On the Meromorphic Continuation of Degree Two L-Functions

To John Coates on the occasion of his 60^{th} birthday, with much gratitude.

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Abstract. We prove that the L-function of any regular (distinct Hodge numbers), irreducible, rank two motive over the rational numbers has meromorphic continuation to the whole complex plane and satisfies the expected functional equation.

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Introduction

In this paper we extend the results of [Tay4] from the ordinary to the crystalline, low weight (i.e. in the Fontaine-Laffaille range) case. The underlying ideas are the same. However this extension allows us to prove the meromorphic continuation and functional equation for the L-function of any regular (i.e. distinct Hodge numbers) rank two "motive" over $\mathbb{Q}$. We avoid having to know what is meant by "motive" by working instead with systems of l-adic representations satisfying certain conditions which will be satisfied by the l-adic realisations of any "motive".
More precisely by a *rank* 2 *weakly compatible system of l-adic representations* $\mathcal{R}$ over $\mathbb{Q}$ we shall mean a 5-tuple $(M, S, \{Q_p(X)\}, \{\rho_\lambda\}, \{n_1, n_2\})$ where

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- M is a number field;
- S is a finite set of rational primes;
- for each prime $p \notin S$ of $\mathbb{Q}$, $Q_p(X)$ is a monic degree 2 polynomial in $M[X]$;
- for each prime λ of M (with residue characteristic l say)

$$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow GL_2(M_\lambda)$$

is a continuous representation such that, if $l \notin S$ then $\rho_\lambda|_{G_l}$ is crystalline, and if $p \notin S \cup \{l\}$ then ρ_λ is unramified at p and $\rho_\lambda(\mathrm{Frob}_p)$ has characteristic polynomial $Q_p(X)$; and

- n_1, n_2 are integers such that for all primes λ of M (lying above a rational prime l) the representation $\rho_\lambda|_{G_l}$ is Hodge-Tate with numbers n_1 and n_2, i.e. $\rho_\lambda \otimes_{\mathbb{Q}_l} \widehat{\mathbb{Q}_l^{ac}} \cong (M_\lambda \otimes_{\mathbb{Q}_l} \widehat{\mathbb{Q}_l^{ac}})(-n_1) \oplus (M_\lambda \otimes_{\mathbb{Q}_l} \widehat{\mathbb{Q}_l^{ac}})(-n_2)$ as $M_\lambda \otimes_{\mathbb{Q}_l} \widehat{\mathbb{Q}_l^{ac}}$-modules with M_λ-linear, $\widehat{\mathbb{Q}_l^{ac}}$-semilinear $G_{\mathbb{Q}_l}$-actions.

We call $\mathcal{R}$ *regular* if $n_1 \neq n_2$ and $\det \rho_\lambda(c) = -1$ for one (and hence all) primes λ of M. We remark that if $\mathcal{R}$ arises from a regular (distinct Hodge numbers) motive then one can use the Hodge realisation to check that $\det \rho_\lambda(c) = -1$ for all λ. Thus we consider this oddness condition part of regularity. It is not difficult to see that if one of the ρ_λ is absolutely reducible so are all the others. In this case we call $\mathcal{R}$ *reducible*, otherwise we call it *irreducible*. (If $\rho_{\lambda_0}^{\mathrm{ss}}$ is the sum of two characters these characters are Hodge-Tate and hence by results of [S1] themselves fit into compatible systems. The elements of these compatible systems provide the Jordan-Hölder factors of the other ρ_λ.)

We will call $\mathcal{R}$ *strongly compatible* if for each rational prime p there is a Weil-Deligne representation $\mathrm{WD}_p(\mathcal{R})$ of $W_{\mathbb{Q}_p}$ such that for primes λ of M not dividing p, $\mathrm{WD}_p(\mathcal{R})$ is equivalent to the Frobenius semi-simplification of the Weil-Deligne representation associated to $\rho_\lambda|_{G_p}$. ($\mathrm{WD}_p(\mathcal{R})$ is defined over $\overline{M}$, but it is equivalent to all its $\mathrm{Gal}\,(\overline{M}/M)$-conjugates.) If $\mathcal{R}$ is strongly compatible and if $i : M \hookrightarrow \mathbb{C}$ then we define an L-function $L(i\mathcal{R}, s)$ as the infinite product

$$L(i\mathcal{R}, s) = \prod_p L_p(i\mathrm{WD}_p(\mathcal{R})^\vee \otimes |\mathrm{Art}^{-1}|_p^{-s})^{-1}$$

which may or may not converge. Fix an additive character $\Psi = \prod \Psi_p$ of $\mathbb{A}/\mathbb{Q}$ with $\Psi_\infty(x) = e^{2\pi\sqrt{-1}x}$, and a Haar measure $dx = \prod dx_p$ on $\mathbb{A}$ with dx_∞ the usual measure on $\mathbb{R}$ and with $dx(\mathbb{A}/\mathbb{Q}) = 1$. If, say, $n_1 > n_2$ then we can also also define an ϵ-factor $\epsilon(i\mathcal{R}, s)$ by the formula

$$\epsilon(i\mathcal{R}, s) = \sqrt{-1}^{1+n_1-n_2} \prod_p \epsilon(i\mathrm{WD}_p(\mathrm{RS})^\vee \otimes |\mathrm{Art}^{-1}|_p^{-s}, \Psi_p, dx_p).$$

(See [Tat] for the relation between l-adic representations of $G_{\mathbb{Q}_p}$ and Weil-Deligne representations of $W_{\mathbb{Q}_p}$, and also for the definition of the local L and ϵ-factors.)

THEOREM A *Suppose that* $\mathcal{R} = (M, S, \{Q_x(X)\}, \{\rho_\lambda\}, \{n_1, n_2\})/\mathbb{Q}$ *is a regular, irreducible, rank* 2 *weakly compatible system of l-adic representations with* $n_1 > n_2$. *Then the following assertions hold.*

1. *If* $i : M \hookrightarrow \mathbb{C}$ *then there is a totally real Galois extension* $F/\mathbb{Q}$ *and a regular algebraic cuspidal automorphic representation* π *of* $GL_2(\mathbb{A}_F)$ *such that* $L(i\mathcal{R}|_{G_F}, s) = L(\pi, s)$.
2. *For all rational primes* $p \notin S$ *and for all* $i : M \hookrightarrow \mathbb{C}$ *the roots of* $i(Q_p(X))$ *have absolute value* $p^{-(n_1+n_2)/2}$.
3. $\mathcal{R}$ *is strongly compatible.*
4. *For all* $i : M \hookrightarrow \mathbb{C}$, *the* L*-function* $L(i\mathcal{R}, s)$ *converges in* $\mathrm{Re}\, s > 1 - (n_1 + n_2)/2$, *has meromorphic continuation to the entire complex plane and satisfies a functional equation*
$$(2\pi)^{-(s+n_1)}\Gamma(s + n_1)L(i\mathcal{R}, s) = \epsilon(i\mathcal{R}, s)(2\pi)^{s+n_2-1}\Gamma(1 - n_2 - s)L(i\mathcal{R}^\vee, 1 - s).$$

More precisely we express $L(i\mathcal{R}, s)$ as a ratio of products of the L-functions associated to Hilbert modular forms over different subfields of F. (See section 6 for more details.)
For example suppose that $X/\mathbb{Q}$ is a rigid Calabi-Yau 3-fold, where by rigid we mean that $H^{2,1}(X(\mathbb{C}), \mathbb{C}) = (0)$. Then the zeta function $\zeta_X(s)$ of X has meromorphic continuation to the entire complex plane and satisfies a functional equation relating $\zeta_X(s)$ and $\zeta_X(4 - s)$. A more precise statement can be found in section six.
Along the way we prove the following result which may also be of interest. It partially confirms the Fontaine-Mazur conjecture, see [FM].

THEOREM B *Let* $l > 3$ *be a prime and let* $2 \le k \le (l + 1)/2$ *be an integer. Let* $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_l^{ac})$ *be a continuous irreducible representation such that*

- ρ *ramifies at only finitely many primes,*
- $\det \rho(c) = -1$,
- $\rho|_{G_l}$ *is crystalline with Hodge-Tate numbers* 0 *and* $1 - k$.

Then the following assertions hold.

1. *There is a Galois totally real field* F *in which* l *is unramified, a regular algebraic cuspidal automorphic representation* π *of* $GL_2(\mathbb{A}_F)$ *and an embedding* λ *of the field of rationality of* π *into* $\mathbb{Q}_l^{ac}$ *such that*
   - $\rho_{\pi,\lambda} \sim \rho|_{G_F}$,

- *π_x is unramified for all places x of E above l, and*
- *π_∞ has parallel weight k.*

2. *If ρ is unramified at a prime p and if α is an eigenvalue of $\rho(\mathrm{Frob}_p)$ then $\alpha \in \mathbb{Q}^{ac}$ and for any isomorphism $i : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$ we have*
$$|i\alpha|^2 = p^{(k-1)/2}.$$
3. *Fix an isomorphism $i : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. There is a rational function $L_{l,i}(X) \in \mathbb{C}(X)$ such that the product*
$$L(i\rho, s) = L_{l,i}(l^{-s})^{-1} \prod_{p \neq l} i \det(1 - \rho_{I_p}(\mathrm{Frob}_p)p^{-s})^{-1}$$
converges in $\mathrm{Re}\, s > (k+1)/2$ and extends to a meromorphic function on the entire complex plane which satisfies a functional equation
$$(2\pi)^{-s}\Gamma(s)L(i\rho, s) = WN(\rho)^{k/2-s}(2\pi)^{s-k}\Gamma(k-s)L(i(\rho^\vee \otimes \epsilon^{k-1}), k-s),$$
where ϵ denotes the cyclotomic character, where $N(\rho)$ denotes the conductor of ρ (which is prime to l), and where W is a complex number. (W is given in terms of local ϵ-factors in the natural way. See section 6 for details.)
4. *If $k = 2$ further assume that for some prime $p \neq l$ we have*
$$\rho|_{G_p} \sim \begin{pmatrix} \epsilon\chi & * \\ 0 & \chi \end{pmatrix}.$$
Then ρ occurs in the l-adic cohomology (with coefficients in some Tate twist of the constant sheaf) of some variety over $\mathbb{Q}$.

Again we actually show that $L(i\rho, s)$ is a ratio of products of the L-functions associated to Hilbert modular forms over different subfields of F. (See section 6 for more details.)

For further discussion of the background to these results and for a sketch of the arguments we use we refer the reader to the introduction of [Tay4].

The first three sections of this paper are taken up generalising results of Wiles [W2] and of Wiles and the author [TW] to totally real fields. Previous work along these lines has been undertaken by Fujiwara [Fu] (unpublished) and Skinner and Wiles [SW2]. However the generalisation we need is not available in the literature, so we give the necessary arguments here. We claim no great originality, this is mostly a technical exercise. We hope, however, that other authors may find theorems 2.6, 3.2 and 3.3 of some use.

In the fourth and fifth sections we generalise some of our results from [Tay4] about a potential version of Serre's conjecture. This is the most original part of this paper. The main result is theorem 5.7. Finally in section six we combine theorems 3.3 and 5.7 to deduce the main results of this paper which we have summarised above.

We would like to apologise for the long delay in submitting this paper (initially made available on the web in 2001) for publication. We would also like to thank the referee for reading the paper very carefully and making several useful suggestions.

Notation

Throughout this paper l will denote a rational prime, usually assumed to be odd and often assumed to be > 3.

If K is a perfect field we will let K^{ac} denote its algebraic closure and G_K denote its absolute Galois group $\mathrm{Gal}\,(K^{ac}/K)$. If moreover p is a prime number different from the characteristic of K then we will let $\epsilon_p : G_K \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character and ω_p the Teichmüller lift of ϵ_p mod p. In the case $p = l$ we will drop the subscripts and write simply $\epsilon = \epsilon_l$ and $\omega = \omega_l$. We will let c denote complex conjugation on $\mathbb{C}$.

If K is an l-adic field we will let $|\ |_K$ denote the absolute value on K normalised to take uniformisers to the inverse of the cardinality of the residue field of K. We will let I_K denote the inertia subgroup of G_K, W_K denote the Weil group of K and $\mathrm{Frob}_K \in W_K/I_K$ denote an arithmetic Frobenius element. We will also let $\mathrm{Art} : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ denote the Artin map normalised to take uniformisers to arithmetic Frobenius elements. Please note these unfotunate conventions. We apologise for making them. (They are inherited from [CDT].) By an n-dimensional Weil-Deligne representation of W_K over a field M we shall mean a pair (r, N) where $r : W_K \to GL_n(M)$ is a homomorphism with open kernel and where $N \in M_n(M)$ satisfies

$$r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}^{-1}\sigma|_K^{-1} N$$

for all $\sigma \in W_K$. We call (r, N) Frobenius semi-simple if r is semi-simple. For $n \in \mathbb{Z}_{>0}$ we define a character $\omega_{K,n} : I_K \to (K^{ac})^\times$ by

$$\omega_{K,n}(\sigma) = \sigma(\sqrt[l^n-1]{l})/\sqrt[l^n-1]{l}.$$

We will often write ω_n for $\omega_{\mathbb{Q}_l,n}$. Note that $\omega_{K,1} = \omega$.

Now suppose that $K/\mathbb{Q}_l$ is a finite unramified extension, that $\mathcal{O}$ is the ring of integers of a finite extension of K with maximal ideal λ and that $2 \le k \le l-1$. Let $\mathcal{MF}_{K,\mathcal{O},k}$ denote the abelian category whose objects are finite length $\mathcal{O}_K \otimes_{\mathbb{Z}_l} \mathcal{O}$-modules D together with a distinguished submodule D^0 and $\mathrm{Frob}_K \otimes 1$-semilinear maps $\varphi_{1-k} : D \to D$ and $\varphi_0 : D^0 \to D$ such that

- $\varphi_{1-k}|_{D^0} = l^{k-1}\varphi_0$, and
- $\mathrm{Im}\,\varphi_{1-k} + \mathrm{Im}\,\varphi_0 = D$.

Also let $\mathcal{MF}_{K,\mathcal{O}/\lambda^n,k}$ denote the full subcategory of objects D with $\lambda^n D = (0)$. If D is an object of $\mathcal{MF}_{K,\mathcal{O},k}$ we define $D^*[1-k]$ by

- $D^*[1-k] = \mathrm{Hom}\,(D, \mathbb{Q}_l/\mathbb{Z}_l)$;
- $D^*[1-k]^0 = \mathrm{Hom}\,(D/D^0, \mathbb{Q}_l/\mathbb{Z}_l)$;
- $\varphi_{1-k}(f)(z) = f(l^{k-1}x + y)$, where $z = \varphi_{1-k}(x) + \varphi_0(y)$;
- $\varphi_0(f)(z) = f(x \bmod D^0)$, where $z \equiv \varphi_{1-k}(x) \bmod (\varphi_0 D^0)$.

There is a fully faithful, $\mathcal{O}$-length preserving, exact, $\mathcal{O}$-additive, covariant functor $\mathbb{M}$ from $\mathcal{MF}_{K,\mathcal{O},k}$ to the category of continuous $\mathcal{O}[G_K]$-modules with essential image closed under the formation of sub-objects. (See [FL], especially section 9. In the notation of that paper $\mathbb{M}(D) = \underline{U}_S(D^*)$, where D^* is $D^*[1-k]$ with its filtration shifted by $k-1$. The reader could also consult section 2.5 of [DDT], where the case $k=2$ and $K=\mathbb{Q}_l$ is discussed.)

If K is a number field and x is a finite place of K we will write K_x for the completion of K at x, $k(x)$ for the residue field of x, ϖ_x for a uniformiser in K_x, G_x for a decomposition group above x, I_x for the inertia subgroup of G_x, and Frob_x for an arithmetic Frobenius element in G_x/I_x. We will also let $\mathcal{O}_K$ denote the integers of K and $\mathfrak{d}_K$ the different of K. If S is a finite set of places of K we will write $K_S^\times$ for the subgroup of $K^\times$ consisting of elements which are units outside S. We will write $\mathbb{A}_K$ for the adeles of K and $||\ ||$ for $\prod_x |\ |_{F_x} : \mathbb{A}_K^\times \to \mathbb{R}^\times$. We also use Art to denote the global Artin map, normalised compatibly with our local normalisations.

We will write μ_N for the group scheme of N^{th} roots of unity. We will write $W(k)$ for the Witt vectors of k. If G is a group, H a normal subgroup of G and ρ a representation of G, then we will let ρ^H (resp. ρ_H) denote the representation of G/H on the H-invariants (resp. H-coinvariants) of ρ. We will also let ρ^{ss} denote the semisimplification of ρ, $\mathrm{ad}\,\rho$ denote the adjoint representation and $\mathrm{ad}^0\rho$ denote the kernel of the trace map from $\mathrm{ad}\,\rho$ to the trivial representation.

Suppose that A/K is an abelian variety over a perfect field K with an action of $\mathcal{O}_M$ defined over K, for some number field M. Suppose also that X is a finite torsion free $\mathcal{O}_M$-submodule. The functor on K-schemes $S \mapsto A(S) \otimes_{\mathcal{O}_M} X$ is represented by an abelian variety $A \otimes_{\mathcal{O}_M} X$. (If X is free with basis $e_1, ..., e_r$ then we can take $A \otimes_{\mathcal{O}_M} X = A^r$. Note that for any ideal $\mathfrak{a}$ of $\mathcal{O}_M$ we then have a canonical isomorphism

$$(A \otimes_{\mathcal{O}_M} X)[\mathfrak{a}] \cong A[\mathfrak{a}] \otimes_{\mathcal{O}_M} X.$$

In general if $Y \supset X \supset \mathfrak{a}Y$ with Y free and $\mathfrak{a}$ a non-zero principal ideal of $\mathcal{O}_M$ prime to the characteristic of K then we can take

$$(A \otimes_{\mathcal{O}_M} X) = (A \otimes_{\mathcal{O}_M} \mathfrak{a}Y)/(A[\mathfrak{a}] \otimes_{\mathcal{O}_M} X/\mathfrak{a}Y).)$$

Again we get an identification

$$(A \otimes_{\mathcal{O}_M} X)[\mathfrak{a}] \cong A[\mathfrak{a}] \otimes_{\mathcal{O}_M} X.$$

If X has an action of some $\mathcal{O}_M$ algebra then $A \otimes_{\mathcal{O}_M} X$ canonically inherits such an action. We also get a canonical identification $(A \otimes_{\mathcal{O}_M} X)^\vee \cong A^\vee \otimes_{\mathcal{O}_M}$

$\mathrm{Hom}(X, \mathcal{O}_M)$. Suppose that $\mu : A \to A^\vee$ is a polarisation which induces an involution c on M. Note that c equals complex conjugation for every embedding $M \hookrightarrow \mathbb{C}$. Suppose also that $f : X \to \mathrm{Hom}_{\mathcal{O}_M}(X, \mathcal{O}_M)$ is c-semilinear. If for all $x \in X - \{0\}$, the totally real number $f(x)(x)$ is totally strictly positive then $\mu \otimes f : A \otimes_{\mathcal{O}_M} X \to (A \otimes_{\mathcal{O}_M} X)^\vee$ is again a polarisation which induces c on M.

If λ is an ideal of $\mathcal{O}_M$ prime to the characteristic of K we will write $\overline{\rho}_{A,\lambda}$ for the representation of G_K on $A[\lambda](K^{ac})$. If λ is prime we will write $T_\lambda A$ for the λ-adic Tate module of A, $V_\lambda A$ for $T_\lambda A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\rho_{A,\lambda}$ for the representation of G_K on $V_\lambda A$. We have a canonical isomorphism $T_\lambda(A \otimes_{\mathcal{O}_M} X) \xrightarrow{\sim} (T_\lambda A) \otimes_{\mathcal{O}_M} X$.

Suppose that M is a totally real field. By an ordered invertible $\mathcal{O}_M$-module we shall mean an invertible $\mathcal{O}_M$-module X together with a choice of connected component X_x^+ of $(X \otimes M_x) - \{0\}$ for each infinite place x of M. If $\mathfrak{a}$ is a fractional ideal in M then we will denote by $\mathfrak{a}^+$ the invertible ordered $\mathcal{O}_M$-module $(\mathfrak{a}, \{(M_x^\times)^0\})$, where $(M_x^\times)^0$ denotes the connected component of 1 in $M_x^\times$. By an M-HBAV ('Hilbert-Blumenthal abelian variety') over a field K we shall mean a triple (A, i, j) where

- A/K is an abelian variety of dimension $[M : \mathbb{Q}]$,
- $i : \mathcal{O}_M \hookrightarrow \mathrm{End}(A/K)$
- and $j : (\mathfrak{d}_M^{-1})^+ \xrightarrow{\sim} \mathcal{P}(A, i)$ is an isomorphism of ordered invertible $\mathcal{O}_M$-modules.

Here $\mathcal{P}(A, i)$ is the invertible $\mathcal{O}_M$ module of symmetric (i.e. $f^\vee = f$) homomorphisms $f : (A, i) \to (A^\vee, i^\vee)$ which is ordered by taking the unique connected component of $(\mathcal{P}(A, i) \otimes M_x)$ which contains the class of a polarisation. (See section 1 of [Rap].)

If λ is a prime of M and if $x \in \mathfrak{d}_M^{-1}$ then $j(x) : A \to A^\vee$ gives rise to an alternating pairing

$$e_{j,x,0} : T_\lambda A \times T_\lambda A \longrightarrow \mathbb{Z}_l(1).$$

This corresponds to a unique $\mathcal{O}_{M,\lambda}$-bilinear alternating pairing

$$e_{j,x} : T_\lambda A \times T_\lambda A \longrightarrow \mathfrak{d}_{M,\lambda}^{-1}(1),$$

which are related by $e_{j,x,0} = \mathrm{tr} \circ e_{j,x}$. The pairing $x^{-1} e_{j,x}$ is independent of x and gives a perfect $\mathcal{O}_{M,\lambda}$-bilinear alternating pairing

$$e_j : T_\lambda A \times T_\lambda A \longrightarrow \mathcal{O}_{M,\lambda}(1),$$

which we will call the j-Weil pairing. (See section 1 of [Rap].) Again using the trace, we can think of e_j as an $\mathcal{O}_{M,\lambda}$-linear isomorphism

$$\widetilde{e}_j : T_\lambda A \otimes \mathfrak{d}_M^{-1} \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(T_\lambda A, \mathbb{Z}_l(1)).$$

More precisely

$$\widetilde{e}_j(a \otimes y)(b) = \mathrm{tr}\,(y e_j(a, b)) = e_{j,x,0}(x^{-1} y a, b).$$

The same formula (for $x \in \mathfrak{d}_M^{-1} - \mathfrak{a}\mathfrak{d}_M^{-1}$) gives rise to an $\mathcal{O}_{M,\lambda}$-linear isomorphism

$$\widetilde{e}_j : A[\mathfrak{a}] \otimes_{\mathcal{O}_M} \mathfrak{d}_M^{-1} \longrightarrow A[\mathfrak{a}]^\vee,$$

which is independent of x and which we will refer to as the j-Weil pairing on $A[\mathfrak{a}]$.

