locus where (Ord) is satisfied. Then the ordinary locus of $X_Q(p)^{\mathrm{ord}}$ of $X_Q(p) \times \mathbb{F}_p$ is equal to $\pi(f^{-1}(M_Q^{\mathrm{ord}})$.
We have a partition $M_Q^{\mathrm{ord}} = M_Q^{\mathrm{ee,ord}} \sqcup M_Q^{\mathrm{mm,ord}} \sqcup M_Q^{\mathrm{em,ord}}$. corresponding to the conditions $(\sigma(s), \tau(s)) \in \{(0,2)$, resp. $(\sigma(s), \tau(s)) \in \{(2,0)$, resp. $(\sigma(s), \tau(s)) \in \{(1,1)$.
Similarly, by taking the inverse images in $\widetilde{M}_Q$ by $h$, we can define a similar partition of $\overline{M}_Q^{\mathrm{ord}}$:

$$\widetilde{M}_Q^{\mathrm{ord}} = \widetilde{M}_Q^{\mathrm{ee,ord}} \sqcup \widetilde{M}_Q^{\mathrm{mm,ord}} \sqcup \widetilde{M}_Q^{\mathrm{em,ord}}.$$

Let $\widetilde{M}_Q^{\rm reg}$ resp. $\widetilde{M}_Q^{\rm sing}$ be the regular locus resp. singular locus of the special fiber of $\widetilde{M}_Q$.
Let us determine the locus $M_Q^{\rm ee,ord}\cap U''$ where $(\sigma(s),\tau(s))=(0,2)$ in $U''$, together with its inverse image $\widetilde{M}_Q^{\rm ee,ord}|\Omega_0$ by $h$. The condition $\tau(s)=2$ translates as $a_{11}=a_{12}=a_{22}=0$; this implies $\lambda_0=0$. On the other hand, $\sigma(s)=0$ implies, using Plücker coordinates, that $\lambda_1 P_1\neq 0$. One checks easily that actually $(\sigma(s),\tau(s))=(0,2)$ if and only if $\lambda_0=0$ and $\lambda_1 P_1\neq 0$.
In particular,$\widetilde{M}_Q^{\rm ee,ord}|\Omega_0$ coincides with the (smooth) irreducible component $\lambda_0=0$ deprived from $\widetilde{M}_Q^{\rm sing}$; moreover, $h$ induces an isomorphism between $\widetilde{M}_Q^{\rm ee,ord}|\Omega_0$ and $M_Q^{\rm ee,ord}\cap U''$.
Similarly for the locus $M_Q^{mm}\cap U''$ where $(\sigma(s),\tau(s))=(2,0)$ in $U''$; the condition $\tau(s)=0$ is given by the equation $a_{11}a_{22}-a_{12}^2\neq 0$, that is, $\lambda_0^2\lambda_1\delta_1\neq 0$; while $\sigma(s)=2$ implies $P_1=0$. Conversely, one sees easily that $(\sigma(s),\tau(s))=(2,0)$ if and only if $P_1=0$ and $\lambda_0\lambda_1\neq 0$.
Therefore,$\widetilde{M}_Q^{\rm mm,ord}|\Omega_0$ coincides with the smooth irreducible component $P_1=0$ minus $\widetilde{M}_Q^{\rm sing}$.
Finally, we consider the locus $M_Q^{em}\cap U''$ where $(\sigma(s),\tau(s))\in\{(1,1)\}$ in $U''$. We see that $\tau(s)=1$ is equivalent to $a_{11}a_{22}-a_{12}^2=0$ and $(a_{11},a_{12},a_{22})\neq(0,0,0)$, that is, $\lambda_0^2\lambda_1\delta_1=0$ and $(\lambda_0 A_{ij}\neq(0,0,0)$. While $\sigma(s)=1$ implies $\lambda_1 P_1^2=0$. Conversely, one sees easily that
$(\sigma(s),\tau(s))\in\{(1,1)\}$ if and only if $\lambda_1=0$ and $\lambda_0 P_1\neq 0$. In other words $\widetilde{M}_Q^{\rm em,ord}|\Omega_0$ coincides with the smooth irreducible component $\lambda_1=0$ minus $\widetilde{M}_Q^{\rm sing}$.
In the three cases, one deduces also from the previous calculations that $h$ induces an isomorphism between $\widetilde{M}_Q^{\alpha\beta,\rm ord}|\Omega_0$ and $M_Q^{\alpha\beta,\rm ord}\cap U''$.

We define then the Zariski closures $M_Q^{\alpha\beta}$ of $M_Q^{\alpha\beta,\rm ord}$ and $\widetilde{M}_Q^{\alpha\beta}$ of $\widetilde{M}_Q^{\alpha\beta,\rm ord}$. Using $G_Q$-equivariance, we define $\widetilde{X}^{\alpha\beta}$ as $(\mathcal{W}_Q\times M_Q^{\alpha\beta})/G_Q$ for all $\alpha,\beta\in\{e,m\}$ (with the convention that $em=me$ )
We can then conclude

Theorem 4 *The scheme $X_Q(p)$ is flat, locally complete intersection over $\mathbb{Z}_p$. The ordinary locus in the special fiber is dense in every irreducible component; it is contained in the regular locus. The special fiber $X_Q(p)\otimes\mathbb{F}_p$ is the union of three irreducible components $X^{mm}$ and $X^{me}$ and $X^{em}$ which are the Zariski closures respectively of the locus where $H_2$ is of multiplicative type, the locus where, locally for the étale topology, $H_2=\mu_p\times\mathbb{Z}/p\mathbb{Z}$ and the locus where $H_2$ is étale. The singular locus of $X_Q(p)\otimes\mathbb{F}_p$ is the locus where $H_2$ étale-locally contains $\alpha_p$.*
*There is a semistable model together with a blowing-up morphism $h_X$ : $\widetilde{X}_Q(p)\to X_Q(p)$ whose center is in the special fiber; the special fiber of $\widetilde{X}_Q(p)$ consists of three smooth irreducible components $\widetilde{X}^{mm}$, $\widetilde{X}^{me}$ and $\widetilde{X}^{em}$*

*crossing transversally. The ordinary locus $\widetilde{X}_Q(p)^{\mathrm{ord}}$ coincides with the regular locus $\widetilde{X}_Q(p)^{\mathrm{reg}}$. The restriction of $h_X$ induces proper surjective morphisms $\widetilde{X}^{\alpha\beta} \to X^{\alpha\beta}$ ($\alpha, \beta \in \{e, m\}$) which are isomorphisms between the respective ordinary loci.*

The irreducibility of the components $X^{\alpha\beta}$ follows from [29] as explained above. This implies the irreducibility of the three components $\widetilde{X}^{\alpha\beta}$ because $h_X$ is an isomorphism between the two dense open subsets $\widetilde{X}^{\alpha\beta,\mathrm{ord}}$ and $X^{\alpha\beta,\mathrm{ord}}$, the latter being irreducible.

Remark: Note that we have thus recovered part of the results of [9]; however, this paper contains extra informations: the singular locus of $X_Q(p) \otimes \mathbb{F}_p$ coincides with the finite set of superspecial abelian surfaces (that is, the cartesian products of supersingular elliptic curves); these isolated singularities are Cohen-Macaulay. The description of the intersections two by two and of the three components is given in Sect.6.2 there.

Finally, we introduce an involution $W$ of the $\mathbb{Z}_p$-schemes $X_Q(p)$ and $\overline{X}_Q(p)$ compatible with $h_X$. The automorphism of the functor $\mathcal{F}_Q$ given by $(A, \lambda, \eta, H_2) \mapsto (\overline{A}, \overline{\lambda}, \overline{\eta}, \overline{H}_2)$ where $\overline{A} = A/H_2$, $\overline{\lambda}$, resp. $\overline{\eta}$ is the quotient polarization resp. $\Gamma$-level structure on $\overline{A}$ deduced from $\lambda$ resp. $\eta$ and $\overline{H}_2 = A[p]/H_2$, induces an involution of the $\mathbb{Z}[1/N]$-scheme $X_Q(p)$, hence of its pull-back to $\mathbb{Z}_p$. If one writes the test objects as $(\alpha : A_0 \to A_2, \eta_0, \eta_2)$ where $A_i$'s are principally polarized abelian varieties, $\alpha$ is an isogeny with lagrangian kernel in $A[p]$ respecting the polarizations and the $\Gamma$-level structures $\eta_i$ on $A_i$, we see that the involution $W$ can be written as the duality $\alpha \mapsto {}^t\alpha$ followed by the identifications of the dual abelian varieties ${}^tA_i$ to $A_i$; hence $W$ maps $(\alpha : A_0 \to A_2, \eta_0, \eta_2)$ to $({}^t\alpha : A_2 \to A_0, \eta_2, \eta_0)$.

