# Higher Fields of Norms and $(\phi, \Gamma)$-Modules

Dedicated to John Coates

on the occasion of his 60th birthday

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Abstract. We describe a generalisation of the Fontaine-Wintenberger theory of the "field of norms" functor to local fields with imperfect residue field, generalising work of Abrashkin for higher dimensional local fields. We also compute the cohomology of associated $p$-adic Galois representations using $(\phi, \Gamma)$-modules.

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## Introduction

Abrashkin [3] has found an analogue of the field of norms functor for higher-dimensional local fields. His construction uses the theory of ramification groups [24] for such fields. As an application of his results (include the transfer of the ramification group structure from characteristic zero to characteristic $p$) he obtains the analogue of Grothendieck's anabelian conjecture for higher-dimensional local fields.
In the first part of this paper we construct an analogue of the field of norms for fairly general[1] local fields with imperfect residue field. Like Abrashkin's, as a starting point it uses the alternative characterisation of the ring of integers of the (classical) field of norms as a subring of Fontaine's ring $\mathcal{R} = \widetilde{\mathbf{E}}^+$ (the perfection of $\mathfrak{o}_{\overline{K}} \otimes \mathbb{F}_p$). However we differ from him, and the original construction by Fontaine and Wintenberger [12], [13], by making no appeal to

[1] The only requirement is that the residue field has a finite $p$-basis.

higher ramification theory. We instead restrict to extensions which are "strictly deeply ramified" (see §1.3 and Remark 1.3.8 below) and appeal instead to the differential characterisation of deeply ramified extensions which forms the basis for Faltings's approach to $p$-adic Hodge theory [10] (although we only use the most elementary parts of Faltings's work). These extensions are (in the classical case) closely related to strictly APF extensions; one may hope that by using Abbès and Saito's higher ramification theory ([1], [2]) a theory for all APF extensions could be developed. We hope to clarify this relation in a subsequent paper. In any case, the theory presented here includes those extensions which arise in the theory of $(\phi, \Gamma)$-modules. It is also perhaps worth noting that in the classical case (perfect residue field), the 2 key propositions on which the theory depends (1.2.1 and 1.2.8) are rather elementary.

In the second part of the paper we begin the study of $(\phi, \Gamma)$-modules in this setting, and prove the natural generalisation of Herr's formula [15] for the cohomology of a $p$-adic Galois representation. We also describe a natural family of (non-abelian) extensions to which this theory applies. We hope to develop this further in a subsequent paper.

This work grew out of the preparation of talks given during a study group at Cambridge in winter 2004, and the author is grateful to the members of the study group, particularly John Coates and Sarah Zerbes, for their comments and encouragement, to Victor Abrashkin, Ivan Fesenko and Jan Nekovář for useful discussions, to Pierre Colmez for letting me have some of his unpublished work, and to the referee for his careful reading of the paper. He also wishes to thank Bilkent University, Ankara, for their hospitality while parts of this paper were being written.

As the referee has pointed out, the possibility of such constructions has been known to the experts for some time (see for example the remarks on page 251 of [11]). After this paper was written the author received a copy of Andreatta and Iovita's preprints [4, 5], which construct rings of norms and compute the cohomology of $(\phi, \Gamma)$-modules for Kummer-like extensions of more general $p$-adic base rings.

## Notation

Throughout this paper $p$ denotes a fixed prime number.

If $A$ is an abelian group and $\xi$ an endomorphism of $A$, or more generally an ideal in a ring of endomorphisms of $A$, we write $A/\xi$ for $A/\xi A$, and $A[\xi]$ for the $\xi$-torsion subgroup of $A$.

If $R$ is a ring of characteristic $p$, we denote by $f = f_R \colon x \mapsto x^p$ the Frobenius endomorphism of $R$.

If $K$ is any $p$-adically valued field and $\lambda \in \mathbb{Q}$ belongs to the value group of $K$, we will by abuse of notation write $p^\lambda$ for the fractional ideal comprised of all $x \in K$ with $v_p(x) \geq \lambda$.

We use the sign $=$ to denote equality or canonical isomorphism, and $A := B$ to indicate that $A$ is by definition $B$.

# 1 Fields of norms

## 1.1 Big local fields

By a *big local field* we mean a complete discretely-valued field, whose residue field $k$ has characteristic $p$ and satisfies $[k : k^p] = p^d$ for some $d \geq 0$ (we then talk of a "$d$-big local field"). If $K$ is such a field we use the usual notations: $\mathfrak{o}_K$ for its valuation ring, $\varpi_K$ for a uniformiser (not always fixed), $k_K$ or (if no confusion is likely) simply $k$ for its residue field, and $v_K$ for the normalised valuation on $K$ with $v_K(\varpi_K) = 1$. When $\operatorname{char} K = 0$, we write $e_K$ for its absolute ramification degree, and $v_p$ for the $p$-adic valuation with $v_p(p) = 1$. Of course, $d = 0$ if and only if $K$ is a local field in the usual sense (i.e., with perfect residue field).

We recall for convenience some facts about big local fields and their extensions, and fix some notation. If $L/K$ is a finite separable extension of $d$-big local fields, then $[L : K] = ef_0p^s$ where $e = e(L/K) = v_L(\varpi_K)$ is the (reduced) ramification degree, and $f_0$ and $p^s$ are the separable and inseparable degrees of the extension $k_L/k_K$, respectively, so that $f = f_0p^s = [k_L : k_K]$.

If $L/K$ is a finite separable extension of big local fields, the valuation ring $\mathfrak{o}_L$ is not necessarily of the form $\mathfrak{o}_K[x]$. There are two particular cases when this is true:

(i) when the residue class extension $k_L/k_K$ is separable [21, III, §6 Lemme 4]. Then there exists $x \in \mathfrak{o}_L$ with $\mathfrak{o}_L = \mathfrak{o}_K[x]$; and if $k_L = k_K$ then $x = \varpi_L$ for any uniformiser $\varpi_L$ will do, and its minimal polynomial is an Eisenstein polynomial.

(ii) when $\varpi_K = \varpi_L$ and the residue class extension is purely inseparable and simple[2]. Let $k_L = k_K(b)$ for some $b$ with $b^q = a \in k_K \setminus k_K^p$, and let $u \in \mathfrak{o}_L$ be any lift of $b$. Then $\mathfrak{o}_L = \mathfrak{o}_K[u]$ where the minimal polynomial of $u$ has the form $g(T) = T^q + \sum_{i=1}^{q-1} c_iT^i - v$, with $\varpi_K | c_i$ and $a = v \bmod \varpi_K$.

Conversely, let $g = T^q + \sum_{i=0}^{q-1} c_iT^i \in \mathfrak{o}_K[T]$ be any polynomial. Let us say that $g$ is a *fake Eisenstein polynomial* if (a) its degree $q$ is a power of $p$; (b) for every $i \geq 1$, $c_i \equiv 0 \pmod{\varpi_K}$; and (c) $c_0$ is a unit whose reduction mod $\varpi_K$ is not a $p^{\text{th}}$ power. Then $g$ is irreducible (since it is irreducible mod $\varpi_K$) and $\mathfrak{o}_K[T]/(g)$ is a discrete valuation ring. It is the valuation ring of a totally fiercely ramified extension of $K$ of degree $q$.

In particular, if $L/K$ is Galois of prime degree then one of (i), (ii) applies, so $\mathfrak{o}_L = \mathfrak{o}_K[x]$.

For any big local field $K$ of characteristic zero there exists a complete subfield $K_u \subset K$ which is absolutely unramified (that is, $p$ is a uniformiser) having the same residue field as $K$. (This holds by the existence of Cohen subrings; see for example [EGA4, 19.8.6] or [18, pp. 211–212]). If $d = 0$ then $K_u$ is unique; otherwise (except when $e_K = 1$) it is non-unique [EGA4, 19.8.7]. If $L/K$ is a finite extension it is not in general possible to find such subfields $K_u \subset K$, $L_u \subset L$ satisfying $K_u \subset L_u$ (even when $K$ itself is absolutely unramified).

[2] In the terminology of [24], $L/K$ is totally fiercely ramified.

Let $K$ be a big local field with residue field $k$, and choose $K_u \subset K$ as above. Then for any $m$ with $0 < m \le e_K$, the quotient $\mathfrak{o}_K/(\varpi_K^m)$ contains $\mathfrak{o}_{K_u}/(p) = k$ and therefore $\mathfrak{o}_K/(\varpi_K^m) \simeq k[\varpi_K]/(\varpi_K^m)$. When $k$ is perfect (but not in general) this isomorphism is canonical, since the projection $\mathfrak{o}_K/(\varpi_K^m) \twoheadrightarrow k$ has a unique section, whose image is the maximal perfect subring of $\mathfrak{o}_K/(\varpi_K^m)$.

If $K$ is a big local field of characteristic $p$ then it contains a coefficient field (non-unique if $d > 0$), so that $K \simeq k_K((\varpi_K))$. If $L/K$ is a finite separable extension then one cannot in general find a coefficient field of $L$ containing one of $K$.

From now on, unless stated explicitly to the contrary, all big local fields will be assumed to have characteristic zero. For a finite extension $L/K$ we then write

$$\delta(L/K) = \sum \delta_i(L/K) = v_p(\mathfrak{D}_{L/K})$$

where the $\delta_i(L/K)$ are the $p$-adic valuations of the primary factors of $\Omega(L/K)$.

## 1.2 Differentials and ramification

If $L/K$ is an extension of big local fields, we usually write $\Omega(L/K) := \Omega_{\mathfrak{o}_L/\mathfrak{o}_K}$ for the module of relative Kähler differentials, which is an $\mathfrak{o}_L$-module of finite length. Then $\Omega(L/K)$ can be generated by $\le (d+1)$ generators (for example, by equation (1.2.2) below). The Fitting ideal of $\Omega(L/K)$ (the product of its primary factors) equals the relative different $\mathfrak{D}_{L/K}$, defined in the usual way as the inverse of the $\mathfrak{o}_K$-dual of $\mathfrak{o}_L$ with respect to the trace form; see for example [10, Lemma 1.1].