Suppose that F is a totally real number field and that π is an algebraic (see for instance [Cl]) cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ with field of definition (or coefficients) $M \subset \mathbb{C}$. (That is M is the fixed field of the group of automorphisms σ of $\mathbb{C}$ with $\sigma\pi^\infty = \pi^\infty$. By the strong multiplicity one theorem this is the same as the fixed field of the group of automorphisms σ of $\mathbb{C}$ with $\sigma\pi_x \cong \pi_x$ for all but finitely many places x of F.) We will say that π_∞ has weight $(\vec{k}, \vec{w}) \in \mathbb{Z}_{>0}^{\mathrm{Hom}\,(F,\mathbb{R})} \times \mathbb{Z}^{\mathrm{Hom}\,(F,\mathbb{R})}$ if for each infinite place $\tau : F \hookrightarrow \mathbb{R}$ the representation π_τ is the $(k_\tau - 1)^{st}$ lowest discrete series representation of $GL_2(F_x) \cong GL_2(\mathbb{R})$ (or in the case $k_\tau = 1$ the limit of discrete series representation) with central character $a \mapsto a^{2-k_\tau-2w_\tau}$. Note that $w = k_\tau + 2w_\tau$ must be independent of τ. If π_∞ has weight $((k, ..., k), (0, ..., 0))$ we will simply say that it has weight k. In some cases, including the cases that π_∞ is regular (i.e. $k_\tau > 1$ for all τ) and the case π_∞ has weight 1, it is known that M is a CM number field and that for each rational prime l and each embedding $\lambda : M \hookrightarrow \mathbb{Q}_l^{ac}$ there is a continuous irreducible representation

$$\rho_{\pi,\lambda} : G_F \to GL_2(M_\lambda)$$

canonically associated to π. For any prime x of F not dividing l the restriction $\rho_{\pi,\lambda}|_{G_x}$ depends up to Frobenius semi-simplification only on π_x (and λ). (See [Tay1] for details. To see that M is a CM field one uses the Peterssen inner product

$$(f_1, f_2) = \int_{GL_2(F)(\mathbb{R}_{>0}^\times)^{\mathrm{Hom}\,(F,\mathbb{R})}} f_1(g)^c(f_2(g))|| \det g||^{w-2} dg.$$

For all $\sigma \in \mathrm{Aut}\,(\mathbb{C})$ the representation ${}^\sigma\pi^\infty$ extends to an algebraic automorphic representation $\pi(\sigma)$ of $GL_2(\mathbb{A}_F)$ with the same value for w. The pairing $(\ ,\ )$ gives an isomorphism ${}^c\pi(\sigma) \cong \pi(\sigma)^\vee || \det ||^{2-w}$. Thus ${}^{\sigma^{-1}c\sigma}\pi^\infty$ is independent of σ and M is a CM field.) We will write $\rho_{\pi,\lambda}|_{W_{F_x}}^{ss} = \mathrm{WD}_\lambda(\pi_x)$, where $\mathrm{WD}_\lambda(\pi_x)$ is a semi-simple two-dimensional representation of W_{F_x}. If π_x is unramified then $\mathrm{WD}_\lambda(\pi_x)$ is also unramified and $\mathrm{WD}_\lambda(\pi_x)(\mathrm{Frob}_x)$ has characteristic polynomial

$$X^2 - t_x X + (\mathrm{N}x)s_x$$

where t_x (resp. s_x) is the eigenvalue of

$$\left[GL_2(\mathcal{O}_{F_x}) \begin{pmatrix} \varpi_x & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathcal{O}_{F_x}) \right]$$

(resp. of

$$\left[GL_2(\mathcal{O}_{F_x})\begin{pmatrix} \varpi_x & 0 \\ 0 & \varpi_x \end{pmatrix} GL_2(\mathcal{O}_{F_x})\right])$$

on $\pi_x^{GL_2(\mathcal{O}_{F_x})}$. An explicit description of some other instances of $\mathrm{WD}_\lambda(\pi_x)$ may be found in section 4 of [CDT].

We may always conjugate $\rho_{\pi,\lambda}$ so that it is valued in $GL_2(\mathcal{O}_{M,\lambda})$ and then reduce it to get a continuous representation $G_F \to GL_2(\mathbb{F}_l^{ac})$. If for one such choice of conjugate the resulting representation is irreducible then it is independent of the choice of conjugate and we will denote it $\overline{\rho}_{\pi,\lambda}$.

1 l-adic modular forms on definite quaternion algebras

In this section we will establish some notation and recall some facts about l-adic modular forms on some definite quaternion algebras.

To this end, fix a prime $l > 3$ and a totally real field F of even degree in which l is unramified. Let D denote the division algebra with centre F which ramifies exactly at the set of infinite places of F. Fix a maximal order $\mathcal{O}_D$ in D and isomorphisms $\mathcal{O}_{D,x} \cong M_2(\mathcal{O}_{F,x})$ for all finite places x of F. These choices allow us to identify $GL_2(\mathbb{A}_F^\infty)$ with $(D \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$. For each finite place x of F also fix a uniformiser ϖ_x of $\mathcal{O}_{F,x}$. Also let A be a topological $\mathbb{Z}_l$-algebra which is either an algebraic extension of $\mathbb{Q}_l$, the ring of integers in such an extension or a quotient of such a ring of integers.

Let $U = \prod_x U_x$ be an open compact subgroup of $GL_2(\mathbb{A}_F^\infty)$ and let $\psi : (\mathbb{A}_F^\infty)^\times/F^\times \to A^\times$ be a continuous character. Also let $\tau : U_l \to \mathrm{Aut}\,(W_\tau)$ be a continuous representation of U_l on a finite A-module W_τ such that

$$\tau|_{U_l \cap \mathcal{O}_{F,l}^\times} = \psi|_{U_l \cap \mathcal{O}_{F,l}^\times}^{-1}.$$

We will write $W_{\tau,\psi}$ for W_τ when we want to think of it as a $U(\mathbb{A}_F^\infty)^\times$-module with U acting via τ and $(\mathbb{A}_F^\infty)^\times$ by ψ^{-1}.

We define $S_{\tau,\psi}(U)$ to be the space of continuous functions

$$f : D^\times \backslash GL_2(\mathbb{A}_F^\infty) \longrightarrow W_\tau$$

such that

- $f(gu) = \tau(u_l)^{-1} f(g)$ for all $g \in GL_2(\mathbb{A}_F^\infty)$ and all $u \in U$, and
- $f(gz) = \psi(z) f(g)$ for all $g \in GL_2(\mathbb{A}_F^\infty)$ and all $z \in (\mathbb{A}_F^\infty)^\times$.

If

$$GL_2(\mathbb{A}_F^\infty) = \coprod_i D^\times t_i U(\mathbb{A}_F^\infty)^\times$$

then

$$\begin{array}{rcl} S_{\tau,\psi}(U) & \xrightarrow{\sim} & \bigoplus_i W_{\tau,\psi}^{(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1} D^\times t_i)/F^\times} \\ f & \longmapsto & (f(t_i))_i. \end{array}$$

The index set over which i runs is finite.

Lemma 1.1 *Each group* $(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1}D^\times t_i)/F^\times$ *is finite and, as we are assuming* $l > 3$ *and* l *is unramified in* F*, the order of* $(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1}D^\times t_i)/F^\times$ *is not divisible by* l.

Proof: Set $V = \prod_{x \not| \infty} \mathcal{O}_{F,x}^\times$. Then we have exact sequences

$$(0) \longrightarrow (UV \cap t_i^{-1}D^{\det=1}t_i)/\{\pm 1\} \longrightarrow (U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1}D^\times t_i)/F^\times \longrightarrow (((\mathbb{A}_F^\infty)^\times)^2 V \cap F^\times)/(F^\times)^2$$

and

$$(0) \longrightarrow \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \longrightarrow (((\mathbb{A}_F^\infty)^\times)^2 V \cap F^\times)/(F^\times)^2 \longrightarrow H[2] \longrightarrow (0),$$

where H denotes the class group of $\mathcal{O}_F$. We see that $(((\mathbb{A}_F^\infty)^\times)^2 V \cap F^\times)/(F^\times)^2$ is finite of 2-power order. Moreover $UV \cap t_i^{-1}D^{\det=1}t_i$ is finite. For $l > 3$ and l unramified in F, $D^\times$ and hence $UV \cap t_i^{-1}D^{\det=1}t_i$ contain no elements of order exactly l. The lemma follows. $\Box$

Corollary 1.2 *If* B *is an* A*-algebra then*

$$S_{\tau,\psi}(U) \otimes_A B \xrightarrow{\sim} S_{\tau \otimes_A B,\psi}(U).$$

If $x \not| l$, or if $x | l$ but $\tau|_{U_x} = 1$, then the Hecke algebra $A[U_x \backslash GL_2(F_x)/U_x]$ acts on $S_{\tau,\psi}(U)$. Explicitly, if

$$U_x h U_x = \coprod_i h_i U_x$$

then

$$([U_x h U_x]f)(g) = \sum_i f(gh_i).$$

Let U_0 denote $\prod_x GL_2(\mathcal{O}_{F,x})$. Now suppose that $\mathfrak{n}$ is an ideal of $\mathcal{O}_F$ and that, for each finite place x of F diving $\mathfrak{n}$, H_x is a quotient of $(\mathcal{O}_{F,x}/\mathfrak{n}_x)^\times$. Then we will write H for $\prod_{x|\mathfrak{n}} H_x$ and we will let $U_H(\mathfrak{n}) = \prod_x U_H(\mathfrak{n})_x$ denote the open subgroup of $GL_2(\mathbb{A}_F^\infty)$ defined by setting $U_H(\mathfrak{n})_x$ to be the subgroup of $GL_2(\mathcal{O}_{F,x})$ consisting of elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \in \mathfrak{n}_x$ and, in the case $x|\mathfrak{n}$, with ad^{-1} mapping to 1 in H_x.
If $x \not| l\mathfrak{n}$ then we will let T_x denote the Hecke operator

$$\left[U_H(\mathfrak{n}) \begin{pmatrix} \varpi_x & 0 \\ 0 & 1 \end{pmatrix} U_H(\mathfrak{n}) \right]$$

and S_x the Hecke operator

$$\left[U_H(\mathfrak{n}) \begin{pmatrix} \varpi_x & 0 \\ 0 & \varpi_x \end{pmatrix} U_H(\mathfrak{n}) \right].$$

If $x|\mathfrak{n}$ and, either $x \not| l$ or $x|l$ but $\tau|_{U_H(\mathfrak{n})} = 1$, then we will set

$$\langle h \rangle = \left[U_H(\mathfrak{n}) \begin{pmatrix} \widetilde{h} & 0 \\ 0 & 1 \end{pmatrix} U_H(\mathfrak{n}) \right]$$

for $h \in H_x$ and $\widetilde{h}$ a lift of h to $\mathcal{O}_{F,x}^\times$; and

$$\mathbf{U}_{\varpi_x} = \left[U_H(\mathfrak{n}) \begin{pmatrix} \varpi_x & 0 \\ 0 & 1 \end{pmatrix} U_H(\mathfrak{n}) \right];$$

and

$$\mathbf{V}_{\varpi_x} = \left[U_H(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} U_H(\mathfrak{n}) \right];$$

and

$$S_{\varpi_x} = \left[U_H(\mathfrak{n}) \begin{pmatrix} \varpi_x & 0 \\ 0 & \varpi_x \end{pmatrix} U_H(\mathfrak{n}) \right].$$

For $x|\mathfrak{n}$ we note the decompositions

$$U_H(\mathfrak{n})_x \begin{pmatrix} \varpi_x & 0 \\ 0 & 1 \end{pmatrix} U_H(\mathfrak{n})_x = \coprod_{a \in k(x)} \begin{pmatrix} \varpi_x & \widetilde{a} \\ 0 & 1 \end{pmatrix} U_H(\mathfrak{n})_x,$$

and

$$U_H(\mathfrak{n})_x \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} U_H(\mathfrak{n})_x = \coprod_{a \in k(x)} \begin{pmatrix} \varpi_x & 0 \\ \varpi_x \widetilde{a} & 1 \end{pmatrix} U_H(\mathfrak{n})_x$$

and

$$U_H(\mathfrak{n})_x \begin{pmatrix} \varpi_x & 0 \\ 0 & \varpi_x \end{pmatrix} U_H(\mathfrak{n})_x = \begin{pmatrix} \varpi_x & 0 \\ 0 & \varpi_x \end{pmatrix} U_H(\mathfrak{n})_x,$$

where $\widetilde{a}$ is some lift of a to $\mathcal{O}_{F,x}$.
We will let $h_{\tau,A,\psi}(U_H(\mathfrak{n}))$ denote the A-subalgebra of $\mathrm{End}_A(S_{\tau,\psi}(U_H(\mathfrak{n})))$ generated by T_x for $x \not| l\mathfrak{n}$ and by $\mathbf{U}_{\varpi_x}$ for $x|\mathfrak{n}$ but $x \not| l$. It is commutative. We will call a maximal ideal $\mathfrak{m}$ of $h_{\tau,A,\psi}(U_H(\mathfrak{n}))$ *Eisenstein* if it contains $T_x - 2$ and $S_x - 1$ for all but finitely many primes x of F which split completely in some finite abelian extension of F. (The following remark may help explain the form of this definition. If $\overline{\rho} : G_F \to GL_2(\overline{\mathbb{F}}_l)$ is a continuous reducible representation, then there is a finite abelian extension L/F such that $\mathrm{tr}\,\overline{\rho}(G_L) = \{2\}$ and $(\epsilon_l^{-1} \det \overline{\rho})(G_L) = \{1\}$.)
For $k \in \mathbb{Z}_{\geq 2}$ and we will let $\mathrm{Symm}^{k-2}(A^2)$ denote the space of homogeneous polynomials of degree $k-2$ in two variables X and Y over A with a $GL_2(A)$-action via

$$\left(\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) f \right)(X,Y) = f\left((X,Y) \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \right) = f(aX + cY, bX + dY).$$

Let A be an $\mathcal{O}_L$ algebra for some extension $L/\mathbb{Q}_l$ containing the images of all embeddings $F \hookrightarrow \mathbb{Q}_l^{ac}$. Suppose that $(\vec{k}, \vec{w}) \in \mathbb{Z}_{>1}^{\mathrm{Hom}\,(F,\mathbb{Q}_l^{ac})} \times \mathbb{Z}^{\mathrm{Hom}\,(F,\mathbb{Q}_l^{ac})}$ is such that $k_\sigma + 2w_\sigma$ is independent of σ. We will write $\tau_{(\vec{k},\vec{w}),A}$ for the representation of $GL_2(\mathcal{O}_{F,l})$ on $W_{(\vec{k},\vec{w}),A} = \bigotimes_{\sigma:F\to\mathbb{Q}_l^{ac}} \mathrm{Symm}^{k_\sigma-2}(A^2)$ via

$$g \longmapsto \otimes_{\sigma:F\to\mathbb{Q}_l^{ac}}(\mathrm{Symm}^{k_\sigma-2}(\sigma g)\otimes \det{}^{w_\sigma}(\sigma g)).$$

We will also write $S_{(\vec{k},\vec{w}),A,\psi}(U)$ for $S_{\tau_{(\vec{k},\vec{w}),A},\psi}(U)$. Let $S^{\mathrm{triv}}_{(\vec{k},\vec{w}),A,\psi}(U)$ denote (0) unless $(\vec{k},\vec{w}) = ((2,...,2),(w,...,w))$, in which case let it denote the subspace of $S_{(\vec{k},\vec{w}),A,\psi}(U)$ consisting of functions f which factor through the reduced norm. Set

$$S_{(\vec{k},\vec{w}),A,\psi}(U_l) = \lim_{\to U^l} S_{(\vec{k},\vec{w}),A,\psi}(U^l \times U_l).$$

It has a smooth action of $GL_2(\mathbb{A}_F^{\infty,l})$ (by right translation). If $(\vec{k},\vec{w}) = ((k,...,k),(0,...,0))$ then we will often write k in place of $(\vec{k},\vec{w})$. Set

$$S_{2,A,\psi} = \lim_{\to U} S_{2,A,\psi}(U)$$

and

$$S^{\mathrm{triv}}_{2,A,\psi} = \lim_{\to U} S^{\mathrm{triv}}_{2,A,\psi}(U).$$

They have smooth actions of $GL_2(\mathbb{A}_F^\infty)$.

LEMMA 1.3 *Suppose that* $(\vec{k},\vec{w}) \in \mathbb{Z}_{>1}^{\mathrm{Hom}(F,\mathbb{Q}_l^{ac})} \times \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{Q}_l^{ac})}$ *and* $w = k_\sigma - 1 + 2w_\sigma$ *is independent of* σ. *Also suppose that* $\psi : \mathbb{A}_F^\times/F^\times \to (\mathbb{Q}_l^{ac})^\times$ *is a continuous character satisfying* $\psi(a) = (\mathrm{N}a)^{1-w}$ *for all* a *in a non-empty open subgroup of* $F_l^\times$. *Choose an isomorphism* $i : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. *Define* $i(\vec{k},\vec{w}) = (i\vec{k}, i\vec{w}) \in \mathbb{Z}_{>1}^{\mathrm{Hom}(F,\mathbb{C})} \times \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})}$ *by* $(i\vec{k})_\tau = \vec{k}_{i^{-1}\tau}$ *and* $(i\vec{w})_\tau = \vec{w}_{i^{-1}\tau}$. *Also define* $\psi_i : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ *by* $\psi_i(z) = i((\mathrm{N}z_l)^{w-1}\psi(z^\infty))(\mathrm{N}z_\infty)^{1-w}$. *Then we have the following assertions.*

1. $S_{(\vec{k},\vec{w}),\mathbb{Q}_l^{ac},\psi}(U_l)$ *is a semi-simple admissible representation of* $GL_2(\mathbb{A}_F^{\infty,l})$ *and* $S_{(\vec{k},\vec{w}),\mathbb{Q}_l^{ac},\psi}(U_l)^{U^l} = S_{(\vec{k},\vec{w}),\mathbb{Q}_l^{ac},\psi}(U_l \times U^l)$.

2. *There is an isomorphism*

$$(S_{(\vec{k},\vec{w}),\mathbb{Q}_l^{ac},\psi}(U_l)/S^{\mathrm{triv}}_{(\vec{k},\vec{w}),\mathbb{Q}_l^{ac},\psi}(U_l)) \otimes_{\mathbb{Q}_l^{ac},i} \mathbb{C} \cong \bigoplus_\pi \pi^{\infty,l} \otimes \pi_l^{U_l}$$

where π *runs over regular algebraic cuspidal automorphic representations of* $GL_2(\mathbb{A}_F)$ *such that* π_∞ *has weight* $(\vec{k},\vec{w})$ *and such that* π *has central character* ψ_i.

3. $S_{2,\mathbb{Q}_l^{ac},\psi}$ *is a semi-simple admissible representation of* $GL_2(\mathbb{A}_F^\infty)$ *and*

$$S^U_{2,\mathbb{Q}_l^{ac},\psi} = S_{2,\mathbb{Q}_l^{ac},\psi}(U).$$

4. There is an isomorphism

$$S_{2,\mathbb{Q}_l^{ac},\psi} \otimes_{\mathbb{Q}_l^{ac},i} \mathbb{C} \cong \bigoplus_{\chi} \mathbb{Q}_l^{ac}(\chi) \oplus \bigoplus_{\pi} \pi^{\infty}$$

where π runs over regular algebraic cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$ such that π_∞ has weight 2 and such that π has central character ψ_i, and where χ runs over characters $(\mathbb{A}_F^\infty)^\times/F_{>>0}^\times \to (\mathbb{Q}_l^{ac})^\times$ with $\chi^2 = \psi$.

Proof: We will explain the first two parts. The other two are similar. Let $C^\infty(D^\times \backslash (D \otimes_{\mathbb{Q}} \mathbb{A})^\times / U_l, \psi_\infty)$ denote the space of smooth functions

$$D^\times \backslash (D \otimes_{\mathbb{Q}} \mathbb{A})^\times / U_l \longrightarrow \mathbb{C}$$

which transform under $\mathbb{A}_F^\times$ by ψ_∞. Let τ_∞ denote the representation of $D_\infty^\times$ on $W_{\tau_\infty} = W_{(\vec{k},\vec{w}),\mathbb{Q}_l^{ac}} \otimes_i \mathbb{C}$ via

$$g \longmapsto \otimes_{\sigma: F \to \mathbb{Q}_l^{ac}} (\mathrm{Symm}^{\,k_\sigma - 2}(i\sigma g) \otimes \det{}^{w_\sigma}(i\sigma g)).$$

Then there is an isomorphism

$$S_{(\vec{k},\vec{w}),\mathbb{Q}_l^{ac},\psi}(U_l) \xrightarrow{\sim} \mathrm{Hom}_{D_\infty^\times}(W_{\tau_\infty}^\vee, C^\infty(D^\times \backslash (D \otimes_{\mathbb{Q}} \mathbb{A})^\times / U_l, \psi_\infty))$$

which sends f to the map

$$y \longmapsto (g \longmapsto y(\tau_\infty(g_\infty))^{-1} \tau_{(\vec{k},\vec{w}),\mathbb{Q}_l^{ac}}(g_l) f(g^\infty))).$$

Everything now follows from the Jacquet-Langlands theorem. □

There is a pairing

$$\mathrm{Symm}^{\,k-2}(A^2) \times \mathrm{Symm}^{\,k-2}(A^2) \longrightarrow A$$

defined by

$$\langle f_1, f_2 \rangle = (f_1(\partial/\partial Y, -\partial/\partial X) f_2(X,Y))|_{X=Y=0}.$$

By looking at the pairing of monomials we see that

$$\langle f_1, f_2 \rangle = (-1)^k \langle f_2, f_1 \rangle$$

and that if $2 \leq k \leq l+1$ then this pairing is perfect. Moreover if

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$$

then

$$\begin{array}{rl} & \langle uf_1, uf_2 \rangle \\ = & (f_1(a\partial/\partial Y - c\partial/\partial X, b\partial/\partial Y - d\partial/\partial X) f_2(aX + cY, bX + dY))|_{X=Y=0} \\ = & (f_1((\det u)\partial/\partial W, -(\det u)\partial/\partial Z) f_2(Z,W))|_{Z=W=0} \\ = & (\det u)^{k-2} \langle f_1, f_2 \rangle, \end{array}$$

where $Z = aX + cY$ and $W = bX + dY$. This extends to a perfect pairing $W_{(\vec{k},\vec{w}),A} \times W_{(\vec{k},\vec{w}),A} \to A$ such that

$$\langle ux, uy \rangle = (\mathrm{N} \det u)^{w-1} \langle x, y \rangle$$

for all $x, y \in W_{(\vec{k},\vec{w}),A}$ and all $u \in GL_2(\mathcal{O}_{F,l})$. Here $w = k_\sigma + 2w_\sigma - 1$, which is independent of σ.
We can define a perfect pairing $S_{k,A,\psi}(U_H(\mathfrak{n})) \times S_{k,A,\psi}(U_H(\mathfrak{n})) \to A$ by setting (f_1, f_2) equal to

$$\sum_{[x]} \langle f_1(x), f_2(x) \rangle \psi(\det x)^{-1} (\#(U_H(\mathfrak{n})(\mathbb{A}_F^\infty)^\times \cap x^{-1} D^\times x)/F^\times)^{-1},$$

where $[x]$ ranges over $D^\times \backslash (D \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times / U_H(\mathfrak{n})(\mathbb{A}_F^\infty)^\times$. (We are using the fact that $\#(U_H(\mathfrak{n})(\mathbb{A}_F^\infty)^\times \cap x^{-1} D^\times x)/F^\times$ is prime to l.) The usual calculation shows that

$$([U_{H'}(\mathfrak{n}') g U_H(\mathfrak{n})] f_1, f_2)_{U_{H'}(\mathfrak{n}')} = \psi(\det g)(f_1, [U_H(\mathfrak{n}) g^{-1} U_{H'}(\mathfrak{n}')] f_2)_{U_H(\mathfrak{n})}.$$

Now specialise to the case that $A = \mathcal{O}$ is the ring on integers of a finite extension of $\mathbb{Q}_l$. We will write simply $h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))$ for $h_{(\vec{k},\vec{w}),\mathcal{O},\psi}(U_H(\mathfrak{n}))$. It follows from lemma 1.3 and the main theorem in [Tay1] that there is a continuous representation

$$\rho : G_F \longrightarrow GL_2(h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n})) \otimes_{\mathcal{O}} \mathbb{Q}_l^{ac})$$

such that

- if $x \not| \mathfrak{n} l$ then ρ is unramified at x and $\mathrm{tr}\, \rho(\mathrm{Frob}_x) = T_x$; and
- $\det \rho = \epsilon(\psi \circ \mathrm{Art}^{-1})$.