This involution $W$ therefore extends to the torsor $\mathcal{W}_Q$ by replacing the diagram $M.(A) = (M(\alpha) : M(A_2) \to M(A_0))$ by its dual $M({}^tA) = (M({}^t\alpha) : M(A_0) \to M(A_2))$ and by interchanging the two isomorphisms $\phi_0$ and $\phi_2$ in the isomorphism of diagrams $\phi : St. \otimes \mathcal{O}_S \to M.(A)$ to obtain $\phi' : St. \otimes \mathcal{O}_S \to M.(A)$.

The involution $W$ on $\mathcal{W}_Q$ is compatible with the forgetful morphism $\mathcal{W}_Q \to M_Q$ where $W$ on $M_Q$ is given by taking the dual of $\alpha^2 : St_2 \to St_0$ with respect to the standard symplectic pairings $\psi_0$ and $\psi_2$, and exchanging $\omega_0$ and $\omega_2$.

Hence, the involution acts on the diagram $X_Q(p) \leftarrow \mathcal{W}_Q \to M_Q$.

Remark: By taking symplectic bases, its matricial interpretation is $\begin{pmatrix} 0 & -s \\ p \cdot s & 0 \end{pmatrix}$; note that this matrix normalizes the automorphism group $G_Q$ of the diagram $St.$.

The involution $W$ exchanges the two extreme irreducible components $X^{ee}$ and $X^{mm}$ of $X_Q(p) \otimes \mathbb{F}_p$ and it leaves the intermediate component $X^{em}$ stable.

Remarks:

1) There is another construction of the morphism $h : \widetilde{M}_Q \to M_Q$ by noticing that the restriction of $h$ above the open subset $U$ introduced at the beginning of the present section coincides with the map $\widetilde{U} \to U$ defined above and is $G_Q$-equivariant. Since $U$, $\widetilde{U}$ and $h|\widetilde{U}$ is symmetric under $(A, C) \mapsto (C, A)$; $W$

extends by $G_Q$-action to an involution of $\widetilde{M}_Q$-still denoted $W$, compatible to $h$. We thus obtain an involution $W$ of the $\mathbb{Z}_p$-scheme $\widetilde{X}_Q(p) = (\mathcal{W}_Q \times \widetilde{M}_Q)/G_Q$ compatible to $h_X : \widetilde{X}_Q(p) \to X_Q(p)$; it exchanges the irreducible components $\widetilde{X}^{ee}$ and $\widetilde{X}^{mm}$ of $\widetilde{X}_Q(p) \otimes \mathbb{F}_p$ and leaves $\widetilde{X}^{em}$ stable.
2) Genestier's construction [11] of the semistable model $\widetilde{M}_B$ of the local model $M_B$ of $X_B(p)$ in a way similar to that of $\widetilde{M}_Q$ implies that the forgetful morphism $M_B \to M_Q$, $(\omega_2, \omega_1, \omega_0) \mapsto (\omega_2, \omega_0)$ extends to the semistable models $\widetilde{M}_B \to \widetilde{M}_Q$; an easy argument provides then a canonical morphism between the Genestier models $\widetilde{X}_B(p) \to \widetilde{X}_Q(p)$. However, it should be noted that the morphism $M_B \to M_Q$ is NOT a local model of the morphism $X_B(p) \to X_Q(p)$. This is already false for the case of the classical modular curve $X_0(p)$ and the classical modular curve $X$ of level prime to $p$.
Finally, note that as explained in the case $* = Q$, there is a Fricke-Weil involution $W$ on $X_B(p)$; it extends to the semistable models and the forgetful morphism $\widetilde{\pi}_{B,Q}$ is compatible with $W$.

### 2.4 Rigid geometry of Siegel varieties

We gather here some informations concerning the rigid geometry of the Siegel varieties $X = X_\emptyset$ and $X_Q(p)$. Some (Prop.2.6, 2) are used in the formulation of the conjecture of Sect.4.3. We hope to develop them in another paper for studying analytic continuation of overconvergent Siegel cusp eigenforms.
Let $X^{\rm rig}$, $X^{*,{\rm rig}}$ resp. $\overline{X}^{\rm rig}$ be the rigid analytic space associated to the $p$-adic completion of its corresponding $\mathbb{Z}_p$-scheme (for the toroidal compactification, we assume throughout this section that we fixed a fine $\Gamma$-admissible polyedral cone decomposition $\Sigma$).
Choosing a $\Gamma_Q(p)$-admissible refinement $\Sigma'$ of $\Sigma$, one can define a smooth toroidal compactification $\overline{X}_Q(p)_{/\mathbb{Q}_p}$ of the $\mathbb{Q}_p$-scheme $X_Q(p) \otimes \mathbb{Q}_p$ (actually, by [10], it exists as a proper smooth scheme over $\mathbb{Z}[\frac{1}{Np}]$). Because of the compatibility of $\Sigma$ and $\Sigma'$, we see that the forgetful morphism $\pi = \pi_{Q\emptyset}$ : $X_Q(p) \to X$ extends uniquely as a morphism $\overline{\pi} : \overline{X}_Q(p) \to \overline{X}$.
Let $\overline{X}_Q(p)^{\rm rig}$ be the rigid space over $\mathbb{Q}_p$ corresponding to the scheme $\overline{X}_Q(p)_{/\mathbb{Q}_p}$ (cf. Chapter 9, Ex.2 of [5]). Let $\overline{\mathcal{X}}$ be the formal completion of $\overline{X}$ along the special fiber. The ordinary locus $\overline{\mathcal{X}}^{\rm ord}$ is an open formal subscheme of $\overline{\mathcal{X}}$; let $\overline{X}^{\rm rig,ord}$ be the corresponding admissible rigid open subset of $\overline{X}^{\rm rig}$. Let $\overline{X}_Q(p)^{\rm rig,ord}$ be the inverse image of $\overline{X}^{\rm rig,ord}$ by $\overline{\pi}^{\rm rig}$.
We want to describe the connected components of this admissible rigid open set and strict neighborhoods thereof, in terms of a suitable model of $\overline{X}_Q(p)^{\rm rig}$. For this purpose, we write simply $X_G$ for the semistable model $\widetilde{X}_Q(p)$ of $X_Q(p)$ over $\mathbb{Z}_p$. We briefly explain the construction of a "toroidal compactifcation of $X_G$" associated to $\Sigma'$, by which we mean a proper regular $\mathbb{Z}_p$-scheme $\overline{X}_G$ together with a toroidal open immersion $X_G \hookrightarrow \overline{X}_G$ such that $\overline{X}_G \otimes \mathbb{Q}_p$ is the (smooth) toroidal compactification $\overline{X}_Q(p)_{/\mathbb{Q}_p}$ associated to $\Sigma'$ mentioned

above. Details on this construction, specific to the genus 2 case, should appear in the thesis of a student of A. Genestier. The model of $\overline{X}_Q(p)^{\mathrm{rig}}$ that we are looking for is then defined as the formal completion $\overline{\mathcal{X}}_G$ of $\overline{X}_G$ along the special fiber.

The construction is as follows. One first takes the normalization of the $\mathbb{Z}_p$-toroidal compactification $\overline{X}$ associated to $\Sigma$, in the finite étale morphism $X_Q(p)_{/\mathbb{Q}_p} \to X_{/\mathbb{Q}_p}$. Let $\overline{X}_Q(p)^{\Sigma}$ be this normalization. The morphism $X_G \to X_Q(p)$ is an isomorphism outside the supersingular locus $X_Q(p)^{\mathrm{ss}}$ and this locus is proper (because we are in genus 2). We can therefore glue the schemes $\overline{X}_Q(p)^{\Sigma}$ and $X_G$ along their common open subscheme $X_Q(p)\backslash X_Q(p)^{\mathrm{ss}}$. We obtain a $\mathbb{Z}_p$-scheme denoted $\overline{X}_G^{\Sigma}$. Let $Z(\Sigma')_{/\mathbb{Q}_p}$ be the closed subscheme of $\overline{X}_Q(p)^{\Sigma}_{/\mathbb{Q}_p}$ which is the center of the blowing-up morphism

$$\overline{X}_Q(p)_{/\mathbb{Q}_p} = \overline{X}_Q(p)^{\Sigma'}_{/\mathbb{Q}_p} \to \overline{X}_Q(p)^{\Sigma}_{/\mathbb{Q}_p}$$

We consider the Zariski closure $Z(\Sigma')$ of $Z(\Sigma')_{/\mathbb{Q}_p}$ in the $\mathbb{Z}_p$-scheme $\overline{X}_G^{\Sigma}$. The blowing-up of $\overline{X}_G^{\Sigma}$ along $Z(\Sigma')$ is the desired scheme. It is denoted $\overline{X}_G$; by restricting the construction to the local charts of Faltings-Chai, it can be proven that $\overline{X}_G$ is regular over $\mathbb{Z}_p$ and that $X_G \hookrightarrow \overline{X}_G$ is toroidal, although the divisor at infinity doesn't have good reduction.