Proposition 1.2.1. *Let $L/K$ be a finite extension of $d$-big local fields with $[L : K] = p^{d+1}$. Assume that there exists a surjection*

$$\Omega(L/K) \twoheadrightarrow (\mathfrak{o}_L/\xi)^{d+1}$$

*for some ideal $\xi \subset \mathfrak{o}_K$ with $0 < v_p(\xi) \le 1$. Then $e(L/K) = p$ and $k_L = k_K^{1/p}$, and the Frobenius endomorphism of $\mathfrak{o}_L/\xi$ has a unique factorisation*

$$\begin{array}{ccc} \mathfrak{o}_L/\xi & \xrightarrow{\ f\ } & \mathfrak{o}_L/\xi \\ {\scriptstyle \text{mod } \xi'} \downarrow & & \uparrow {\scriptstyle \text{inclusion}} \\ \mathfrak{o}_L/\xi' & \overset{\simeq}{\dashrightarrow} & \mathfrak{o}_K/\xi \end{array}$$

*where $\xi' \subset \mathfrak{o}_L$ is the ideal with valuation $p^{-1}v_p(\xi)$. In particular, Frobenius induces a surjection $f\colon \mathfrak{o}_L/\xi \twoheadrightarrow \mathfrak{o}_K/\xi$.*

*Proof.* Let $\varpi_L$ be a uniformiser. We have $[L : K] = p^{d+1} = ef_0p^s$, and if $p^r = [k_L : k_L^p k]$ then $\dim_{k_L} \Omega_{k_L/k} = r \le s$. We have the exact sequence of differentials

$$(\varpi_L)/(\varpi_L^2) \longrightarrow \Omega(L/K) \otimes_{\mathfrak{o}_L} k_L \longrightarrow \Omega_{k_L/k} \longrightarrow 0 \qquad (1.2.2)$$

and if $e=1$ the first map is zero (taking $\varpi_L=\varpi_k$). It follows that

$$\dim_{k_L}\big(\Omega(L/K)\otimes_{\mathfrak{o}_L}k_L\big)\ \begin{cases}\le 1+r & \text{in general}\\ = r & \text{if } e=1.\end{cases}$$

By definition, $d=[k_L:k_L^p]\ge r$ and by hypothesis $\dim_{k_L}\big(\Omega(L/K)\otimes_{\mathfrak{o}_L}k_L\big)\ge d+1$, so we must have $r=s=d$, $f_0=1$, $e=p$ and $k_L=k^{1/p}$.
Let $\{t_\alpha\mid 1\le\alpha\le d\}\subset\mathfrak{o}_L^*$ be a lift of a $p$-basis for $k_L$. Then $d\varpi_L,\{dt_\alpha\}$ is a basis for $\Omega(L/K)\otimes k_L$. Introduce a multi-index notation $I=(i_1,\dots,i_d)$, $t^I=\prod t_\alpha^{i_\alpha}$. Then the $k$-vector space $\mathfrak{o}_L/(\varpi_K)$ has as a basis the reduction mod $\varpi_K$ of the $p^{d+1}$ monomials $\{t^I\varpi_L^j\mid 0\le j<p,\ 0\le i_\alpha<p\}$. So by Nakayama's lemma,

$$\mathfrak{o}_L=\mathfrak{o}_K[\varpi_L,\{t_\alpha\}]=\bigoplus_{\substack{0\le j<p\\0\le i_\alpha<p}}t^I\varpi_L^j\mathfrak{o}_K. \tag{1.2.3}$$

Lemma 1.2.4. *If* $x=\sum_{0\le j<p,\,0\le i_\alpha<p}x_{I,j}t^I\varpi_L^j$ *with* $x_{I,j}\in\mathfrak{o}_K$, *then*

$$v_p(x)=\min_{I,j}\Big(v_p(x_{I,j})+\frac{j}{e_L}\Big).$$

*Proof.* If $y_I\in\mathfrak{o}_K$ for $0\le i_\alpha<p$, then since the elements $t^I$ are linearly independent mod $(\varpi_L)$, we have

$$\varpi_L\Big|\sum_I y_It^I \iff \text{for all } I,\ y_I\equiv 0 \pmod{\varpi_K} \iff \varpi_K\Big|\sum_I y_It^I$$

from which we see that

$$v_K\Big(\sum_I y_It^I\Big)=\min_I v_K(y_I) \tag{1.2.5}$$

and that this is an integer. Therefore

$$v_K\Big(\varpi_L^j\sum_I x_{I,j}t^I\Big)\equiv\frac{j}{p}\pmod{\mathbb{Z}}$$

and so

$$v_p(x)=v_p\Big(\sum_{j=0}^{p-1}\varpi_L^j\sum_I x_{I,j}t^I\Big)=\min_j\Big\{v_p\big(\varpi_L^j\sum_I x_{I,j}t^I\big)\Big\}.$$

Then the lemma follows from (1.2.5). □

From (1.2.3) we obtain $(d+1)$ relations in $\mathfrak{o}_L$ of the shape:

$$\varpi_L^p=\sum_{j=0}^{p-1}A_j(t)\varpi_L^j,\qquad t_\alpha^p=\sum_{j=0}^{p-1}B_{\alpha,j}(t)\varpi_L^j\quad(1\le\alpha\le d) \tag{1.2.6}$$

where $A_j$, $B_{\alpha,j} \in \mathfrak{o}_K[X_1, \dots, X_d]$ are polynomials of degree $< p$ in each variable. Write $D_\gamma$ for the derivative with respect to $X_\gamma$, and $\delta_{\alpha\gamma}$ for Kronecker delta. Therefore in $\Omega(L/K)$ the following relations hold:

$$\Big(-p\varpi_L^{p-1} + \sum_{j=1}^{p-1} jA_j(t)\varpi_L^{j-1}\Big)d\varpi_L + \sum_\gamma \Big(\sum_{j=0}^{p-1} D_\gamma A_j(t)\varpi_L^j\Big)dt_\gamma = 0$$
$$\Big(\sum_{j=1}^{p-1} jB_{\alpha,j}(t)\varpi_L^{j-1}\Big)d\varpi_L - pt_\alpha^{p-1}dt_\alpha + \sum_\gamma \Big(\sum_{j=0}^{p-1} D_\gamma B_{\alpha,j}(t)\varpi_L^j\Big)dt_\gamma = 0$$

The condition on $\Omega(L/K)$ forces all the coefficients in these identities to be divisible by $\xi$. From (1.2.4) this implies that for all $j > 0$, $A_j(t)\varpi_L^{j-1} \equiv 0 \equiv B_{\alpha,j}(t)\varpi_L^{j-1} \pmod{\xi}$. Therefore

$$\varpi_L^p \equiv A_0(t) \quad \text{and} \quad t_\alpha^p \equiv B_{\alpha,0}(t) \pmod{\varpi_L \xi}.$$

Similarly, for every $\gamma$ and every $j \geq 0$,

$$D_\gamma A_j(t) \equiv D_\gamma B_{\alpha,j}(t) \equiv 0 \pmod{\varpi_L^{-j}\xi}.$$

This last congruence implies that the nonconstant coefficients of $A_j$ and $B_{\alpha,j}$ are divisible by $\varpi_L^{-j}\xi$, so especially

$$A_0(t) \equiv A_0(0), \quad B_{\alpha,0}(t) \equiv B_{\alpha,0}(0) \pmod{\xi}.$$

The first of these congruences, together with 1.2.4 and the first equation of (1.2.6), implies that $v_L(A_0(0)) = p$. We will therefore choose $\varpi_K = A_0(0)$ as the uniformiser of $K$. Then

$$\varpi_L^p \equiv \varpi_K, \quad t_\alpha^p \equiv b_\alpha \pmod{\xi}$$

where $b_\alpha = B_{\alpha,0}(0) \in \mathfrak{o}_K^*$. If $m = v_K(\xi)$ then, as noted just before the statement of this Proposition, $\mathfrak{o}_K/\xi \xrightarrow{\sim} k[\varpi_K]/(\varpi_K^m)$. We fix such an isomorphism. If $\bar{b}_\alpha \in k$ denotes the reduction of $b_\alpha$ mod $\varpi_K$, then by (1.2.3) there are compatible isomorphisms

$$\mathfrak{o}_L/\xi \xrightarrow{\sim} k[\varpi_L, \{t_\alpha\}]/(\varpi_L^{mp}, \{t_\alpha^p - \bar{b}_\alpha\})$$
$$\mathfrak{o}_L/\xi' \xrightarrow{\sim} k[\varpi_L, \{t_\alpha\}]/(\varpi_L^m, \{t_\alpha^p - \bar{b}_\alpha\})$$

such that the inclusion $\mathfrak{o}_K/\xi \hookrightarrow \mathfrak{o}_L/\xi$ induces the identity on $k$ and maps $\varpi_K$ to $\varpi_L^p$. Therefore

$$\mathfrak{o}_L/\xi' \xrightarrow[f]{\sim} (\mathfrak{o}_L/\xi)^p = \mathfrak{o}_K/\xi \subset \mathfrak{o}_L/\xi$$

as required. □

*Remark* 1.2.7. It is perhaps worth noting that in the case $d = 0$ the proof just given simplifies greatly; in this case $L/K$ is totally ramified by hypothesis, so $\varpi_L$ satisfies an Eisenstein polynomial over $K$, whose constant term we may take to be $-\varpi_K$. We then have *canonical* isomorphisms $\mathfrak{o}_K/\xi = k[\varpi_K]/(\varpi_K^m)$, $\mathfrak{o}_L/\xi = k[\varpi_L]/(\varpi_L^{mp})$, and the minimal polynomial of $\varpi_L$ gives at once the congruence $\varpi_L^p \equiv \varpi_K \pmod{\xi}$ — cf. [21], Remark 1 after Proposition 13 of §III.6.

Recall now the key lemma in the theory ([9], [10], [22]) of deep ramification of local fields:

Proposition 1.2.8. *(Faltings) Let $L$ and $K'$ be linearly disjoint finite extensions of a $d$-big local field $K$, and set $L' = LK' \simeq L \otimes_K K'$. Assume there exists a surjection $\Omega(K'/K) \twoheadrightarrow (\mathfrak{o}_{K'}/p^\lambda)^{d+1}$ for some $\lambda \geq 0$. Then*

$$\delta(L'/K') \leq \delta(L/K) - \frac{1}{d+2}\min(\lambda, \delta(L/K)).$$

*Proof.* (expanded from the proof of [10, Theorem 1.2]). For simplicity of notation write:

$$R = \mathfrak{o}_K,\ S = \mathfrak{o}_L,\ R' = \mathfrak{o}_{K'},\ S' = \mathfrak{o}_{L'}$$

$$\delta = \delta(L/K),\ \delta_i = \delta_i(L/K),\ \delta' = \delta(L'/K'),\ \delta_i' = \delta_i(L'/K').$$

If $M$ is an $S'$-module of finite length, write $\ell_p(M)$ for $1/e_{L'}$ times the length of $M$ (so $\ell_p(M)$ also equals the $p$-adic valuation of the Fitting ideal of $M$). Consider the homomorphism $\gamma = \beta\alpha$, which links the two exact[3] sequences of differentials in the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & S' \otimes_S \Omega_{S/R} & & & & \\
 & & & & \downarrow{\scriptstyle\alpha} & \overset{\gamma}{\searrow} & & & \\
0 & \longrightarrow & S' \otimes_{R'} \Omega_{R'/R} & \longrightarrow & \Omega_{S'/R} & \xrightarrow{\ \beta\ } & \Omega_{S'/R'} & \longrightarrow & 0
\end{array}$$

In this diagram, all entries are torsion $S'$-modules which can be generated by $\leq (d+1)$ elements. We then have the following inequalities:

(a) $\ell_p(\ker\gamma) \geq \min(\lambda, \delta)$

(b) $\ell_p(\operatorname{im}\gamma) \geq (d+2)\delta' - (d+1)\delta$

[3] See [20, p.420, footnote] or [10, Lemma 1.1]

Since $\ell_p(\operatorname{im}\gamma)+\ell_p(\ker\gamma) = \ell_p(S'\otimes\Omega_{S/R}) = \delta$, combining (a) and (b) gives the desired inequality.