From the theory of pseudo-representations (or otherwise, see [Ca2]) we deduce that if $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))$ then ρ gives rise to a continuous representation

$$\rho_{\mathfrak{m}} : G_F \longrightarrow GL_2(h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}})$$

such that

- if $x \not| \mathfrak{n} l$ then $\rho_{\mathfrak{m}}$ is unramified at x and $\mathrm{tr}\, \rho_{\mathfrak{m}}(\mathrm{Frob}_x) = T_x$; and
- $\det \rho_{\mathfrak{m}} = \epsilon(\psi \circ \mathrm{Art}^{-1})$.

From the Cebotarev density theorem we see that $h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$ is generated by $\mathbf{U}_{\varpi_x}$ for $x | \mathfrak{n}$ but $x \not| l$ and by T_x for all but finitely many $x \not| l\mathfrak{n}$. (For let h denote the $\mathcal{O}$-subalgebra of $h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$ generated by $\mathbf{U}_{\varpi_x}$ for $x | \mathfrak{n}$ but

$x \not| l$ and by T_x for all but finitely many $x \not| l\mathfrak{n}$. The Cebotarev densitry theorem implies that $\operatorname{tr} \rho_{\mathfrak{m}}$ is valued in h and hence

$$T_x = \operatorname{tr} \rho_{\mathfrak{m}}(\mathrm{Frob}_x) \in h$$

for all $x \not| \mathfrak{n}l$. Thus $h = h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$.)

We will write $\overline{\rho}_{\mathfrak{m}}$ for $(\rho_{\mathfrak{m}} \bmod \mathfrak{m})$. If $\phi : h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}} \to R$ is a map of local $\mathcal{O}$-algebras then we will write ρ_ϕ for $\phi\rho_{\mathfrak{m}}$. If R is a field of characteristic l we will sometimes write $\overline{\rho}_\phi$ instead of ρ_ϕ.

Lemma 1.4 *Let $(\vec{k},\vec{w})$ be as above. Suppose that $x \not| \mathfrak{n}$ is a split place of F above l such that $2 \leq k_x \leq l-1$. If $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))$ and if I is an open ideal of $h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$ (for the l-adic topology) then $((\rho_{\mathfrak{m}} \otimes \epsilon^{-w_x}) \bmod I)|_{G_x}$ is of the form $\mathbb{M}(D)$ for some object D of $\mathcal{MF}_{F_x,\mathcal{O},k_x}$ with $D \neq D^0 \neq (0)$.*

Proof: Combining the construction of $\rho_{\mathfrak{m}}$ with the basic properties of $\mathbb{M}$ listed in the section of notation, we see that it suffices to prove the following.

Suppose that π is a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ such that π_∞ is regular algebraic of weight $(\vec{k},\vec{w})$. Let M denote the field of definition of π. Suppose that x is a split place of F above l with π_x unramified. Let M^{ac} denote the algebraic closure of M in $\mathbb{C}$ and fix an embedding $\lambda : M^{ac} \hookrightarrow \mathbb{Q}_l^{ac}$. Let $\tau : F \hookrightarrow M^{ac}$ be the embedding so that $\lambda \circ \tau$ gives rise to x. Suppose that $2 \leq k_\tau \leq l-1$. If I is a power of the prime of $\mathcal{O}_M$ induced by λ, then $(\rho_{\pi,\lambda} \otimes \epsilon^{-w_\tau})|_{G_x} \bmod I$ is of the form $\mathbb{M}(D)$ for some object D of $\mathcal{MF}_{F_x,\mathcal{O}_{M,\lambda},k_\tau}$ with $D \neq D^0 \neq (0)$.

By the construction of $\rho_{\pi,\lambda}$ in [Tay1], our assumption that $(\rho_{\pi,\lambda} \bmod \lambda)$ is irreducible, and the basic properties of $\mathbb{M}$, we see that it suffices to treat the case that π_y is discrete series for some finite place y (cf [Tay2]). Because $2 \leq k_\tau \leq l-1$, it follows from [FL] that we need only show that $\rho_{\pi,\lambda}$ is crystalline with Hodge-Tate numbers $-w_\tau$ and $1-k_\tau-w_\tau$. In the case π_y is discrete series for some finite place y this presumably follows from Carayol's construction of $\rho_{\pi,\lambda}$ [Ca1] and Faltings theory [Fa], but for a definite reference we refer the reader to theorem VII.1.9 of [HT] (but note the different, more sensible, conventions in force in that paper). $\Box$

Corollary 1.5 *Suppose that $x \not| \mathfrak{n}$ is a split place of F. Suppose that $(\vec{k},\vec{w})$ is as above and that $2 \leq k_x \leq l-1$. If $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $h_{(\vec{k},\vec{w}),\psi}(U_H(\mathfrak{n}))$ then $\overline{\rho}_{\mathfrak{m}}|_{I_x} \sim \omega_2^{k_x-1+(l+1)w_x} \oplus \omega_2^{l(k_x-1)+(l+1)w_x}$ or*

$$\begin{pmatrix} \omega^{k_x+w_x-1} & * \\ 0 & \omega^{w_x} \end{pmatrix}.$$

Proof: This follows easily from the above lemma together with theorem 5.3, proposition 7.8 and theorem 8.4 of [FL]. $\Box$

The following lemma is well known.

LEMMA 1.6 *Suppose that x is a finite place of F and that π is an irreducible admissible representation of $GL_2(F)$. If χ_1 and χ_2 are two characters of $F^\times$, let $\pi(\chi_1,\chi_2)$ denote the induced representation consisting of locally constant functions $GL_2(F) \to \mathbb{C}$ such that*

$$f\left(\left(\begin{array}{cc} a & b \\ 0 & d \end{array}\right) g\right) = \chi_1(a)\chi_2(b)|a/b|_x^{1/2} f(g)$$

(with $GL_2(F)$-action by right translation). Let U_1 (resp. U_2) denote the subgroup of elements in $GL_2(\mathcal{O}_{F,x})$ which are congruent to a matrix of the form

$$\left(\begin{array}{cc} 1 & * \\ 0 & 1 \end{array}\right) \bmod (\varpi_x)$$

(resp.

$$\left(\begin{array}{cc} * & * \\ 0 & * \end{array}\right) \bmod (\varpi_x^2)).$$

1. *If $\pi^{U_1} \neq (0)$ then π is a subquotient of some $\pi(\chi_1,\chi_2)$ where the conductors of χ_1 and χ_2 are ≤ 1.*

2. *If the conductors of χ_1 and χ_2 are ≤ 1 then*

$$\pi(\chi_1,\chi_2)^{U_1}$$

   is two dimensional with a basis e_1, e_2 such that

$$\mathbf{U}_{\varpi_x} e_i = (\mathrm{N}x)^{1/2}\chi_i(\varpi_x)e_i$$

   and

$$\langle h\rangle e_i = \chi_i(h)e_i$$

   for $h \in (\mathcal{O}_{F,x}/x)^\times$.

3. *If $\pi^{U_2} \neq (0)$ then π is either cuspidal or a subquotient of some $\pi(\chi_1,\chi_2)$ where the conductors of χ_1 and χ_2 are equal and ≤ 1.*

4. *If π is cuspidal then $\dim \pi^{U_2} \leq 1$ and $\mathbf{U}_{\varpi_x}$ acts as zero on π^{U_2}.*

5. *If χ_1 and χ_2 have conductor 1 then $\pi(\chi_1,\chi_2)^{U_2}$ is one dimensional and $\mathbf{U}_{\varpi_x}$ acts on it as 0.*

6. *If χ_1 and χ_2 have conductor 0 then $\pi(\chi_1,\chi_2)^{U_2}$ is three dimensional and $\mathbf{U}_{\varpi_x}$ acts on it with characteristic polynomial*

$$X(X-(\mathrm{N}x)^{1/2}\chi_1(\varpi_x))(X-(\mathrm{N}x)^{1/2}\chi_2(\varpi_x)).$$

As a consequence we have the following lemma.

LEMMA 1.7 *Suppose that $\xi : h_{k,\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}} \to \mathbb{Q}_l^{ac}$ and that $x \nmid l$.*

1. *If* $x(\mathfrak{n}) = 1$ *and if* ξ' *is any extension of* ξ *to the subalgebra of* $\mathrm{End}\,(S_{k,\mathcal{O},\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}})$ *generated by* $h_{k,\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$ *and* $\langle h \rangle$ *for* $h \in H$, *then*
$$\xi(\rho_{\mathfrak{m}})|_{G_x} \sim \begin{pmatrix} * & * \\ 0 & \chi_x \end{pmatrix}$$
where $\chi_x(\mathrm{Art}\,\varpi_x) = \xi(\mathbf{U}_{\varpi_x})$ *and, for* $u \in \mathcal{O}_{F,x}^{\times}$, *we have* $\chi_x(\mathrm{Art}\,u) = \xi'(\langle u \rangle)$.

2. *If* $x(\mathfrak{n}) = 2$ *and* $H_x = \{1\}$ *then either* $\xi(\mathbf{U}_{\varpi_x}) = 0$ *or* $\xi(\mathbf{U}_{\varpi_x})$ *is an eigenvalue of* $\xi(\rho_{\mathfrak{m}})|_{G_x}(\sigma)$ *for any* $\sigma \in G_x$ *lifting* Frob_x.

We also get the following corollary.

Corollary 1.8 1. *If* $x \nmid l$, $x(\mathfrak{n}) = 1$ *and* $\mathbf{U}_{\varpi_x}^2 - (\mathrm{N}x)\psi(\varpi_x) \notin \mathfrak{m}$ *then*
$$\rho_{\mathfrak{m}}|_{G_x} \sim \begin{pmatrix} * & * \\ 0 & \chi_x \end{pmatrix}$$
where $\chi_x(\mathrm{Art}\,\varpi_x) = \mathbf{U}_{\varpi_x}$ *and* $\chi_x(\mathrm{Art}\,u) = \langle u \rangle$ *for* $u \in \mathcal{O}_{F,x}^{\times}$. *In particular* $\langle h \rangle \in h_{k,\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$ *for all* $h \in H_x$.

2. *If* $x \nmid l$, $x(\mathfrak{n}) = 2$, $H_x = \{1\}$ *and* $\mathbf{U}_{\varpi_x} \in \mathfrak{m}$ *then* $\mathbf{U}_{\varpi_x} = 0$ *in* $h_{k,\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$.

3. *If* l *is coprime to* $\mathfrak{n}$ *and for all* $x|\mathfrak{n}$ *we have* $x(\mathfrak{n}) = 2$, $H_x = \{1\}$ *and* $\mathbf{U}_{\varpi_x} \in \mathfrak{m}$, *then the algebra* $h_{k,\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$ *is reduced.*

Proof: The first part follows from the previous lemma via a Hensel's lemma argument. For the second part one observes that by the last lemma $\xi(\mathbf{U}_{\varpi_x}) = 0$ for all $\xi : h_{k,\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}} \to \mathbb{Q}_l^{ac}$. Hence by lemma 1.6 we have that $\mathbf{U}_{\varpi_x} = 0$ on $S_{k,\mathbb{Q}_l^{ac},\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$. The third part follows from the second (because the algebra $h_{k,\psi}(U_H(\mathfrak{n}))_{\mathfrak{m}}$ is generated by commuting semi-simple elements). □

2 Deformation rings and Hecke algebras I

In this section we extend the method of [TW] to totally real fields. This relies crucially on the improvement to the argument of [TW] found independently by Diamond [Dia] and Fujiwara (see [Fu], unpublished). Following this advance it has been clear to experts that some extension to totally real fields would be possible, the only question was the exact extent of the generalisation. Fujiwara has circulated some unpublished notes [Fu]. Then Skinner and Wiles made a rather complete analysis of the ordinary case (see [SW2]). We will treat the low weight, crystalline case. As will be clear to the reader, we have not tried to work in maximal generality, rather we treat the case of importance for this paper. We apologise for this. It would be very helpful to have these results documented in the greatest possible generality.
In this section and the next let F denote a totally real field of even degree in which a prime $l > 3$ splits completely. (As the reader will be able to check

without undue difficulty it would suffice to assume that l is unramified in F.) Let D denote the quaternion algebra with centre F which is ramified at exactly the infinite places, let $\mathcal{O}_D$ denote a maximal order in D and fix isomorphisms $\mathcal{O}_{D,x} \cong M_2(\mathcal{O}_{F,x})$ for all finite places x of F. Let $2 \leq k \leq l-1$. Let $\psi : \mathbb{A}_F^\times/F^\times \to (\mathbb{Q}_l^{ac})^\times$ be a continuous character such that

- if $x \not| l$ is a prime of F then $\psi|_{\mathcal{O}_{F,x}^\times} = 1$,
- $\psi|_{\mathcal{O}_{F,l}^\times}(u) = (\mathrm{N}u)^{2-k}$.

For each finite place x of F choose a uniformiser ϖ_x of $\mathcal{O}_{F,x}$. Suppose that $\phi : h_{k,\mathbb{F}_l^{ac},\psi}(U_0) \to \mathbb{F}_l^{ac}$ is a homomorphism with non-Eisenstein kernel, which we will denote $\mathfrak{m}$. Let $\mathcal{O}$ denote the ring of integers of a finite extension $K/\mathbb{Q}_l$ with maximal ideal λ such that

- K contains the image of every embedding $F \hookrightarrow \mathbb{Q}_l^{ac}$,
- ψ is valued in $\mathcal{O}^\times$,
- there is a homomorphism $\widetilde{\phi} : h_{k,\mathcal{O},\psi}(U_0)_\mathfrak{m} \to \mathcal{O}$ lifting ϕ, and
- all the eigenvalues of all elements of the image of $\overline{\rho}_\phi$ are rational over $\mathcal{O}/\lambda$.

For any finite set Σ of finite places of F not dividing l we will consider the functor $\mathcal{D}_\Sigma$ from complete noetherian local $\mathcal{O}$-algebras with residue firld $\mathcal{O}/\lambda$ to sets which sends R to the set of $1_2 + M_2(\mathfrak{m}_R)$-conjugacy classes of liftings $\rho : G_F \to GL_2(R)$ of $\overline{\rho}_\phi$ such that

- ρ is unramified outside l and Σ,
- $\det \rho = \epsilon(\psi \circ \mathrm{Art}^{-1})$, and
- for each place x of F above l and for each finite length (as an $\mathcal{O}$-module) quotient R/I of R the $\mathcal{O}[G_x]$-module $(R/I)^2$ is isomorphic to $\mathbb{M}(D)$ for some object D of $\mathcal{MF}_{F_x,\mathcal{O},k}$.

This functor is represented by a universal deformation

$$\rho_\Sigma : G_F \longrightarrow GL_2(R_\Sigma).$$

(This is now very standard, see for instance appendix A of [CDT].)
Now let Σ be a finite set of finite places of F not dividing l such that if $x \in \Sigma$ then

- $\mathrm{N}x \equiv 1 \bmod l$,
- $\overline{\rho}_\phi$ is unramified at x and $\overline{\rho}_\phi(\mathrm{Frob}_x)$ has distinct eigenvalues $\alpha_x \neq \beta_x$.

By Hensel's lemma the polynomial $X^2 - T_x X + (\mathrm{N}x)\psi(\varpi_x)$ splits as $(X - A_x)(X - B_x)$ in $h_{k,\mathcal{O},\psi}(U_0)_{\mathfrak{m}}$, where $A_x \bmod \mathfrak{m} = \alpha_x$ and $B_x \bmod \mathfrak{m} = \beta_x$. For $x \in \Sigma$ we will let Δ_x denote the maximal l-power quotient of $(\mathcal{O}_F/x)^\times$. We will let $\mathfrak{n}_\Sigma = \prod_{x\in\Sigma} x$; $\Delta_\Sigma = \prod_{x\in\Sigma}\Delta_x$; $U_{0,\Sigma} = U_{\{1\}}(\mathfrak{n}_\Sigma)$; and $U_{1,\Sigma} = U_{\Delta_\Sigma}(\mathfrak{n}_\Sigma)$. We will let $\mathfrak{m}_\Sigma$ denote the ideal of either $h_{k,\psi}(U_{0,\Sigma})$ or $h_{k,\psi}(U_{1,\Sigma})$ generated by

- l;
- $T_x - \operatorname{tr}\overline{\rho}_\phi(\mathrm{Frob}_x)$ for $x \not| l\mathfrak{n}_\Sigma$; and
- $\mathbf{U}_{\varpi_x} - \alpha_x$ for $x \in \Sigma$.

Lemma 2.1 *Let Σ satisfy the assumptions of the last paragraph.*

1. *If $x \in \Sigma$ then $\rho_\Sigma|_{G_x} \sim \chi_{\alpha,x} \oplus \chi_{\beta,x}$ where $\chi_{\alpha,x} \bmod \mathfrak{m}_{R_\Sigma}$ is unramified and takes Frob_x to α_x.*
2. *$\chi_{\alpha,x} \circ \mathrm{Art}\,|_{\mathcal{O}_{F,x}^\times}$ factors through Δ_x, and these maps make R_Σ into a $\mathcal{O}[\Delta_\Sigma]$-module.*
3. *The universal property of R_Σ gives rise to a surjection of $\mathcal{O}[\Delta_\Sigma]$-algebras*

$$R_\Sigma \twoheadrightarrow h_{k,\psi}(U_{1,\Sigma})_{\mathfrak{m}_\Sigma}$$

under which ρ_Σ pushes forward to $\rho_{\mathfrak{m}_\Sigma}$.

Proof: The first part is proved in exactly the same manner as lemma 2.44 of [DDT]. The second part is then clear. The third part is clear because for $x \not| \mathfrak{n}_\Sigma l$ we have $\operatorname{tr}\rho_\Sigma(\mathrm{Frob}_x) \mapsto T_x$ while for $x \in \Sigma$ we have $\chi_{\alpha,x}(\varpi_x) \mapsto \mathbf{U}_{\varpi_x}$. $\Box$

Lemma 2.2 *The map*

$$\begin{array}{rcl} \eta : S_{k,\mathcal{O},\psi}(U_{0,\Sigma-\{x\}})_{\mathfrak{m}_{\Sigma-\{x\}}} & \longrightarrow & S_{k,\mathcal{O},\psi}(U_{0,\Sigma})_{\mathfrak{m}_\Sigma} \\ f & \longmapsto & A_x f - \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} f \end{array}$$

is an isomorphism which induces an isomorphism

$$\eta^* : h_{k,\psi}(U_{0,\Sigma})_{\mathfrak{m}_\Sigma} \xrightarrow{\sim} h_{k,\psi}(U_{0,\Sigma-\{x\}})_{\mathfrak{m}_{\Sigma-\{x\}}}.$$

Proof: The map η is well defined because $\mathbf{U}_{\varpi_x} \circ \eta = \eta \circ A_x$. It is injective with torsion free cokernel because the composition of η with the adjoint of the natural inclusion $S_{k,\mathcal{O},\psi}(U_{0,\Sigma-\{x\}}) \hookrightarrow S_{k,\mathcal{O},\psi}(U_{0,\Sigma})$ is $(\mathrm{N}x)A_x - B_x \notin \mathfrak{m}_\Sigma$. As $\alpha_x/\beta_x \neq (\mathrm{N}x)^{\pm 1}$, no lift of $\overline{\rho}_\phi$ with determinant $\epsilon(\psi \circ \mathrm{Art}^{-1})$ has conductor at x exactly x. Thus

$$S_{k,\mathcal{O},\psi}(U_{0,\Sigma})_{\mathfrak{m}_\Sigma} = (S_{k,\mathcal{O},\psi}(U_{0,\Sigma-\{x\}}) + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} S_{k,\mathcal{O},\psi}(U_{0,\Sigma-\{x\}}))_{\mathfrak{m}_\Sigma}.$$

As

$$\mathbf{U}_{\varpi_x}(f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} f_2) = (T_x f_1 + (\mathrm{N}x)\psi(\varpi_x) f_2) - \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} f_1$$

and the matrix

$$\begin{pmatrix} T_x & (\mathrm{N}x)\psi(\varpi_x) \\ -1 & 0 \end{pmatrix}$$

has eigenvalues A_x and B_x which are distinct mod $\mathfrak{m}$, the lemma follows. □

We remark that $S_{k,\mathcal{O},\psi}(U_{1,\Sigma})$ is a Δ_Σ-module via $h \mapsto \langle h \rangle$.