Remark: For the sake of completion, let us mention another abstract construction. Let $\mathcal{X}_G$ be the formal completion of $X_G$ along the special fiber. One can apply the notion of normalization studied in[4] to define the "normalization" $\overline{\mathcal{X}}_G^{(\mathcal{U}_i)}$ of $\mathcal{X}_G$ along $\overline{X}_Q(p)^{\mathrm{rig}}$ associated to an admissible affinoid cover of $\overline{X}_Q(p)^{\mathrm{rig}}$ (we denote by $\mathcal{U}_i$ the formal scheme associated to the affinoid $U_i$). The $\mathbb{Z}_p$-formal scheme $\overline{\mathcal{X}}_G^{(\mathcal{U}_i)}$ is endowed with an open immersion of formal schemes $\mathcal{X}_G \to \overline{\mathcal{X}}_G^{(\mathcal{U}_i)}$. However, this construction does depend on the choice of the covering. This is why the specific construction described above is better suited for our purpose.

We still denote by $\overline{\pi}$ the morphism $\overline{X}_G \to \overline{X}$ as well as its $p$-adic completion $\overline{\mathcal{X}}_G \to \overline{\mathcal{X}}$. We define the ordinary locus $\overline{\mathcal{X}}_G^{\mathrm{ord}}$ as the inverse image in $\overline{\mathcal{X}}_G$ of the ordinary locus $\overline{\mathcal{X}}^{\mathrm{ord}}$ of $\overline{\mathcal{X}}$.

We observe that $\overline{\mathcal{X}}_G^{\mathrm{ord}}$ is smooth. Its underlying $\mathbb{F}_p$-scheme is denoted by $\overline{X}_G^{\mathrm{ord}}$. Let $\widetilde{X}^{\alpha\beta,\mathrm{ord}}$ ($\alpha,\beta \in \{e,m\}$) be the three connected components of $X_G \otimes \mathbb{F}_p$. We denote by $\overline{X}_G^{\alpha\beta,\mathrm{ord}}$ the Zariski closure of $\widetilde{X}^{\alpha\beta,\mathrm{ord}}$ in $\overline{X}_G^{\mathrm{ord}}$. We have a partition into three smooth open subschemes

$$\overline{X}_G^{\mathrm{ord}} = \overline{X}_G^{mm,\mathrm{ord}} \sqcup \overline{X}_G^{me,\mathrm{ord}} \sqcup \overline{X}_G^{ee,\mathrm{ord}}$$

Therefore, by taking the inverse image by the specialization map associated to the model $\overline{\mathcal{X}}_G$, we obtain three connected components of the open admissible subset $\overline{X}_Q(p)^{\mathrm{rig,ord}}$:

$$\overline{X}_Q(p)^{\mathrm{rig,ord}} = ]\overline{X}_G^{mm,\mathrm{ord}}[ \sqcup ]\overline{X}_G^{me,\mathrm{ord}}[ \sqcup ]\overline{X}_G^{ee,\mathrm{ord}}[$$

We need to extend this to admissible quasi-compact neighborhoods of $\overline{X}_Q(p)^{\mathrm{rig,ord}}$. First we fix a lifting $E$ of the Hasse invariant (see [15] Sect.3, or see next section below). Let $\mathcal{G}^{\mathrm{rig}} \to \overline{X}^{\mathrm{rig}}$ be the rigid analytification of the semi-abelian scheme $\mathcal{G} \to \overline{X}$ (as in Chap.9, ex.2 of [5]). By a Theorem of Abbès and Mokrane [1] Prop.8.2.3 (and [2] for an improved radius of convergence), the open subdomain $\overline{X}^{\mathrm{rig}}(p^{-a})$ of $\overline{X}^{\mathrm{rig}}$ defined as the locus where the lifting $E$ of the Hasse invariant satisfies $|E|_p > p^{-a}$ ($a = \frac{1}{p(p-1)}$ for [1], and $a = \frac{p-1}{2p-1}$ for [2]) is endowed with a finite flat group scheme $C_{can}$ of rank $p^2$ whose restriction to the ordinary locus is canonically isomorphic to $\mathcal{G}[p]^0$. For each $r \in ]p^{-a}, 1[ \cap p^{\mathbb{Q}}$, we define

$$\overline{X}\{r\} = \{x \in \overline{X}^{\mathrm{rig}}(L); |E|_p \geq r\}$$

These domains are admissible, quasi-compact relatively compact neighborhoods of $\overline{X}_Q(p)^{\mathrm{rig,ord}}$ (cf.[19] Sect.3.1.6). Let $\overline{X}_Q(p)\{r\}$ be the inverse image of $\overline{X}\{r\}$ by $\overline{\pi}^{\mathrm{rig}}$.

Proposition 2.8 *1) For any $r$ sufficiently close to 1, the neighborhood $\overline{X}_Q(p)\{r\}$ has still three connected components denoted $\overline{X}_G^{\alpha\beta}\{r\}$ ($\alpha, \beta \in \{e, m\}$); $\overline{X}_G^{\alpha\beta}\{r\}$ is defined as the largest connected subset of $\overline{X}_Q(p)\{r\}$ containing $]\overline{X}_G^{\alpha\beta,\mathrm{ord}}[$.*
*2) For any $r \in ]p^{-a}, 1[$, the isomorphism $]\overline{X}_G^{mm,\mathrm{ord}}[ \cong \overline{X}^{\mathrm{rig,ord}}$ induced by the forgetful morphism extends to an isomorphism $\overline{X}_G^{mm,\mathrm{rig}}\{r\} \cong \overline{X}\{r\}$ (the inverse morphism being given by the canonical subgroup).*

Proof: Since we won't need the first part of the proposition, we won't prove it in this paper. For the second statement, which is crucial to our conjecture, we notice that by definition, the morphism $\overline{\pi}$ sends $\overline{X}_G^{mm}\{r\}$ into $\overline{X}\{r\}$ while the inverse map is provided by the canonical subgroup as in [1] Prop.8.2.3.
Finally, we note that the involution $W$ extends to the toroidal compactifications hence defines an involution of $\overline{X}_Q(p)^{\mathrm{rig}}$ which exchanges $]\overline{X}_G^{mm,\mathrm{ord}}[$ and $]\overline{X}_G^{ee,\mathrm{ord}}[$ resp. $\overline{X}_G^{mm}\{r\}$ and $\overline{X}_G^{ee}\{r\}$ and leaves stable the middle component $]\overline{X}_G^{em,\mathrm{ord}}[$ resp. $\overline{X}_G^{em}\{r\}$.

Finally, we can consider in a similar way the extension to compatible toroidal compactifications $\overline{X}_{U_B}(p)$ and $\overline{X}_B(p)$ of the morphisms $\pi_{U_B,B}$ and $\pi_{B,Q}$. We shall consider the inverse image by

$$\pi_{B,Q} \circ \pi_{U_B,B} : \overline{X}_{U_B}(p)^{\mathrm{rig}} \to \overline{X}_Q(p)^{\mathrm{rig}}$$

of $\overline{X}_G^{mm}\{r\}$.