*Proof of (a):*

We have $\alpha\colon \ker\gamma \xrightarrow{\sim} \operatorname{im}\alpha\cap\ker\beta$. Therefore as there is a surjection $\Omega_{R'/R} \to (R'/p^\lambda)^{d+1}$, and as $\Omega_{S'/R}$ can be generated by $(d+1)$ elements, we have

$$\ker\beta \supset \Omega_{S'/R}[p^\lambda] \simeq (S'/p^\lambda)^{d+1}$$

and so

$$\ker\gamma \supset S'\otimes_S \Omega_{S/R}[p^\lambda] \simeq \bigoplus_{i=0}^{d} S'/p^{\min(\lambda,\delta_i)}.$$

Therefore

$$\ell_p(\ker\gamma) \geq \sum \min(\lambda,\delta_i) \geq \min(\lambda, \sum\delta_i) = \min(\lambda,\delta).$$

*Proof of (b):*

Evidently $\operatorname{im}\gamma = S'd(S) = S'd(R'S)$. Now since under the trace form we have $\mathfrak{D}^{-1}_{L/K} = \operatorname{Hom}_R(S,R)$, it follows that

$$R'\,\mathfrak{D}^{-1}_{L/K} = \operatorname{Hom}_{R'}(R'\otimes S, R') \supset \operatorname{Hom}_{R'}(S',R') = \mathfrak{D}^{-1}_{L'/K'}$$

and so $S' \supset R'S \supset \mathfrak{D}_{L/K}\mathfrak{D}^{-1}_{L'/K'} = \varpi^j S'$ say, where $\varpi = \varpi_{L'}$ is a uniformiser and $j = e_{L'}(\delta-\delta')$. Therefor we have inclusions

$$\operatorname{im}\gamma \supset S'd(\varpi^j S') \supset \varpi^j\Omega_{S'/R'} = p^{\delta-\delta'}\Omega_{S'/R'} \simeq \bigoplus_{i=0}^{d} S'/(p^{\max(0,\delta'_i-\delta+\delta')})$$

and therefore

$$\ell_p(\operatorname{im}\gamma) \geq \sum_{i=0}^{d}(\delta'_i-\delta+\delta') = (d+2)\delta' - (d+1)\delta.$$

□

## 1.3 Deep ramification and norm fields

In this section we will work with towers $K_0 \subset K_1 \subset \dots$ of finite extensions of $d$-big local fields. If $K_\bullet = \{K_n\}$ is such a tower, write $K_\infty = \bigcup K_n$. We abbreviate $\mathfrak{o}_n = \mathfrak{o}_{K_n}$, $\varpi_n = \varpi_{K_n}$ and $k_n = k_{K_n}$. Define an equivalence relation on towers by setting $K_\bullet \sim K'_\bullet$ if there exists $r\in\mathbb{Z}$ such that for every $n$ sufficiently large, $K'_n = K_{n+r}$.

We shall say that a tower $K_\bullet$ is *strictly deeply ramified* if there exists $n_0 \geq 0$ and an ideal $\xi \subset \mathfrak{o}_{n_0}$ with $0 < v_p(\xi) \leq 1$, such that the following condition holds:

> For every $n \geq n_0$, the extension $K_{n+1}/K_n$ has degree $p^{d+1}$, and there exists a surjection $\Omega(K_{n+1}/K_n) \twoheadrightarrow (\mathfrak{o}_{n+1}/\xi)^{d+1}$.

(1.3.1)

If $K_\bullet$ is strictly deeply ramified then so is any equivalent tower (with the same $\xi$ and possible different $n_0$). See 1.3.8 below for some comments on this definition. Let $K_\bullet$ be a strictly deeply ramified tower, and $(n_0, \xi)$ a pair for which (1.3.1) holds. Then by 1.2.1, for every $n \geq n_0$ we have $e(K_{n+1}/K_n) = p$, and Frobenius induces a surjection $f\colon \mathfrak{o}_{n+1}/\xi \twoheadrightarrow \mathfrak{o}_n/\xi$. We can then choose uniformisers $\varpi_n$ of $K_n$ such that $\varpi_{n+1}^p \equiv \varpi_n \pmod{\xi}$ for every $n \geq n_0$. Define

$$X^+ = X^+(K_\bullet, \xi, n_0) := \varprojlim_{n \geq n_0} (\mathfrak{o}_n/\xi, f)$$

and wite $pr_n\colon X^+ \twoheadrightarrow \mathfrak{o}_n/\xi$ for the $n^{\text{th}}$ projection in the inverse limit. Set $\Pi = (\varpi_n \bmod \xi) \in X^+$.

Let $k' = \varprojlim_{n \geq n_0} (k_n, f)$; since $k_{n+1} = k_n^{1/p}$, the projections $pr_n\colon k' \to k_n$ for any $n \geq n_0$ are isomorphisms. (Note that the residue field $k_\infty$ of $K_\infty$ is then the perfect closure $(k')^{1/p^\infty}$ of $k'$.)

Theorem 1.3.2. *$X(K_\bullet, \xi, n_0)$ is a complete discrete valuation ring of characteristic $p$, with uniformiser $\Pi$, and residue field $k'$. Up to canonical isomorphism (described in the proof below) $X^+(K_\bullet, \xi, n_0)$ depends only on the equivalence class of the tower $K_\bullet$, and not on the choices of $\xi$ and $n_0$ satisfying* (1.3.1).

*Proof.* Define a partial order on triples $(K_\bullet, \xi, n_0)$ satisfying (1.3.1) by setting $(K'_\bullet, \xi', n'_0) \geq (K_\bullet, \xi, n_0)$ if and only if $v_p(\xi') \leq v_p(\xi)$ and for some $r \geq 0$ one has $n'_0 + r \geq n_0$ and $K'_n = K_{n+r}$ for every $n \geq 0$. It is obvious that under this order any two triples have an upper bound if and only if the associated towers of extensions are equivalent.

If $(K'_\bullet, \xi', n'_0) \geq (K_\bullet, \xi, n_0)$ and $r$ is as above then there is a canonical map

$$X^+(K_\bullet, \xi, n_0) \to X^+(K'_\bullet, \xi', n'_0)$$
$$g\colon (x_n)_{n \geq n_0} \mapsto (x_{n+r} \bmod \xi')_{n \geq n'_0}.$$

If $\xi = \xi'$, $g$ is obviously an isomorphism. In general we can define a map $h$ in the other direction by

$$h\colon (y_n)_{n \geq n'_0} \mapsto (y_{n+s-r}^{p^s})_{n \geq n_0}$$

which is well-defined and independent of $s$ for $s$ sufficiently large. Then $g$ and $h$ are mutual inverses. For three triples $(K''_\bullet, \xi'', n''_0) \geq (K'_\bullet, \xi', n'_0) \geq (K_\bullet, \xi, n_0)$ the isomorphisms just described are obviously transitive, so we obtain the desired independence on choices.

Truncating $K_\bullet$ if necessary we may therefore assume that $n_0 = 0$ and $\xi = \varpi_0$. We then have by 1.2.1

$$X^+/(\Pi^{p^m}) = \varprojlim \mathfrak{o}_n/(\varpi_0, \varpi_n^{p^m}) \xrightarrow[pr_m]{\sim} \mathfrak{o}_m/(\varpi_0).$$

Therefore $\varprojlim X^+/(\Pi^{p^m}) = \varprojlim \mathfrak{o}_m/(\varpi_0) = X^+$, so $X^+$ is $\Pi$-adically complete and separated, and $\Pi$ is not nilpotent. Since $X^+/(\Pi)$ is a field, $X^+$ is therefore a discrete valuation ring with uniformiser $\Pi$. $\square$

To make the definition of $X^+$ truly functorial, we define for an equivalence class $\mathcal{K}$ of towers

$$X^+_{\mathcal{K}} := \varinjlim X^+(K_\bullet, \xi, n_0)$$

where the limit is taken over triples $(K_\bullet, \xi, n_0)$ with $K_\bullet \in \mathcal{K}$ and $(\xi, n_0)$ satisfying (1.3.1), and the transition maps are the isomorphisms $g$ in the preceding proof. We let $\Pi_{\mathcal{K}}$ denote any uniformiser of $X^+_{\mathcal{K}}$, and define $k_{\mathcal{K}} = X^+_{\mathcal{K}}/(\Pi_{\mathcal{K}})$ to be its residue field.

Definition. The field of fractions $X_{\mathcal{K}}$ of $X^+_{\mathcal{K}}$ is the *norm field* of $\mathcal{K}$.

Of course this is illogical terminology, because when $d > 0$ this has nothing to do with norms. But when $d = 0$ it is just the field of norms $X_K(K_\infty)$ for the extension $K_\infty/K$ in the sense of Fontaine and Wintenberger ([12], [13], and [23] — especially 2.2.3.3), and for $d > 0$ see also remark 1.3.9 below.
Let $K_\bullet$ be a tower of $d$-big local fields, $\mathcal{K}$ its equivalence class, and $L_\infty/K_\infty$ a finite extension. Then there exists a finite extension $L_0/K_0$ contained in $L_\infty$ such that $L_\infty = K_\infty L_0$; write $L_n = K_n L_0$. The equivalence class $\mathcal{L}$ of $L_\bullet$ depends only on $L_\infty$.

Theorem 1.3.3. *Let $\mathcal{K}$ and $\mathcal{L}$ be as above. Then if $\mathcal{K}$ is strictly deeply ramified so is $\mathcal{L}$.*

*Proof.* The condition on the extension degrees is clear. By Proposition 1.2.8 with $(K, K', L, L') = (K_n, K_{n+1}, L_n, L_{n+1})$ we have

$$\delta(L_{n+1}/K_{n+1}) \le \delta(L_n/K_n) - \frac{1}{d+2}\min(v_p(\xi), \delta(L_n/K_n))$$

and so $\delta(L_n/K_n) \to 0$ as $n \to \infty$. Using the exact sequences of differentials for the extensions $L_{n+1}/L_n/K_n$ and $L_{n+1}/K_{n+1}/K_n$, it follows that the annihilators of the kernel and cokernel of the canonical map

$$\mathfrak{o}_{L_{n+1}} \otimes_{\mathfrak{o}_{K_{n+1}}} \Omega(K_{n+1}/K_n) \to \Omega(L_{n+1}/L_n)$$

have $p$-adic valuation tending to zero as $n \to \infty$. Therefore $L_\bullet$ satisfies (1.3.1) for any $\xi'$ with $0 < v_p(\xi') < v_p(\xi)$ (and suitable $n_0$). $\square$

Theorem 1.3.4. *Let $K_\bullet$ be strictly deeply ramified, $\mathcal{K}$ its equivalence class and $L_\infty/K_\infty$ a finite extension.*

*(i) $X_{\mathcal{L}}$ is a finite separable extension of $X_{\mathcal{K}}$. More generally, if $L'_\infty/K_\infty$ is another finite extension and $\tau\colon L_\infty \to L'_\infty$ is a $K_\infty$-homomorphism, the maps $\tau\colon \mathfrak{o}_{L_n}/\xi \hookrightarrow \mathfrak{o}_{L'_n}/\xi$, for $n$ sufficiently large and $v_p(\xi)$ sufficiently small, induce an injection $X_{\mathcal{K}}(\tau)\colon X^+_{\mathcal{L}} \hookrightarrow X^+_{\mathcal{L}'}$ which makes $X_{\mathcal{L}'}/X_{\mathcal{L}}$ a separable extension of degree $[L'_\infty : \tau L_\infty]$.*

(ii) *The sequences* $(e(L_n/K_n))$, $(s(L_n/K_n))$ *and* $(f_0(L_n/K_n))$ *are stationary for* $n$ *sufficiently large. Their limits equal* $e(X_{\mathcal{L}}/X_{\mathcal{K}})$, $s(X_{\mathcal{L}}/X_{\mathcal{K}})$ *and* $f_0(X_{\mathcal{L}}/X_{\mathcal{K}})$ *respectively.*

(iii) *There exists a constant* $c \geq 0$ *such that* $\delta(L_n/K_n) = cp^{-n}$ *for* $n$ *sufficiently large.*