Lemma 2.3 *1.* $\sum_{h\in\Delta_\Sigma} \langle h \rangle : S_{k,\mathcal{O},\psi}(U_{1,\Sigma})_{\Delta_\Sigma} \xrightarrow{\sim} S_{k,\mathcal{O},\psi}(U_{0,\Sigma})$.

2. $S_{k,\mathcal{O},\psi}(U_{1,\Sigma})$ *is a free* $\mathcal{O}[\Delta_\Sigma]$*-module.*

Proof: The second assertion follows from the first as we can compute that

$$\dim S_{k,\mathcal{O},\psi}(U_{1,\Sigma}) \otimes_{\mathcal{O}} K = [U_{0,\Sigma} : U_{1,\Sigma}] \dim S_{k,\mathcal{O},\psi}(U_{0,\Sigma}) \otimes_{\mathcal{O}} K.$$

(We use the fact that $[U_{0,\Sigma} : U_{1,\Sigma}]$ is coprime to $\#(U_{0,\Sigma}(\mathbb{A}_F^\infty)^\times \cap x^{-1} D^\times x)/F^\times$ for all $x \in (D \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$.)

Using the duality introduced above it suffices to check that the natural map

$$S_{k,\mathcal{O},\psi}(U_{0,\Sigma}) \otimes_{\mathcal{O}} K/\mathcal{O} \longrightarrow (S_{k,\mathcal{O},\psi}(U_{1,\Sigma}) \otimes_{\mathcal{O}} K/\mathcal{O})^{\Delta_\Sigma}$$

is an isomorphism. This is immediate from the definitions and the fact that $l \not| \#(U_{0,\Sigma}(\mathbb{A}_F^\infty)^\times \cap x^{-1} D^\times x)/F^\times$ for all $x \in (D \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$. □

As $S_{k,\mathcal{O},\psi}(U_{1,\Sigma})_{\mathfrak{m}_\Sigma}$ is a direct summand of $S_{k,\mathcal{O},\psi}(U_{1,\Sigma})$, we deduce the following corollary.

Corollary 2.4 *1.* $S_{k,\mathcal{O},\psi}(U_{1,\Sigma})_{\mathfrak{m}_\Sigma,\Delta_\Sigma} \xrightarrow{\sim} S_{k,\mathcal{O},\psi}(U_0)_{\mathfrak{m}}$ *compatibly with a map* $h_{k,\psi}(U_{1,\Sigma})_{\mathfrak{m}_\Sigma} \to h_{k,\psi}(U_0)_{\mathfrak{m}}$ *sending* T_x *to* T_x *for* $x \not| l\mathfrak{n}_\Sigma$, $\langle h \rangle$ *to* 1 *for* $h \in \Delta_\Sigma$ *and* $\mathbf{U}_{\varpi_x}$ *to* A_x *for* $x \in \Sigma$.

2. $S_{k,\mathcal{O},\psi}(U_{1,\Sigma})_{\mathfrak{m}_\Sigma}$ *is a free* $\mathcal{O}[\Delta_\Sigma]$*-module.*

Suppose that $\rho : G_F \to GL_2(\mathcal{O}/\lambda^n)$ is a lifting of $\overline{\rho}_\phi$ corresponding to some map $R_\emptyset \to \mathcal{O}/\lambda^n$. If x is a place of F above l and if $(\mathcal{O}/\lambda^n)^2 \cong \mathbb{M}(D)$ as a G_x-module, then we set

$$H^1_f(G_x, \mathrm{ad}^0 \rho) = H^1(G_x, \mathrm{ad}^0 \rho) \cap \mathrm{Im}\,(\mathrm{Ext}^1_{\mathcal{MF}_{F_x,\mathcal{O}/\lambda^n,k}}(D,D) \longrightarrow H^1(G_x, \mathrm{ad}\,\rho)).$$

Exactly as in section 2.5 of [DDT] we see that

$$\mathrm{Im}\,(\mathrm{Ext}^1_{\mathcal{MF}_{F_x,\mathcal{O}/\lambda^n,k}}(D,D) \longrightarrow H^1(G_x, \mathrm{ad}\,\rho)) \cong (\mathcal{O}/\lambda^n)^2 \oplus H^0(G_x, \mathrm{ad}^0 \rho).$$

If two continuous $\mathcal{O}[G_x]$-modules have the same restriction to I_x, then one is in the image of $\mathbb{M}$ if and only if the other is. We conclude that the image of the composite

$$\mathrm{Ext}^1_{\mathcal{MF}_{\mathcal{O}/\lambda^n,k}}(D,D) \longrightarrow H^1(G_x, \mathrm{ad}\,\rho) \xrightarrow{\mathrm{tr}} H^1(G_x, \mathcal{O}/\lambda^n)$$

is at least one dimensional (coming from unramified twists) and hence that

$$\#H^1_f(G_x, \mathrm{ad}^0\overline{\rho}_\phi) | \#(\mathcal{O}/\lambda^n)\#H^0(G_x, \mathrm{ad}^0\overline{\rho}_\phi).$$

We will let $H^1_\Sigma(G_F, \mathrm{ad}^0\rho)$ denote the kernel of the map

$$H^1(G_F, \mathrm{ad}^0\rho) \longrightarrow \bigoplus_{x \not| \mathfrak{n}_\Sigma l} H^1(I_x, \mathrm{ad}^0\rho) \oplus \bigoplus_{x|l} H^1(G_x, \mathrm{ad}^0\rho)/H^1_f(G_x, \mathrm{ad}^0\rho).$$

The trace pairing $(a, b) \mapsto \mathrm{tr}\, ab$ gives a perfect duality on $\mathrm{ad}^0\overline{\rho}_\phi$. For $x|l$ we will let $H^1_f(G_x, \mathrm{ad}^0\overline{\rho}_\phi(1))$ denote the annihilator in $H^1(G_x, \mathrm{ad}^0\overline{\rho}_\phi(1))$ of $H^1_f(G_x, \mathrm{ad}^0\overline{\rho}_\phi)$ under Tate local duality. We will also let $H^1_\Sigma(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1))$ denote the kernel of the restriction map from $H^1(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1))$ to

$$\bigoplus\nolimits_{x \not| \mathfrak{n}_\Sigma l} H^1(I_x, \mathrm{ad}^0\overline{\rho}_\phi(1)) \oplus \bigoplus\nolimits_{x\in\Sigma} H^1(G_x, \mathrm{ad}^0\overline{\rho}_\phi(1)) \oplus \\ \bigoplus\nolimits_{x|l} H^1(G_x, \mathrm{ad}^0\overline{\rho}_\phi)/H^1_f(G_x, \mathrm{ad}^0\overline{\rho}_\phi(1)))$$

so that

$$H^1_\Sigma(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1)) = \ker(H^1_\emptyset(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1)) \longrightarrow \bigoplus_{x\in\Sigma} H^1(G_x/I_x, \mathrm{ad}^0\overline{\rho}_\phi(1))).$$

A standard calculation (see for instance section 2.7 of [DDT]) shows that

$$H^1_\Sigma(G_F, \mathrm{ad}^0\overline{\rho}_\phi) \cong \mathrm{Hom}_{\mathcal{O}}(\mathfrak{m}_{R_\Sigma}/\mathfrak{m}^2_{R_\Sigma}, \mathcal{O}/\lambda),$$

so that R_Σ can be topologically generated by $\dim H^1_\Sigma(G_F, \mathrm{ad}^0\overline{\rho}_\phi)$ elements as an $\mathcal{O}$-algebra. A formula of Wiles (see theorem 2.19 of [DDT]) then tells us that R_Σ can be topologically generated as an $\mathcal{O}$-algebra by

$$\#\Sigma + \dim H^1_\Sigma(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1))$$

elements.

Lemma 2.5 *Suppose that the restriction of $\overline{\rho}_\phi$ to $F(\sqrt{(-1)^{(l-1)/2}l})$ is irreducible. Then for any $m \in \mathbb{Z}_{>0}$ we can find a set Σ_m of primes such that*

1. $\#\Sigma_m = \dim H^1_\emptyset(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1))$,
2. R_{Σ_m} *can be topologically generated by* $\dim H^1_\emptyset(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1))$ *elements as an $\mathcal{O}$-algebra,*
3. *if $x \in \Sigma_m$ then $\mathrm{N}x \equiv 1 \bmod l^m$ and $\overline{\rho}_\phi(\mathrm{Frob}_x)$ has distinct eigenvalues α_x and β_x.*

Proof: By the above calculation we may replace the second requirement by the requirement that $H^1_{\Sigma_m}(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1)) = (0)$ (for then R_{Σ_m} is generated by $\#\Sigma_m = \dim H^1_\emptyset(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1))$ elements). Then we may suppress the first requirement, because any set satisfying the modified second requirement and the third requirement can be shrunk to one which also satisfies the first requirement. (Note that for x satisfying the third requirement $H^1(G_x/I_x, \mathrm{ad}^0\overline{\rho}_\phi(1))$ is one dimensional.) Next, by the Cebotarev density theorem, it suffices to show that for $[\gamma] \in H^1_\emptyset(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1))$ we can find $\sigma \in G_F$ such that

- $\sigma|_{F(\zeta_{l^m})} = 1$,
- $\overline{\rho}_\phi(\sigma)$ has distinct eigenvalues, and
- $\gamma(\sigma) \notin (\sigma - 1)\mathrm{ad}^0\overline{\rho}_\phi$.

Let F_m denote the extension of $F(\zeta_{l^m})$ cut out by $\mathrm{ad}^0\overline{\rho}$. Finally it will suffice to show that

1. $H^1(\mathrm{Gal}\,(F_m/F), \mathrm{ad}^0\overline{\rho}(1)) = (0)$; and
2. for any non-trivial irreducible $\mathrm{Gal}\,(F_m/F)$-submodule V of $\mathrm{ad}^0\overline{\rho}_\phi$ we can find $\sigma \in \mathrm{Gal}\,(F_m/F(\zeta_{l^m}))$ such that $\mathrm{ad}^0\overline{\rho}_\phi(\sigma)$ has an eigenvalue other than 1 but σ does have an eigenvalue 1 on V.

(Given $[\gamma] \in H^1_\emptyset(G_F, \mathrm{ad}^0\overline{\rho}_\phi(1))$ the first assertion tells us that the $\mathcal{O}/\lambda$-span of γG_{F_m} contains some non-trivial irreducible $\mathrm{Gal}\,(F_m/F)$-submodule V of $\mathrm{ad}^0\overline{\rho}_\phi$. Let σ be as in the second assertion for this V. Then for some $\sigma' \in G_{F_m}$ we will have

$$\gamma(\sigma'\sigma) = \gamma(\sigma') + \gamma(\sigma) \notin (\sigma - 1)\mathrm{ad}^0\overline{\rho}_\phi.)$$

Because $l > 3$ is unramified in F, we see that $[F(\zeta_l) : F] > 2$ and so, by the argument of the penultimate paragraph of the proof of theorem 2.49 of [DDT], $H^1(\mathrm{Gal}\,(F_m/F), \mathrm{ad}^0\overline{\rho}(1)) = (0)$.

Suppose that V is an irreducible $\mathrm{Gal}\,(F_m/F)$-submodule of $\mathrm{ad}^0\overline{\rho}_\phi$ and write $\mathrm{ad}^0\overline{\rho}_\phi = V \oplus W$. If $W = (0)$ any $\sigma \in \mathrm{Gal}\,(F_m/F(\zeta_{l^m}))$ with an eigenvalue other than 1 on $\mathrm{ad}^0\overline{\rho}_\phi$ will suffice to prove the second assertion. Thus suppose that $W \neq (0)$. If $\dim W = 1$ then $\overline{\rho}_\phi$ is induced from a character of some quadratic extension E/F and any $\sigma \notin G_E$ will suffice to prove the second assertion (as E is not a subfield of $F(\zeta_{l^m})$). If $\dim W = 2$ then G_F acts on V via a quadratic character corresponding to some quadratic extension E/F and $\overline{\rho}_\phi$ is induced from some character χ of G_E. Let χ' denote the $\mathrm{Gal}\,(E/F)$-conjugate of χ. Then any $\sigma \in G_{E(\zeta_{l^m})}$ with $\chi/\chi'(\sigma) \neq 1$ will suffice to prove the second assertion. (Such a σ will exist unless the restriction of $\mathrm{ad}^0\overline{\rho}_\phi$ to $E(\sqrt{(-1)^{(l-1)/2}l})$ is trivial in which case $\overline{\rho}_\phi$ becomes reducible over $F(\sqrt{(-1)^{(l-1)/2}l})$, which we are assuming is not the case.) $\Box$

Combining lemma 2.5, corollary 2.4 and theorem 2.1 of [Dia] we obtain the following theorem.

Theorem 2.6 *Keep the notation and assumptions of the second and fourth paragraphs of this section and suppose that the restriction of $\overline{\rho}_\phi$ to the absolute Galois group of $F(\sqrt{(-1)^{(l-1)/2}l})$ is irreducible. Then the natural map*

$$R_\emptyset \longrightarrow h_{k,\psi}(U_0)_{\mathfrak{m}}$$

is an isomorphism of complete intersections and $S_{k,\mathcal{O},\psi}(U_0)_{\mathfrak{m}}$ is finite free as a $h_{k,\psi}(U_0)_{\mathfrak{m}}$-module.

3 Deformation rings and Hecke algebras II

In this section we use analogues of Wiles' arguments from [W2] to extend the isomorphism of theorem 2.6 from $\emptyset$ to any Σ.
We will keep the notation and assumptions of the last section. (Σ will again be *any* finite set of finite places of F not dividing l.) Let $\rho_{\widetilde{\phi}} : G_F \to GL_2(\mathcal{O})$ denote the Galois representation corresponding to $\widetilde{\phi}$ (a chosen lift of ϕ). The universal property of R_Σ gives maps

$$R_\Sigma \twoheadrightarrow R_\emptyset \xrightarrow{\widetilde{\phi}} \mathcal{O}.$$

We will denote the kernel by $\wp_\Sigma$. A standard calculation (see section 2.7 of [DDT]) shows that

$$\mathrm{Hom}_{\mathcal{O}}(\wp_\Sigma/\wp_\Sigma^2, K/\mathcal{O}) \cong H^1_\Sigma(G_F, (\mathrm{ad}^0\rho)\otimes K/\mathcal{O}),$$

where

$$H^1_\Sigma(G_F, (\mathrm{ad}^0\rho)\otimes K/\mathcal{O}) = \varinjlim_n H^1_\Sigma(G_F, (\mathrm{ad}^0\rho)\otimes \lambda^{-n}/\mathcal{O}).$$

In particular we see that

$$\#\ker(\wp_\Sigma/\wp_\Sigma^2 \twoheadrightarrow \wp_\emptyset/\wp_\emptyset^2) = \#(H^1_\Sigma(G_F,(\mathrm{ad}^0\rho)\otimes K/\mathcal{O})/H^1_\emptyset(G_F,(\mathrm{ad}^0\rho)\otimes K/\mathcal{O}))$$

divides

$$\begin{array}{ll} & \prod_{x\in\Sigma} \#H^1(I_x, (\mathrm{ad}^0\rho)\otimes K/\mathcal{O})^{G_x} \\ = & \prod_{x\in\Sigma} \#H^0(G_x, (\mathrm{ad}^0\rho)\otimes K/\mathcal{O}(-1)) \\ = & \prod_{x\in\Sigma} \#\mathcal{O}/(1-\mathrm{N}x)((1+\mathrm{N}x)^2\det\rho(\mathrm{Frob}_x) - (\mathrm{N}x)(\mathrm{tr}\,\rho\mathrm{Frob}_x))\mathcal{O}. \end{array}$$

Let $\mathfrak{n}'_\Sigma$ denote the product of the squares of the primes in Σ and set $U_\Sigma = U_{\{1\}}(\mathfrak{n}'_\Sigma)$. Let $h_\Sigma = h_{k,\psi}(U_\Sigma)_{\mathfrak{m}'_\Sigma}$ and $S_\Sigma = S_{k,\mathcal{O},\psi}(U_\Sigma)_{\mathfrak{m}'_\Sigma}$, where $\mathfrak{m}'_\Sigma$ is the maximal ideal of $h_{k,\psi}(U_\Sigma)$ generated by

- λ,
- $T_x - \mathrm{tr}\,\overline{\rho}_\phi(\mathrm{Frob}_x)$ for $x \nmid l\mathfrak{n}'_\Sigma$, and

- $\mathbf{U}_{\varpi_x}$ for $x \in \Sigma$.

The Galois representation $\rho_{\mathfrak{m}'_\Sigma}$ induces a homomorphism $R_\Sigma \to h_\Sigma$ which takes $\operatorname{tr}\rho_\Sigma(\mathrm{Frob}_x)$ to T_x for all $x \not| \mathfrak{n}'_\Sigma l$. Corollary 1.8 tells us that for $x \in \Sigma$ we have $\mathbf{U}_{\varpi_x} = 0$ in h_Σ and that h_Σ is reduced. In particular the map $R_\Sigma \to h_\Sigma$ is surjective.
From lemma 1.3, lemma 1.6 and the strong multiplicity one theorem for $GL_2(\mathbb{A}_F)$ we see that $\dim(S_\Sigma \otimes_{\mathcal{O}} K)[\wp_\Sigma] = 1$.
We can write

$$S_\Sigma \otimes_{\mathcal{O}} K = (S_\Sigma \otimes_{\mathcal{O}} K)[\wp_\Sigma] \oplus (S_\Sigma \otimes_{\mathcal{O}} K)[\mathrm{Ann}_{h_\Sigma}(\wp_\Sigma h_\Sigma)].$$

We set

$$\Omega_\Sigma = S_\Sigma/(S_\Sigma[\wp_\Sigma] \oplus S_\Sigma[\mathrm{Ann}_{h_\Sigma}(\wp_\Sigma h_\Sigma)]).$$

By theorem 2.4 of [Dia] and theorem 2.6 above, we see that

$$\#\Omega_\emptyset = \#\wp_\emptyset/\wp_\emptyset^2.$$

Let $w_\Sigma \in GL_2(\mathbb{A}_F^\infty)$ be defined by $w_{\sigma,x} = 1_2$ if $x \notin \Sigma$ and

$$w_{\Sigma,x} = \begin{pmatrix} 0 & 1 \\ \varpi_x^2 & 0 \end{pmatrix}$$

if $x \in \Sigma$. Then w_Σ normalises U_Σ. We define a new pairing on $S_{k,\mathcal{O},\psi}(U_\Sigma)$ by

$$(f_1, f_2)' = (\prod_{x\in\Sigma} \psi(\varpi_x))^{-1}(f_1, w_\Sigma f_2).$$

Because $(\ ,\ )$ is a perfect pairing so is $(\ ,\ )'$. Moreover the action of any element of $h_{k,\psi}(U_\Sigma)$ is self adjoint with respect to $(\ ,\ )'$, so that $(\ ,\ )'$ restricts to a perfect pairing on S_Σ. Choose a perfect $\mathcal{O}$-bilinear pairing on $S_\Sigma[\wp_\Sigma]$, let j_Σ denote the natural inclusion

$$j_\Sigma : S_\Sigma[\wp_\Sigma] \hookrightarrow S_\Sigma,$$

and let $j_\Sigma^\dagger$ denote the adjoint of j_Σ with respect to $(\ ,\ )'$ on S_Σ and the chosen pairing on $S_\Sigma[\wp_\Sigma]$. Then one sees that

$$j_\Sigma^\dagger : \Omega_\Sigma \xrightarrow{\sim} S_\Sigma[\wp_\Sigma]/j_\Sigma^\dagger S_\Sigma[\wp_\Sigma].$$

If $x \not| l\mathfrak{n}'_\Sigma$ then define

$$i_x : S_{k,\mathcal{O},\psi}(U_\Sigma) \longrightarrow S_{k,\mathcal{O},\psi}(U_{\Sigma\cup\{x\}})$$

by

$$i_x(f) = (\mathrm{N}x)\psi(\varpi_x)f - \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} T_x f + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x^2 \end{pmatrix} f.$$

It is easy to check that i_x commutes with T_y for $y \not| l\mathfrak{n}'_{\Sigma\cup\{x\}}$ and with $\mathbf{U}_{\varpi_y}$ for $y \in \Sigma$. Moreover $\mathbf{U}_{\varpi_x} i_x = 0$ and so

$$i_x : S_\Sigma \longrightarrow S_{\Sigma\cup\{x\}}.$$

Moreover $i_x S_\Sigma[\wp_\Sigma] \subset S_{\Sigma\cup\{x\}}[\wp_{\Sigma\cup\{x\}}]$. We will let $i_x^\dagger$ denote the adjoint of i_x with respect to the pairings $(\ ,\ )'$ on S_Σ and $S_{\Sigma\cup\{x\}}$. (We warn the reader that the former is not simply the restriction of the latter.) An easy calculation shows that $i_x^\dagger$ equals

$$\psi(\varpi_x)(\mathrm{N}x)[U_\Sigma U_{\Sigma\cup\{x\}}] - T_x[U_\Sigma \begin{pmatrix} \varpi_x & 0 \\ 0 & 1 \end{pmatrix} U_{\Sigma\cup\{x\}}] + [U_\Sigma \begin{pmatrix} \varpi_x^2 & 0 \\ 0 & 1 \end{pmatrix} U_{\Sigma\cup\{x\}}]$$

and hence that

$$i_x^\dagger \circ i_x = \psi(\varpi_x)(\mathrm{N}x)(1-\mathrm{N}x)(T_x^2 - (1+\mathrm{N}x)^2\psi(\varpi_x)).$$

The following key lemma is often referred to as Ihara's lemma.

Lemma 3.1 *$S_{\Sigma\cup\{x\}}/i_x S_\Sigma$ is l-torsion free.*

Proof: It suffices to check that

$$i_x : S_{k,\mathcal{O}/\lambda,\psi}(U_\Sigma)_{\mathfrak{m}'_\Sigma} \longrightarrow S_{k,\mathcal{O}/\lambda,\psi}(U_{\Sigma\cup\{x\}})_{\mathfrak{m}'_{\Sigma\cup\{x\}}}$$

is injective, or even that the localisation at $\mathfrak{m}'_\Sigma$ of the kernel of

$$\begin{array}{rcl} S_{k,\mathcal{O}/\lambda,\psi}(U_\Sigma)^3 & \longrightarrow & S_{k,\mathcal{O}/\lambda,\psi}(U_{\Sigma\cup\{x\}}) \\ (f_1,f_2,f_3) & \longmapsto & f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} f_2 + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x^2 \end{pmatrix} f_3 \end{array}$$

vanishes.