## 3 Siegel modular forms

### 3.1 Arithmetic Siegel modular forms and $q$-expansion

In [26], care has been taken to define the arithmetic Siegel varieties and modular forms adelically. However, here for simplicity, we restrict our attention to one connected component $X$ corresponding to a discrete subgroup $\Gamma \subset Sp_4(\mathbb{Z})$. We assume that $X$ has a geometrically connected model over $\mathbb{Z}[1/N]$. We also assume that $\Gamma$ is neat, so that the problem of classifying principally polarized abelian surfaces with $\Gamma$-level structure is a fine moduli problem (if it is not the case, see [26] Section 3 where $X$ is only a coarse moduli problem).
Let $f: A \to X$ be the universal principally polarized abelian surface with $\Gamma$-level structure $\eta$ over $\mathbb{Z}[1/N]$. We put $\underline{\omega} = e^*\Omega_{A/X}$, where $e$ denotes the unit section.
For any pair of integers $\kappa = (k, \ell)$ ($k \geq \ell$), we consider the rational representation of $GL(2)$: $W_\kappa(\mathbb{Q}) = \mathrm{Sym}^{k-\ell} \otimes \det^\ell St_2$. Here, $St_2$ denotes the standard two-dimensional representation of $GL(2)$; the standard Levi $M$ of the Siegel parabolic of $Sp_4$ is identified to $GL(2)$ by
(4.1.1) $U \mapsto \mathrm{diag}(U, s^tU^{-1}s)$
The twist by $s$ occurs because our choice of the symplectic matrix $J$ defining $G$ involves the matrix $s$ instead of $1_2$. We use (4.1.1) to identify $M$ to $GL(2)$. Let $B_M = TN_M$ be the Levi decomposition of the standard Borel of $M$ (corresponding to the group of upper triangular matrices in $GL(2)$). In order to define integral structures on the space of Siegel modular forms, it will be useful to consider an integral structure of $W_\kappa(\mathbb{Q})$. Since there is in general an ambiguity for such an integral structure, we need to make our choice explicit: following [15] Sect.3, we take it to be the induced $\mathbb{Z}$-module $W_\kappa = \mathrm{Ind}_{B_M}^M \kappa$. For any ring $R$, we put $W_\kappa(R) = W_\kappa \otimes R$.
Let $\mathcal{T} = \mathrm{Isom}_X(\mathcal{O}_X^2, \underline{\omega})$ be the right $GL(2)$-torsor over $X$ of isomorphisms $\phi: \mathcal{O}_X^2 \to \underline{\omega}$. By putting $\omega_1 = \phi((1,0))$ and $\omega_1 = \phi((0,1))$, it can also be viewed as the moduli scheme classifying quintuples $(A, \lambda, \eta, \omega_1, \omega_2)$ where $A, \lambda$ is a principally polarized abelian varieties with a $\Gamma$ level structure $\eta$ over a base $S$, endowed with a basis $(\omega_1, \omega_2)$ of $\underline{\omega}_{A/S}$. One writes $\pi: \mathcal{T} \to X$ for the structural map. Note that $\pi_*\mathcal{O}_\mathcal{T}$ carries a left action (by right translation) of $GL(2)$.
Then, for any $\kappa = (k, \ell) \in \mathbb{Z}^2$, one defines the locally free sheaf $\underline{\omega}^\kappa$ over $X$ as $(\pi_*\mathcal{O}_\mathcal{T})^{N_M}[\kappa^{-1}]$. Its sections are functions on $\mathcal{T}$ such that for any $\phi \in \mathrm{Isom}_X(\mathcal{O}_X^2, \underline{\omega})$, for any $t \in T$ and any $n \in N_M$, $f(A, \lambda, \eta, \phi \circ tn) = \kappa(t)^{-1} f(A, \lambda, \eta, \phi)$.
One sees easily that $\pi^*\underline{\omega}^\kappa = W_\kappa(\mathcal{O}_\mathcal{T})$, so that $\underline{\omega}^\kappa$ is a locally free sheaf which is non zero if and only if $k \geq \ell$.

We briefly recall some notations concerning toroidal compactifications, canonical extensions of sheaves and $q$-expansions. It will allow us in particular to define the cuspidal subsheaf $\underline{\omega}_\kappa$ of the canonical extension of $\underline{\omega}^\kappa$.
For any ring $R$, let $S_2(R)$ be the module of symmetric $2 \times 2$-matrices with

entries in $R$. Recall that the bilinear form $Tr : S_2(\mathbb{R}) \times S_2(\mathbb{R}) \to \mathbb{R}$ identifies the dual of $S_2(\mathbb{Z})$ to the module $\mathcal{S}$ of matrices $\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$, $a, b, c \in \mathbb{Z}$.

Let $S_2(\mathbb{R})^+$ be the cone of definite positive matrices in $S_2(\mathbb{R})$ and $\widetilde{S}_2$ the cone of semi-definite positive matrices whose kernel is $\mathbb{Q}$-rational.

A standard rational boundary component of level $N$ is a pair $(Z, \phi : \frac{1}{N}Z/Z \to (\mathbb{Z}/N\mathbb{Z})^r)$ where $Z$ is a free non zero quotient of $\mathbb{Z}^2$ (of rank $r$) and $\phi$ is an isomorphism. Let us view $\mathbb{Z}^2$ as the standard lagrangian $\langle e_1, e_2\rangle$ of $\mathbb{Z}^4$ endowed with the symplectic pairing ${}^t xJy$, Then, a general rational boundary component of level $N$ is the image of a standard one by the action of $Sp_4(\mathbb{Z})$ on the space of lagrangians and on the projective space of $\mathbb{Z}^4$.

We denote by $RBC_1$, $RBC_N$, resp. $SRBC_1$, $SRBC_N$, the set of rational boundary components, resp. the set of standard rational boundary components. We can partition $\widetilde{S}_2 \cap S_2(\mathbb{Z})$ as $\sqcup_{Z \in SRBC_1} S(Z)^+$ where $S(Z)^+$ denotes the set of semidefinite symmetric matrices of $S_2(\mathbb{Z})$ which induce a positive definite quadratic form on $Z$.

Let $\Sigma = \{\Sigma_Z\}_{Z \in RBC_1}$ be an $Sp_4(\mathbb{Z})$-admissible family of rational polyhedral cone decompositions $\Sigma_Z$ of $S(Z)^+$ (see [8] Chapt.I Def.5.8.2). As explained in [10] p.126, this decomposition can be used for any level $N$ congruence subgroup $\Gamma$, since it is *a fortiori* $\Gamma$-admissible. To $\Sigma$, one can associate a toroidal compactification $\overline{X}$ over $\mathbb{Z}[\frac{1}{N}]$ of $X$ as in [10] IV.6.7; it is smooth if $\Sigma$ is sufficiently fine; this is assumed in the sequel.

The compactification $\overline{X}$ carries a degenerating semi-abelian scheme $\mathcal{G}$ extending $A$ (see [10] Th.IV.5.7 and IV.6.7). One still denotes by $\underline{\omega}$ the sheaf $e^*\Omega_{\mathcal{G}/\overline{X}}$ where $e$ is the unit section of $\mathcal{G} \to \overline{X}$.

Recall that $\overline{X}$ is a projective smooth, geometrically connected scheme over $\mathbb{Z}[\frac{1}{N}]$. It is endowed with a projection map $b$ to the minimal compactification $X^*_{\mathbb{Z}[\frac{1}{N}]}$. Let $D = \overline{X} \backslash X = b^{-1}(\partial X^*)$; it is a relative Cartier divisor with normal crossings; its irreducible components are smooth.

The rank two vector bundle $\underline{\omega}$ over $\overline{X}$ does not descend as a vector bundle on $X^*$; however its determinant $\omega = \det \underline{\omega}$ descends as an ample line bundle.

The $GL(2)$-torsor $\overline{\mathcal{T}} = \mathrm{Isom}_{\overline{X}}(\mathcal{O}^2_{\overline{X}}, \underline{\omega})$ (with structural map $\overline{\pi} : \overline{\mathcal{T}} \to \overline{X}$) allows to define "the canonical extension" of the vector bundles $\underline{\omega}^\kappa$ to $\overline{X}$: one can either define this extension as

$$\underline{\omega}^\kappa = (\overline{\pi}_* \mathcal{O}_{\overline{\mathcal{T}}})^{N_M}[\kappa^{-1}]$$

$(k \geq \ell)$. Or one can also use the $\mathbb{Z}$-structure $W_\kappa = \mathrm{Ind}_{B_M}^M \kappa$ of the rational representation $W_\kappa(\mathbb{Q})$ of $GL(2)$ in order to give an equivalent definition of $\underline{\omega}^\kappa$ as the sections of the $\overline{X}$-vector bundle $\overline{\mathcal{T}} \overset{GL_2}{\times} W_\kappa$; here, as usual, the contraction product is the quotient of the product by the equivalence relation $(\phi \circ g, w) \sim (\phi, g \cdot w)$ for any $\phi \in \overline{\mathcal{T}}$, $g \in GL_2$ and $w \in W_\kappa$. For details see [10] Chapter 4 and 6, [21] Sect.4 and [15] Sect.3.

Let $\underline{\omega}_\kappa = \underline{\omega}^\kappa(-D)$ the sub-vector bundle of $\underline{\omega}^\kappa$ on $\overline{X}$ whose sections vanish along $D$. Recall the Koecher principle: $H^0(\overline{X} \otimes \mathbb{C}, \underline{\omega}^\kappa) = H^0(X \otimes \mathbb{C}, \underline{\omega}^\kappa)$. We define

Definition 3.1 *For any $\mathbb{Z}[1/N]$-algebra $R$ one defines the $R$-module of arithmetic Siegel modular forms resp. cusp forms, as $H^0(X \otimes R, \underline{\omega}^\kappa)$ resp. $H^0(\overline{X} \otimes R, \underline{\omega}_\kappa)$ which we write also $H^0(X \otimes R, \underline{\omega}_\kappa)$ by convention.*