*Proof.* It suffices in (i) to consider the case of a single extension $L_\infty/K_\infty$. Let $m = [L_\infty : K_\infty]$. Changing $\xi$ and $n_0$ if necessary, we can assume that (1.3.1) holds for both $K_\bullet$ and $L_\bullet$ with the same $\xi$ and $n_0$, and that $[L_n : K_n] = [L_\infty : K_\infty] = m$ for $n \geq n_0$. Then for every $n \geq n_0$, $\mathfrak{o}_{L_n}/\xi$ is a finite flat $\mathfrak{o}_n/\xi$-algebra of rank $m$. Therefore by Nakayama's lemma $X_{\mathcal{L}}^+$ is a finite flat $X_{\mathcal{K}}^+$-algebra of rank $m$, so $X_{\mathcal{L}}/X_{\mathcal{K}}$ is a finite extension of degree $m$.
Consider the discriminant $\mathfrak{d} = \mathfrak{d}_{X_{\mathcal{L}}/X_{\mathcal{K}}} \subset X_{\mathcal{K}}^+$ of $X_{\mathcal{L}}^+/X_{\mathcal{K}}^+$. The projection of $\mathfrak{d}$ to $\mathfrak{o}_n/\xi$ equals the discriminant of $\mathfrak{o}_{L_n}/\xi$ over $\mathfrak{o}_n/\xi$. Since $\delta(L_n/K_n) \to 0$ the latter is nonzero for $n$ sufficiently large. So $X_{\mathcal{L}}/X_{\mathcal{K}}$ is separable. Its residue field extension is isomorphic to $k_{L_n}/k_n$ for $n$ sufficiently large. So the sequences $(f_0(L_n/K_n))$ and $(s(L_n/K_n))$ are ultimately stationary, hence the same holds for $e(L_n/K_n) = [L_n : K_n]/f(L_n/K_n)$.
Let $v_{X_{\mathcal{K}}}(\mathfrak{d}) = r$; then for $n \geq n_0$, $(\varpi_n^r)$ equals the discriminant of $\mathfrak{o}_{L_n}/\xi$ over $\mathfrak{o}_n/\xi$. So for $n$ sufficiently large, $v_p(\varpi_n^r) = m\delta(L_n/K_n)$. Therefore $\delta(L_n/K_n) = p^{-n}c$ where $c$ equals $rp^n/me_{K_n}$, which is constant for $n$ sufficiently large. □

So if $\mathcal{K}$ is strictly deeply ramified, for any finite $L_\infty/K_\infty$ we may define

$$X_{\mathcal{K}}^+(L_\infty) := X_{\mathcal{L}}^+, \quad X_{\mathcal{K}}(L_\infty) := X_{\mathcal{L}}$$

which by the above is a functor from the category of finite extensions of $K_\infty$ to that of $X_{\mathcal{K}}$.

Theorem 1.3.5. *The functor* $X_{\mathcal{K}}(-)$ *defines an equivalence between the category of finite extensions of* $K_\infty$ *and the category of finite separable extensions of* $X_{\mathcal{K}}$.

*Proof.*
*The functor is fully faithful.* It is enough to show that if $L_\infty/K_\infty$ is a finite Galois extension then any non-trivial $\sigma \in \mathrm{Gal}(L_\infty/K_\infty)$ induces a non-trivial automorphism $X_{\mathcal{K}}(\sigma)$ of $X_{\mathcal{K}}(L_\infty) = X_{\mathcal{L}}$. In that case since $[X_{\mathcal{L}} : X_{\mathcal{K}}] = [L_\infty : K_\infty]$ it follows that $X_{\mathcal{L}}/X_{\mathcal{K}}$ is a Galois extension, and that $X_{\mathcal{K}}(-)\colon \mathrm{Gal}(L_\infty/K_\infty) \xrightarrow{\sim} \mathrm{Gal}(X_{\mathcal{L}}/X_{\mathcal{K}})$, from which the fully faithfulness is formal by Galois theory.
Assume that $X_{\mathcal{K}}(\sigma) = 1$. Then replacing $\sigma$ by a suitable power, we may assume it has prime order. Replacing $K_\infty$ by the fixed field of $\sigma$, and truncating the tower if necessary we may then assume that $L_\infty/K_\infty$ is cyclic of prime degree $\ell$, with Galois group $G$ say.
In this case for $n$ sufficiently large, $L_n/K_n$ is cyclic of degree $\ell$ and so $\mathfrak{o}_{L_n} = \mathfrak{o}_{K_n}[x_n]$ for some $x_n \in \mathfrak{o}_{L_n}$. If $g_n \in \mathfrak{o}_{K_n}[T]$ is the minimal polynomial of $x_n$

then

$$\mathfrak{D}_{L_n/K_n} = (g_n'(x_n)) = \prod_{1\neq\sigma\in G} (x_n - \sigma x_n).$$

So since $\delta(L_n/K_n) \to 0$, it follows that if $1 \neq \sigma \in G$ and $n$ is sufficiently large, then $\sigma x_n \not\equiv x_n \pmod{\xi}$. So $\sigma$ acts nontrivially on $\mathfrak{o}_{L_n}/\xi$ hence also on $X_{\mathcal{L}}$.

*The functor is essentially surjective.*

Using fully faithfulness, it is enough to show that if $Y/X_{\mathcal{K}}$ is a finite Galois extension then there exists $L_\infty/K_\infty$ and a $X_{\mathcal{K}}$-isomorphism $X_{\mathcal{K}}(L_\infty) \xrightarrow{\sim} Y$. Let $Y^+ \subset Y$ be the valuation ring of $Y$. Building the extension step-by-step we are reduced to the cases:

(a) $Y/X_{\mathcal{K}}$ is unramified. The categories of finite unramified extensions of $X_{\mathcal{K}}$ and $K_\infty$ are equivalent to the categories of finite separable extensions of their respective residue fields $k_{\mathcal{K}}$ and $k_\infty$. But as $k_\infty$ is the perfect closure of $k_{\mathcal{K}}$ these categories are equivalent.

(b) $Y/X_{\mathcal{K}}$ is ramified and of prime degree $\ell$. There are two subcases:

(b1) $e(Y/X_{\mathcal{K}}) = \ell$. Then $Y^+ = X_{\mathcal{K}}^+[\Pi_Y]$ where the uniformiser $\Pi_Y$ satisfies an Eisenstein polynomial $G(T) \in X_{\mathcal{K}}^+[T]$.

Choose $n_0$ such that (1.3.1) holds and $v_p(\xi) > v_p(\varpi_{n_0})$. For every $n \geq n_0$, let $g_n \in \mathfrak{o}_n[T]$ be any monic polynomial such that $\bar{g}_n = pr_n(G) \in (\mathfrak{o}_n/\xi)[T]$. Then $g_n$ is an Eisenstein polynomial, and $g_n(T^p) \equiv g_{n+1}(T)^p \pmod{\xi}$. Fix an algebraic closure $\overline{K}$ of $K_\infty$ and let $\bar{\mathfrak{o}}$ be its valuation ring.

*Claim:* There exist $n_1 \geq n_0$, $\xi' \in \mathfrak{o}_{n_1}$ with $v_p(\xi') \leq v_p(\xi)$, and roots $x_n \in \bar{\mathfrak{o}}$ of $g_n$, such

(i) For every $n \geq n_1$, $x_{n+1}^p \equiv x_n \pmod{\xi'}$

(ii) If $L_n := K_n(x_n) \subset \overline{K}$ then $L_{n+1} = K_{n+1}L_n$ for all $n \geq n_1$.

(iii) If $n \geq n_1$ then $(\mathfrak{o}_{L_{n+1}}/\xi')^p = \mathfrak{o}_{L_n}/\xi'$, and there is an isomorphism of $X_{\mathcal{K}}^+$-algebras

$$Y^+ \xrightarrow{\sim} \varprojlim_{n\geq n_1} \left(\mathfrak{o}_{L_n}/\xi', f\right)$$

mapping $\Pi_Y$ to $(x_n \bmod \xi')_n$.

Granted this claim, $L_\infty := \bigcup L_n$ is an extension with $X_{\mathcal{K}}(L_\infty) \simeq Y$.

*Proof of claim.* (i) Let $S_n = \{x_{n,i} \mid 1 \leq i \leq \ell\} \subset \bar{\mathfrak{o}}$ be the set of roots of $g_n$. Then for all $n \geq 0$ and all $i$ we have

$$\prod_{j=1}^{\ell} (x_{n+1,i}^p - x_{n,j}) = g_n(x_{n+1,i}^p) \equiv g_{n+1}(x_{n+1,i})^p \equiv 0 \pmod{\xi}.$$

Choose $n_1 \geq n_0$ and $\xi' \subset \mathfrak{o}_{n_1}$ such that $0 < v_p(\xi') \leq \ell^{-1}v_p(\xi)$. Then for each $i$ there exists $j$ with $x_{n+1,i}^p \equiv x_{n,j} \pmod{\xi'}$. Choosing such a $j$ for each $i$ then

determines a map $S_{n+1} \to S_n$, and by compactness $\varprojlim S_n$ is nonempty. Let $(x_n)$ be any element of the inverse limit; then (i) is satisfied.
If $L_n = K_n(x_n)$, then $[L_n : K_n] = e(L_n/K_n) = \ell$. Since it satisfies an Eisenstein polynomial, $x_n$ is a uniformiser of $L_n$, and $\mathfrak{o}_{L_n}/\xi' = (\mathfrak{o}_n/\xi')[x_n] = (\mathfrak{o}_n/\xi')[T]/\bar{g}_n(T)$. Therefore for each $n$ there is a unique surjection

$$f\colon \mathfrak{o}_{L_{n+1}}/\xi' \twoheadrightarrow \mathfrak{o}_{L_n}/\xi' \tag{1.3.6}$$

which is Frobenius on $\mathfrak{o}_{n+1}/\xi$ and maps $x_{n+1}$ to $x_n$ (mod $\xi'$).
Let $\mu_n\colon Y^+ \twoheadrightarrow \mathfrak{o}_{L_n}/\xi'$ be the map taking $\Pi_Y$ to $x_n$, and whose restriction to $X_{\mathcal{K}}^+$ is $pr_n$. The different of $Y/X_{\mathcal{K}}$ is $(G'(\Pi_Y))$, and it is nonzero since $Y/X_{\mathcal{K}}$ is separable. Let $r = v_Y(G'(\Pi_Y))$. Then $\bar{g}_n'(x_n) = \mu_n(G'(\Pi_Y))$ equals $x_n^r$ times a unit. Therefore if $n$ is large enough so that $v_{L_n}(\xi) > r$, we have $v_{L_n}(g_n'(x_n)) = r$. Therefore $\delta(L_n/K_n) = v_p(g_n'(x_n)) \to 0$. Order the roots of $g_n$ so that $x_n = x_{n,1}$. Since

$$\prod_{i\neq 1}(x_{n+1}^p - x_{n,i}) \equiv \prod_{i\neq 1}(x_n - x_{n,i}) \equiv g_n'(x_n) \pmod{\xi}'$$

it follows that for $n$ sufficiently large, $x_{n+1}^p$ is closer to $x_n$ than to any of the other roots $\{x_{n,i} \mid i \neq 1\}$ of $g_n$. By Krasner's lemma, $x_n \in K_n(x_{n+1}^p)$, so $L_n \subset L_{n+1}$ and the map (1.3.6) is induced by the Frobenius endomorphism of $\mathfrak{o}_{L_{n+1}}/\xi'$ (by its uniqueness).
We have to check that $L_{n+1} = K_{n+1}L_n$ for $n$ sufficiently large. Since $[L_{n+1} : K_{n+1}] = \ell = [L_n : K_n]$ it is enough to show that the extensions $L_n/K_n$ and $K_{n+1}/K_n$ are linearly disjoint. If not, since $[L_n : K_n]$ is prime, there exists a $K_n$-homomorphism $\tau\colon L_n \to K_{n+1}$, and so $\ell = p$. But as $\delta(L_n/K_n) \to 0$ and $\Omega(K_{n+1}/K_n)$ surjects onto $(\mathfrak{o}_{n+1}/\xi)^{d+1}$ this implies that for $n$ sufficiently large, $\Omega(K_{n+1}/\tau L_n)$ surjects onto $k_{n+1}^{d+1}$, which is impossible as $[K_{n+1} : \tau L_n] = p^d$.
Finally, making $n_1$ sufficiently large, we have a commutative diagram

$$\begin{array}{ccccc}
X_{\mathcal{K}}^+ & \hookrightarrow & Y^+ & & \\
\downarrow\!\!\downarrow{\scriptstyle pr_{n+1}} & & {\scriptstyle \mu_{n+1}}\downarrow\!\!\downarrow & \searrow{\scriptstyle \mu_n} & \\
\mathfrak{o}_{n+1}/\xi' & \hookrightarrow & \mathfrak{o}_{L_{n+1}}/\xi' & & \\
& \searrow{\scriptstyle f} & & \searrow{\scriptstyle f} & \\
& & \mathfrak{o}_n/\xi' & \hookrightarrow & \mathfrak{o}_{L_n}/\xi'
\end{array} \tag{1.3.7}$$

where $L_{n+1} = K_{n+1}L_n$ for $n \geq n_1$, inducing a $X_{\mathcal{K}}^+$-homorphism

$$Y^+ \to X_{\mathcal{K}}^+(L_\infty) = \varprojlim_{n\geq n_1} (\mathfrak{o}_{L_n}/\xi', f).$$