Let V denote the subgroup of elements $u \in U_\Sigma$ with

$$u_x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \varpi_x.$$

We see that

$$V \cap \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} V \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}^{-1} = U_{\Sigma\cup\{x\}}$$

and that U_Σ is the subgroup of $GL_2(\mathbb{A}_F^\infty)$ generated by V and

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}^{-1} V \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}.$$

Thus the sequence

$$\begin{array}{cccccc} (0) & \to & S_{k,\mathcal{O}/\lambda,\psi}(U_\Sigma) & \to & S_{k,\mathcal{O}/\lambda,\psi}(V) \oplus S_{k,\mathcal{O}/\lambda,\psi}(V) & \to S_{k,\mathcal{O}/\lambda,\psi}(U_{\Sigma\cup\{x\}}) \\ & & f & \mapsto & (\begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} f, -f) & \\ & & & & (f_1,f_2) & \mapsto f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} f_2 \end{array}$$

is exact.
Hence it suffices to show that the localisation at $\mathfrak{m}'_\Sigma$ of the kernel of

$$\begin{array}{rcl} S_{k,\mathcal{O}/\lambda,\psi}(U_\Sigma)^2 & \longrightarrow & S_{k,\mathcal{O}/\lambda,\psi}(V) \\ (f_1,f_2) & \longmapsto & f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} f_2 \end{array}$$

vanishes. However if (f_1,f_2) is in the kernel then f_1 is invariant by the subgroup of $GL_2(\mathbb{A}_F^\infty)$ generated by U_Σ and

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} U_\Sigma \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}^{-1},$$

i.e. by $U_\Sigma SL_2(F_x)$.
First suppose that $k=2$. Then, by the strong approximation theorem, we see that f_1 is invariant by right translation by any element of $SL_2(\mathbb{A}_F^\infty)$, so that $f_1 \in S^{\mathrm{triv}}_{k,\mathcal{O}/\lambda,\psi}(U_\sigma)$. Any maximal ideal of $h_{2,\psi}(U_\Sigma)$ in the support of $S^{\mathrm{triv}}_{k,\mathcal{O}/\lambda,\psi}(U_\sigma)$ is Eisenstein.
Now suppose that $3 \leq k \leq l-1$. By the strong approximation theorem, given any $g \in GL_2(\mathbb{A}_F^\infty)$ and any $u \in GL_2(\mathcal{O}_{F,l})$, we can find a $\delta \in D^\times \cap gU_\Sigma SL_2(F_x)g^{-1}$ such that

$$g_l^{-1}\delta g_l \equiv u \bmod l.$$

Then

$$f_1(g) = f_1(\delta g) = f_1(g(g^{-1}\delta g)) = f_1(gu) = u^{-1}f_1(g),$$

so that

$$f_1(g) \in (\bigotimes_{\mathcal{O}_{F,l}\to\mathcal{O}/\lambda} \mathrm{Symm}^{\,k-2}((\mathcal{O}/\lambda)^2))^{GL_2(\mathcal{O}_{F,l})} = (0).$$

Thus $f_1 = 0$. $\Box$

In particular we see that $i_x S_\Sigma[\wp_\Sigma] = S_{\Sigma\cup\{x\}}[\wp_{\Sigma\cup\{x\}}]$. Thus

$$\begin{array}{rcl} \Omega_{\Sigma\cup\{x\}} & \cong & S_\Sigma[\wp_\Sigma]/j_\Sigma^\dagger i_x^\dagger S_\Sigma[\wp_\Sigma] \\ & \cong & S_\Sigma[\wp_\Sigma]/j_\Sigma^\dagger(1-\mathrm{N}x)(\mathrm{N}x)(T_x^2-(1+\mathrm{N}x)^2\psi(\varpi_x))S_\Sigma[\wp_\Sigma], \end{array}$$

and so

$$\#\Omega_{\Sigma\cup\{x\}} = \#\Omega_\Sigma \# \left(\mathcal{O}/(1-\mathrm{N}x)((\mathrm{N}x)\mathrm{tr}\,\rho(\mathrm{Frob}_x)^2-(1+\mathrm{N}x)^2\det\rho(\mathrm{Frob}_x))\right).$$

We conclude that

$$\#(\wp_\Sigma/\wp_\Sigma^2)|\#\Omega_\Sigma$$

for all Σ (which contains no prime above l). Combining this with theorem 2.4 of [Dia] we see obtain the following theorem.

Theorem 3.2 *Keep the notation and assumptions of the second and fourth paragraphs of section 2 and suppose that the restriction of $\overline{\rho}_\phi$ to the absolute Galois group of $F(\sqrt{(-1)^{(l-1)/2}l})$ is irreducible. If Σ is a finite set of finite places of F not dividing l then the natural map*

$$R_\Sigma \longrightarrow h_\Sigma$$

is an isomorphism of complete intersections and S_Σ is a free h_Σ-module.

As an immediate consequence we have the following theorem.

Theorem 3.3 *Let $l > 3$ be a prime and let $2 \leq k \leq l-1$ be an integer. Let F be a totally real field of even degree in which l splits completely. Let $\rho : G_F \to GL_2(\mathcal{O}_{\mathbb{Q}_l^{ac}})$ be a continuous irreducible representation unramified outside finitely many primes and such that for each place x of F above l the restriction $\rho|_{G_x}$ is crystalline with Hodge-Tate numbers 0 and $1-k$. Let $\overline{\rho}$ denote the reduction of ρ modulo the maximal ideal of $\mathcal{O}_{\mathbb{Q}_l^{ac}}$. Assume that the restriction of $\overline{\rho}$ to $F(\sqrt{(-1)^{(l-1)/2}l})$ is irreducible and that there is a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ and an embedding $\lambda : M_\pi \hookrightarrow \mathbb{Q}_l^{ac}$ such that*

- $\overline{\rho}_{\pi,\lambda} \sim \overline{\rho}$,
- *π_x is unramified for every finite place x of F, and*
- *π_∞ has weight k.*

Then there is a regular algebraic cuspidal automorphic representation π' of $GL_2(\mathbb{A}_F)$ and an embedding $\lambda' : M_{\pi'} \to \mathbb{Q}_l^{ac}$ such that $\rho \sim \rho_{\pi',\lambda'}$ and π'_∞ has weight k.

Proof: We need only remark that $\det \rho / \det \rho_{\pi,\lambda}$ has finite l-power order and so by twisting π we may suppose that $\det \rho = \det \rho_{\pi,\lambda}$ (as $l > 2$). $\square$

4 A potential version of Serre's conjecture

In this section we will prove the following result, which we will improve somewhat in section 5.

Proposition 4.1 *Let $l > 2$ be a prime. Suppose that $\overline{\rho} : G_\mathbb{Q} \to GL_2(\mathbb{F}_l^{ac})$ is a continuous odd representation with $\overline{\rho}|_{I_l} \sim \omega_2^{k-1} \oplus \omega_2^{l(k-1)}$ for some integer $2 \leq k \leq l$. (In particular $\overline{\rho}|_{G_l}$ is absolutely irreducible.) Then there is a Galois totally real field F of even degree in which l splits completely, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ and an embedding $\lambda : M_\pi \hookrightarrow \mathbb{Q}_l^{ac}$ such that*

1. $\overline{\rho}|_{G_F} \sim \overline{\rho}_{\pi,\lambda}$;

2. π_∞ *has weight* 2*; and*

3. *for each place* x *of* F *above* l*,* $\mathrm{WD}_\lambda(\pi_x)$ *is tamely ramified and*

$$\mathrm{WD}_\lambda(\pi_x)|_{I_x} = \omega_2^{k-(l+1)} \oplus \omega_2^{lk-(l+1)}.$$

We remark that the key improvement of this over results in [Tay4] is the condition that l split completely in F. This may seem minor but it will be crucial for the arguments in section 5 and the proof of theorem 5.7. We now turn to the proof of the proposition.

Suppose that $\overline{\rho}$ is valued in $GL_2(k')$ for some finite field $k' \subset \mathbb{F}_l^{ac}$ and let k denote the unique quadratic extension of k' in $\mathbb{F}_l^{ac}$. We must have that $\overline{\rho}|_{G_l} = \mathrm{Ind}_{\mathbb{Q}_{l^2}}^{\mathbb{Q}_l}\theta$, where $\theta|_{I_l} = \omega_2^{k-1}$ with $2 \le k \le l$ and so θ is not equal to its $\mathrm{Gal}\,(\mathbb{Q}_{l^2}/\mathbb{Q}_l)$-conjugate. Set $\overline{\mu} = \epsilon^{-1}\det\overline{\rho}$, let N denote the minimum splitting field for $\overline{\mu}$ and $\mathfrak{f}_\mu$ its conductor. Thus N is a cyclic totally real extension of $\mathbb{Q}$. Choose an imaginary quadratic field M in which l remains prime and which contains only two roots of unity. Let δ_M denote the unique non-trivial character of $\mathbb{A}^\times/\mathbb{Q}^\times \mathrm{N}\mathbb{A}_M^\times$ and let $\mathfrak{f}_M$ denote the conductor of δ_M. Choose a Galois totally real field E'' such that $E''M$ contains a primitive root of unity ζ of order $2\#k^\times$. Note that the degree over $\mathbb{F}_l$ of every residue field of a prime of E'' above l is even.

Choose a continuous character $\chi_0 : M^\times(M_\infty^\times \times \prod_q \mathcal{O}_{M,q}^\times) \to M^\times$ which extends the canonical inclusion on $M^\times$ (use the fact that M has a prime $x \nmid 2$ with $-1 \notin (k(x)^\times)^2$) and let $\mathfrak{f}_0$ denote the conductor of χ_0. Also choose two distinct odd primes p_1 and p_2 such that for both $i = 1, 2$

- χ_0 is unramified above p_i;
- $p_i \neq l$;
- $\overline{\rho}$ is unramified at p_i;
- $\overline{\rho}(\mathrm{Frob}_{p_i})$ has distinct eigenvalues; and
- p_i splits in the Hilbert class field of M.

(We explain why this is possible. Let M' denote the extension of M cut out by $\overline{M}^{\ker\overline{\rho}}$ and by the Hilbert class field H of M. By the Cebotarev density theorem it suffices to find $\sigma \in \mathrm{Gal}\,(M'/H)$ so that $\overline{\rho}(\sigma)$ has distinct eigenvalues. A fortiori it suffices to find $\sigma \in I_l$ so that $\overline{\rho}(\sigma)$ has distinct eigenvalues. This is possible because $(l+1) \nmid k-1$.) Set $w = 2w_{E''M}\#(\mathcal{O}_M/l\mathfrak{f}_M\mathfrak{f}_\mu\mathfrak{f}_0\mathfrak{f}_0^c\mathcal{O}_M)^\times$, where $w_{E''M}$ denotes the number of roots of unity in $E''M$. Let S_1 denote set of rational primes dividing $\mathfrak{f}_M\mathfrak{f}_\mu\mathfrak{f}_0\mathfrak{f}_0^c$, let S_2 be a finite set of rational primes disjoint from S_1 which split in M and such that the primes of M above S_2 generate the class group of M, and set $S_0 = S_1 \cup S_2 \cup \{l, p_1, p_2\}$. As in the proof of lemma 1.1 of [Tay4] we can find an open subgroup W_0 of $\prod_{q \notin S_0} \mathcal{O}_{M,q}^\times/\mathbb{Z}_q^\times$ such that $W_0 \cap M_{S_0}^\times/\mathbb{Q}_{S_0}^\times \subset (M_{S_0}^\times/\mathbb{Q}_{S_0}^\times)^w$. Let w' denote the index of W_0 in

$\prod_{q\not\in S_0}\mathcal{O}_{M,q}^\times/\mathbb{Z}_q^\times$. Then we can choose a Galois (over $\mathbb{Q}$) totally real field E' such that

- $E'\supset E''$;
- E' contains a primitive root of 1 of order ww'; and
- χ_0 extends to a continuous character $\chi_0:\mathbb{A}_M^\times\to(E'M)^\times$.

(If $\widetilde{\chi}_0:\mathbb{A}_M^\times\to\mathbb{C}^\times$ is any extension of χ_0 then $\widetilde{\chi}_0 c(\widetilde{\chi}_0)\prod_{x\not|\infty}|\;|_x$ has finite order and is valued in $\mathbb{R}_{>0}^\times$ and so is identically 1. Hence $c(\widetilde{\chi}_0)=\widetilde{\chi}_0^{-1}\prod_{x\not|\infty}|\;|_x$ and $\widetilde{\chi}_0$ is valued in a CM field.)
Let E denote the maximal totally real extension of E' which is unramified outside lp_1p_2 and tamely ramified at these primes. Choose primes $\wp_1$ and $\wp_2$ of EM above p_1 and p_2 respectively. Also choose a prime λ of EM above l and an embedding $k\hookrightarrow\mathcal{O}_{EM}/\lambda$ such that the composite of the Artin map $I_l\to\mathcal{O}_{M,l}^\times$ with the natural map $\mathcal{O}_{M,l}^\times\to(\mathcal{O}_{EM}/\lambda)^\times$ coincides with $\omega_2^{-1}:I_l\to k^\times\subset(\mathcal{O}_{EM}/\lambda)^\times$. Let $\mu:\mathrm{Gal}\,(N/\mathbb{Q})\to(EM)^\times$ be the unique character reducing modulo λ to $\overline{\mu}$. For $i=1,2$ we can find $\alpha_i\in(\wp_i\cap M)\mathcal{O}_{E''M}$ which reduces modulo λ to an eigenvalue of $\overline{\rho}(\mathrm{Frob}_{p_i})$ and which satisfies $\alpha_i\alpha_i^c=p_i$. (First choose $\alpha_i'\in M\cap\wp_i$ satisfying $\alpha_i'(\alpha_i')^c=p_i$ and then multiply α_i' by a suitable root of unity in $E''M$.)

Lemma 4.2 *Let $\mathfrak{a}'$ denote the product of all primes of E above lp_1p_2 and factor $\mathfrak{a}'\mathcal{O}_{ME}=\mathfrak{a}\mathfrak{a}^c$, where $\wp_1\wp_2\lambda|\mathfrak{a}$ (which is possible as p_1 and p_2 split in M and as the degree over $\mathbb{F}_l$ of the residue field of every prime of E above l is even). There is a unit $\eta\in\mathcal{O}_E^\times$ with $\eta\equiv\zeta\bmod\mathfrak{a}$.*

Proof: Let $\overline{\zeta}$ denote the image of ζ in $\mathcal{O}_{E'}/(\mathfrak{a}'\cap\mathcal{O}_{E'})=\mathcal{O}_{E'M}/(\mathfrak{a}\cap\mathcal{O}_{E'M})$. Let H denote the maximal totally real abelian extension of E which is unramified outside lp_1p_2 and which is tamely ramified above each of these three primes. Thus H/E' is Galois and $\mathrm{Gal}\,(H/E)$ is the commutator subgroup of $\mathrm{Gal}\,(H/E')$. In particular the transfer map $\mathrm{Gal}\,(E/E')\to\mathrm{Gal}\,(H/E)$ vanishes. By class field theory we can identify $(\mathcal{O}_{E'}/\mathfrak{a}')^\times/\mathcal{O}_{E'}^\times$ as a subgroup of $\mathrm{Gal}\,(E/E')$ and $(\mathcal{O}_E/\mathfrak{a})^\times/\mathcal{O}_E^\times$ as a subgroup of $\mathrm{Gal}\,(H/E)$ in such a way that the natural map

$$(\mathcal{O}_{E'}/\mathfrak{a}')^\times/\mathcal{O}_{E'}^\times\longrightarrow(\mathcal{O}_E/\mathfrak{a})^\times/\mathcal{O}_E^\times$$

corresponds to the transfer map on Galois groups and so is trivial. The lemma follows. □

Lemma 4.3 *There is a continuous character $\chi:\mathbb{A}_M^\times\to(EM)^\times$ such that*

- *$\chi|_{M^\times}$ is the canonical inclusion;*

- $\chi|_{\mathcal{O}_{M,l}^\times}$ *is the unique character of order prime to* l *with*

$$\chi|_{\mathcal{O}_{M,l}^\times}(x) \equiv x^{l+1-k} \bmod \lambda$$

for all $x \in \mathcal{O}_{M,l}^\times$ *and all primes* λ *of* EM *above* l *(and where* $\mathcal{O}_M \hookrightarrow \mathcal{O}_{EM}$ *via the natural map);*

- *for* $i = 1, 2$, χ *is non-ramified above* p_i *and* $\chi|_{M_{\wp_i}}(p_i) = \alpha_i$*; and*

- $\chi|_{\mathbb{A}^\times} = \mu\delta_{M/\mathbb{Q}}||\ ||^{-1} i_\infty$ *where* $\delta_{M/\mathbb{Q}}$ *is the unique non-trivial character of* $\mathbb{A}^\times/\mathbb{Q}^\times \mathrm{N}\mathbb{A}_M^\times$, $||\ ||$ *is the product of the usual absolute values and* i_∞ *is the projection onto* $\mathbb{R}^\times$.

Proof: Note that $\chi_0|_{\mathbb{A}^\times} = \nu\delta_{M/\mathbb{Q}}||\ ||^{-1} i_\infty$, where ν is a finite order character of $\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times$ with conductor dividing $\mathfrak{f}_0\mathfrak{f}_0^c\mathfrak{f}_M$. We look for $\chi = \chi_0\chi_1$. Thus we are required to find a finite order continuous character $\chi_1 : \mathbb{A}_M^\times/M^\times \longrightarrow (EM)^\times$ such that

- $\chi_1|_{\mathbb{A}^\times} = \mu\nu^{-1}$, and
- χ_1 has prescribed, finite order restriction to $M_{\wp_1}^\times$, $M_{\wp_2}^\times$ and $\mathcal{O}_{M,l}^\times$, the latter compatible with $\mu\nu^{-1}|_{\mathbb{Z}_l^\times}$ (because $\overline{\mu}|_{\mathbb{Z}_l^\times}$ takes x to $(x \bmod l)^{2-k}$).

Note that $\mu\nu^{-1}$ has conductor dividing $\mathfrak{f}_M\mathfrak{f}_\mu\mathfrak{f}_0\mathfrak{f}_0^c$. Also note that for $i = 1, 2$ the unit $a_i = \alpha_i\chi_0(\varpi_{\wp_i})^{-1}$ satisfies $a_i a_i^c = 1$ for all complex conjugations c and so is a root of unity. Thus the specified restrictions have orders dividing $w_{E''M}$ in the first two cases and $\#(\mathcal{O}_M/l\mathfrak{f}_M\mathfrak{f}_\mu\mathfrak{f}_0\mathfrak{f}_0^c\mathcal{O}_M)^\times$ in the third case.
We can find a character

$$\chi_{1,S_0} : \prod_{q\in S_0} M_q^\times \longrightarrow (EM)^\times$$

with the desired restrictions to $\prod_{q\in S_0}\mathbb{Q}_q^\times$, $M_{\wp_1}^\times$, $M_{\wp_2}^\times$ and $\mathcal{O}_{M,l}^\times$, and with order dividing w. As

$$(\prod_{q\in S_0} M_q^\times \times \prod_{q\notin S_0} \mathcal{O}_{M,q}^\times)/M_{S_0}^\times \xrightarrow{\sim} \mathbb{A}_M^\times/M^\times M_\infty^\times,$$

it suffices to find a character

$$\chi_1^{S_0} : \prod_{q\notin S_0} \mathcal{O}_{M,q}^\times/\mathbb{Z}_q^\times \longrightarrow (EM)^\times$$

which coincides with χ_{1,S_0}^{-1} on $M_{S_0}^\times/\mathbb{Q}_{S_0}^\times$. One can choose such a character which is trivial on W_0 and so has order dividing w'. □

We remark that as $\chi(c\circ\chi)||\ ||(i_\infty\circ \mathrm{N}_{M/\mathbb{Q}})^{-1}$ has finite image contained in the totally positive elements of $E^\times$ we must have $\chi(c\circ\chi) = ||\ ||^{-1}(i_\infty\circ\mathrm{N}_{M/\mathbb{Q}})$.

If x is a place of EM above a place x' of M, let χ_x denote the character

$$\begin{array}{rcl} \mathbb{A}_M^\times/M^\times & \longrightarrow & (EM)_x^\times \\ a & \longmapsto & \chi(a)a_{x'}^{-1} \end{array}$$

where $a_{x'}$ denotes the x' component of a embedded in $(EM)_x^\times$ via the natural map $M_{x'} \to (EM)_x$.

Set $\mathfrak{b} = \lambda\wp_1\wp_2$ and $\mathfrak{b}_0 = \mathfrak{b} \cap E$, so that $\mathcal{O}_E/\mathfrak{b}_0 \cong \mathcal{O}_{EM}/\lambda \times \mathcal{O}_{EM}/\wp_1 \times \mathcal{O}_{EM}/\wp_2$. Let $W_{\mathfrak{b}_0,0}/\mathbb{Q}$ denote the finite free group scheme with $\mathcal{O}_E$-action which has

$$W_{\mathfrak{b}_0,0}(\mathbb{Q}^{ac}) \cong \mathcal{O}_E/\mathfrak{b}_0(1) \oplus \mathcal{O}_E/\mathfrak{b}_0.$$

By the standard pairing on $W_{\mathfrak{b}_0,0}$ we shall mean the map $W_{\mathfrak{b}_0,0} \otimes_{\mathcal{O}_E} \mathfrak{d}_E^{-1} \to W_{\mathfrak{b}_0,0}^\vee$ which corresponds to the pairing

$$\begin{array}{ccccc} (\mathcal{O}_E/\mathfrak{b}_0(1) \oplus \mathcal{O}_E/\mathfrak{b}_0) & \times & (\mathcal{O}_E/\mathfrak{b}_0(1) \oplus \mathcal{O}_E/\mathfrak{b}_0) & \longrightarrow & \mathcal{O}_E/\mathfrak{b}_0(1) \\ (x_1, y_1) & \times & (x_2, y_2) & \longmapsto & y_2x_1 - y_1x_2. \end{array}$$

We will let $X/\mathbb{Q}$ denote the moduli space for quadruples (A, i, j, α), where (A, i, j) is an E-HBAV and $\alpha : W_{\mathfrak{b}_0,0} \xrightarrow{\sim} A[\mathfrak{b}_0]$ takes the standard pairing on $W_{\mathfrak{b}_0,0}$ to the j-Weil pairing on $A[\mathfrak{b}_0]$. As $\mathfrak{b}_0$ is divisible by two primes with coprime residue characteristic we see that X is a fine moduli space. As in section 1 of [Rap] we see that X is smooth and geometrically connected (because of the analytic uniformization of its complex points by a product of copies of the upper half complex plane).