For $R = \mathbb{C}$, these vector spaces canonically identify to the corresponding spaces of classical Siegel modular forms of level $\Gamma$ and weight $\kappa$ (see [15] Th.3.1).
The arithmetic $q$-expansion (at the $\infty$ cusp) is defined as follows.
Let $\eta = (Z, \phi) \in SRBC_N$ with $Z = \mathbb{Z}^2$ and with $\phi$ the canonical identification $\frac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2 = \mathbb{Z}/N\mathbb{Z}^2$ (it is called the infinity cusp).
Consider the rational polyhedral cone decomposition (RPCD) $\Sigma_\eta$ of $S_2(\mathbb{R})^+$ corresponding to $\eta$. Let $D_\eta = D \cap b^{-1}(\{\eta\})$. By definition, the completion of $\overline{X}$ along $D_\eta$ admits an open cover by affine formal schemes $\mathcal{U}_\sigma$ ($\sigma \in \Sigma_\eta$) with a canonical surjective finite etale cover $\phi_\sigma : \mathcal{S}_\sigma \to \mathcal{U}_\sigma$ where $\mathcal{S}_\sigma = \operatorname{Spf} \mathbb{Z}[1/N][[q^T; T \in \mathcal{S} \cap \sigma^\vee]]$. The morphism $\phi_\sigma$ is Galois; its group is the stabilizer $\overline{\Gamma}_\sigma$ of $\sigma$ in the image $\overline{\Gamma}$ of $\Gamma \cap Q$ by the projection $Q \to Q/U = M$. Recall that $M(\mathbb{Z}) = GL(2, \mathbb{Z})$ acts on $S_2(\mathbb{Z})^+$ by $g \cdot S = gS^tg$. Moreover, $\phi_\sigma$ is uniquely determined by the property that the pull-back by $\phi_\sigma$ of the restriction of $\mathcal{G}$ to $\mathcal{U}_\sigma$ is the canonical Mumford family

$$f_\sigma : \mathcal{G}_\sigma \to \mathcal{S}_\sigma$$

deduced by Mumford's construction (see [10] p.54) from the canonical degenerescence data in $\mathrm{DD}_{\mathrm{ample}}$ on the global torus $\tilde{G}_\sigma = \mathbf{G}_m^2$ over $\mathcal{S}_\sigma$, together with the standard level $N$ structure $\mu_N^2 \times (\mathbb{Z}/N\mathbb{Z})^2 \to \tilde{G}_\sigma[N]$;
Given $f \in H^0(\overline{X}, \omega^\kappa)$, for any rational polyedral cone $\sigma$, we restrict $f$ to $\mathcal{U}_\sigma$ and pull it back to $\mathcal{S}_\sigma$ by $\phi_\sigma$. The bundle $\underline{\omega}_{\mathcal{G}_\sigma/\mathcal{S}_\sigma}$ of the Mumford family is trivial, hence the pull-back of the torsor $\overline{\mathcal{T}}$ to $\mathcal{S}_\sigma$ is trivial too; it is isomorphic to $\mathcal{S}_\sigma \times GL(2)$. In consequence, $\phi_\sigma^* \underline{\omega}^\kappa$ is the trivial bundle $W_\kappa \otimes \mathcal{S}_\sigma$. Hence $\phi_\sigma^* f$ yields a series in $W_\kappa[[q^T; T \in \mathcal{S} \cap \sigma^\vee]]$ which is invariant by $\overline{\Gamma}$ where the action of $\gamma \in \overline{\Gamma}$ is given by $\gamma \cdot (\sum_T a_T q^T) = \sum_T \rho_\kappa(\gamma)(a_T) q^{\gamma T^t \gamma}$. These series are compatible when one varies the cone $\sigma$ either by restricting to its faces of by letting $\overline{\Gamma}$ act (this action permutes the cones in $\Sigma_\eta$); recall that $\bigcap_{\sigma \in \Sigma_\eta} \sigma^\vee = \widetilde{S}_2$; this implies that there exists one well-defined series which belongs to the intersection $W_\kappa[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]$ of the rings $W_\kappa[[q^T; T \in \mathcal{S} \cap \sigma^\vee]]$ and which is fixed by $\overline{\Gamma}$. It is called the $q$-expansion or Fourier expansion (at the infinity cusp) of $f$:

$$FE(f) \in W_\kappa[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]^{\overline{\Gamma}}$$

For any $\mathbb{Z}[1/N]$-algebra $R$ and any form $f \in H^0(X \times R, \omega^\kappa)$ defined over $R$, one defines an analogue series $FE_R(f)$ with coefficients in $W_\kappa(R) = W_\kappa \otimes R$.

Proposition 3.2 *1) (q-expansion principle) If $f$ is any form defined over $R$, if the coefficients of its $q$-expansion vanish in $W_\kappa(R)$, then $f=0$.*
*2) The map $FE$ sends the submodule of cusp forms over any ring $R$ to the submodule of $W_\kappa(R)[[q^T;T\in\mathcal{S}\cap\widetilde{S}_2]]^{\overline{\Gamma}}$ of series whose coefficients $a_T\in W_\kappa(R)$ vanish unless $T\in\mathcal{S}\cap S_2(\mathbb{R})^+$.*

The first point follows from the irreducibility of the modular scheme; the second from the examination of $\phi_\sigma^*(f)$ along $\phi_\sigma^*D$.

Remark: By comparing the two definitions of $\omega^\kappa$ given above, one sees that

$$W_\kappa[[q^T;T\in\mathcal{S}\cap\sigma^\vee]]=(\overline{\pi}_*\mathcal{O}_{\overline{T}})^{N_M}[\kappa^{-1}]\otimes_{\mathcal{O}_{\overline{X}}}\mathbb{Z}[[q^T;T\in\mathcal{S}\cap\sigma^\vee]]$$

We shall use this when comparing $q$-expansion of classical forms to $q$-expansion of $p$-adic forms.

### 3.2 $p$-adic Siegel modular forms and $q$-expansion

Let $X$ as in the previous subsection. We fix a fine $\Gamma$-admissible family of rational polyedral cone decompositions $\Sigma_\xi$; we denote by For any integer $m\geq 1$, let $X_m$ be the pull-back of $X$ to $\mathbb{Z}/p^m\mathbb{Z}$. Let $S_m$ be the ordinary locus and for each $n\geq 1$, consider $T_{m,n}=\mathrm{Isom}_{S_m}(\mu_{p^n}^2,A[p^n]^0)=\mathrm{Isom}_{S_m}(A[p^n]^{et},(\mathbb{Z}/p^n\mathbb{Z})^2)$; for any $n\geq 1$, $T_{m,n}$ is a connected Galois cover of $S_m$ of Galois group $GL_2(\mathbb{Z}/p^n\mathbb{Z})$ (see [10] Prop.7.2).
Let $V_{m,n}=H^0(T_{m,n},\mathcal{O}_{T_{m,n}})$, $V_{m,\infty}=\bigcup_{n\geq 1}V_{m,n}$. One can define the $\Sigma$-"toroidal compactification" $\overline{S}_m$ of $S_m$ as the locus of $\overline{X}_m$ over which $\mathcal{G}[p]^0$ is of multiplicative type; similarly, define $\overline{T}_{m,n}$ as

$$\mathrm{Isom}_{\overline{X}\otimes\mathbb{Z}/p^m\mathbb{Z}}(\mu_{p^n}^2,\mathcal{G}[p^n]^0)$$

We still denote by $D$ the pull-back to $\overline{T}_{m,n}$ of the divisor at $\infty$. We can now define $V_{!,m,n}=H^0(\overline{T}_{m,n},\mathcal{O}_{\overline{T}_{m,n}}(-D))$ and $V_{!,m,\infty}=\bigcup_m V_{!,m,n}$. We also consider the corresponding $p$-adic limits: $S_\infty=\varinjlim S_m$, $T_{\infty,\infty}=\varinjlim T_{m,\infty}$, $V=\varprojlim V_{m,\infty}$ and $V_!=\varprojlim V_{!,m,\infty}$. These last two spaces are respectively the space of generalized $p$-adic modular forms resp. cusp forms.
Let $\mathbf{M}$ resp. $\mathbf{N_M}$ be the group of $\mathbb{Z}_p$-points of $M=GL_2$ resp. $N_M$ the unipotent radical of the standard Borel $B_M$ of $M$. Then, $T_{\infty,\infty}\to S_\infty$ is a right $\mathbf{M}$ étale torsor, hence $\mathbf{M}$ acts on the left (by right translations) on $V$ (and $V_!$) by $m\cdot f(\psi)=f(\psi\circ m)$. Let $LC(\mathbf{M}/\mathbf{N_M},\mathbb{Z}/p^m\mathbb{Z})$ resp. $\mathcal{C}(\mathbf{M}/\mathbf{N_M},\mathbb{Z}_p)$ be the ring of $\mathbb{Z}/p^m\mathbb{Z}$-valued locally constant, resp. $\mathbb{Z}_p$-valued continous functions on $\mathbf{M}/\mathbf{N_M}$, viewed as a left $\mathbf{M}$-module via the left translation action. In particular, these modules are $\overline{\Gamma}$-modules. Note that $\mathcal{C}(\mathbf{M}/\mathbf{N_M},\mathbb{Z}_p)=\mathrm{projlim}\,LC(\mathbf{M}/\mathbf{N_M},\mathbb{Z}/p^m\mathbb{Z})$.
Let us define now the $p$-adic $q$-expansion map. It is a ring homomorphism

$$\mathrm{FE}:V^{\mathbf{N_M}}\to\Big(\mathcal{C}(\mathbf{M}/\mathbf{N_M},\mathbb{Z}_p)[[q^T;T\in\mathcal{S}\cap\widetilde{S}_2]]\Big)^{\overline{\Gamma}}$$

given as follows.