Since $Y^+$ and $X_{\mathcal{K}}^+(L_\infty)$ are both valuation rings of extensions of $X_{\mathcal{K}}$ of the same degree, this is an isomorphism. □

(b2) $e=1$ and $s=1$. Then $Y^+ = X^+_{\mathcal{K}}[U]$ for some $U \in (Y^+)^*$, whose reduction mod $\Pi_{\mathcal{K}}$ generates $k_Y/k_{\mathcal{K}}$. As in (b1), let $G$ be the minimal polynomial of $U$, and get $\bar{g}_n \in (\mathfrak{o}_n/\xi)[T]$ be its image, and $g_n \in \mathfrak{o}_n[T]$ any monic lift. Then $g_n$ is a fake Eisenstein polynomial (cf. §1.1) hence is irreducible; just as above we find roots $u_n \in \bar{\mathfrak{o}}$ of $g_n$ such that $u_{n+1}^p \equiv u_n \pmod{\xi'}$ for $n$ sufficiently large and suitable $\xi'$. The remainder of the argument proceeds exactly as for (b1). □

*Remark* 1.3.8. The condition 1.3.1 is closely related, in the case $d=0$, to that of strictly arithmetically profinite extension [23, §1.2.1]. It is possible to weaken the condition without affecting the results: one could instead just require that there exist surjections $\Omega(K_{n+1}/K_n) \twoheadrightarrow (\mathfrak{o}_{n+1}/\xi_{n+1})^{d+1}$ where $\xi_n \subset \mathfrak{o}_n$ is a sequence of ideals whose $p$-adic valuations do not tend too rapidly to zero.

*Remark* 1.3.9. Suppose that $K$ (and therefore also $X_{\mathcal{K}}$) is a $(d+1)$-dimensional local field. Then, as Fesenko and Zerbes have remarked to the author, local class field theory for higher dimensional local fields [17] gives a reciprocity homomorphism $K^M_{d+1}(K) \to \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$, where $K^M_*()$ is Milnor $K$-theory, which becomes an isomorphism after passing to a suitable completion $\widehat{K^M_{d+1}(K)}$. Therefore there is a commutative diagram

$$\begin{array}{ccc} \varprojlim_{\text{norms}} \widehat{K^M_{d+1}(K_n)} & \xrightarrow{\sim} & \widehat{K^M_{d+1}(X_{\mathcal{K}})} \\ \| & & \| \\ \varprojlim \mathrm{Gal}(\overline{K}/K_n)^{\mathrm{ab}} = \mathrm{Gal}(\overline{K}/K_\infty)^{\mathrm{ab}} & \xrightarrow{\sim} & \mathrm{Gal}(\overline{X_{\mathcal{K}}}/X_{\mathcal{K}})^{\mathrm{ab}} \end{array}$$

which may be viewed as the generalisation of the Fontaine-Wintenberger definition (for $d=0$) of $X_{\mathcal{K}}$ as the inverse limit of the $K_n$ with respect to the norm maps.

## 2 $(\phi, \Gamma)$-modules

### 2.1 Definitions

We review Fontaine's definition [11] of the $(\phi,\Gamma)$-module associated to a $p$-adic representation, in an appropriately axiomatic setting. The key assumptions making the theory possible are (2.1.1) and (2.1.2) below.

We begin with a strictly deeply ramified tower $K_\bullet$ of $d$-big local fields (always of characteristic zero) such that $K_n/K_0$ is Galois for each $n$, and set $K=K_0$, $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Fixing an algebraic closure $\overline{K}$ of $K$ containing $K_\infty$, write $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K) \supset \mathcal{H}_K = \mathrm{Gal}(\overline{K}/K_\infty)$. All algebraic extensions of $K$ will be tacitly assumed to be subfields of $\overline{K}$.

Let $\mathbf{E}_K = X_{\mathcal{K}}$ be the norm field of the tower $K_\bullet$, and $\mathbf{E}^+_K$ its valuation ring. To be consistent with the notation established in [8], we write $\bar{\pi}$, or when there is no confusion simply $\pi$, for a uniformiser of $\mathbf{E}_K$. Then $\mathbf{E}^+_K$ is (noncanonically)

isomorphic to $k_K[[\bar{\pi}]]$. For a finite extension $L/K$, one writes $\mathbf{E}_L$ for the norm field of the tower $LK_\bullet$, and $\mathbf{E}$ for $\varinjlim \mathbf{E}_L$ (the limit over all finite extensions $L/K$). The group $\mathcal{G}_K$ then acts continuously (for the valuation topology) on $\mathbf{E} = \mathbf{E}_K^{\mathrm{sep}}$, and this action identifies the subgroup $\mathcal{H}_K$ with $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_K)$.
If $E$ is any of these rings of characteristic $p$, write $E^{\mathrm{rad}}$ for the perfect closure $\sqrt[p^\infty]{E}$ of $E$, and $\widetilde{\mathbf{E}}$ for the completion of $E^{\mathrm{rad}}$. In particular, $\widetilde{\mathbf{E}}^+$ is the valuation ring of the algebraic closure of $\mathbf{E}_K$, and can be alternatively described as $\varprojlim(\mathfrak{o}_{\overline{K}}/p, f)$, also known as $\mathcal{R}$. By continuity the action of $\mathcal{G}_K$ on $\mathbf{E}$ extends uniquely to a continuous action on $\mathbf{E}^{\mathrm{rad}}$ and $\widetilde{\mathbf{E}}$, and for any $L$ on has $\widetilde{\mathbf{E}}_L = \widetilde{\mathbf{E}}^{\mathcal{H}_L}$.
In the theory of $(\phi, \Gamma)$-modules there are two kinds of rings of characteristic zero which appear. The first are those with perfect residue ring, which are completely canonical. These are:

- $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+) \subset \widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$;
- $\widetilde{\mathbf{A}}_L = W(\widetilde{\mathbf{E}}_L)$, for any finite $L/K$;
- $\widetilde{\mathbf{A}}_L^+ = W(\widetilde{\mathbf{E}}_L^+) = \widetilde{\mathbf{A}}^+ \cap \widetilde{\mathbf{A}}_L$

They carry a unique lifting of Frobenius (namely the Witt vector endomorphism $F$), and the action of $\mathcal{G}_K$ on $\widetilde{\mathbf{E}}$ defines an action on $\widetilde{\mathbf{A}}$. The ring $\widetilde{\mathbf{A}}$ has a canonical topology (also called the weak topology) which is the weakest structure of topological ring for which $\widetilde{\mathbf{A}} \to \widetilde{\mathbf{E}}$ is continuous (for the valuation topology on $\widetilde{\mathbf{E}}$). Equivalently, in terms of the definition of $W(\widetilde{\mathbf{E}})$ as $\widetilde{\mathbf{E}}^{\mathbb{N}}$ with Witt vector multiplication and addition, it is the product of the valuation topologies on the factors. The $\mathcal{G}_K$-action is evidently continuous with respect to the canonical topology. The other natural topology to put on $\widetilde{\mathbf{A}}$ is the $p$-adic (or strong) topology.
The other rings of characteristic zero have imperfect residue rings, and depend on certain choices. Let $\mathbf{A}_K^+$ be a complete regular local ring of dimension 2, together with an isomorphism $\mathbf{A}_K^+/(p) \simeq \mathbf{E}_K^+$. Such a lift of $\mathbf{E}_K^+$ exists and is unique up to nonunique isomorphism. If $C$ is a $p$-Cohen ring with residue field $k$, then any $\mathbf{A}_K^+$ is (non-canonically) isomorphic to $C[[\pi]]$. Define $\mathbf{A}_K$ to be the $p$-adic completion of $(\mathbf{A}_K^+)_{(p)}$; it is a $p$-Cohen ring with residue field $\mathbf{E}_K$.
Fix a principal ideal $I = (\pi)$ of $\mathbf{A}_K^+$ lifting $(\bar{\pi}) \subset \mathbf{E}_K^+$. Then $\mathbf{A}_K$ is the $p$-adic completion of $\mathbf{A}_K^+[1/\pi]$. The essential choice to be made is a lifting $\phi\colon \mathbf{A}_K^+ \to \mathbf{A}_K^+$ of the absolute Frobenius endomorphism of $\mathbf{E}_K^+$, which is required to satisfy two conditions. The first is simply

$$\phi(I) \subset I. \tag{2.1.1}$$

It is clear that $\phi$ extends to an endomorphism of $\mathbf{A}_K$, whose reduction mod $p$ is the absolute Frobenius of $\mathbf{E}_K$.
For any finite extension $L/K$ there exists a finite étale extension $\mathbf{A}_L/\mathbf{A}_K$, unique up to unique isomorphism, with residue field $\mathbf{E}_L$. Let $\mathbf{A}_{\overline{K}} = \varinjlim \mathbf{A}_L$, the

direct limit taken over finite extensions $L/K$, and let $\mathbf{A}$ be the $p$-adic completion of $\mathbf{A}_{\overline{K}}$. Then $\mathbf{A}_{\overline{K}}$ is the maximal unramified extension of $\mathbf{A}_K$, and the isomorphism $\mathcal{H}_K \simeq \mathrm{Gal}(\mathbf{E}/\mathbf{E}_K)$ extends to an isomorphism with $\mathrm{Aut}(\mathbf{A}_{\overline{K}}/\mathbf{A}_K)$. This in turn extends to a unique action of $\mathcal{H}_K$ on $\mathbf{A}$, continuous for both the canonical and $p$-adic topologies, and for any finite $L/K$ one has $\mathbf{A}^{\mathcal{H}_L} = \mathbf{A}_L$ by the Ax-Sen-Tate theorem [6].

Since $\mathbf{A}_L/\mathbf{A}_K$ is étale there is a unique extension of $\phi$ to an endomorphism of $\mathbf{A}_L$ whose reduction mod $p$ is Frobenius; by passage to the limit and completion it extends to an endomorphism of $\mathbf{A}$. We use $\phi$ to denote any of these endomorphisms.