Let Γ denote the set of pairs

$$(\gamma, \varepsilon) \in GL_2(\mathcal{O}_E/\mathfrak{b}_0) \times \mathcal{O}_{E,\gg 0}^\times/(\mathcal{O}_{E,\equiv 1\ (\mathfrak{b}_0)}^\times)^2$$

such that

$$\varepsilon \det\gamma \equiv 1 \bmod \mathfrak{b}_0.$$

Here $\mathcal{O}_{E,\gg 0}^\times$ denotes the set of totally positive elements of $\mathcal{O}_E^\times$, and $\mathcal{O}_{E,\equiv 1\ (\mathfrak{b}_0)}^\times$ denotes the set of elements of $\mathcal{O}_E^\times$ which are congruent to 1 modulo $\mathfrak{b}_0$. The group Γ acts faithfully on X via

$$(\gamma, \varepsilon)(A, i, j, \alpha) = (A, i, j \circ \varepsilon^{-1}, \alpha \circ \gamma^{-1}).$$

The action of $G_\mathbb{Q}$ on the group of automorphisms of X preserves Γ and we have

$$\sigma(\gamma, \varepsilon) = \left(\begin{pmatrix} \epsilon(\sigma) & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} \epsilon(\sigma)^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \varepsilon \right).$$

The set $H^1(G_\mathbb{Q}, \Gamma)$ is in bijection with the set of pairs (R, ψ) where $R : G_\mathbb{Q} \to GL_2(\mathcal{O}_E/\mathfrak{b}_0)$ is a continuous representation and $\psi : G_\mathbb{Q} \to \mathcal{O}_{E,\gg 0}^\times/(\mathcal{O}_{E,\equiv 1\ (\mathfrak{b}_0)}^\times)^2$ is a continuous homomorphism with

$$\epsilon^{-1} \det R \equiv \psi^{-1} \bmod \mathfrak{b}_0.$$

This pair corresponds to the cocycle

$$(R,\psi)(\sigma) = \left(R(\sigma) \begin{pmatrix} \epsilon(\sigma)^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \psi(\sigma) \right).$$

Thus to any such pair we can associated a twist $X_{R,\psi}/\mathbb{Q}$ of $X/\mathbb{Q}$.
Next we will give a description of the F-rational points of $X_{R,\psi}$ for any number field F. Let N' denote the splitting field of ψ. Let $W_R/\mathbb{Q}$ denote the finite free group scheme with an action of $\mathcal{O}_E$ such that

$$W_R(\mathbb{Q}^{ac}) \cong \mathcal{O}_E/\mathfrak{b}_0 \oplus \mathcal{O}_E/\mathfrak{b}_0$$

with Galois action via R. By the standard pairing on W_R/N' we shall mean the map $W_R \otimes_{\mathcal{O}_E} \mathfrak{d}_E^{-1} \to W_R^\vee$ (defined over N') which corresponds to the pairing

$$\begin{array}{ccccc} (\mathcal{O}_E/\mathfrak{b}_0 \oplus \mathcal{O}_E/\mathfrak{b}_0) & \times & (\mathcal{O}_E/\mathfrak{b}_0 \oplus \mathcal{O}_E/\mathfrak{b}_0) & \longrightarrow & \mathcal{O}_E/\mathfrak{b}_0 \\ (x_1,y_1) & \times & (x_2,y_2) & \longmapsto & y_2x_1 - y_1x_2. \end{array}$$

Then F-rational points of $X_{R,\psi}$ correspond to quadruples (A,i,j,β), where $(A,i,j)/N'F$ is an E-HBAV and where $\beta : W_R \xrightarrow{\sim} A[\mathfrak{b}_0]$ such that

- under β the standard pairing on W_R and the j-Weil pairing on $A[\mathfrak{b}_0]$ correspond, and
- for all $\sigma \in \mathrm{Gal}\,(N'F/F)$ there is an isomorphism

  $$\kappa_\sigma : \sigma(A,i) \xrightarrow{\sim} (A,i)$$

  such that $\sigma(j) = \kappa_\sigma^* \circ j \circ \psi(\sigma)^\sim$ for some lifting $\psi(\sigma)^\sim \in \mathcal{O}_E^\times$ of $\psi(\sigma)$ and such that for some lifting $\sigma^\sim \in G_F$ of σ

  $$\begin{array}{ccc} \sigma A[\mathfrak{b}_0] & \xrightarrow{\kappa_\sigma} & A[\mathfrak{b}_0] \\ \uparrow & & \uparrow \\ W_R & \xrightarrow{R(\sigma^\sim)} & W_R \end{array}$$

  commutes, where the left vertical arrow is $\sigma^\sim \circ \beta$ and the right one is β.

We will be particularly interested in two pairs (R,ψ) defined as follows. For $\sigma \in \mathrm{Gal}\,(N/\mathbb{Q})$ we can write $\mu(\sigma) = \zeta^{-2m_\sigma}$ for some integer m_σ. Define $\eta_\sigma = (\eta\zeta^{-1})^{m_\sigma} \in \mathcal{O}_{EM,\equiv 1\ (\mathfrak{b})}^\times$ and $\psi(\sigma) = \mathrm{N}_{EM/E}\eta_\sigma = \eta^{2m_\sigma}$. As

$$\eta^{2\#k^\times} = (-\eta^{\#k^\times})^2 \in (\mathcal{O}_{E,\equiv 1\ (\mathfrak{b}_0)}^\times)^2,$$

we see that

$$\psi : \mathrm{Gal}\,(N/\mathbb{Q}) \longrightarrow \mathcal{O}_{E,\gg 0}^\times/(\mathcal{O}_{E,\equiv 1\ (\mathfrak{b}_0)}^\times)^2$$

is a homomorphism. Let

$$R_{\overline{\rho}} = \overline{\rho} \oplus \mathrm{Ind}\,_{G_M}^{G_\mathbb{Q}} \chi_{\wp_1} \oplus \mathrm{Ind}\,_{G_M}^{G_\mathbb{Q}} \chi_{\wp_2}$$

and

$$R_{Dih} = \mathrm{Ind}\,_{G_M}^{G_{\mathbb{Q}}}\chi_\lambda \oplus \mathrm{Ind}\,_{G_M}^{G_{\mathbb{Q}}}\chi_{\wp_1} \oplus \mathrm{Ind}\,_{G_M}^{G_{\mathbb{Q}}}\chi_{\wp_2},$$

so that $\epsilon^{-1}\det R_\rho = \epsilon^{-1}\det R_{Dih} = \mu$. Then $(R_{\overline{\rho}}, \psi)$ and (R_{Dih}, ψ) define elements of $H^1(G_{\mathbb{Q}}, \Gamma)$ and we will denote the corresponding twists of X by $X_{\overline{\rho}}$ and X_{Dih} respectively. Note that $X_{\overline{\rho}}$ and X_{Dih} become isomorphic over $\mathbb{Q}_l$, $\mathbb{Q}_{p_1}$, $\mathbb{Q}_{p_2}$ and $\mathbb{R}$.

Lemma 4.4 *Suppose that F is a number field. If $X_{\overline{\rho}}$ has an F-rational point then there exists an abelian variety B/F of dimension $[EM:\mathbb{Q}]$, an embedding $i' : \mathcal{O}_{EM} \hookrightarrow \mathrm{End}\,(B/F)$, and an isomorphism β' between $B[\mathfrak{b}](F^{ac})$ and $R_{\overline{\rho}}$.*

Proof: Suppose that $(A, i, j, \beta)/FN$ is a quadruple corresponding to an F-rational point of $X_{\overline{\rho}}$ as above. Also, for $\sigma \in \mathrm{Gal}\,(NF/F)$ let $\kappa_\sigma : \sigma A \xrightarrow{\sim} A$ be the maps of the last but one paragraph. Set $B = A \otimes_{\mathcal{O}_E} \mathcal{O}_{EM}$ and let i' denote the natural map $\mathcal{O}_{EM} \to \mathrm{End}\,(B)$. Let β' denote the composite

$$W_{R_{\overline{\rho}}} \xrightarrow{\beta} A[\mathfrak{b}_0] \longrightarrow A[\mathfrak{b}_0] \otimes_{\mathcal{O}_E} \mathcal{O}_{EM}/\mathfrak{b} = B[\mathfrak{b}].$$

Define $f_0 : \mathcal{O}_{EM} \to \mathrm{Hom}\,_{\mathcal{O}_E}(\mathcal{O}_{EM}, \mathcal{O}_E)$ by $f_0(a)(b) = \mathrm{tr}\,_{EM/E}ab^c$ and set $f = j(1) \otimes f_0$ a polarisation of B. Also set

$$\kappa'_\sigma = \kappa_\sigma \otimes \eta_\sigma : \sigma B \longrightarrow B.$$

We see that κ'_σ commutes with the action of $\mathcal{O}_{EM}$, that $\sigma f = (\kappa'_\sigma)^\vee f \kappa'_\sigma$ and that for any lifting $\sigma^\sim \in G_F$ of σ

$$\begin{array}{ccc} \sigma B[\mathfrak{b}] & \stackrel{\kappa'_\sigma}{\longrightarrow} & B[\mathfrak{b}] \\ \uparrow & & \uparrow \\ W_{R_{\overline{\rho}}} & \stackrel{R_{\overline{\rho}}(\sigma^\sim)}{\longrightarrow} & W_{R_{\overline{\rho}}} \end{array}$$

commutes, where the left vertical arrow is $\sigma^\sim \circ \beta'$ and the right one is β'. As the quadruple (B, i', f, β') has no non-trivial automorphisms (because any automorphism of (B, i', f) has finite order and because $\mathfrak{b}$ is divisible by two primes with distinct residual characteristic), we see that $\kappa'_\sigma \sigma(\kappa'_\tau) = \kappa'_{\sigma\tau}$. Thus we can descend (B, i') to F in such a way that β' also descends to an isomorphism $\beta' : W_{R_{\overline{\rho}}} \xrightarrow{\sim} B[\mathfrak{b}]$ over F. $\Box$

Lemma 4.5 *X_{Dih} has a $\mathbb{Q}$-rational point and hence $X_{\overline{\rho}}$ has rational points over $\mathbb{Q}_l$, $\mathbb{Q}_{p_1}$, $\mathbb{Q}_{p_2}$ and over $\mathbb{R}$.*

Proof: Fix an embedding $\tau : M \hookrightarrow \mathbb{C}$ and let Φ denote the CM-type for EM consisting of all embeddings $EM \hookrightarrow \mathbb{C}$ which restrict to τ on M. Let $(\mathfrak{d}_{EM}^{-1})^-$ denote the ordered $\mathcal{O}_E$-module $\{d \in \mathfrak{d}_{EM}^{-1} : \ \mathrm{tr}\,_{EM/E}d = 0\}$ with $(\mathfrak{d}_{EM}^{-} \otimes_{E,\sigma} \mathbb{R})^+$ the subset with positive imaginary part under $\sigma \otimes \tau$. From the theory of complex multiplication (see [Lang], particularly theorem 5.1 of chapter 5) we see that there is

- an abelian variety A/M of dimension $[E:\mathbb{Q}]$;
- an embedding $i:\mathcal{O}_{EM}\hookrightarrow \mathrm{End}\,(A/M)$;
- an isomorphism $j:(\mathfrak{d}_{EM}^{-1})^- \xrightarrow{\sim} \mathcal{P}(A,i|_{\mathcal{O}_E})$; and
- for each prime $\mathfrak{q}$ of E a Galois invariant isomorphism $\alpha_\mathfrak{q}:\mathcal{O}_{EM,\mathfrak{q}}(\chi_\mathfrak{q}) \xrightarrow{\sim} T_\mathfrak{q}A$

such that

- the action of EM on $Lie\,\tau A$ is $\bigoplus_{\sigma\in\Phi}\sigma$; and
- for any $d\in(\mathfrak{d}_{EM}^{-1})^-$ which is totally positive the $j(d)$-Weil pairing on $T_\mathfrak{q}A$ is given by
$$x\times y\longmapsto \mathrm{tr}\,_{EM/E}dxy^c.$$

(For the existence of j note that if f is a polarisation of $\tau A/\mathbb{C}$ such that the f-Rosati involution stabilises and acts trivially on E, then the f-Rosati involution also stabilises EM and acts on it via complex conjugation. This follows from the fact that EM is the centraliser of E in $\mathrm{End}\,(\tau A/\mathbb{C})$.) As $\chi(c\circ\chi)=(||\ \ ||^{-1}i_\infty)\circ \mathrm{N}_{M/\mathbb{Q}}$, we see that for $\sigma\in G_M$ we have

$$\mathrm{tr}\,_{EM/M}d\alpha_\mathfrak{q}(\sigma x)\alpha_\mathfrak{q}(\sigma y)^c=\epsilon_q(\sigma)\mathrm{tr}\,_{EM/M}d\alpha_\mathfrak{q}(x)\alpha_\mathfrak{q}(y)^c.$$

Thus the quadruple $(A,i|_{\mathcal{O}_E},j,(\prod_\mathfrak{q}\alpha_\mathfrak{q}) \bmod \mathfrak{b}_0)$ defines a point in $X_{Dih}(M)$. As $\chi(\chi\circ c)=(||\ \ ||^{-1}i_\infty\mu)\circ\mathrm{N}_{M/\mathbb{Q}}$, we see that $c\circ\chi\circ\mathrm{N}_{NM/M}=\chi\circ c\circ\mathrm{N}_{NM/M}$ and so over NM there is an isomorphism between $(A,i,j,\{\alpha_\mathfrak{q}\})$ and $(cA,c\circ i\circ c,c\circ j,\{c\circ\alpha_\mathfrak{q}\circ c\})$. Thus the point in $X_{Dih}(M)\subset X_{Dih}(NM)$ defined by $(A,i|_{\mathcal{O}_E},j,(\prod_\mathfrak{q}\alpha_\mathfrak{q}) \bmod \mathfrak{b}_0)$ is invariant under c and so lies in $X_{Dih}(\mathbb{Q})$. $\Box$

Combining the last two lemmas with a theorem of Moret-Bailly (see theorem G of [Tay4]) we see that we can find a Galois totally real field F of even degree in which l, p_1 and p_2 split completely, an abelian variety B/F of dimension $[EM:M]$ and an embedding $i:\mathcal{O}_{EM}\hookrightarrow\mathrm{End}\,(B/F)$ such that $B[\lambda]$ realises $\overline{\rho}$ and, for $i=1,2$, $B[\wp_i]$ realises $\mathrm{Ind}\,_{G_M}^{G_\mathbb{Q}}(\chi_{\wp_i} \bmod \wp_i)$. As $B[\lambda]$ is unramified at any prime above p_1 we see that the action of inertia at such a prime on $T_\lambda B$ has l-power order. As $B[\wp_2]$ is unramified at any prime above p_1 we see that the action of inertia at such a prime on $T_{\wp_2}B$ has p_2-power order. Hence the action of inertia at a prime above p_1 on $T_\lambda B$ has both l-power order and p_2-power order. We conclude that $T_\lambda B$ is unramified at primes above p_1 and hence B has semi-stable reduction at such primes. As p_1 splits completely in F and as $B[\wp_1]$ is reducible as a representation of the decomposition group of any prime of F above p_1, we see that $T_{\wp_1}B$ is an ordinary representation of the decomposition group at any prime of F above p_1. If x is a prime of F above l then I_x acts on both $B[\wp_1]$ and $B[\wp_2]$ via $\widetilde{\omega}_2^{k-(l+1)}\oplus\widetilde{\omega}_2^{lk-(l+1)}$ where $\widetilde{\omega}_2:I_x\to\mathcal{O}_{EM}^\times$ is tamely ramified and reduces mod λ to ω_2. Thus I_x acts on $T_{\wp_1}B$ by $\widetilde{\omega}_2^{k-(l+1)}\oplus\widetilde{\omega}_2^{lk-(l+1)}$. Because $\mathrm{Ind}\,_{G_{FM}}^{G_F}(\chi_{\wp_1})$ is modular, theorem 5.1

of [SW2] tells us that there is a algebraic, cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ of weight 2 and an embedding $M_\pi \hookrightarrow EM$ such that $\rho_{\pi,\wp_1}$ is equivalent to $T_{\wp_1}B$. (Alternatively one may appeal to the main theorem of [SW1], theorem 3.3 of this paper and a standard descent argument.) It follows that in addition $\rho_{\pi,\lambda}$ is equivalent to $T_\lambda B$. This completes the proof of proposition 4.1.

Using Langlands base change [Langl] we immediately obtain the following corollary.

Corollary 4.6 *Let $l > 2$ be a prime. Suppose that $\overline{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_l^{ac})$ is a continuous odd representation with $\overline{\rho}|_{I_l} \sim \omega_2^{k-1} \oplus \omega_2^{l(k-1)}$ for some integer $2 \leq k \leq l$. Then there is a Galois totally real field F of even degree in which l splits completely, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ and an embedding $\lambda : M_\pi \hookrightarrow \mathbb{Q}_l^{ac}$ such that*

1. *$\overline{\rho}|_{G_F} \sim \overline{\rho}_{\pi,\lambda}$;*
2. *π_∞ has weight 2;*
3. *the central character of $\pi^{\infty,l}$ is unramified; and*
4. *for each place x of F above l, $\mathrm{WD}_\lambda(\pi_x)$ is tamely ramified and*
$$\mathrm{WD}_\lambda(\pi_x)|_{I_x} = \omega_2^{k-(l+1)} \oplus \omega_2^{lk-(l+1)}.$$

5 Change of weight

In this section we will prove various refinements of proposition 4.1, but first we shall discuss some results about congruences between modular forms.

Let F be a totally real field of even degree in which a prime $l > 3$ splits completely. Let $\mathfrak{n}$ denote an ideal of $\mathcal{O}_F$ coprime to l. Let $\psi : (\mathbb{A}_F^\infty)^\times/F^\times \to (\mathbb{Q}_l^{ac})^\times$ be a continuous character trivial on $\mathcal{O}_{F,x}^\times$ if $x \not| l$ and on $(1 + l\mathcal{O}_{F,x})$ if $x|l$. Suppose further that there exists $i \in (\mathbb{Z}/(l-1)\mathbb{Z})$ such that for $a \in \mathcal{O}_{F,l}^\times$, $\psi(a)$ is congruent to $(\mathrm{N}a)^{-i}$ modulo the maximal ideal of $\mathcal{O}_{\mathbb{Q}_l^{ac}}$.

Let D denote the division algebra with centre F ramified at exactly the infinite places of F. Let $\mathcal{O}_D$ be a maximal order in D and fix an isomorphism $\mathcal{O}_{D,x} \cong M_2(\mathcal{O}_{F,x})$ for each finite place x of F. We will write

- $U_0(\mathfrak{n}, l)$ for $U_{\{1\}}(\mathfrak{n}l)$, and
- $U_1(\mathfrak{n}, l)$ for $U_{(\mathcal{O}_F/l\mathcal{O}_F)^\times}(\mathfrak{n}l)$.

(See section 1, in particular the paragraph after corollary 1.2, for this notation.)

We will let $\overline{\eta}^i$ denote the character $U_0(\mathfrak{n}, l)/U_1(\mathfrak{n}, l) \to (\mathbb{F}_l^{ac})^\times$ which sends u, with
$$u_l = \begin{pmatrix} * & * \\ * & d \end{pmatrix},$$

to $(\mathrm{N}d \bmod l)^i$. We will also let η^i denote the Teichmüller lift of $\overline{\eta}^i$. For any $\mathcal{O}_{\mathbb{Q}_l^{ac}}$-algebra R, there is a natural embedding

$$S_{\eta^i\otimes R,\psi}(U_0(\mathfrak{n},l)) \hookrightarrow S_{\eta^i\otimes R,\psi}(U_1(\mathfrak{n},l)) = S_{2,R,\psi}(U_1(\mathfrak{n},l)),$$

which is equivariant for the action of T_x and S_x for all $x \not| l\mathfrak{n}$, and for $\mathbf{U}_{\varpi_x}$ for $x|\mathfrak{n}$. The image is the subset of $S_{2,R,\psi}(U_1(\mathfrak{n},l))$ where $\langle h\rangle = 1$ for all $h \in (\mathcal{O}_F/l\mathcal{O}_F)^\times$. If $\phi : h_{\eta^i,\mathbb{F}_l,\psi}(U_0(\mathfrak{n},l)) \to \mathbb{F}_l^{ac}$ has non-Eisenstein kernel then for $x|l$ we have

$$\det \overline{\rho}_\phi|_{I_x} = \omega^{1+i}.$$

The operators $\mathbf{U}_{\varpi_x}$ and $\mathbf{V}_{\varpi_x}$ on $S_{2,\mathbb{F}_l^{ac},\psi}(U_1(\mathfrak{n},l))$ commute with the action of $\langle h\rangle$ for $h \in (\mathcal{O}_F/l\mathcal{O}_F)^\times$ and hence preserve $S_{\overline{\eta}^i,\psi}(U_0(\mathfrak{n},l))$. We will let $h_{\overline{\eta}^i,\mathbb{F}_l,\psi}(U_0(\mathfrak{n},l))'$ (resp. $h_{\overline{\eta}^i,\mathbb{F}_l,\psi}(U_0(\mathfrak{n},l))''$) denote the commutative subalgebra of the endomorphisms of $S_{\overline{\eta}^i,\psi}(U_0(\mathfrak{n},l))$ generated by $h_{\overline{\eta}^i,\mathbb{F}_l,\psi}(U_0(\mathfrak{n},l))$ and $\mathbf{U}_{\varpi_x}$ (resp. $\mathbf{V}_{\varpi_x}$) for all $x|l$. If $\phi : h_{\overline{\eta}^i,\mathbb{F}_l,\psi}(U_0(\mathfrak{n},l))' \to \mathbb{F}_l^{ac}$ and $\phi(\mathbf{U}_{\varpi_x}) \neq 0$ then

$$\overline{\rho}_\phi|_{G_x} \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

where χ_2 is unramified and $\chi_2(\mathrm{Frob}_x) = \phi(\mathbf{U}_{\varpi_x})$ (see [W1]).
If $f_1 \in S_{\overline{\eta}^i,\psi}(U_0(\mathfrak{n},l))$ and $f_2 \in S_{\overline{\eta}^{-i},\psi^{-1}}(U_0(\mathfrak{n},l))$ then define (f_1,f_2) to be

$$\sum_{[x]\in D^\times\backslash(D\otimes_{\mathbb{Q}}\mathbb{A}^\infty)/U_0(\mathfrak{n},l)(\mathbb{A}_F^\infty)^\times} f_1(x)f_2(xw)(\#(U_0(\mathfrak{n},l)(\mathbb{A}_F^\infty)^\times \cap x^{-1}D^\times x)/F^\times)^{-1},$$