For the infinity cusp $\eta$ defined above, and for any $\sigma \in \Sigma_\eta$, we consider the base change $\phi_{\sigma,m}$ of the morphism $\phi_\sigma : \mathcal{S}_\sigma \to \mathcal{U}_\sigma$ to $\mathbb{Z}/p^m\mathbb{Z}$. As noticed above, the canonical Mumford family $f_\sigma : \mathcal{G}_\sigma \to \mathcal{S}_\sigma$ admits a canonical rigidification $\psi_{can} : \mu^2_{p^\infty} \cong \mathcal{G}_\sigma[p^\infty]^0$ induced from the tautological rigidification of $\tilde{G}_\sigma = \mathbf{G}^2_m$. This provides a canonical lifting $\Phi_{\sigma,m} : \mathcal{S}_\sigma \to \overline{T}_{m,\infty}$ of $\phi_{\sigma,m}$. These liftings are compatible when $m$ grows, this gives rise to a lifting $\Phi_\sigma : \mathcal{S}_{\sigma\widehat{(p)}} \to \overline{T}_{\infty,\infty}$ of $\phi_\sigma : \mathcal{S}_{\sigma\widehat{(p)}} \to \mathcal{U}_{\sigma\widehat{(p)}}$ (the hat means $p$-adic completion).

For $f \in V$, one can therefore take the pull-back of $f \bmod p^m$ by $\Phi_{\sigma,m}$ (resp. of $f$ by $\Phi_\sigma$). The resulting series belongs to $\mathcal{O}_{\mathcal{S}_\sigma} \otimes \mathbb{Z}/p^m\mathbb{Z} = \mathbb{Z}/p^m\mathbb{Z}[[q^T; T \in \mathcal{S} \cap \sigma^\vee]]$ resp. $\mathcal{O}_{\mathcal{S}_\sigma} \otimes \mathbb{Z}_p = \mathbb{Z}_p[[q^T; T \in \mathcal{S} \cap \sigma^\vee]]$. It is however useful for further use to view it as belonging to $\mathcal{O}_{\mathcal{S}_\sigma} \otimes LC(\mathbf{M}, \mathbb{Z}/p^m\mathbb{Z})$ resp. to $\mathcal{O}_{\mathcal{S}_\sigma} \widehat{\otimes} \mathcal{C}(\mathbf{M}, \mathbb{Z}_p) = \mathcal{C}(\mathbf{M}, \mathbb{Z}_p)[[q^T; T \in \mathcal{S} \cap \sigma^\vee]]$ in the following way: the map $x \in \mathbf{M} \mapsto \Phi^*_{\sigma,m}(x \cdot f)$ is an $\mathcal{O}_{\mathcal{S}_\sigma} \otimes \mathbb{Z}/p^m\mathbb{Z}$-valued locally constant map on $\mathbf{M}$. The evaluation of this function at $1 \in \mathbf{M}$ gives the $\mathbb{Z}/p^m\mathbb{Z}[[q^T; T \in \mathcal{S} \cap \sigma^\vee]]$-valued $q$-expansion mentioned above. By taking the inverse limit over $m$, one gets the desired $q$-expansion with coefficients in $\mathcal{C}(\mathbf{M}, \mathbb{Z}_p)$. Both $\mathbb{Z}_p$-coefficient and $\mathcal{C}(\mathbf{M}, \mathbb{Z}_p)$-coefficient $q$-expansions are compatible to restriction to faces; however, only the $\mathcal{C}(\mathbf{M}, \mathbb{Z}_p)$-coefficient expansion is compatible to the action of $\overline{\Gamma}$; we conclude that the functions $x \in \mathbf{M} \mapsto \Phi^*_\sigma(x \cdot f)$ for all $\sigma$'s give rise to an element of the submodule $H^0(\overline{\Gamma}, \mathcal{C}(\mathbf{M}, \mathbb{Z}_p)[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]])$ of $\overline{\Gamma}$- invariants of $\mathcal{C}(\mathbf{M}, \mathbb{Z}_p)[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]$. We finally restrict our attention to $f \in V^{\mathbf{N_M}}$; thus we obtain a $q$-expansion in

$$\mathcal{C}(\mathbf{M}/\mathbf{N_M}, \mathbb{Z}_p)[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]$$

We list below some well-known facts for which we refer to [15].

Proposition 3.3 *1) (p-adic q-expansion principle) For any $\sigma \in \Sigma_\eta$, for any $m \geq 1$, $V/p^mV \subset V_{m,\infty} \hookrightarrow \mathbb{Z}/p^m\mathbb{Z}[[q^T; T \in \mathcal{S} \cap \sigma^\vee]]$ is injective with flat cokernel. In particular, the ring homomorphism* FE *is injective.*
*2) The restriction of $FE$ to the ideal $V_!$ of cusp forms takes values in the ideal generated by $q^T$ for $T \in \mathcal{S} \cap S_2(\mathbb{R})^+$. and the q-expansion principle holds for cusp forms for any cone $\sigma$ and any $m \geq 1$ as above.*

We simply recall that the first point results from the irreducibility of $\overline{T}_{m,\infty}$ (Igusa irreducibility theorem, [10] V.7.2) and the second from direct examination of $\Phi^*_\sigma(f)$.

It remains to compare the classical and $p$-adic modular forms resp. $q$-expansions. The embedding of classical forms into $V$ comes from the canonical morphism $\iota : T_{\infty,\infty} \to \mathcal{T}|S_\infty$ given by the fact that for an abelian variety $A$ (of dimension 2) over a base $S$ where $p$ is nilpotent, any rigidification $\psi : \mu^2_{p^\infty} \cong A[p^\infty]^0$ gives rise to an isomorphism $\mathcal{O}^2_S \cong \underline{\omega}_{A/S}$. One checks easily that $\iota^* : H^0(X, \underline{\omega}^\kappa) \to V^{\mathbf{N_M}}[\kappa]$ and $\iota^* : H^0(X, \underline{\omega}_\kappa) \to V_!^{\mathbf{N_M}}[\kappa]$.

Thus given a classical form, we first view it as a section of $(\pi_*\mathcal{O}_\mathcal{T})^{N_M}$, then one restricts it to the ordinary locus and one takes its pull-back by the morphism $\iota$.

The comparison of the two definitions of $\underline{\omega}^\kappa$ provides a commutative square expressing the compatibility of classical and $p$-adic $q$-expansions:

$$\begin{array}{ccc} V^{\mathbf{N_M}} & \to & H^0(\overline{\Gamma}, \mathcal{C}(\mathbf{M}/\mathbf{N_M}, \mathbb{Z}_p)[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]) \\ \uparrow & & \uparrow \\ H^0(X, \underline{\omega}^\kappa) & \to & H^0(\overline{\Gamma}, W_\kappa(\mathbb{Z}_p)[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]) \end{array}$$

In the case where $\kappa$ is diagonal so that $W_\kappa(\mathbb{Z}_p)$ is free of rank one, one can formulate more simply the diagram by composing both horizontal maps by the evaluation of functions on $\mathbf{M}/\mathbf{N_M}$ at 1, sending $f : \mathbf{M}/\mathbf{N_M} \to \mathbb{Z}_p$ to $f(1)$. We thus get a commutative square

$$\begin{array}{ccc} V^{\mathbf{N_M}} & \to & \mathbb{Z}_p[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]) \\ \uparrow & & \uparrow \\ H^0(X, \underline{\omega}^\kappa) & \to & H^0(\overline{\Gamma}, W_\kappa(\mathbb{Z}_p)[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]) \end{array}$$

Let $H \in H^0(X_1, \det^{p-1}\underline{\omega})$ be the Hasse invariant on $X_1$. We fix an integer $t \geq 1$ sufficiently large such that $H^t$ lifts to $X$ over $\mathbb{Z}_p$. This can be achieved because $\det\underline{\omega}$ is ample. We denote by $E$ such a lifting. Recall that $\mathrm{FE}(E) \equiv 1 \pmod p$; this is because $\mathrm{FE}(H) = 1$ in $\mathbb{Z}/p\mathbb{Z}[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]$.