The lifting $\phi$ of Frobenius determines (see [7, Ch,IX, §1, ex.14] and [11, 1.3.2]) a unique embedding

$$\mu_K\colon \mathbf{A}_K \longrightarrow W(\mathbf{E}_K)$$

such that $\mu\circ\phi = F\circ\mu$, which maps $\mathbf{A}_K^+$ into $W(\mathbf{E}_K^+)$. We identify $\mathbf{A}_K$ with its image under this map. An alternative description of $\mu_K$ is as follows: consider the direct limit

$$\phi^{-\infty}\mathbf{A}_K = \varinjlim(\mathbf{A}_K,\phi)$$

on which $\phi$ is an automorphism. Its $p$-adic completion is a complete unramified DVR of characteristic zero, with perfect residue field $\mathbf{E}_K^{\mathrm{rad}}$, hence is canonically isomorphic to $W(\mathbf{E}_k^{\mathrm{rad}})$. Likewise the action of $\phi$ on $\mathbf{A}$ determines an embedding $\mu\colon \mathbf{A} \hookrightarrow W(\mathbf{E})$, which is uniquely characterised by the same properties as $\mu_K$. The embeddings $\mathbf{A}_K \hookrightarrow \mathbf{A} \hookrightarrow W(\widetilde{\mathbf{E}})$ induce topologies on $\mathbf{A}_K$ and $\mathbf{A}$. One writes $\mathbf{A}^+ = \mathbf{A}\cap\widetilde{\mathbf{A}}^+$. Then $\mathbf{A}^+/p\mathbf{A}^+ \simeq \mathbf{E}^+$ by [11, 1.8.3], and a basis of neighbourhoods of 0 for the canonical topology on $\mathbf{A}$ is the collection of $\mathbf{A}^+$-submodules

$$p^m\mathbf{A} + \pi^n\mathbf{A}^+, \quad m,n\geq 0.$$

The reduction map $\mathbf{A}\to\mathbf{E}$ is $\mathcal{H}_K$-equivariant by construction, and so $\mu$ is $\mathcal{H}_K$-equivariant. The second, and much more serious, condition to be satisfied by $\phi$ is:

$$\mathbf{A}\subset\widetilde{\mathbf{A}} \text{ is stable under the action of } \mathcal{G}_K. \tag{2.1.2}$$

In particular, $\mathbf{A}$ inherits an action of $\mathcal{G}_K$, and $\mathbf{A}_K$ and $\mathbf{A}_K^+$ inherit an action of $\Gamma_K$, continuous for the canonical topology.

A $\mathbb{Z}_p$-representation of $\mathcal{G}_K$ is by definition a $\mathbb{Z}_p$-module of finite type with a continuous action of $\mathcal{G}_K$. Assuming (2.1.1) and (2.1.2) above are satisfied, Fontaine's theory associates to a $\mathbb{Z}_p$-representation of $\mathcal{G}_K$ the $\mathbf{A}_K$-module of finite type

$$\mathbf{D}(V) = \mathbf{D}_K(V) := (\mathbf{A}\otimes_{\mathbb{Z}_p} V)^{\mathcal{H}_K}.$$

The functor $\mathbf{D}$ is faithful and exact. The $\mathbf{A}_K$-module $\mathbf{D}(V)$ has commuting semilinear actions of $\phi$ and $\Gamma_K$. Being a finitely-generated $\mathbf{A}_K$-module, $\mathbf{D}(V)$ has a natural topology (which is the quotient topology for any surjection $\mathbf{A}_K^d \to \mathbf{D}(V)$), for which the action of $\Gamma_K$ is continuous. Therefore $\mathbf{D}(V)$ has the structure of an étale $(\phi,\Gamma_K)$-module, and just as in [11] we have:

Theorem 2.1.3. *Assume conditions* (2.1.1) *and* (2.1.2) *are satisfied. The functor* $\mathbf{D}$ *is an equivalence of categories*

$$(\mathbb{Z}_p\text{-representations of } \mathcal{G}_K) \longrightarrow (\text{étale } (\phi,\Gamma_K)\text{-modules over } \mathbf{A}_K)$$

*and an essential inverse is given by* $D \mapsto (\mathbf{A} \otimes_{\mathbf{A}_K} D)^{\phi=1}$.

Lemma 2.1.4. *(i) The sequences*

$$0 \to \mathbb{Z}_p \to \mathbf{A} \xrightarrow{\phi-1} \mathbf{A} \to 0 \tag{2.1.5}$$

$$0 \to \mathbb{Z}_p \to \mathbf{A}^+ \xrightarrow{\phi-1} \mathbf{A}^+ \to 0 \tag{2.1.6}$$

*are exact, and for every* $n > 0$*, the map*

$$\phi - 1 \colon \pi^n \mathbf{A}^+ \to \pi^n \mathbf{A}^+ \tag{2.1.7}$$

*is an isomorphism.*
*(ii) For any* $n > 0$ *and for any* $L/K$*, the map* $\phi - 1 \colon \mathbf{E}_L^+ \to \mathbf{E}_L^+$ *is an isomorphism.*

*Proof.* It suffices (by passage to the limit) to prove the corresponding statements mod $p^m$. By dévissage it is enough to check them mod $p$. Therefore (2.1.5), (2.1.6) follow from the Artin–Schreier sequences for $\mathbf{E}$ and $\mathbf{E}^+$, and (2.1.7) follows from (ii), since $\mathbf{A}^+/p\mathbf{A}^+ = \mathbf{E}^+$. Rewriting the map as $\pi^{n(p-1)}\phi - 1 \colon \mathbf{E}_L^+ \to \mathbf{E}_L^+$, by Hensel's lemma it is an isomorphism. ☐

## 2.2 Cohomology

We assume that we are in the situation of the previous subsection. In particular, we assume that conditions (2.1.1) and (2.1.2) are satisfied. If $G$ is a profinite group and $M$ a topological abelian group with a continuous $G$-action, by $H^*(G,M)$ we shall always mean continuous group cohomology. Write $\mathcal{C}^\bullet(G,M)$ for the continuous cochain complex of $G$ with coefficients in $M$, so that $H^*(G,M) = H^*(\mathcal{C}^\bullet(G,M))$. If $\phi \in \mathrm{End}_G(M)$ write $\mathcal{C}^\bullet_\phi(G,M)$ for the simple complex associated to the double complex $[\mathcal{C}^\bullet(G,M) \xrightarrow{\phi-1} \mathcal{C}^\bullet(G,M)]$. Write $H^*_\phi(G,M)$ for the cohomology of $\mathcal{C}^\bullet_\phi(G,M)$, and $H^*_\phi(M)$ for the cohomology of the complex $M \xrightarrow{\phi-1} M$ (in degrees 0 and 1).
If $H \subset G$ is a closed normal subgroup and $M$ is discrete then there are two Hochschild–Serre spectral sequences converging to $H^*_\phi(G,M)$, whose $E_2$-terms are respectively

$$H^a(G/H, H^b_\phi(H,M)) \quad \text{and} \quad H^a_\phi(G/H, H^b(H,M)),$$

and which reduce when $H = \{1\}$ and $H = G$ respectively to the long exact sequence

$$H^a(G, M^{\phi=1}) \to H^a_\phi(G,M) \to H^{a-1}(G, M/(\phi-1)) \to H^{a+1}(G, M^{\phi=1})$$

and the short exact sequences

$$0 \to H^{b-1}(G,M)/(\phi-1) \to H^b_\phi(G,M) \to H^b(G,M)^{\phi=1} \to 0.$$

Theorem 2.2.1. *Let $V$ be a $\mathbb{Z}_p$-representation of $\mathcal{G}_K$, and set $D = \mathbf{D}_K(V)$. There are isomorphisms*

$$H^*(\mathcal{G}_K, V) \xrightarrow{\sim} H^*_\phi(\Gamma_K, D) \tag{2.2.2}$$

$$H^*(\mathcal{H}_K, V) \xrightarrow{\sim} H^*_\phi(D) \tag{2.2.3}$$

*which are functorial in $V$, and compatible with restriction and corestriction.*

*Remarks.* (i) In the case when $K$ has perfect residue field, and $K_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension, we recover Théorème 2.1 of [15], since taking $\gamma$ to be a topological generator of $\Gamma_K \simeq \mathbb{Z}_p$, the complex

$$D \xrightarrow{\binom{\phi-1}{\gamma-1}} D \oplus D \xrightarrow{(\gamma-1,1-\phi)} D$$

computes $H^*_\phi(\Gamma_K, D)$.
(ii) An oversimplified version of the proof runs as follows: from the short exact sequence (2.1.5) we have, tensoring with $V$ and applying the functor $R\Gamma(\mathcal{H}_K, -)$, an isomorphism (in an unspecified derived category)

$$R\Gamma(\mathcal{H}_K, V) \xrightarrow{\sim} R\Gamma(\mathcal{H}_K, \mathbf{A}\otimes V \xrightarrow{\phi-1} \mathbf{A}\otimes V). \tag{2.2.4}$$

But for $i>0$, $H^i(\mathcal{H}_K, \mathbf{A}\otimes V) = 0$, and $H^0(\mathcal{H}_K, \mathbf{A}\otimes V) = D$, so the right-hand side of (2.2.4) is isomorphic to $[D \xrightarrow{\phi-1} D]$. Applying $R\Gamma(\Gamma_K, -)$ then would give

$$R\Gamma(\mathcal{G}_K, V) \xrightarrow{\sim} R\Gamma(\Gamma_K, D \xrightarrow{\phi-1} D).$$

Since the formalism of derived categories in continuous cohomology requires extra hypotheses (see for example [16] or [19, Ch.4]) which do not hold in the present situation, we fill in this skeleton by explicit reduction to discrete modules. (Note that in general these Galois cohomology groups will not be of finite type over $\mathbb{Z}_p$, hence need not commute with inverse limits.)