where

$$w_x = \begin{pmatrix} 0 & 1 \\ \varpi_x^{x(\mathfrak{n})} & 0 \end{pmatrix}$$

if $x \not| l$ and $w_x = 1_2$ if $x|l$. This is easily seen to be a perfect pairing. Moreover a standard calculation shows that the adjoint of S_x is S_x^{-1}, the adjoint of T_x is $S_x^{-1}T_x$, the adjoint of $\mathbf{U}_{\varpi_x}$ for $x|\mathfrak{n}$ is $S_{\varpi_x}^{-1}\mathbf{U}_{\varpi_x}$ and the adjoint of $\mathbf{U}_{\varpi_x}$ for $x|l$ is $S_{\varpi_x}^{-1}\mathbf{V}_{\varpi_x}$. Thus if $\phi : h_{\overline{\eta}^i,\mathbb{F}_l,\psi}(U_0(\mathfrak{n},l)) \to \mathbb{F}_l^{ac}$ then there is also a homomorphism $\phi^* : h_{\overline{\eta}^{-i},\mathbb{F}_l,\psi^{-1}}(U_0(\mathfrak{n},l)) \to \mathbb{F}_l^{ac}$ satisfying $\phi^*(T_x) = \phi(S_x)^{-1}\phi(T_x)$ and $\phi^*(S_x) = \phi(S_x)^{-1}$. Moreover if ϕ extends to $h_{\overline{\eta}^i,\mathbb{F}_l,\psi}(U_0(\mathfrak{n},l))''$ so that $\phi(\mathbf{V}_{\varpi_x}) \neq 0$ then ϕ^* extends to $h_{\overline{\eta}^{-i},\mathbb{F}_l,\psi^{-1}}(U_0(\mathfrak{n},l))'$ with $\phi^*(\mathbf{U}_{\varpi_x}) \neq 0$. We deduce that $\overline{\rho}_{\phi^*} = \overline{\rho}_\phi^\vee(1)$. Hence if $\phi : h_{\overline{\eta}^i,\mathbb{F}_l,\psi}(U_0(\mathfrak{n},l))'' \to \mathbb{F}_l^{ac}$ and $\phi(\mathbf{V}_{\varpi_x}) \neq 0$ then

$$\overline{\rho}_\phi|_{G_x} \sim \begin{pmatrix} \epsilon\chi_1 & * \\ 0 & \omega^i\chi_2 \end{pmatrix}$$

where χ_1 and χ_2 are unramified.
We will denote by I^i the induced representation from $U_0(\mathfrak{n},l)$ to $U_H(\mathfrak{n})$ of $\overline{\eta}^i$. It is a tesnor product $\bigotimes_{x|l} I_x^i$ where I_x^i is the induction from $U_0(\mathfrak{n},l)_x$ to $GL_2(\mathcal{O}_{F,x})$ of $\overline{\eta}^i$. We can realise I_x^i concretely as the space of functions

$$\theta : k(x)^2 - \{(0,0)\} \longrightarrow \mathbb{F}_l^{ac}$$

such that $\theta(a(x,y)) = a^i\theta(x,y)$ for all $a \in k(x)^\times$. The action of $GL_2(\mathcal{O}_{F,x})$ is via $(u\theta)(x,y) = \theta((x,y)u)$. We have an isomorphism

$$S_{\overline{\eta}^i,\psi}(U_0(\mathfrak{n},l)) \cong S_{I^i,\psi}(U_H(\mathfrak{n}))$$

under which $f \in S_{\overline{\eta}^i,\psi}(U_0(\mathfrak{n},l))$ corresponds to $F \in S_{I^i,\psi}(U_H(\mathfrak{n}))$ if

$$f(g) = F(g)((0,1)_x)$$

and

$$F(g)(a_x,b_x) = f(gu^{-1})$$

where $u \in GL_2(\mathcal{O}_{F,l})$ and

$$u \bmod x = \begin{pmatrix} * & * \\ a_x & b_x \end{pmatrix}$$

for all $x|l$.
Now suppose that $0 \le i \le l-2$. If x is a prime of F above l then we have an exact sequence

$$(0) \longrightarrow \operatorname{Symm}^i((\mathbb{F}_l^{ac})^2) \longrightarrow I_x^i \longrightarrow \operatorname{Symm}^{l-1-i}((\mathbb{F}_l^{ac})^2) \otimes \det{}^i \longrightarrow (0).$$

The first map is just the natural inclusion of homogeneous polynomials of degree i into the space of homogeneous functions of degree i. The second map sends a homogeneous function θ onto the polynomial

$$\sum_{(s,t)\in\mathbb{P}^1(k(x))} \theta(s,t)(tX-sY)^{l-1-i}.$$

Thus for any subset T of the set of places of F above l we have a submodule $I_T^i \subset I^i$ with

$$I_T^i \cong \bigotimes_{x\notin T} \operatorname{Symm}^i((\mathbb{F}_l^{ac})^2) \otimes \bigotimes_{x\in T} I_x^i.$$

These give rise to subspaces

$$S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l)) \subset S_{\overline{\eta}^i,\psi}(U_0(\mathfrak{n},l))$$

with

$$S_{\overline{\eta}^i,\psi,\emptyset}(U_0(\mathfrak{n},l)) \cong S_{i+2,\mathbb{F}_l^{ac},\psi}(U_H(\mathfrak{n}))$$

as a module for the Hecke operators T_x and S_x for all $x \not| l\mathfrak{n}$ and for $\mathbf{U}_{\varpi_x}$ for all $x|\mathfrak{n}$.
The following lemma is a variant of an unpublished result of Buzzard (see [Bu]).

Lemma 5.1 *For any set T of places of F above l and for any place $x \notin T$ of F above l there is an injection*

$$\kappa_x : S_{\overline{\eta}^i,\psi,T\cup\{x\}}(U_0(\mathfrak{n},l))/S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l)) \hookrightarrow S_{\overline{\eta}^i,\psi,T\cup\{x\}}(U_0(\mathfrak{n},l))$$

which is equivariant for the actions of T_y and S_y for all $y \not| l$ and for $\mathbf{U}_{\varpi_x}$ for $x|\mathfrak{n}$, and such that the composite

$$S_{\overline{\eta}^i,\psi,T\cup\{x\}}(U_0(\mathfrak{n},l)) \xrightarrow{\kappa_x} S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l)) \hookrightarrow S_{\overline{\eta}^i,\psi,T\cup\{x\}}(U_0(\mathfrak{n},l))$$

coincides with $\mathbf{V}_{\varpi_x}$.

Proof: Define $U_0(T) \subset U_H(\mathfrak{n})$ by $U_0(T)_y = U_0(\mathfrak{n},l)_y$ if $y \in T$ and $U_0(T)_y = U_H(\mathfrak{n})_y$ otherwise. Let τ_T denote the representation

$$(\bigotimes_{y\in T} \overline{\eta}_y^i) \otimes (\bigotimes_{y\notin T} \mathrm{Symm}^{2+i}((\mathbb{F}_l^{ac})^2)$$

of $U_0(T)_l$. If $x \notin T$ is a place of F above l, let $\tau_{T,x}$ denote the representation

$$(\bigotimes_{y\in T} \overline{\eta}_y^i) \otimes (\bigotimes_{y\notin T\cup\{x\}} \mathrm{Symm}^{i+2}((\mathbb{F}_l^{ac})^2)) \otimes (\mathrm{Symm}^{l-1-i}((\mathbb{F}_l^{ac})^2) \otimes \det{}^i)$$

of $U_0(T)_l$. Then the exact sequence

$$\begin{aligned}(0) \longrightarrow S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l)) \longrightarrow S_{\overline{\eta}^i,\psi,T\cup\{x\}}(U_0(\mathfrak{n},l)) \longrightarrow \\ \longrightarrow S_{\overline{\eta}^i,\psi,T\cup\{x\}}(U_0(\mathfrak{n},l))/S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l)) \longrightarrow (0)\end{aligned}$$

is identified to the exact sequence

$$(0) \longrightarrow S_{\tau_T,\psi}(U_0(T)) \xrightarrow{\alpha} S_{\tau_{T\cup\{x\}},\psi}(U_0(T\cup\{x\})) \xrightarrow{\beta} S_{\tau_{T,x},\psi}(U_0(T)) \longrightarrow (0),$$

where

$$\alpha(f)(g) = f(g)(0,1)_x$$

and

$$\beta(f)(g)(X,Y)_x = \sum_{(s:t)\in\mathbb{P}^1(k(x))} f(gu(s,t)^{-1})(tX-sY)^{l-1-i}$$

with $u(s,t) \in GL_2(\mathcal{O}_{F,x})$ congruent to

$$\begin{pmatrix} * & * \\ s & t \end{pmatrix}$$

modulo x.
Now define

$$\kappa : S_{\tau_{T,x},\psi}(U_0(T)) \longrightarrow S_{\tau_{T\cup\{x\}},\psi}(U_0(T\cup\{x\}))$$

by

$$\kappa(f)(g) = f(g\gamma)(1,0)_x$$

where

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} \in GL_2(F_x).$$

To see this is well defined the only slightly subtle point is that if $u \in U_0(\mathfrak{n}, l)_x$ then

$$\begin{array}{rcl} \kappa(f)(gu) & = & f(g\gamma(\gamma^{-1}u\gamma))(1,0)_x \\ & = & (\det u)^i f(g\gamma)((1,0)_x(\gamma^{-1}u^{-1}\gamma)) \\ & = & (\det u)^i f(g\gamma)(\eta_x(u)/\det u, 0)_x \\ & = & \eta_x(u)^{-i} f(g\gamma)(1,0)_x \\ & = & \eta_x(u)^{-i}\kappa(f)(g). \end{array}$$

Moreover κ is clearly injective and equivariant for the action of T_y and S_y if $y \not| l\mathfrak{n}$ and for $\mathbf{U}_{\varpi_x}$ for $x|\mathfrak{n}$. Finally we have

$$\begin{array}{rcl} (\kappa \circ \beta)(f)(g) & = & \sum_{(s:t)\in\mathbb{P}^1(k(x))} f(g\gamma u(s,t)^{-1})t^{l-1-i} \\ & = & \sum_{s\in k(x)} f(g\gamma u(s,1)^{-1}) \\ & = & (\mathbf{V}_{\varpi_x} f)(g). \end{array}$$

as we can take

$$u(s,1) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

$\square$

Corollary 5.2 *There is a natural surjection*

$$h_{\overline{\eta}^i,\mathbb{F}_l^{ac},\psi}(U_0(\mathfrak{n},l)) \twoheadrightarrow h_{i+2,\mathbb{F}_l^{ac},\psi}(U_H(\mathfrak{n}))$$

which takes T_y to T_y and S_y to S_y for all $y \not| l\mathfrak{n}$ and which takes $\mathbf{U}_{\varpi_x}$ to $\mathbf{U}_{\varpi_x}$ for all $x|\mathfrak{n}$. If $\mathfrak{m}$ is a maximal ideal of $h_{\overline{\eta}^i,\mathbb{F}_l^{ac},\psi}(U_0(\mathfrak{n},l))$ such that for any $x|l$ and any maximal ideal $\mathfrak{m}_x''$ of $h_{\overline{\eta}^i,\mathbb{F}_l^{ac},\psi}(U_0(\mathfrak{n},l))''$ extending $\mathfrak{m}$ one has $\mathbf{V}_{\varpi_x} \in \mathfrak{m}_x''$, then $h_{i+2,\mathbb{F}_l^{ac},\psi}(U_H(\mathfrak{n}))_\mathfrak{m} \neq (0)$. This assumption will be verified if $\mathfrak{m}$ is non-Eisenstein and the kernel of a homomorphism $\phi : h_{\overline{\eta}^i,\psi}(U_0(\mathfrak{n},l)) \to \mathbb{F}_l^{ac}$ such that for all $x|l$

$$\overline{\rho}_\phi|_{G_x} \not\sim \begin{pmatrix} \epsilon\chi_1 & * \\ 0 & \omega^i\chi_2 \end{pmatrix},$$

with χ_1 and χ_2 unramified.

Proof: Choose a minimal T such that $S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l))_\mathfrak{m} \neq (0)$. If $T = \emptyset$ then $S_{k,\mathbb{F}_l^{ac},\psi}(U_H(\mathfrak{n}))_\mathfrak{m} \neq (0)$ and the corollary follows. Thus suppose that $x \in T$ and set $T' = T - \{x\}$. By our minimality assumption we see that

$$S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l))_\mathfrak{m} \xrightarrow{\sim} (S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l))/S_{\overline{\eta}^i,\psi,T'}(U_0(\mathfrak{n},l)))_\mathfrak{m} \overset{\kappa_x}{\hookrightarrow} S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l))_\mathfrak{m}$$

and the composite coincides with $\mathbf{V}_{\varpi_x}$. Thus $\mathbf{V}_{\varpi_x}$ is an isomorphism on the space $S_{\overline{\eta}^i,\psi,T}(U_0(\mathfrak{n},l))_\mathfrak{m}$ and $\mathbf{V}_{\varpi_x}$ does not lie in some maximal ideal of $h_{\overline{\eta}^i,\mathbb{F}_l^{ac},\psi}(U_0(\mathfrak{n},l))''$ above $\mathfrak{m}$, a contradiction. $\square$

We also have the following lemma, which generalises results of Ash and Stevens [AS]. We write U_0 for $\prod_y GL_2(\mathcal{O}_{F,y})$.

Lemma 5.3 *If $k \in \mathbb{Z}_{\geq 2}$ and if $\phi : h_{k,\mathbb{F}_l^{ac},\psi}(U_0) \to \mathbb{F}_l^{ac}$ is a homomorphism, then there is a homomorphism $(D\phi) : h_{k+l+1,\mathbb{F}_l^{ac},\psi(\epsilon\circ\mathrm{Art}^{-1})}(U_0) \to \mathbb{F}_l^{ac}$ such that for all places $y \not| l$ we have $(D\phi)(T_y) = \phi(T_y)(\mathrm{N}y)$ and $(D\phi)(S_y) = \phi(S_y)(\mathrm{N}y)^2$.*

Proof: If $f \in S_{k,\mathbb{F}_l^{ac},\psi}(U_0)$ then the function

$$(Df)(g) = f(g)(||\mathrm{N}\det g||(\mathrm{N}\det g_l))^{-1},$$

where $||\ || : (\mathbb{A}^\infty)^\times \to \mathbb{Q}_{>0}^\times$ denotes the product of the usual p-adic absolute values, lies in $S_{\tau_{k,\mathbb{F}_l^{ac}}\otimes(\mathrm{N}\det),\psi(\epsilon\circ\mathrm{Art}^{-1})}(U_0)$. Moreover if $T_y f = af$ (resp. $S_y f = bf$) then $T_y(Df) = a(\mathrm{N}y)(Df)$ (resp. $S_y(Df) = b(\mathrm{N}y)(Df)$). Thus it suffices to exhibit an embedding

$$S_{\tau_{k,\mathbb{F}_l^{ac}}\otimes(\mathrm{N}\det),\psi(\epsilon\circ\mathrm{Art}^{-1})}(U_0) \hookrightarrow S_{k+l+1,\mathbb{F}_l^{ac},\psi(\epsilon\circ\mathrm{Art}^{-1})}(U_0)$$

compatible with the action of T_y and S_y for all $y \not| l$. By lemma 1.1 it suffices to exhibit a $GL_2(\mathcal{O}_{F,l})$-equivariant embedding

$$\bigotimes_x (\mathrm{Symm}^{k-2}(k(x)^2) \otimes \det) \hookrightarrow \bigotimes_x \mathrm{Symm}^{k+l-1}(k(x)^2),$$

or simply $GL_2(\mathcal{O}_{F,x})$-equivariant embeddings

$$\mathrm{Symm}^{k-2}(k(x)^2) \otimes \det \hookrightarrow \mathrm{Symm}^{k+l-1}(k(x)^2)$$

for all $x|l$. Because l splits completely in F such an embedding simply results from multiplication by $X^lY - XY^l$, as we see from the following calculation. For $a, b, c, d \in \mathbb{F}_l$ we have

$$\begin{array}{rl} & (aX+cY)^l(bX+dY) - (aX+cY)(bX+dY)^l \\ = & (aX^l+cY^l)(bX+dY) - (aX+cY)(bX^l+dY^l) \\ = & (ad-bc)(X^lY - XY^l). \end{array}$$

□

We now turn to our improvements to proposition 4.1. First we have the following lemma.

Lemma 5.4 *Let $l > 3$ be a prime. Suppose that $\overline{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_l^{ac})$ is a continuous odd representation with $\overline{\rho}|_{I_l} \sim \omega_2^{k-1} \oplus \omega_2^{l(k-1)}$ for some integer $2 \leq k \leq l$. Then there is a Galois totally real field F in which l splits completely, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ and an embedding $\lambda : M_\pi \hookrightarrow \mathbb{Q}_l^{ac}$ such that*

1. *$\overline{\rho}|_{G_F} \sim \overline{\rho}_{\pi,\lambda}$;*
2. *π_∞ has weight 2; and*

3. for each place x of F above l, π_x has conductor dividing x.

Proof: Let F, π, λ be as provided by corollary 4.6. Let $\psi_0 : (\mathbb{A}_F^\infty)^\times/F^\times \to (\mathbb{Q}_l^{ac})^\times$ be the character such that $\epsilon(\psi_0 \circ \mathrm{Art}^{-1})$ equals the determinant of $\rho_{\pi,\lambda}$. Thus ψ_0 is unramified away from l. Let $\mathfrak{n}_0$ denote the prime to l part of the conductor of π. Let D be the division algebra with centre F which is ramified at exactly the infinite places of F. Let $\mathcal{O}_D$ be a maximal order in D and fix an isomorphism $\mathcal{O}_{D,x} \cong M_2(\mathcal{O}_{F,x})$ for each finite place x of F. Let $\mathcal{O}$ denote the ring of integers of $\mathbb{Q}_l^{ac}$.
Let χ_k denote the character $\mathbb{F}_{l^2}^\times \to \mathcal{O}^\times$ which sends a to the Teichmüller lift of a^{k-l-1}. Let $\Theta(\chi_k)$ denote a model over $\mathcal{O}$ of the representation of $GL_2(\mathbb{Z}_l) \to \to GL_2(\mathbb{F}_l)$ denoted the same way in section 3.1 of [CDT]. Let Θ_k denote the representation $\bigotimes_{x|l} \Theta(\chi_k)$ of $GL_2(\mathcal{O}_{F,l})$. From proposition 4.1, lemma 1.3 and lemma 4.2.4 of [CDT] we see that there is a homomorphism

$$\phi_1 : h_{\Theta_k,\mathcal{O},\psi_0}(U_{H_0}(\mathfrak{n}_0)) \longrightarrow \mathbb{F}_l^{ac}$$

such that $\ker \phi_1$ is non-Eisenstein and $\overline{\rho}_{\phi_1} \sim \overline{\rho}|_{G_F}$.
By lemma 3.1.1 of [CDT] we see that $\Theta_k \otimes \mathbb{F}_l^{ac}$ has a Jordan-Hölder sequence with subquotients

$$R_T = \bigotimes_{x \notin T} \mathrm{Symm}^{k-2}((\mathbb{F}_l^{ac})^2) \otimes \bigotimes_{x\in T} (\mathrm{Symm}^{l-1-k}((\mathbb{F}_l^{ac})^2) \otimes \det^{k-1})$$

where T runs over sets of places of F above l, and where, if $k = l$, we only have one subquotient namely $T = \emptyset$. Thus for some T, ϕ_1 factors through $h_{R_T,\mathbb{F}_l^{ac},\psi_0}(U_{H_0}(\mathfrak{n}_0))$. It then follows from corollary 1.5 that for $x \in T$ we must have $\overline{\rho}|_{I_x} \sim \omega_2^{k-l} \oplus \omega_2^{kl-1}$ or

$$\begin{pmatrix} 1 & * \\ 0 & \omega^{k-1} \end{pmatrix}.$$

Thus in fact ϕ_1 must factor through $h_{R_\emptyset,\mathbb{F}_l^{ac},\psi_0}(U_{H_0}(\mathfrak{n}_0)) = h_{k,\mathbb{F}_l^{ac},\psi_0}(U_{H_0}(\mathfrak{n}_0))$. It follows from the first part of corollary 5.2 that ϕ_1 gives rise to a map

$$\phi_0 : h_{\eta^{k-2},\mathcal{O},\psi_0}(U_{H_0}(\mathfrak{n}_0)) \longrightarrow \mathbb{F}_l^{ac}$$

such that $\ker \phi_0$ is non-Eisenstein and $\overline{\rho}_{\phi_0} \sim \overline{\rho}|_{G_F}$. The proposition follows. □

Combining the lemma 5.4 with the main theorem of [SW1], we immediately obtain the following corollary.

Corollary 5.5 *Let $l > 3$ be a prime. Suppose that $\overline{\rho} : G_\mathbb{Q} \to GL_2(\mathbb{F}_l^{ac})$ is a continuous odd representation with $\overline{\rho}|_{I_l} \sim \omega_2^{k-1} \oplus \omega_2^{l(k-1)}$ for some integer $2 \leq k \leq l$. Then there is a Galois totally real field F in which l splits completely, a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ and an embedding $\lambda : M_\pi \hookrightarrow \mathbb{Q}_l^{ac}$ such that*

1. $\overline{\rho}|_{G_F} \sim \overline{\rho}_{\pi,\lambda}$;
2. π_∞ *has weight* 2;
3. *for each finite place* x *of* F *not dividing* l, π_x *is unramified; and*
4. *for each place* x *of* F *above* l, *the conductor of* π_x *divides* x.

Now we can use corollary 5.2 to obtain a further refinement of proposition 4.1.

LEMMA 5.6 *Let* $l > 3$ *be a prime. Suppose that* $\overline{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_l^{ac})$ *is a continuous odd representation with* $\overline{\rho}|_{I_l} \sim \omega_2^{k-1} \oplus \omega_2^{l(k-1)}$ *for some integer* $2 \leq k \leq l$. *Then there is a Galois totally real field* F *of even degree in which* l *splits completely, a regular algebraic cuspidal automorphic representation* π *of* $GL_2(\mathbb{A}_F)$ *and an embedding* $\lambda : M_\pi \hookrightarrow \mathbb{Q}_l^{ac}$ *such that*

1. $\overline{\rho}|_{G_F} \sim \overline{\rho}_{\pi,\lambda}$;
2. π_∞ *has weight* k; *and*
3. π_x *is unramified at every finite place* x *of* F.