By [15] Sect.3.6, the Hecke operators $U_{p,1} = [\mathbf{N_M}\mathrm{diag}(1,1,p,p)\mathbf{N_M}]$ and $U_{p,2} = p^{-3}[\mathbf{N_M}\mathrm{diag}(1,p,p,p^2)\mathbf{N_M}]$ do act on $V_!^{\mathbf{N_M}}$. Let $e = \lim (U_{p,1}U_{p,2})^{n!}$ be the corresponding idempotent of $\mathrm{End}_{\mathbb{Z}_p} V_!^{\mathbf{N_M}}$. The module $eV_!^{\mathbf{N_M}}$ is called the module of ordinary $p$-adic cusp forms (with strict Iwahori $p$-level). Hida's control theorem [15] Th.1.1 says that for any weight $\kappa$ (not necessarily cohomological), the cokernel of the inclusion $eH^0(S_\infty, \underline{\omega}_\kappa) \subset eV_!^{\mathbf{N_M}}[\kappa]$ is finite.
Comment: Actually, Th.1.1 of [15] also contains a "classicity statement", but only for very regular weights. Since we need in [26] an analogue of this statement including all cohomological weights (including those such that $k = \ell$), we prove it there for all cohomological weights after localisation to a non-Eisenstein maximal ideal of the Hecke algebra.
This theorem is crucial for us in [26] in order to produce overconvergent cusp forms $g$ satisfying
(LIM) The $q$-expansion of $g$ is the $p$-adic limit of $q$-expansions of cusp eigenforms of cohomological weight.
This condition provides the framework for the conjecture stated in the present paper. On the other hand, it would be very interesting to generalize Hida theorem to $p$-adic forms with finite slope for $U_{p,1}$ different from 0. Such a generalization would produce new overconvergent forms satisfying (LIM).

### 3.3 Overconvergence

We endow $\mathbb{C}_p$ with the $p$-adic norm such that $||_p = p^{-1}$. For any extension $L$ of $\mathbb{Q}_p$ contained in $\mathbb{C}_p$ and for any real number $r \in ]0,1[$, we consider the $L$-vector space of $r$-overconvergent Siegel modular forms

$$S_\kappa(\Gamma; r) = H^0(\overline{X}\{r\} \times L, \underline{\omega}_\kappa)$$

If $r$ is in $|L^\times|_p$, this is a Banach space for the norm $|f| = \sup_{x \in \overline{X}\{r\}(L)} |f(x)|_p$ by [5] Th.4.1.6. In particular, for any $r < r'$ in $]p^{-a}, 1[\cap |L^\times|_p$, the inclusions

$$\mathrm{res}_{r,r'} : S_\kappa(K; r) \hookrightarrow S_\kappa(K; r')$$

are completely continuous by [19] 2.4.1.
It should be noted that the above fact does not require the assumption that the weight $\kappa$ be cohomological (that is $k_1 \geq k_2 \geq 3$). In [26], we indeed apply this to $\kappa = (2,2)$.

Let $a$ be either the Abbès-Mokrane bound ($a = \frac{1}{p(p-1)}$) or the Andreatta-Gasbarri's bound ($a = \frac{p-1}{2p-1}$). By [1] Lemma 8.2.1 and [2], for any $r \in ]p^{-a}, 1[$, the canonical lifting $F_{can}$ of the Frobenius endomorphism is defined as a rigid morphism $\overline{X}\{r\} \to \overline{X}^{\mathrm{rig}}$.
The following two results are contained in [26] Sect.4.5

Proposition 3.4 *There exists $r \in ]p^{-a}, 1[\cap p^{\mathbb{Q}}$ such that $F_{can}$ maps $\overline{X}\{r\}$ into $\overline{X}\{r^p\}$ and is finite flat of degree $p^3$. It yields a continous homomorphism of Banach spaces $\phi = F_{can}^* : S_\kappa(\Gamma; r^p) \to S_\kappa(K; r)$ and a trace homomorphism $Tr_\phi : S_\kappa(\Gamma; r) \to S_\kappa(\Gamma; r^p)$.*

Corollary 3.5 *There exists $r \in ]p^{-a/p}, 1[\cap |L^\times|_p$, the composition $\psi = \mathrm{res}_{r^p,r} \circ Tr_\phi$ defines a completely continuous endomorphism of the Banach space $S_\kappa(\Gamma; r)$.*

The evaluation on the rigid Mumford families $\mathcal{G}_\eta^{rig} \to \mathcal{S}_\sigma$ (for all polyedral cones $\sigma$ in $\Sigma_\eta$ as above) defines a $L$-linear homomorphism

$$\mathrm{FE} : S_\kappa(K; r) \hookrightarrow L[[q^T; T \in \mathcal{S} \cap S_2(\mathbb{R})^+]].$$

The overconvergent $q$-expansion principle says that FE is injective. It follows directly from the connectedness of $\overline{X}^{rig}$.
We define $U_{p,1}$ as $p^{-3}\psi$ the operator corresponding to the weight $\kappa = (2,2)$. We denote by $S_2(\Gamma; r)$ the $L$-Banach space of $r$-overconvergent forms of weight $(2,2)$. Then it follows immediately from Cor.3.3 that

Corollary 3.6 *There exists $r \in ]p^{-a/p}, 1[\cap |L^\times|_p$ such that the operator $U_{p,1}$ is completely continuous on the Banach space $S_2(\Gamma; r)$ of weight 2 overconvergent $p$-adic cusp forms.*

Recall that by [24] Prop.7, one can define a Fredholm determinant $P(t) = \det(1 - tU_{p,1})$ which is a $p$-adic entire function of $t$ and such that $\lambda \in \overline{\mathbb{Q}}_p$ is a non-zero eigenvalue of $U_{p,1}$ if and only if $P(\lambda^{-1}) = 0$; so that the non-zero eigenvalues of $U_{p,1}$ form a sequence decreasing to 0. By Prop.12 and Remark 3 following this proposition in [24], each spectral subspace associated to a non-zero eigenvalue is finite dimensional (its dimension being equal to the multiplicity of the root $\lambda^{-1}$ of $P$) and there is a direct sum decomposition of the Banach space as the sum of the (finite dimensional) spectral subspace and the largest closed subspace on which $U_{p,1} - \lambda$ is invertible.
In particular, for any positive number $\alpha$, the set of eigenvalues $\lambda_i \in \overline{\mathbb{Q}}_p$ of $U_{p,1}$ such that $ord_p(\lambda_i) \leq \alpha$ is finite. Moreover one has a direct sum decomposition of the Banach space $S_2(\Gamma, r)$ as $S_2(\Gamma, r)^{\leq \alpha} \oplus S_2(\Gamma, r)^{>\alpha}$, where the first space is finite dimensional, defined as the direct sum of the spectral subspaces for all eigenvalues $\lambda_i$ with $ord_p(\lambda_i) \leq \alpha$, and the second is the (closed) largest subspace on which all the operators $U_{p,1} - \lambda_i$ are invertible.

## 4 Galois representations of low weight and overconvergent modular forms

### 4.1 Eichler-Shimura maps

Let $\kappa = (k, \ell)$ be a cohomological weight, that is, a pair of integers such that $k \geq \ell \geq 3$. Let $k = a + 3$, $\ell = b + 3$. Then, $(a, b)$, $a \geq b \geq 0$ is a dominant weight for $(G, B, T)$; let $V_{a,b}$ be the local system on the Siegel variety associated to the irreducible representation of $G$ of highest weight $(a, b)$; recall that the central character of this representation is $z \mapsto z^{a+b}$. For any (neat) compact open subgroup $L$ of $G_f$, for $a' \geq b' \geq 0$ and $k' = a' + 3$, $\ell' = b' + 3$, there is a canonical Hecke-equivariant linear injection

$$H^0(S_L, \underline{\omega}_\kappa) \hookrightarrow H^3(S_L, V_{a,b}(\mathbb{C}))$$

See Section 3.8 of [15] where it is explained how to make it canonical, and where it is called the Eichler-Shimura map. Actually the image is contained in $H^3_! = \mathrm{Im}(H^3_c \to H^3)$. It follows for instance from Th.5.5, Chapter VI of [10].

### 4.2 Galois representation associated to a cohomological cusp eigenform

Let $f$ be a cusp eigenform of cohomological weight $\kappa = (k, \ell)$. Let $k = a + 3$, $\ell = b + 3$. By the EIchler-Shimura injection, the Hecke eigensystem associated to $f$ occurs in $H^3(X(\mathbb{C}), V_{a,b}(\mathbb{C}))$. For any prime $q$ prime to $N$, let $P_{f,q} \in \mathbb{C}[X]$ be the degree four Hecke polynomial at $q$ for the eigensystem of $f$ (see [26]).
Let $E$ be the number field generated by the eigenvalues of the Hecke operators outside $N$. We fix a $p$-adic embedding $\iota_p$ of $\overline{\mathbb{Q}}$; let $F \subset \overline{\mathbb{Q}}_p$ be a $p$-adic field containing $\iota_p(E)$ ( big enough but of finite degree).