*Proof.* We construct a functorial isomorphism (2.2.2); once one knows that it is compatible with restriction, one may obtain (2.2.3) by passage to the limit over finite extensions $L/K$; alternatively it can be proved directly (and more simply) by the same method as (2.2.2). The compatibility of the constructed isomorphisms with restriction and corestriction is an elementary verification which we leave to the interested reader.
Write $V_m = V/p^mV$ and $D_m = D/p^mD$; we have $D_m = \mathbf{D}_K(V_m)$ since $\mathbf{D}_K$ is exact. A basis of neighbourhoods of 0 in $D_m$ is given by the open subgroups

$$D_m \cap (\pi^n\mathbf{A}^+ \otimes V_m) = (\pi^n\mathbf{A}^+ \otimes V_m)^{\mathcal{H}_K}$$

which are stable under $\Gamma_K$ and $\phi$. Write also

$$D_{m,n} = D_m/(\pi^n \mathbf{A}^+ \otimes V_m)^{\mathcal{H}_K}$$

which is a discrete $\Gamma_K$-module; we have topological isomorphisms

$$D_m = \varprojlim_n (D_{m,n}), \quad D = \varprojlim_m (D_m)$$

and $H^*_\phi(\Gamma_K, D)$ is the cohomology of $\varprojlim_{m,n} \mathcal{C}^\bullet_\phi(\Gamma_K, D_{m,n})$.
From 2.1.4 we obtain for every $m, n \geq 1$ a short exact sequence

$$0 \to V_m \to (\mathbf{A}/\pi^n \mathbf{A}^+) \otimes V_m \xrightarrow{\phi - 1} (\mathbf{A}/\pi^n \mathbf{A}^+) \otimes V_m \to 0$$

and so the canonical map

$$\mathcal{C}^\bullet(\mathcal{G}_K, V_m) \to \mathcal{C}^\bullet_\phi(\mathcal{G}_K, (\mathbf{A}/\pi^n \mathbf{A}^+) \otimes V_m) \tag{2.2.5}$$

is a quasi-isomorphism, for every $m, n \geq 1$.
The inclusion $D_{m,n} \hookrightarrow (\mathbf{A}/\pi^n \mathbf{A}^+) \otimes V_m$ induces a morphism of complexes

$$\alpha_{m,n} \colon \mathcal{C}^\bullet_\phi(\Gamma_K, D_{m,n}) \to \mathcal{C}^\bullet_\phi(\mathcal{G}_K, (\mathbf{A}/\pi^n \mathbf{A}^+) \otimes V_m).$$

Passing to the inverse limit and taking cohomology, this together with (2.2.5) defines a functorial map

$$H^*_\phi(\Gamma_K, D) \to H^*(\mathcal{G}_K, V) \tag{2.2.6}$$

whose inverse will be (2.2.2). To prove it is an isomorphism, it is enough to show:

Proposition 2.2.7. *For every $m \geq 1$, $\varprojlim_n (\alpha_{m,n})$ is a quasi-isomorphism.*

*Proof.* First note that the exactness of $\mathbf{D}$ implies that there is a short exact sequence

$$0 \to D_m \to D_{m+1} \to D_1 \to 0$$

which clearly has a continuous set-theoretical splitting (it is enough to give a continuous section of the surjection $\mathbf{A}_K \to \mathbf{E}_K$ which is easy), so gives rise to a long exact sequence of continuous cohomology. Suppose the result is shown for $m = 1$. Then (2.2.6) is an isomorphism for every $V$ with $pV = 0$, and so by the 5-lemma it is an isomorphism for every $V$ of finite length, whence the result holds for all $m \geq 1$. So we may assume for the rest of the proof that $pV = 0$ and $m = 1$, and therefore replace $\mathbf{A}$ by $\mathbf{E}$.
Fix a finite Galois extension $L/K$ such that $\mathcal{H}_L$ acts trivially on $V$. We then have a natural map

$$D_{1,n} = \frac{(\mathbf{E} \otimes V)^{\mathcal{H}_K}}{(\pi^n \mathbf{E}^+ \otimes V)^{\mathcal{H}_K}} \to \frac{(\mathbf{E} \otimes V)^{\mathcal{H}_L}}{(\pi^n \mathbf{E}^+ \otimes V)^{\mathcal{H}_L}} = \mathbf{E}_L/\pi^n \mathbf{E}_L^+ \otimes V.$$

The map $\alpha_{1,n}$ therefore factors as the composite of two maps

$$\begin{aligned} \mathcal{C}^\bullet_\phi(\Gamma_K, D_{1,n}) &\xrightarrow{\beta_n} \mathcal{C}^\bullet_\phi(\mathrm{Gal}(L_\infty/K), \mathbf{E}_L/\pi^n\mathbf{E}^+_L \otimes V) \\ &\xrightarrow{\gamma_n} \mathcal{C}^\bullet_\phi(\mathcal{G}_K, \mathbf{E}/\pi^n\mathbf{E}^+ \otimes V) \end{aligned}$$

which we treat in turn:

(a) $\gamma_n$ *is a quasi-isomorphism.* We may compute the induced map $H^*(\gamma_n)$ on cohomology using the morphism of associated spectral sequences, which on $E_2$-terms is the map

$$\begin{aligned} &H^a(\mathrm{Gal}(L_\infty/K), H^b_\phi(\mathbf{E}_L/\pi^n\mathbf{E}^+_L) \otimes V) \\ &\qquad \to H^a(\mathrm{Gal}(L_\infty/K), H^b_\phi(\mathcal{H}_L, \mathbf{E}_L/\pi^n\mathbf{E}^+_L) \otimes V) \end{aligned} \tag{2.2.8}$$

We then have a commutative square (where $\mathbf{E}$ is regarded as a discrete $\mathcal{H}_L$-module)

$$\begin{array}{ccc} H^b_\phi(\mathbf{E}_L) & \longrightarrow & H^b_\phi(\mathbf{E}_L/\pi^n\mathbf{E}^+_L) \\ \downarrow & & \downarrow \\ H^b_\phi(\mathcal{H}_L, \mathbf{E}) & \longrightarrow & H^b_\phi(\mathcal{H}_L, \mathbf{E}/\pi^n\mathbf{E}^+) \end{array}$$

in which all the arrows are isomorphisms; in fact by 2.1.4(ii), the horizontal arrows are isomorphisms, and since $H^b(\mathcal{H}_L, \mathbf{E}) = 0$ for $b > 0$ the same is true of the left vertical arrow. Therefore the maps (2.2.8) are isomorphisms, and hence $\gamma_n$ is a quasi-isomorphism, for every $n \geq 1$.

(b) $\varprojlim(\beta_n)$ *is a quasi-isomorphism.* We consider the cohomology of the finite group $\Delta = \mathrm{Gal}(L_\infty/K_\infty)$ acting on the short exact sequence

$$0 \to \pi^n\mathbf{E}^+_L \otimes V \to \mathbf{E}_L \otimes V \to \mathbf{E}_L/\pi^n\mathbf{E}^+_L \otimes V \to 0. \tag{2.2.9}$$

Lemma 2.2.10. *(i)* $H^j(\Delta, \mathbf{E}_L \otimes V) = 0$ *for* $j > 0$.
*(ii) There exists* $r \geq 0$ *such that for all* $j > 0$ *and* $n \in \mathbb{Z}$*, the group* $H^j(\Delta, \pi^n\mathbf{E}^+_L \otimes V)$ *is killed by* $\pi^r$.

*Proof.* It is enough to prove (ii) for $n = 0$ (since $\pi$ is fixed by $\Delta$) and since $\mathbf{E}_L = \varinjlim \pi^{-n}\mathbf{E}^+_L$, (ii) implies (i). It is therefore enough to know that if $M$ is any $\mathbf{E}^+_L$-module with a semilinear action of $\Delta$, then there exists $r \geq 0$ such that $\pi^r H^j(\Delta, M) = 0$ for any $j > 0$, which is standard.[4] □

To complete the computation of $\beta_n$, we next recall [16, 1.9] that an inverse system $(X_n)$ of abelian groups is *ML-zero* if for every $n$ there exists $r = r(n) \geq 0$

[4] Let $M \to N^\bullet$ be the standard resolution. Choose $y \in \mathbf{E}^+_L$ such that $x = \mathrm{tr}_{\mathbf{E}_L/\mathbf{E}_K}(y) \neq 0$, and let $\lambda(m) = \sum_{g \in \Delta} g(ym)$. Then the composite $(N^\bullet)^\Delta \hookrightarrow N^\bullet \xrightarrow{\lambda} (N^\bullet)^\Delta$ is multiplication by $x$, hence by passing to cohomology, multiplication by $x$ kills $H^j(\Delta, M)$ for $j > 0$.

such that $X_{n+r} \to X_n$ is the zero map. The class of ML-zero inverse systems is a Serre subcategory [16, 1.12]. A morphism $(X_n) \to (Y_n)$ is said to be an *ML-isomorphism* if its kernel and cokernel are ML-zero, and if this is so, the induced maps

$$\varprojlim X_n \to \varprojlim Y_n, \quad R^1 \varprojlim X_n \to R^1 \varprojlim Y_n \tag{2.2.11}$$

are isomorphisms. This implies that if $(f_n)\colon (X_n^\bullet) \to (Y_n^\bullet)$ is a morphism of inverse systems of complexes with surjective transition maps $X_{n+1}^i \to X_n^i$, $Y_{n+1}^i \to Y_n^i$, then if $(H^*(f_n))\colon (H^*(X_n^\bullet)) \to (H^*(Y_n^\bullet))$ is an ML-isomorphism, the map $\varprojlim(f_n)\colon \varprojlim X_n^\bullet \to \varprojlim Y_n^\bullet$ is a quasi-isomorphism. (Consider the induced map between the exact sequences [16, (2.1)] for $X_n^\bullet$ and $Y_n^\bullet$.)
From the exact sequence of cohomology of (2.2.9) and the lemma, we deduce that:

- for all $j > 0$, the inverse system $(H^j(\Delta, \mathbf{E}_L/\pi^n \mathbf{E}_L^+ \otimes V))_n$ is ML-zero;
- the map of inverse systems

$$(D_{1,n})_n \to (H^0(\Delta, \mathbf{E}_L/\pi^n \mathbf{E}_L^+ \otimes V))_n$$

  is an ML-isomorphism.

We now have a spectral sequence of inverse systems of abelian groups $({}_nE_2^{ij})_n \Rightarrow ({}_nE_\infty^{i+j})_n$ with

$${}_nE_2^{ij} = H_\phi^i(\Gamma_K, H^j(\Delta, \mathbf{E}_L/\pi^n \mathbf{E}_L^+ \otimes V))$$
$${}_nE_\infty^k = H_\phi^k(\mathrm{Gal}(L_\infty/K), \mathbf{E}_L/\pi^n \mathbf{E}_L^+ \otimes V).$$

such that, for all $i \geq 0$ and $j > 0$, the inverse system $({}_nE_2^{ij})_n$ are ML-zero. Therefore the edge homomorphism

$$({}_nE_2^{i0})_n \to ({}_nE_\infty^i)_n$$

is an ML-isomorphism. Moreover for all $i \geq 0$ the map of inverse systems

$$(H_\phi^i(\Gamma_K, D_{m,n}))_n \to ({}_nE_2^{i0})_n$$

is an ML-isomorphism, so composing with the edge homomorphism gives an ML-isomorphism

$$(H_\phi^i(\Gamma_K, D_{1,n}))_n \to (H_\phi^i(\mathrm{Gal}(L_\infty/K), \mathbf{E}_L/\pi^n \mathbf{E}_L^+ \otimes V))_n.$$

Hence $\varprojlim(\beta_n)_n$ is a quasi-isomorphism. □

2.3 Kummer towers

Let $F$ be any local field of characteristic 0, with perfect residue field. Set $\varpi = \varpi_F$, $k = k_F$, $\mathfrak{o} = \mathfrak{o}_F$. (Later in this section we will require further that $F$ is absolutely unramified.)

Let $K \supset F$ be any $d$-big local field such that $\mathfrak{o}_K/\mathfrak{o}_F$ is formally smooth (i.e., $\varpi$ is a uniformiser of $K$). Let $\{t_\alpha \mid 1 \le \alpha \le d\} \subset \mathfrak{o}_K^*$ be a set of units whose reductions $\{\bar{t}_\alpha\} \subset k_K$ form a $p$-basis for $k_K$.

Fix an algebraic closure $\overline{K}$ of $K$. Let $(\varepsilon_n)_{n\ge 0}$ be a compatible system of primitive $p^n$-th roots of unity in $\overline{K}$, and for each $\alpha$ let $(t_{\alpha,n})_{n\ge 0}$ be a compatible system of $p^n$-th roots of $t_\alpha$.