Proof: Now let F, π, λ be as provided by corollary 5.5. Also denote by $\psi_0 : (\mathbb{A}_F^\infty)^\times / F^\times \to (\mathbb{Q}_l^{ac})^\times$ be the character such that $\epsilon(\psi_0 \circ \mathrm{Art}^{-1})$ equals the determinant of $\rho_{\pi,\lambda}$. Thus ψ_0 is unramified away from l. Note also that if $a \in \mathcal{O}_{F,l}^\times$ then $\psi_0(a)$ is the Teichmüller lift of $(\mathrm{N}a)^{2-k} \bmod l$. Let D be the division algebra with centre F which is ramified at exactly the infinite places of F. Let $\mathcal{O}_D$ be a maximal order in D and fix an isomorphism $\mathcal{O}_{D,x} \cong M_2(\mathcal{O}_{F,x})$ for each finite place x of F. Let $U_0 = \prod_y GL_2(\mathcal{O}_{F,y})$. There is a homomorphism

$$\phi_0 : h_{\overline{\eta}^{k-2},\mathbb{F}_l^{ac},\psi_0}(U_0(\mathcal{O}_F, l)) \to \mathbb{F}_l^{ac}$$

with $\ker \phi_0$ non-Eisenstein and $\overline{\rho}_{\phi_0} \sim \overline{\rho}|_{G_F}$. By corollary 5.2 this factors through $h_{k,\mathbb{F}_l^{ac},\psi_0}(U_0)$ and the proposition follows. $\Box$

Finally we have the following version of our potential version of Serre's conjecture.

THEOREM 5.7 *Let* $l > 3$ *be a prime. Suppose that* $\overline{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_l^{ac})$ *is a continuous irreducible odd representation with* $\overline{\rho}|_{G_l}$ *irreducible. Then there is a Galois totally real field* F *of even degree in which* l *splits completely, a regular algebraic cuspidal automorphic representation* π *of* $GL_2(\mathbb{A}_F)$ *and an embedding* $\lambda : M_\pi \hookrightarrow \mathbb{Q}_l^{ac}$ *such that*

1. $\overline{\rho}|_{G_F} \sim \overline{\rho}_{\pi,\lambda}$;
2. π_∞ *has weight* $k_{\overline{\rho}}$, *where* $k_{\overline{\rho}}$ *is the weight associated to* $\overline{\rho}|_{G_l}$ *by Serre in [S2]; and*
3. π_x *is unramified for every finite place* x *of* F.

Proof: From the definition of $k_{\overline{\rho}}$ we see that there is an integer $0 \leq c < l-1$ such that $2 \leq k_{\overline{\rho}} - c(l+1) \leq l$ and $(\overline{\rho} \otimes \epsilon^{-c})|_{I_l} \sim \omega_2^{k_{\overline{\rho}}-1-c(l+1)} \oplus \omega_2^{l(k_{\overline{\rho}}-1)-c(l+1)}$. By lemma 5.6 we can find a Galois totally real field F of even degree in which l splits completely and a regular algebraic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ such that

1. $(\overline{\rho} \otimes \epsilon^{-c})|_{G_F}$ is equivalent to $\overline{\rho}_{\pi,\lambda}$ for some prime $\lambda | l$ and some embedding $k(\lambda) \hookrightarrow \mathbb{F}_l^{ac}$;
2. π_∞ has weight $k_{\overline{\rho}} - c(l+1)$; and
3. π_x is unramified at every finite place x of F.

By lemma 1.3 we can find, for some character ψ, a homomorphism

$$\phi : h_{k_{\overline{\rho}}-c(l+1),\mathbb{F}_l^{ac},\psi}(U_0) \to \mathbb{F}_l^{ac}$$

with non-Eisenstein kernel such that $\overline{\rho}_\phi \cong (\overline{\rho} \otimes \epsilon^{-c})|_{G_F}$. The theorem now follows from lemma 5.3. □

6 Applications

Combining theorem 2.1 of [Tay4], theorem 5.7, theorem 3.3 and a standard descent argument (see for example the proof of theorem 2.4 of [Tay3]) we obtain our main theorem.

Theorem 6.1 *Let $l > 3$ be a prime and let $2 \leq k \leq l-1$ be an integer. Let $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_{\mathbb{Q}_l^{ac}})$ be a continuous irreducible representation such that*

- *ρ is ramifies at only finitely many primes,*
- $\det \rho(c) = -1$,
- *$\rho|_{G_l}$ is crystalline with Hodge-Tate numbers 0 and $1-k$.*

Let $\overline{\rho}$ denote the reduction of ρ modulo the maximal ideal of $\mathcal{O}_{\mathbb{Q}_l^{ac}}$. If $\overline{\rho}|_{G_l}$ is irreducible assume that $\overline{\rho}$ restricted to $\mathbb{Q}_l(\sqrt{(-1)^{(l-1)/2}l})$ is irreducible. (This will be the case if, for instance, $2k \neq l+3$.) Then there is a Galois totally real field F in which l is unramified with the following property. For each subfield $E \subset F$ with $\mathrm{Gal}\,(F/E)$ *soluble there is a regular algebraic cuspidal automorphic representation π_E of $GL_2(\mathbb{A}_E)$ and an embedding λ of the feild of coefficients of π_E into $\mathbb{Q}_l^{ac}$ such that*

- $\rho_{\pi_E,\lambda} \sim \rho|_{G_E}$,
- *$\pi_{E,x}$ is unramified for all places x of E above l, and*
- *$\pi_{E,\infty}$ has weight k.*

(We explain the parenthetical comment. In the case discussed in that comment $\overline{\rho}|_{I_l} = \omega_2^{k-1} \oplus \omega_2^{l(k-1)}$ and so $\overline{\rho}|_{I_{\mathbb{Q}_l(\sqrt{\pm l})}} = (\omega_2')^{2(k-1)} \oplus (\omega_2')^{2l(k-1)}$ where ω_2' is the fundamental character of level 2 of $I_{\mathbb{Q}_l(\sqrt{\pm l})}$. The assumption $k \neq (l+3)/2$ tells us that $(l+1) \not| 2(k-1)$ so that $\overline{\rho}|_{G_{\mathbb{Q}_l(\sqrt{\pm l})}}$ is absolutely irreducible.)
Combining this with the main theorem of [Bl] we deduce the following corollary.

COROLLARY 6.2 *Keep the assumptions of theorem 6.1. If ρ is unramified at a prime p and if α is an eigenvalue of $\rho(\mathrm{Frob}_p)$ then $\alpha \in \mathbb{Q}^{ac}$ and for any isomorphism $i : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$ we have*

$$|i\alpha|^2 \leq p^{(k-1)/2}.$$

(We remark that we can deduce this corollary for all but finitely many primes p by appealing to theorem 3.4.6 of [BL] instead of the main theorem of [Bl].)
Continue to assume that ρ satisfies the hypotheses of theorem 6.1. If $p \neq l$ and if $i : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$ then we define

$$L_p(i\rho, X) = i \det(1 - \rho_{I_p}(\mathrm{Frob}_p)X) \in \mathbb{C}[X].$$

Corollary 6.2 tells us that

$$L^l(i\rho, s) = \prod_{p \neq l} L_p(i\rho, p^{-s})^{-1}$$

defines a meromorphic function in $\mathrm{Re}\, s > (k+1)/2$.
Choose a non-trivial additive character $\Psi = \prod \Psi_p : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ with $\ker \Psi_l = \mathbb{Z}_l$ and $\Psi_\infty(x) = e^{2\pi\sqrt{-1}x}$. Also choose a Haar measure $dx = \prod dx_p$ on $\mathbb{A}_F$ with dx_∞ the usual measure on $\mathbb{R}$, with $dx_l(\mathbb{Z}_l) = 1$ and with $dx(\mathbb{A}_F/F) = 1$. If $p \neq l$ we will let $\mathrm{WD}(\rho|_{G_p})$ denote the Weil-Deligne representation associated to $\rho|_{G_p}$. Then we define

$$\epsilon(i\rho, s) = \sqrt{-1}^k \prod_{p \neq l} \epsilon(i\mathrm{WD}(\rho^\vee|_{G_p}) \otimes |\mathrm{Art}^{-1}|_p^{-s}, \Psi_p, dx_p).$$

(See [Tat].) Note that $\epsilon(i\rho, s) = WN^{k/2-s}$ where W is independent of s, and where N is the (prime to l) conductor of ρ. The proof of corollary 2.2 of [Tay4] then gives the following corollary.

COROLLARY 6.3 *Keep the assumptions of theorem 6.1 and let $i : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. There is a rational function $L_l(\rho, X)$ such that if we set*

$$L(i\rho, s) = L^l(i\rho, s) L_l(i\rho, l^{-s})^{-1}$$

then $L(i\rho, s)$ has meromorphic extension to the entire complex plane and satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L(i\rho, s) = \epsilon(i\rho, s)(2\pi)^{s-k}\Gamma(k-s)L(i(\rho^\vee \otimes \epsilon^{k-1}), k-s).$$

The proof of corollary 2.4 of [Tay4] also gives us the following result.

Corollary 6.4 *Keep the assumptions of theorem 6.1 and if $k = 2$ further assume that for some prime $p \neq l$ we have*

$$\rho|_{G_p} \sim \begin{pmatrix} \epsilon\chi & * \\ 0 & \chi \end{pmatrix}.$$

Then ρ occurs in the l-adic cohomology (with coefficients in some Tate twist of the constant sheaf) of some variety over $\mathbb{Q}$.

By a *rank d weakly compatible system of l-adic representations* $\mathcal{R}$ over $\mathbb{Q}$ we shall mean a 5-tuple $(M, S, \{Q_p(X)\}, \{\rho_\lambda\}, \{n_1, ..., n_d\})$ where

- M is a number field;
- S is a finite set of rational primes;
- for each prime $p \notin S$ of $\mathbb{Q}$, $Q_p(X)$ is a monic degree d polynomial in $M[X]$;
- for each prime λ of M (with residue characteristic l say)

  $$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow GL_d(M_\lambda)$$

  is a continuous representation such that, if $l \notin S$ then $\rho_\lambda|_{G_l}$ is crystalline, if $p \notin S \cup \{l\}$ then ρ_λ is unramified at p and $\rho_\lambda(\mathrm{Frob}_p)$ has characteristic polynomial $Q_p(X)$; and
- $\{n_1, ..., n_d\}$ is a multiset (i.e. set with multiplicities) of integers such that for all primes λ of M (lying above a rational prime l) the representation $\rho_\lambda|_{G_l}$ is Hodge-Tate with numbers $\{n_1, ..., n_d\}$.

We will call $\{n_1, ..., n_d\}$ the *Hodge numbers* of $\mathcal{R}$. We will call $\mathcal{R}$ *strongly compatible* if for each rational prime p there is a Weil-Deligne representation $\mathrm{WD}_p(\mathcal{R})$ of $W_{\mathbb{Q}_p}$ such that for primes λ of M not dividing p, $\mathrm{WD}_p(\mathcal{R})$ is equivalent to the Frobenius semi-simplification of the Weil-Deligne representation associated to $\rho_\lambda|_{G_p}$. We will call a rank 2 weakly compatible system $\mathcal{R}$ *regular* if the Hodge numbers are distinct and for one, and hence all, primes λ of M we have $\det \rho_\lambda(c) = -1$.

We remark that whatever is meant by a "motive", the l-adic realisations of a "motive" would give rise to weakly compatible systems of l-adic representations which are generally expected to be strongly compatible. Moreover one can use the Hodge realisation to see that if the Hodge numbers of a rank 2 "motive" are distinct then the associated system of l-adic representations is regular in the above sense. This explains the perhaps somewhat unnatural definition of regularity given above.

The following lemma is an easy consequence of the characterisation of one dimensional Hodge-Tate representations of $G_{\mathbb{Q}}$.

Lemma 6.5 *If $\mathcal{R}/\mathbb{Q}$ is a rank 2 weakly compatible system of l-adic representations and if ρ_λ is absolutely reducible for one λ, then ρ_λ is absolutely reducible for all λ.*

We will call a rank 2 weakly compatible system of l-adic representations *reducible* if the hypothesis (and hence the conclusion) of the previous lemma holds. Otherwise we call it *irreducible*.

Theorem 6.6 *Suppose that $\mathcal{R} = (M, S, \{Q_x(X)\}, \{\rho_\lambda\}, \{n_1, n_2\})/\mathbb{Q}$ is a regular, irreducible, rank 2 weakly compatible system of l-adic representations with $n_1 > n_2$.*

1. *There is a Galois totally real field such that for any $i : M \hookrightarrow \mathbb{C}$ there is a regular algebraic cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ with $L(i\mathcal{R}|_{G_F}, s) = L(\pi, s)$.*
2. *For all rational primes $p \notin S$ and for all $i : M \hookrightarrow \mathbb{C}$ the roots of $i(Q_p(X))$ have absolute value $p^{-(n_1+n_2)/2}$.*
3. *$\mathcal{R}$ is strongly compatible.*
4. *Fix $i : M \hookrightarrow \mathbb{C}$. If we define*

$$L(i\mathcal{R}, s) = \prod_p L_p(i\mathrm{WD}_p(\mathcal{R})^\vee, s)^{-1}$$

and

$$\epsilon(i\mathcal{R}, s) = i^{1+n_1-n_2} \prod_p \epsilon(i\mathrm{WD}_p(\mathcal{R})^\vee \otimes |\mathrm{Art}^{-1}|_p^{-s}, \Psi_p, dx_p)$$

then the product defining $L(i\mathcal{R}, s)$ converges to a meromorphic function in $\mathrm{Re}\, s > 1 - (n_1 + n_2)/2$ and $L(i\mathcal{R}, s)$ has meromorphic continuation to the entire complex plane and satisfies a functional equation

$$(2\pi)^{-(s+n_1)}\Gamma(s + n_1)L(i\mathcal{R}, s) = \epsilon(i\mathcal{R}, s)(2\pi)^{s+n_2-1}\Gamma(1 - n_2 - s)L(i\mathcal{R}^\vee, 1 - s).$$

Proof: We may assume that $n_1 = 0$. For all but finitely many primes λ of M the representation ρ_λ satisfies the hypotheses of theorem 6.1. The first part follows immediately from that theorem and the second part from corollary 6.2. Choose one such prime λ and fix an embedding $M_\lambda \subset \mathbb{Q}_l^{ac}$. Let F be as in theorem 6.1 and write

$$1 = \sum_j m_j \mathrm{Ind}\,^{\mathrm{Gal}\,(F/\mathbb{Q})}_{\mathrm{Gal}\,(F/E_j)} \chi_j$$

where $m_j \in \mathbb{Z}$, $\mathrm{Gal}\,(F/E_j)$ is soluble and χ_j is a character of $\mathrm{Gal}\,(F/E_j)$. For each j we have a regular algebraic cuspidal automorphic representation π_j of

$GL_2(\mathbb{A}_{E_j})$ with field of coefficients M_j and an embedding $\lambda_j : M_j \hookrightarrow \mathbb{Q}_l^{ac}$ such that
$$\rho_{\pi_j,\lambda_j} \sim \rho_\lambda|_{G_{E_j}}.$$
We see in particular that $\lambda_j : M_j \hookrightarrow M$. Thus any embedding $\lambda' : M \hookrightarrow \mathbb{Q}_{l'}^{ac}$ gives rise to an embedding $\lambda_j' : M_j \hookrightarrow \mathbb{Q}_{l'}^{ac}$. From the Cebotarev density theorem we see that
$$\rho_{\pi_j,\lambda_j'} \sim \rho_{\lambda'}|_{G_{E_j}}$$
and hence that
$$\rho_{\lambda'} = \sum_j m_j \mathrm{Ind}_{\mathrm{Gal}\,(\mathbb{Q}^{ac}/E_i)}^{\mathrm{Gal}\,(\mathbb{Q}^{ac}/\mathbb{Q})} \rho_{\pi_j,\lambda_j'} \otimes \chi_j.$$
As the $\rho_{\pi_j,\lambda_j'}$ are strongly compatible (see [Tay1]), the same is true for the $\rho_{\lambda'}$. (To check compatibility of the nilpotent operators in the Weil-Deligne representations one notices that it suffices to check that they are equal after any finite base change.) Moreover we see that
$$L(i\mathcal{R}, s) = \prod_j L(\pi_j \otimes (\chi_j \circ \mathrm{Art} \circ \det), s)^{m_j}$$
and that
$$\epsilon(i\mathcal{R}, s) = \prod_j \epsilon(\pi_j \otimes (\chi_j \circ \mathrm{Art} \circ \det), s)^{m_j},$$
and the fourth part of the theorem follows. □

As an example suppose that $X/\mathbb{Q}$ is a rigid Calabi-Yau 3-fold. Let $\mathcal{X}/\mathbb{Z}$ denote a model for X. Also let $\zeta_X(s)$ denote the zeta function of X, so that
$$\zeta_X(s) = \prod_p \zeta_{X,p}(p^{-s})^{-1},$$
where $\zeta_{X,p}(T)$ is a rational function of T and for all but finitely many p we have
$$\zeta_{X,p}(T) = \prod_x (1 - T^{[k(x):\mathbb{F}_p]})$$
where x runs over closed points of $\mathcal{X} \times \mathbb{F}_p$. If we set
$$Z_X(s) = ((s-1)(s-3))^{-1}(2\pi)^{-s\dim H^2(X(\mathbb{C}),\mathbb{R})}\Gamma(s-1)^{\dim H^2(X(\mathbb{C}),\mathbb{R})^{c=1}} \Gamma(s-2)^{\dim H^2(X(\mathbb{C}),\mathbb{R})^{c=-1}}\zeta_X(s),$$
then we have that
$$Z_X(s) = AB^{s-2}Z_X(4-s)$$
where B is a non-zero rational number and where $A = \pm 1$. (To see this note that

- $H^0(X \times \mathbb{Q}^{ac}, \mathbb{Q}_l) = \mathbb{Q}_l$ and $H^6(X \times \mathbb{Q}^{ac}, \mathbb{Q}_l) = \mathbb{Q}_l(-3)$;

- $H^1(X \times \mathbb{Q}^{ac}, \mathbb{Q}_l) = H^5(X \times \mathbb{Q}^{ac}, \mathbb{Q}_l) = (0)$;
- $H^{2,0}(X(\mathbb{C}), \mathbb{C}) = H^{0,2}(X(\mathbb{C}), \mathbb{C}) = (0)$ and so by Lefschetz's theorem there is finite dimensional $\mathbb{Q}$-vector space W with a continuous action of $G_{\mathbb{Q}}$ such that
$$H^2(X \times \mathbb{Q}^{ac}, \mathbb{Q}_l) \cong W \otimes_{\mathbb{Q}} \mathbb{Q}_l(-1)$$
and
$$H^4(X \times \mathbb{Q}^{ac}, \mathbb{Q}_l) \cong W^\vee \otimes_{\mathbb{Q}} \mathbb{Q}_l(-2)$$
for all rational primes l; and
- $\{H^3(X \times \mathbb{Q}^{ac}, \mathbb{Q}_l)\}$ forms a regular, rank two weakly compatible system in the above sense.

Thus it suffices to combine the above theorem with the functional equation for Artin L-series.)

Corrections to [Tay4].

We are extremely grateful to Laurent Clozel for raising the following points. All the references below are to [Tay4].

- The third bulleted point on page 130 should read $\det \overline{\rho} = \epsilon$. (Without this change the choice of a_λ at the top of page 136 becomes impossible.)
- It would be clearer if the parenthetical comment "(as $\beta_v - \beta_v^c$ is coprime to p)" read "(as $\beta_v\beta_v^c = \psi(\phi_v)\psi^c(\phi_v) = p$ and $\beta_v - \beta_v^c$ is coprime to p)".
- Before the "i.e." in the middle of page 135 it would be clearer to add a parenthetical explanation "(note that $\mathrm{End}_{\mathcal{O}_M}(A_1)$ is the centraliser of $\mathcal{O}_M$ in $M_{[N:\mathbb{Q}(\beta_v)]}(\mathcal{O}_{\mathbb{Q}(\beta_v)})$, which is just $\mathcal{O}_N$)".
- The superscript $\mathrm{Gal}\,(L/K)$ in the fourth displayed formula on page 135 should read $\mathrm{Gal}\,(\widetilde{F}_v/F_v)$.
- After the fourth displayed formula on page 135 it would be clearer to add the parenthetical comment: "(N.B. Because $[\widetilde{F}_v : F_v] | \#\chi_v(I_v) | \#k^\times$ and because N_0 contains a primitive $\#k^\times$ root of one, N contains a primitive $[\widetilde{F}_v : F_v]$ root of one.)".
- The proof of lemma 1.4 is wrong. A correct proof can be given as follows. "Choose $z \in (i\mathbb{R}_{>0})^{\mathrm{Hom}\,(M,\mathbb{R})}$. Let M act on $\mathbb{C}^{\mathrm{Hom}\,(M,\mathbb{R})}$ by acting via τ on the τ-component. Set $A = \mathbb{C}^{\mathrm{Hom}\,(M,\mathbb{R})}/(\mathfrak{d}_M^{-1}1 + \mathcal{O}_M z)$ (where 1 denotes the vector $(1, ..., 1)$). This complex torus is an abelian variety with an action of $\mathcal{O}_M$, which is actually defined over $\mathbb{R}$ (in such a way that complex conjugation on $A(\mathbb{C})$ corresponds to complex conjugation

on $\mathbb{C}^{\mathrm{Hom}\,(M,\mathbb{R})}$). Moreover $\mathcal{P}(A,i) \cong \mathcal{O}_M^+$, where $\alpha \in \mathcal{O}_M$ corresponds to the alternating Riemannian form

$$E(x+yz, u+vz) = \mathrm{tr}_{M/\mathbb{Q}}\alpha(yu - xv)$$

for $x, y, u, v \in M \otimes_{\mathbb{Q}} \mathbb{R}$."

- At the end of the second sentence of the paragraph before theorem 1.6 add "and $\det \overline{\rho} = \epsilon$" after "the case that $\overline{\rho}$ has insoluble image".
- With the above changes, specifically adding $\det \overline{\rho} = \epsilon$ in two places, theorem 1.6 requires some further proof. The following will suffice: "We may assume that $\overline{\rho}$ has insoluble image. Choose a totally real quadratic extension F'/F in which all primes above l split and a finite extension k'/k and a character $\xi : G_{F'} \to (k')^\times$ such that $\det \overline{\rho}|_{G_{F'}} = \epsilon\xi^2$. (This is possible as the obstruction to taking the square root of a character lies in the two part of the Brauer group.) Now work with $\overline{\rho}' = \overline{\rho} \otimes \xi^{-1} : G_{F'} \to GL_2(k')$, and find p, N, M, λ, $\wp$, L, ψ, E'/F' and A' as above. Let E be the normal closure of E'/F. Then l and p split completely in E/F. Take $A = A' \times_{E'} E$ and argue as above."

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