The Galois representation $W_f = H^3(S_K, V_{a,b}(F)))_f$ (largest subspace where Hecke acts as on $f$) is $E$-rational and pure of Deligne weight $\mathbf{w} = 3 + a + b$.
Let $S$ be the set of prime divisors of $N$, and $\Gamma$ be the Galois group of the maximal algebraic extension of $\mathbb{Q}$ unramified outside $S$ and $p$. By a series of papers (due to R. Taylor, Laumon and Weissauer) there exists a degree four Galois representation $R_{f,p} : \Gamma \to GL_4(\overline{\mathbb{Q}}_p)$ such that for any $\ell \notin S \cup \{p\}$, $\det(X \cdot 1_4 - R_{f,p}(Fr_\ell)) = P_{\pi,\ell}(X)$.
Its relation to $W_f$ is: $W_f^4 = R_{f,p}^m$, where $m = \dim W_f$.
We take $F$ big enough for $R_{f,p}$ to be defined over it.
Remark: Let $\epsilon : \Gamma \to \mathbb{Z}_p^\times$ be the $p$-adic cyclotomic character. With the convention above, we have $\nu \circ \rho_{\pi,p} = \epsilon^{-\mathbf{w}} \cdot \omega_f$, where $\omega_f$ is a finite order character modulo $N$, given as the Galois avatar of the companion character of $f$ (this can viewed using Poincaré duality for $W_f$, see for instance [25], beginning of Sect.2).
Remark: Given a classical cusp eigenform $g \in H^0(X, \omega_{(2,2)})$, there is no geometric construction of an associated Galois representation (there is no Eichler-Shimura map to transport the eigensystem to the étale cohomology). See below for a $p$-adic construction, if the $q$-expansion of $g$ is a $p$-adic limit of $q$-expansions of cohomological weight cusp eigenforms.

### 4.3 A conjecture

Let $g \in H^0(\overline{X}\{r\}, \omega_{(2,2)})$ be an overconvergent cusp eigenform of weight $(2, 2)$ and auxiliary level group $K$ (unramified at $p$). By Prop.2.6, 2, since $\overline{X}\{r\}$ is canonically identified to $\overline{X}_G^{mm}\{r\} \subset \overline{X}_Q(p)^{\mathrm{rig}}$, one can view $g$ as an element of $H^0(\overline{X}_G^{mm}\{r\}, \omega_{(2,2)})$, where $\overline{X}_G^{mm}\{r\}$ is a strict neighborhood of $]\overline{X}_G^{mm}[$ in $\overline{X}_Q(p)^{\mathrm{rig}}$. We shall actually need to consider the pull-back of $g$ by $\pi_{B,Q} \circ \pi_{U_B,B}$ as a section of $\underline{\omega}_{2,2}$ over the quasi-compact relatively compact rigid open

$$(\pi_{B,Q} \circ \pi_{U_B,B})^{-1}(\overline{X}_G^{mm}\{r\})$$

in $\overline{X}_{U_B}(p)^{\mathrm{rig}}$.
Assume that

(LIM-EIG) there exists a sequence $(g_i)$ of classical cusp eigenforms $g_i \in H^0(\overline{X}_{U_B}(p), \omega_{\kappa_i})$ with cohomological weights $\kappa_i = (k_i, \ell_i)$ and level $K$ (that is, prime to $p$, equal to the auxiliary level of $g$) such that the $q$-expansions of the $g_i$'s converge $p$-adically to that of $g$.

Let $\Pi_{U_B}$ be the subgroup of matrices in $G(\mathbb{Z}_p)$ whose reduction modulo $p$ belongs to $U_B(\mathbb{Z}/p\mathbb{Z})$.

Comments: 1) Note that the key-point in this assumption is that the forms $g_i$ are eigenforms. If we insist that the sequence of $p$-adic weights satisfies $\kappa_i \equiv (2, 2) \pmod{p-1)p^i}$, we cannot assume in general that the level of the $g_i$'s is prime to $p$; then we simply need to replace $K_p = G(\mathbb{Z}_p)$ by $\Pi_{U_B}$ as

$p$-component of the level group in (LIM-EIG). We can motivate the choice of the $p$-level group $\Pi_{U_B}$ by recalling that both in the proof of the main theorem of [6] and in the Control Theorem for the Iwahori levels for $GSp(4)$ of [27], it has been natural to consider the pull-back of $g$ by $\pi_{B,Q} \circ \pi_{U_B,B}$ as a section of $\underline{\omega}_{2,2}$ over the strict neighborhood

$$(\pi_{B,Q} \circ \pi_{U_B,B})^{-1}(\overline{X}_G^{mm}\{r\})$$

in $\overline{X}_{U_B}(p)^{\mathrm{rig}}$. This is the analogue of Hida's $p$-stabilization for $p$-adic modular forms.
2) Note also that it is a well-known theorem [15] that any $p$-adic cusp form is the $p$-adic limit (in the sense of $q$-expansions) of prime-to-$p$ level classical cusp forms of weights $\kappa_i$ satisfying $\kappa_i \equiv (2,2) \pmod{p-1}$, where however, the forms $g_i$'s are not necessarily eigen even if $g$ is.

Recall then that for any weight $\kappa$, there is a $q$-expansion map (always at the infinity cusp)

$$H^0((\pi_{B,Q} \circ \pi_{U_B,B})^{-1}(\overline{X}_G^{mm}\{r\}), \underline{\omega}_\kappa) \to W_\kappa(\mathbb{Q}_p)[[q^T; T \in \mathcal{S} \cap \widetilde{S}_2]]^{\overline{\Gamma}}$$

These maps are compatible with the $p$-adic $q$-expansion map via the canonical injection of $H^0((\pi_{B,Q} \circ \pi_{U_B,B})^{-1}(\overline{X}_G^{mm}\{r\}), \underline{\omega}_\kappa)$ into the space of $p$-adic cusp forms.
We give below a conjectural criterion for the analytic continuation of $g$ to $\overline{X}_{U_B}(p)^{\mathrm{rig}}$.
Let $\rho_{g,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_4(\overline{\mathbb{Q}}_p)$ be the Galois representation associated to the limit of the pseudo-representations of the $g_i$'s. We call it the Galois representation associated to $g$. Note that by Sen theory (Bull. Soc. Math. de France 1999), if the $\kappa_i$ converge to $(2,2)$ in $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ and if $\rho_{g,p}$ is Hodge-Tate, its $p$-adic Hodge-Tate weights should be $0,0,1,1$. Our conjecture reads as follows.

Conjecture: *Let $g \in S_{2,2}(K,r)$ be an overconvergent cusp eigenform satisfying* (LIM-EIG)*; assume that there exists an abelian surface $A$ defined over $\mathbb{Q}$ such that $\rho_{g,p}$ is isomorphic to the contragredient $\rho_{A,p}^\vee$ of the representation on the $p$-adic Tate module of $A$. Then, $g$ extends to a global section $g \in H^0(\overline{X}_{U_B}(p)^{\mathrm{rig}}, \omega_{(2,2)})$ thus defining by the rigid GAGA principle a classical cusp form of weight* $(2,2)$ *and level $K^p \times \Pi_{U_B}$.*
*If the abelian variety has good reduction at $p$, the cusp eigenform has level prime to $p$.*

Remark: The minimal level group $\Pi_{g,p}$ at $p$ of the classical cusp eigenform $g$ satisfies $\Pi_{U_B} \subset \Pi_{g,p} \subset G(\mathbb{Z}_p)$; the compatibility between gobal and local Langlands correspondences predicts that the (local) Weil-Deligne representation associated to $D_{pst}(\rho_{g,p})$ determines $\Pi_{g,p}$.

The main result (Theorem 4) of [26] provides under certain assumptions (primarily the assumption of near ordinarity) such pairs of an overconvergent cusp eigenform $g$ with a converging sequence $(g_i)$ of cusp eigenforms, together with an abelian surface $A$ defined over $\mathbb{Q}$ with potential good ordinary reduction at $p$.

Actually one starts there from an abelian surface satisfying certain condition, the most stringent being that the Galois representation $\rho^\vee_{A,p}$ must be congruent modulo $p$ to the representation $\rho_{f,p}$ associated to a cusp eigenform of level $K$ prime to $p$, ordinary at $p$ with cohomological weight. Then Hida theory ([26] Lemma 4.2) yields a sequence $(g_i)$ converging to a limit $g$ which is overconvergent of weight $(2,2)$ and auxiliary level $K$.

Note that once a generalization of Coleman Families Theory to the Siegel case is available, there might be new examples of such forms $g$.

In the situation treated in [26], the representation $\rho_{g,p} = \rho^\vee_{A,p}$ is potentially crystalline but not crystalline, which implies that the eigenforms $g_i$ are indeed $p$-new of $p$-level $\Pi_{U_B}$, hence the presence of $\Pi_{U_B}$ as conjectural $p$-level group of $g$.

The conjecture above would imply that the $L$ function of the motive $h^1(A)$ is automorphic: $L(h^1(A), s) = L_{\rm spin}(g, s)$, hence, by a classical theorem of Piatetskii-Shapiro, it would have analytic continuation and functional equation.

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