Set $F_n = F(\varepsilon_n)$, $\mathfrak{o}_n = \mathfrak{o}_{F_n}$, $k_n = k_{F_n}$, $K'_n = K(t_{1,n}, \dots, t_{d,n})$ and $K_n = K'_n(\varepsilon_n)$. The tower $\{F_n\}$ is strictly deeply ramified; choose $n_0 \ge 0$, $\xi \in K_{n_0}$ with $0 < v_p(\xi) \le 1$, and uniformisers $\varpi_n \in \mathfrak{o}_n$ such that $\varpi_{n+1}^p \equiv \varpi_n^p \pmod{\xi}$ for all $n \ge n_0$. Let $X_{\mathcal{F}}$ be the field of norms of $\{F_n\}$ and $k_{\mathcal{F}} = \varprojlim(k_n, f)$ its residue field. Put $\bar{\pi} = \Pi_{\mathcal{F}}$, so that $X_{\mathcal{F}} \simeq k_{\mathcal{F}}[[\bar{\pi}]]$, and the isomorphism is canonical once the uniformisers $\varpi_n$ are fixed (since $k_{\mathcal{F}}$ is perfect). Write $^{-}$ for reduction mod $\xi$.

We have $\mathfrak{o}_{K'_n} = \mathfrak{o}_K[t_{1,n}, \dots, t_{d,n}]$ since this ring is a DVR, and so $\varpi_n$ satisfies an Eisenstein polynomial over $K'_n$ as well as over $F$. Hence $\mathfrak{o}_{K_n} = \mathfrak{o}_{K'_n}[\varpi_n] = \mathfrak{o}_n \otimes_{\mathfrak{o}} \mathfrak{o}_K[\{t_{\alpha,n}\}]$, and so

$$\mathfrak{o}_{K_n}/\xi = \mathfrak{o}_n/\xi \otimes_k k_K[\bar{t}_{1,n}, \dots, \bar{t}_{d,n}] = \mathfrak{o}_n/\xi \otimes_k k_K^{1/p^n}$$

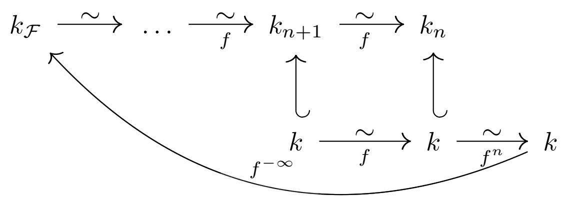
and we have a commutative diagram

$$\begin{array}{ccc}
\mathfrak{o}_{n+1}/\xi \otimes_k k_K^{1/p^{n+1}} & \xrightarrow[\sim]{1\otimes f^{n+1}} & \mathfrak{o}_{n+1}/\xi \underset{f^{-n-1},\,k}{\otimes} k_K \\
\Big\downarrow f & & \Big\downarrow f\otimes 1 \\
\mathfrak{o}_n/\xi \otimes_k k_K^{1/p^n} & \xrightarrow[\sim]{1\otimes f^{n}} & \mathfrak{o}_n/\xi \underset{f^{-n},\,k}{\otimes} k_K \\
 & & \Big\downarrow \wr \\
 & & k_n[\varpi_n]/(\varpi_n^{rp^n}) \underset{f^{-n},\,k}{\otimes} k_K
\end{array}$$

Therefore

$$\mathbf{E}_K^+ = X_{\mathcal{K}}^+ = \varprojlim_{n\ge n_0} k_n[\varpi_n]/(\varpi_n^{rp^n}) \underset{f^{-n},\,k}{\otimes} k_K = k_{\mathcal{F}}[[\bar{\pi}]] \underset{f^{-\infty},\,k}{\widehat{\otimes}} k_K$$

where $f^{-\infty}\colon k \hookrightarrow k_{\mathcal{F}}$ is the homomorphism making the diagram




commute. In other words, if we view $k_{\mathcal{F}}$ as an extension of $k$ via the map $f^{-\infty}$ just defined, we have $\mathbf{E}_K^+ = k_{\mathcal{F}}[[\bar{\pi}]]\widehat{\otimes}_k k_K$.
Set $K'_\infty = \bigcup K'_n \subset K_\infty$. Define the various Galois groups

$$\begin{aligned}\Gamma_K &= \mathrm{Gal}(K_\infty/K) = \Gamma_F \ltimes \Delta_{K/F}\\ \Gamma_F &= \mathrm{Gal}(K_\infty/K'_\infty) = \mathrm{Gal}(F_\infty/F) \hookrightarrow \mathbb{Z}_p^*\\ \Delta_{K/F} &= \mathrm{Gal}(K_\infty/F_\infty) \simeq \mathbb{Z}_p^d\end{aligned}$$

acting on $K_\infty$ as follows: if $a \in \mathbb{Z}_p^*$ is the image of $\gamma_a \in \Gamma_F$ and $\underline{b} \in \mathbb{Z}_p^d$ the image of $\delta_{\underline{b}} \in \Delta_{K/F}$ then

$$\begin{aligned}\gamma_a\colon \varepsilon_n &\mapsto \varepsilon_n^a & \qquad \delta_{\underline{b}}\colon \varepsilon_n &\mapsto \varepsilon_n\\ t_{\alpha,n} &\mapsto t_{\alpha,n} & t_{\alpha,n} &\mapsto \varepsilon_n^{b_\alpha} t_{\alpha,n}.\end{aligned}$$

To be more precise we suppose from now on that $F/\mathbb{Q}_p$ is unramified, so that $\mathfrak{o}_n = \mathfrak{o}[\varepsilon_n]$, and we may choose $\varpi_n = \varepsilon_n - 1$. Then the projections $k_{\mathcal{F}} \to k$, $k_{\mathcal{K}} \to k_K$ are isomorphisms, and $\Gamma_K$ acts on $\mathbf{E}_K^+ = k_K[[\bar{\pi}]]$ as follows: for $a \in \mathbb{Z}_p^*$,

$$\gamma_a\colon \Pi \mapsto (1+\Pi)^a - 1, \quad \gamma_a = \text{ identity on } k_c K$$

and for $\underline{b} \in \mathbb{Z}_p^d$, $\delta_{\underline{b}}$ is the unique automorphism of $\mathbf{E}_K^+$ whose reduction mod $(\bar{\pi})$ is the identity, and which satisfies

$$\delta_{\underline{b}}\colon \bar{\pi} \mapsto \bar{\pi}, \quad \bar{t}_\alpha \mapsto (1+\bar{\pi})^{b_\alpha}\bar{t}_\alpha.$$

Such a unique automorphism exists since $k_K$ is formally étale over $\mathbb{F}_p(\bar{t}_1, \dots, \bar{t}_d)$.
To lift to characteristic 0, set $\mathbf{A}_K^+ = \mathfrak{o}_K[[\pi]]$, with the obvious surjection to $\mathbf{E}_K^+ = k_K[[\bar{\pi}]]$. The lifting $\phi$ of Frobenius is given as follows: on $\mathfrak{o}_K$ it is the unique lifting of Frobenius for which $\phi(t_i) = t_i^p$; and $\phi(\pi) = (1+\pi)^p - 1$. It is then immediate that the conditions (2.1.1), (2.1.2) hold, and the action of $\Gamma_K$ on $\mathbf{A}_K^+$ satisfies

$$\begin{aligned}\gamma_a\colon \pi &\mapsto (1+\pi)^a - 1 & \qquad \delta_{\underline{b}}\colon \pi &\mapsto \pi\\ \gamma_a &= \text{identity on } \mathfrak{o}_K & t_\alpha &\mapsto (1+\pi)^{b_\alpha} t_\alpha.\end{aligned}$$

*Remark* 2.3.1. There is a natural generalisation of this construction for a Lubin-Tate formal group $G$ over $\mathfrak{o}_F$ associated to a distinguished polynomial $g \in \mathfrak{o}_F[X]$. One takes $F_\infty/F$ to be the Lubin-Tate extension generated by the division points of $G$, and $K'_n = K(\{t_{\alpha,n}\})$ where $g(t_{\alpha,n+1}) = t_{\alpha,n}$. Then $\mathbf{A}_K^+$ is the affine algebra of $G$ over $\mathfrak{o}_K$; the lifting of Frobenius is given by $g$. For some details when $d = 0$, and indications of what does and what does not extend, see Lionel Fourquaux's Ph.D. thesis [14, §1.4.1].

## References


[1] A. Abbès, T. Saito: *Ramification of local fields with imperfect residue fields.* American J. of Math. 124 (2002), 879–920

[2] — , — : *Ramification of local fields with imperfect residue fields II.* In: Kazuya Kato's fiftieth birthday, Doc. Math. 2003, Extra Vol., 5–72

[3] V. Abrashkin: *An analogue of the field-of-norms functor and the Grothendieck conjecture.* `arXiv:math.NT/0503200`

[4] F. Andreatta: *Generalized ring of norms and generalised $(\phi, \Gamma)$-modules.* To appear in Ann. Sci. Éc. Norm. Sup.

[5] F. Andreatta, A. Iovita: *Cohomology of generalised $(\phi, \Gamma)$-modules.* Preprint.

[6] J. Ax: *Zeros of Polynomials over Local Fields—The Galois Action.* J. of Algebra 15 (1970), 417–428

[7] N. Bourbaki: *Algèbre Commutative, Chapitres 8 et 9.* Masson, Paris, 1983

[8] F. Cherbonnier, P. Colmez: *Représentations p-adiques surconvergentes.* Invent. math. 133 (1998), 581–611

[9] J. Coates, R. Greenberg: *Kummer theory for abelian varieties over local fields.* Invent. math. 124 (1996), 129–174

[10] G. Faltings: *p-adic Hodge theory.* J. Amer. Math. Soc. 1 (1988), 255–299

[11] J.-M. Fontaine: *Représentation p-adiques des corps locaux.* In: The Grothendieck Festschrift Vol.II, (P. Cartier *et al.*, ed.), 249–309. Birkhäuser, 1990

[12] J.-M. Fontaine, J. P. Wintenberger: *Le ≪corps des normes≫ de certaines extensions algébriques de corps locaux.* CRAS 288 série A (1979), 367–370

[13] — , — : *Extensions algébriques et corps des normes des extensions APF des corps locaux.* CRAS 288 série A (1979), 441–444

[14] L. Fourquaux: Ph.D. thesis, Université Paris 6, December 2005.

[15] L. Herr: *Sur la cohomologie galoisienne des corps p-adiques*. Bull. Soc. Math. France 126 (1998), 563–600

[16] U. Jannsen: *Continuous étale cohomology*. Math. Ann. 280 (1988), 207–245

[17] K. Kato: *A generalization of local class field theory by using $K$-groups II*. J. Fac. Sci. Univ. Tokyo 27 (1980), 603–683

[18] H. Matsumura: *Commutative Algebra*. 2nd edition, Benjamin/Cummings, 1980

[19] J. Nekovář: *Selmer complexes*. Preprint (March 2006), available at `http://www.math.jussieu.fr/~nekovar/pu/`

[20] A. J. Scholl: *An introduction to Kato's Euler systems*. Galois Representations in Arithmetic Algebraic Geometry (A. J. Scholl & R. L. Taylor, ed.), 379–460. Cambridge University Press, 1998

[21] J-P. Serre: *Corps locaux*. Hermann, 1968

[22] J. Tate: *p-divisible groups*. Proceedings of a conference on local fields, Driebergen (T. A. Springer, ed.), 158–183, Springer-Verlag (1967)

[23] J. P. Wintenberger: *Le corps des normes de certaines extensions infinies de corps locaux; applications*. Annales scientifiques de l'É.N.S 4e série 16 no1 (1983), 59–89

[24] I. B. Zhukov: *On ramification theory in the case of an imperfect residue field*. Mat. Sb. 194 (2003), no. 12, 3–30; translation in Sb. Math. 194 (2003), no. 11-12, 1747–1774

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